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Research article

Some fixed point results for nonlinear contractive conditions in ordered proximity spaces with an application

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Abstract: In this article, we use the concept of proximity spaces to prove common fixed point results for mappings satisfying generalized (ψ,β)-Geraghty contraction type mapping in partially ordered proximity spaces. Finally, we investigate an application to endorse our results.

Keywords: fixed point; common fixed point; (ψ,β)-Geraghty contraction; proximity spaces
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Frigyes Riesz [1] first discovered the fundamental ideas of proximity spaces in 1908. Later, in 1934, Efremovich [2] resurrected and axiomatized this theory, which was later printed in 1951. Numerous studies on proximity spaces have been conducted over the years [3–6]. Smirnov [5] explained the relationship between proximities and uniformities, as well as the relationship between the proximity relation and topological spaces.

Inspired by [7], Kostic [8] introduced fixed point theory and defined the concepts of w-distance and w_0-distance in proximity space. For more details on w-distance, see [9]. Naimpally et al. and Sharma [10, 11] have also done studies on proximity spaces and their examples. Qasim et al. [12] give the theorems of Matkowski and Boyd-Wong in proximity spaces.

Geraghty [13] established a class of functions in 1973, which he designated as the set of functions. Khan et al. [14] introduced the idea of an altering distance function. Alsamir et al. [15] gave some common fixed point teorems in partially ordered metric-like spaces.

Moreover an important development is reported in fixed point theory via some applications. Hammad et al. [16–18] utilized some fixed point techniques to solve differential and integral equations.

In this study, we give some common fixed point results via generalized (ψ,β)-Geraghty contraction mappings in proximity spaces and an application of the existence of a unique solution of an integral equation.

2. Preliminaries

In this section, we will include the basic definitions and theorems that will be necessary in the following parts of our work.

Definition 2.1. [5] Suppose Ω is a set and δ is a relation on the set 2^{Ω}. If the following hold, the pair (Ω, δ) is considered to be in a proximity space: for any $A, B, C \in 2^{\Omega}$, where 2^{Ω} is the power set of Ω.

(p_1) $A\delta B \Rightarrow B\delta A$,

(p_2) $A\delta B \Rightarrow A, B \neq \emptyset$,

(p_3) $A\delta(B \cup C) \Leftrightarrow A\delta B$ or $A\delta C$,

(p_4) $A \cap B \neq \emptyset \Rightarrow A\delta B$,

(p_5) For all $\gamma \subseteq \Omega$, $A\delta\gamma$ or $B\delta(\Omega - \gamma)$ implies $A\delta B$.

We shall write all $\xi \in \Omega$ and $A \subseteq \Omega$ as $\xi\delta A$ and $A\delta\xi$ rather than $\{\xi\}\delta A$ and $A\delta\{\xi\}$, respectively. If $\xi\delta\mu$ means that $\xi = \mu$ for every $\xi, \mu \in \Omega$, then the proximity space (Ω, δ) is said to be separated. Generalizations of uniform features are used to describe the characteristics of proximity spaces, metric and topological continuity qualities, respectively.

Any proximity relation on a non-empty set Ω induces a topology τ_δ through the Kuratowski closure operator. When applied to all $A \subseteq \Omega$, the Kuratowski closure operator can be described as $cl(A) = \{\xi \in \Omega : \xi\delta A\}$. The topology τ_δ in this situation is always completely regular and if (Ω, δ) is separated, it is Tychonoff.

If (Ω, τ) is a topological space and δ is a proximity on Ω such that $\tau_\delta = \tau$, it is said that τ and δ are compatible. Every completely regular topology on a nonempty set Ω, has a compatible proximity. Also, we obtain $\xi\delta\{\xi_n\}$ if a sequence $\{\xi_n\}$ converges to a point $\xi \in \Omega$ with regard to the induced topology τ_δ. Additionally, each uniform space $(\Omega, \mathcal{U})$ is associated with a proximity structure that is described by for all $A, B \subseteq \Omega$, $A\delta B$ if $(A \times B) \cap C \neq \emptyset$ for all $C \in \mathcal{U}$. See [10, 11] for more information.

Example 2.1. [12] Give us a metric space (Ω, p). Take into account the relation δ on 2^{Ω},

$$A\delta B \Leftrightarrow p(A, B) = 0 \text{ and } p(A, B) = \inf\{p(u, v) : u \in A, v \in B\}.$$

δ is thus a proximity on Ω. Additionally, the metric topologies τ_p and δ are compatible.

In order to get the proximity space version of the Banach fixed-point theorem, Kostic [8] defined the concepts of w-distance and w_0-distance, which were inspired by [7].

Definition 2.2. [8] Let $w : \Omega \times \Omega \to [0, \infty)$ be a function and (Ω, δ) be a proximity space. Then w is a w-distance on Ω, if the axiom below is true:

(w_1) if $w(\eta, A) = 0$ and $w(\eta, B) = 0$ imply $A\delta B$ for all $\eta \in \Omega$ and $A, B \subseteq \Omega$, when $w(\eta, A) = \inf\{w(\eta, \xi) : \xi \in A\}$.

Definition 2.3. [8] A w-distance on a proximity space (Ω, δ) is also referred to as a w_0-distance if the axioms below are true:

($\mathbf{w}_2$) For any $\xi, \mu, \eta \in \Omega$, $w(\xi, \mu) \leq w(\xi, \eta) + w(\eta, \mu)$,

($\mathbf{w}_3$) Since w is lower semicontinuous in both variables with regard to τ_δ, we get

$$w(\xi, \mu) \leq \liminf_{\xi' \to \xi} w(\xi', \mu) = \sup_{B \in \mathcal{U}_\xi} \inf_{\xi' \in B} w(\xi', \mu),$$

and

$$w(\mu, \xi) \leq \liminf_{\xi' \to \xi} w(\mu, \xi') = \sup_{B \in \mathcal{U}_\xi} \inf_{\xi' \in B} w(\mu, \xi'),$$

where $\mathcal{U}_\xi$ is a base of neighborhoods of the point $\xi \in \Omega$.

Remark 2.1. [12] It is evident that for every sequence $\{\xi_n\}$ convergent to ξ with respect to τ_δ, $w(\xi, \mu) \leq \liminf_{n \to \infty} w(\xi_n, \mu)$ and $w(\mu, \xi) \leq \liminf_{n \to \infty} w(\mu, \xi_n)$ exist. This is true if w is lower semicontinuous in both variables with respect to τ_δ.

Example 2.2. [12] Let $\Omega = \mathbb{R}$ possess the usual metric as well as the proximity δ specified in Example 2.1. Definition of w_1, w_2:$\Omega \times \Omega \to [0, \infty)$ by

$$w_1(\xi, \mu) = \max\{|\xi|, |\mu|\} \text{ and } w_2(\xi, \mu) = \frac{|\xi| + |\mu|}{2},$$

both w_1 and w_2 are w_0-distance on Ω.

Lemma 2.1. *[7, 8] Let (Ω, δ) be a space of proximity with w-distance w. The following properties are then true:*

(i) *If (Ω, δ) is separated, then $w(\eta, \xi) = 0$ and $w(\eta, \mu) = 0$ imply $\xi = \mu$,*

(ii) *If $w(\eta, \xi) = 0$ and $w(\eta, \xi_n) \to 0$ as $n \to \infty$, then $\{\xi_n\}$ subsequently converges to ξ with respect to τ_δ.*

Definition 2.4. [15] Let $(\Omega, \preceq)$ be a partially ordered set and $g, h : \Omega \to \Omega$ be two mappings. Then

(i) The elements $\xi, \mu \in \Omega$ are called comparable if $\xi \preceq \mu$ or $\mu \preceq \xi$ holds,

(ii) g is called nondecreasing i.e., if $\xi \preceq \mu$ implies $g\xi \preceq g\mu$,

(iii) The pair (g, h) is weakly increasing if $g\xi \preceq hg\xi$ and $h\xi \preceq gh\xi$ for all $\xi \in \Omega$,

(iv) The mapping g is weakly increasing if the pair (g, I) is weakly increasing, where I is denoted to the identity mapping on Ω.

Definition 2.5. [15] Let $(\Omega, \preceq)$ be a partially ordered set. Ω is called regular, if whenever $\{\eta_n\}$ is a nondecreasing sequence in Ω w.r.t. $\preceq$ such that $\eta_n \to \eta$, then $\eta_n \preceq \eta$ for $\forall n \in \mathbb{N}$.

Definition 2.6. [13] If $\{x_n\}$ is a sequence in $[0, \infty)$ with $\alpha(x_n) \to 1$, then $x_n \to 0$. The set of functions $\alpha : [0, \infty) \to [0, 1)$ which holds the condition is denoted with a class of functions Π.

Definition 2.7. [14] If the circumstances below are true;

(i) ψ is continuous and non-decreasing,

(ii) $\psi(x) = 0 \Leftrightarrow x = 0$,

afterward, the function $\psi : [0, \infty) \to [0, \infty)$ is referred to as an altering distance function.

3. Main results

The following lemma is introduced at the beginning of this section and will be used to prove our main results.

Lemma 3.1. *Let* (Ω, δ) *be a separated proximity space with* w_0-*distance* w *and* $\{\eta_n\}$ *be a sequence in* Ω *such that* $\lim_{n\to+\infty} w(\eta_n, \eta_{n+1}) = 0$. *If* $\lim_{n,m\to+\infty} w(\eta_n, \eta_m) \neq 0$, *then there exist* $\varepsilon > 0$ *and two sequences* $\{n_k\}$ *and* $\{m_k\}$ *of positive integers with* $n_k > m_k > k$ *such that following three sequences* $\{w(\xi_{2n_k}, \xi_{2m_k})\}$, $\{w(\xi_{2n_k-1}, \xi_{2m_k})\}$, $\{w(\xi_{2n_k}, \xi_{2m_k+1})\}$ *converge to* r^+ *when* $k \to \infty$.

Proof. Let $\{\eta_n\} \subseteq \Omega$ be a sequence such that

$$\lim_{n\to+\infty} w(\eta_n, \eta_{n+1}) = 0 \text{ and } \lim_{n,m\to+\infty} w(\eta_n, \eta_m) \neq 0.$$

Then there exist $r > 0$ and two sequences $\{n_k\}$, $\{m_k\}$ of positive integers such that the lowest positive integer, n_k, for which $n_k > m_k > k$, $w(\eta_{2n_k}, \eta_{2m_k}) \geq r$. This means that $w(\eta_{2n_k-2}, \eta_{2m_k}) < r$. The triangular inequality implies that

$$\begin{aligned} r &\leq w(\eta_{2n_k}, \eta_{2m_k}) \\ &\leq w(\eta_{2n_k}, \eta_{2n_k-1}) + w(\eta_{2n_k-1}, \eta_{2n_k-2}) + w(\eta_{2n_k-2}, \eta_{2m_k}) \\ &< w(\eta_{2n_k}, \eta_{2n_k-1}) + w(\eta_{2n_k-1}, \eta_{2n_k-2}) + r. \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, implies that

$$\lim_{n\to+\infty} w(\xi_{2n_k}, \xi_{2m_k}) = r^+.$$

Once more, we may determine that

$$\left| w(\xi_{2n_k}, \xi_{2m_k+1}) - w(\xi_{2n_k}, \xi_{2m_k}) \right| \leq w(\xi_{2m_k}, \xi_{2m_k+1})$$

from the triangular inequality. In the inequality above, if we let $k \to \infty$ go, we get

$$\lim_{k\to+\infty} w(\xi_{2n_k}, \xi_{2m_k+1}) = r^+.$$

Similarly, one can easily show that

$$\lim_{k\to+\infty} w(\xi_{2n_k-1}, \xi_{2m_k}) = r^+.$$

□

Definition 3.1. Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g, h : \Omega \to \Omega$ be two mappings. If $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ exists with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that

$$\psi(w(g\xi, h\mu)) \leq \alpha(K_{\xi,\mu})\beta(K_{\xi,\mu}), \tag{3.1}$$

holds for all comparable elements $\xi, \mu \in \Omega$, where

$$K_{\xi,\mu} = \max\{w(\xi, \mu), w(g\xi, \xi), w(\mu, h\mu)\},$$

we may then state that the pair (g, h) is of the generalized (ψ,β)-Geraghty contraction type.

Theorem 3.2. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g, h : \Omega \to \Omega$ be two mappings that meet the requirements listed below:*

(i) *The pair (g, h) is weakly increasing,*

(ii) *The pair (g, h) is generalized (ψ,β)-Geraghty contraction type,*

(iii) *Either g or h is continuous,*

(iv) *For all $\eta \in \Omega$, any iterative sequences $\{g^n\eta\}$ and $\{h^n\eta\}$ have convergent subsequences with respect to τ_δ.*

Then g and h have a common fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if η, $u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the common fixed point of g and h is unique.

Proof. Let $\xi_0 \in \Omega$, $\xi_1 = g\xi_0$ and $\xi_2 = h\xi_1$. By continuing in this manner, we create a sequence $\{\xi_n\} \subseteq \Omega$ defined by $\xi_{2n+1} = g\xi_{2n}$ and $\xi_{2n+2} = h\xi_{2n+1}$. Since the pair (g, h) is weakly increasing

$$\xi_1 = g\xi_0 \preceq hg\xi_0 = \xi_2 = g\xi_1 \preceq ... \preceq hg\xi_{2n} = \xi_{2n+2} \preceq ....$$

Thus $\xi_n \preceq \xi_{n+1}$ for all $n \in \mathbb{N}$. If there exists some $l \in \mathbb{N}$ such that $w(\xi_{2n_l}, \xi_{2n_l+1}) = 0$. Hence $\xi_{2l} = \xi_{2l+1}$ and $g\xi_{2l} = \xi_{2l}$. To show that $h\xi_{2l} = \xi_{2l}$ it is enough to show that $\xi_{2l} = \xi_{2l+1} = \xi_{2l+2}$. Assume

$$w(\xi_{2l+1}, \xi_{2l+2}) \neq 0 \text{ and } w(\xi_{2l+2}, \xi_{2l+1}) \neq 0.$$

Since $\xi_{2l} \preceq \xi_{2l+1}$, then by (3.1) we have

$$\begin{aligned}
\psi(w(\xi_{2l+1}, \xi_{2l+2})) &= \psi(w(g\xi_{2l}, h\xi_{2l+1})) \\
&\leq \alpha(K_{\xi_{2l},\xi_{2l+1}})\beta(K_{\xi_{2l},\xi_{2l+1}}) \\
&= \alpha(\max\{w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l}, g\xi_{2l}), w(\xi_{2l+1}, h\xi_{2l+1})\}) \\
&\quad \beta(\max\{w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l}, g\xi_{2l}), w(\xi_{2l+1}, h\xi_{2l+1})\}\} \\
&= \alpha(\max\{w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l+1}, \xi_{2l+2})\}) \\
&\quad \beta(\max\{w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l}, \xi_{2l+1}), w(\xi_{2l+1}, \xi_{2l+2})\}) \\
&= \alpha(w(\xi_{2l+1}, \xi_{2l+2}))\beta(w(\xi_{2l+1}, \xi_{2l+2})) \\
&< \beta(w(\xi_{2l+1}, \xi_{2l+2})) \\
&\leq \psi(w(\xi_{2l+1}, \xi_{2l+2})),
\end{aligned}$$

which is a contradiction. So $w(\xi_{2l+1}, \xi_{2l+2}) = 0$ and similarly $w(\xi_{2l+2}, \xi_{2l+1}) = 0$. That is $\xi_{2l} = \xi_{2l+1} = \xi_{2l+2}$. Thus ξ_{2l} is a common fixed point for g and h. We now presume that

$$w(\xi_n, \xi_{n+1}) \neq 0 \text{ and } w(\xi_{n+1}, \xi_n) \neq 0$$

for all $n \in \mathbb{N}$. When n is even, $n = 2t$ follows some $t \in \mathbb{N}$

$$\begin{aligned}
\psi(w(\xi_n, \xi_{n+1})) &= \psi(w(\xi_{2t}, \xi_{2t+1})) \\
&= \psi(w(g\xi_{2t}, h\xi_{2t-1}))
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha(\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t-1},h\xi_{2t-1}), w(\xi_{2t},g\xi_{2t})\})\\
&\quad \beta(\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t-1},h\xi_{2t-1}), w(\xi_{2t},g\xi_{2t})\})\\
&= \alpha(\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t},\xi_{2t+1})\})\\
&\quad \beta(\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t},\xi_{2t+1})\})\\
&< \beta(\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t},\xi_{2t+1})\}). \qquad (3.2)
\end{aligned}
$$

Assume

$$\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t},\xi_{2t+1})\} = w(\xi_{2t},\xi_{2t+1}).$$

By (3.2), we get

$$\psi(w(\xi_{2t},\xi_{2t+1})) < \psi(w(\xi_{2t},\xi_{2t+1})),$$

which is a contradiction. Thus,

$$\max\{w(\xi_{2t-1},\xi_{2t}), w(\xi_{2t},\xi_{2t+1})\} = w(\xi_{2t-1},\xi_{2t}).$$

Therefore

$$\psi(w(\xi_{2n},\xi_{2n+1})) < \psi(w(\xi_{2n-1},\xi_{2n})). \qquad (3.3)$$

Because ψ is an altering distance function, we draw the conclusion that

$$w(\xi_{2n},\xi_{2n+1}) < w(\xi_{2n-1},\xi_{2n})$$

is true for every $n \in \mathbb{N}$. $n = 2t+1$ for some $t \in \mathbb{N}$ if n is odd. By (3.1) we have

$$
\begin{aligned}
\psi(w(\xi_n,\xi_{n+1})) &= \psi(w(\xi_{2t+1},\xi_{2t+2}))\\
&= \psi(w(g\xi_{2t},h\xi_{2t+1}))\\
&\leq \alpha(\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t},g\xi_{2t}), w(\xi_{2t+1},h\xi_{2t+1})\})\\
&\quad \beta(\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t},g\xi_{2t}), w(\xi_{2t+1},h\xi_{2t+1})\})\\
&= \alpha(\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t+1},\xi_{2t+2})\})\\
&\quad \beta(\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t+1},\xi_{2t+2})\})\\
&< \beta(\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t+1},\xi_{2t+2})\}). \qquad (3.4)
\end{aligned}
$$

Assume that

$$\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t+1},\xi_{2t+2})\} = w(\xi_{2t+1},\xi_{2t+2}).$$

By (3.4) we get

$$\psi(w(\xi_{2t+1},\xi_{2t+2})) < \psi(w(\xi_{2t+1},\xi_{2t+2})),$$

which is a contradiction. Then,

$$\max\{w(\xi_{2t},\xi_{2t+1}), w(\xi_{2t+1},\xi_{2t+2})\} = w(\xi_{2t},\xi_{2t+1}).$$

Thus,

$$\psi(w(\xi_n,\xi_{n+1})) < \psi(w(\xi_{n-1},\xi_n)).$$

We conclude that

$$w(\xi_{2n+1},\xi_{2n+2}) \leq w(\xi_{2n},\xi_{2n+1}) \qquad (3.5)$$

holds for all $n \in \mathbb{N}$ since ψ is an altering distance function. The result of combining (3.3) and (3.5) is that

$$w(\xi_n, \xi_{n+1}) \leq w(\xi_{n-1}, \xi_n)$$

holds for all $n \in \mathbb{N}$. The sequence $\{w(\xi_n, \xi_{n+1})\}$ is hence a decreasing sequence. Therefore, $\nu \geq 0$ exists such that $\lim_{n\to\infty} w(\xi_n, \xi_{n+1}) = \nu$ and the sequence $\{w(\xi_n, \xi_{n+1})\}$ is a decreasing sequence.

We now have proof that $\nu = 0$. Consider the contrary, which is $\nu > 0$. We get

$$\psi(w(\xi_n, \xi_{n+1})) \leq \alpha(w(\xi_{n-1}, \xi_n))\beta(w(\xi_{n-1}, \xi_n))$$

from (3.2) and (3.4). The inequality above indicates that $\psi(\nu) < \beta(\nu) \leq \psi(\nu)$ if the lim sup is taken. This is a contradiction. Therefore, $\nu = 0$. This implies that

$$w(\xi_n, \xi_{n+1}) \to 0 \text{ as } n \to \infty.$$

In a similar way, we can get

$$w(\xi_{n+1}, \xi_n) \to 0 \text{ as } n \to \infty.$$

We can now prove that

$$\lim_{n,m\to\infty} w(\xi_n, \xi_m) = 0 \text{ and } \lim_{n,m\to\infty} w(\xi_m, \xi_n) = 0.$$

Assume that

$$\lim_{n,m\to\infty} w(\xi_n, \xi_m) \neq 0.$$

Using Lemma 3.1, there exist $r > 0$, two sequences $\{\xi_{n_k}\}$ and $\{\xi_{m_k}\}$ of $\{\xi_n\}$ with $2n_k > 2m_k \geq k$ such that the three sequences $\{w(\xi_{2n_k}, \xi_{2m_k})\}$, $\{w(\xi_{2n_k-1}, \xi_{2m_k})\}$, $\{w(\xi_{2n_k}, \xi_{2m_k+1})\}$ converge to r^+ when $k \to \infty$. From (3.1) we have

$$\begin{aligned} \psi(w(\xi_{2n_k}, \xi_{2m_k+1})) &= \psi(w(g\xi_{2m_k}, h\xi_{2n_k-1})) \\ &\leq \alpha(K_{\xi_{2m_k}, \xi_{2n_k-1}})\beta(K_{\xi_{2m_k}, \xi_{2n_k-1}}), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} K_{\xi_{2m_k}, \xi_{2n_k-1}} &= \max\{w(\xi_{2n_k-1}, \xi_{2m_k}), w(\xi_{2n_k-1}, h\xi_{2n_k-1}), w(\xi_{2m_k}, g\xi_{2m_k})\} \\ &= \max\{w(\xi_{2n_k-1}, \xi_{2m_k}), w(\xi_{2n_k-1}, \xi_{2n_k}), w(\xi_{2m_k}, \xi_{2m_k+1})\}. \end{aligned}$$

Letting $k \to \infty$ in (3.6) and using the properties of ψ, α and β, we deduce that

$$\begin{aligned} \psi(r) &\leq \alpha(r)\beta(r) \\ &< \beta(r) \\ &\leq \psi(r), \end{aligned}$$

a contradiction. Therefore

$$\lim_{n,m\to\infty} w(\xi_n, \xi_m) = 0.$$

We can get $\lim_{n,m\to\infty} w(\xi_m, \xi_n) = 0$ in a similar way. According to (iv), if the sequence $\{\xi_n\}$ has a subsequence $\{\xi_{n_l}\}$ that is convergent with regard to τ_δ to some $\eta \in \Omega$, then we get

$$w(\eta, \xi_{n_l}) \leq \lim_{k\to\infty} \inf w(\xi_{n_k}, \xi_{n_l}) = 0 \tag{3.7}$$

and symmetrically, we obtain

$$w(\xi_{n_l}, \eta) \leq \lim_{k\to\infty} \inf w(\xi_{n_l}, \xi_{n_k}) = 0.$$

Since g or h is continuous, we get

$$\lim_{n\to\infty} w(\xi_{n+1}, h\eta) = \lim_{n\to\infty} w(g\xi_n, h\eta) = w(g\eta, h\eta), \tag{3.8}$$

$$\lim_{n\to\infty} w(g\eta, \xi_{n+1}) = \lim_{n\to\infty} w(g\eta, h\xi_n) = w(g\eta, h\eta). \tag{3.9}$$

Thus,

$$\lim_{n\to\infty} w(\xi_{n+1}, h\eta) = w(\eta, h\eta), \tag{3.10}$$

$$\lim_{n\to\infty} w(g\eta, \xi_{n+1}) = w(g\eta, \eta). \tag{3.11}$$

Combining (3.8) and (3.10), we conclude that $w(\eta, h\eta) = w(g\eta, h\eta)$. Also, by (3.9) and (3.10) we deduce that $w(g\eta, \eta) = w(g\eta, h\eta)$. So,

$$w(\eta, h\eta) = w(g\eta, \eta) = w(g\eta, h\eta). \tag{3.12}$$

We now demonstrate how $w(g\eta, \eta) = 0$ and $w(\eta, h\eta) = 0$. Suppose that, on the contrary, is $w(\eta, h\eta) > 0$ and $w(g\eta, \eta) > 0$. Thus, we get

$$\begin{aligned} \psi(w(\eta, h\eta)) &= \psi(w(g\eta, h\eta)) \\ &\leq \alpha(K_{\eta,\eta})\beta(K_{\eta,\eta}) \\ &< \beta(K_{\eta,\eta}) \\ &\leq \psi(K_{\eta,\eta}), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \psi(w(g\eta, \eta)) &= \psi(w(g\eta, h\eta)) \\ &\leq \alpha(K_{\eta,\eta})\beta(K_{\eta,\eta}) \\ &< \beta(K_{\eta,\eta}) \\ &\leq \psi(K_{\eta,\eta}), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} K_{\eta,\eta} &= \max\{w(\eta, \eta), w(g\eta, \eta), w(\eta, h\eta)\} \\ &= \max\{w(g\eta, \eta), w(\eta, h\eta)\}, \end{aligned}$$

which is a contradiction. Thus we obtain $w(g\eta, \eta) = 0$ and $w(\eta, h\eta) = 0$. Hence $g\eta = \eta$, $\eta = h\eta$. So, η is a common fixed point of g, h.

We assume that u is yet another fixed point of g and h in order to demonstrate the uniqueness of the common fixed point.

We now show that $w(u,u) = 0$. On the contrary, suppose that is $w(u,u) > 0$.

$$\begin{aligned}\psi(w(u,u)) &= \psi(w(gu,hu)) \\ &\leq \alpha(w(u,u))\beta(w(u,u)) \\ &< \beta(w(u,u)) \\ &\leq \psi(w(u,u))\end{aligned}$$

is a contradiction because of $u \preceq u$. Hence $w(u,u) = 0$.

So get the conclusion that η and u are comparable on the additional requirements on Ω.

We suppose that $w(\eta,u) \neq 0$.

$$\begin{aligned}\psi(w(\eta,u)) &= \psi(w(g\eta,hu)) \\ &\leq \alpha(w(\eta,u))\beta(w(\eta,u)) \\ &< \beta(w(\eta,u)) \\ &\leq \psi(w(\eta,u)),\end{aligned}$$

and

$$\begin{aligned}\psi(w(u,\eta)) &= \psi(w(gu,h\eta)) \\ &\leq \alpha(w(u,\eta))\beta(w(u,\eta)) \\ &< \beta(w(u,\eta)) \\ &\leq \psi(w(u,\eta)),\end{aligned}$$

this is a contradiction. Thus $w(\eta,u) = 0$ and $w(u,\eta) = 0$. Hence $u = \eta$. Thus g and h have a unique common fixed point. □

Theorem 3.3. *Let $(\Omega,\preceq)$ be a partially ordered set, (Ω,δ) be a separated proximity space with w_0−distance w and $g,h:\Omega\to\Omega$ be two mappings that meet the requirements listed below:*

(i) *The pair (g,h) is weakly increasing,*

(ii) *The pair (g,h) is generalized (ψ,β)-Geraghty contraction type,*

(iii) *Ω is regular,*

(iv) *For all $\eta\in\Omega$, any iterative sequences $\{g^n\eta\}$ and $\{h^n\eta\}$ have convergent subsequences with respect to τ_δ.*

Then g and h have a common fixed point $\nu\in\Omega$ with $w(\nu,\nu) = 0$. Furthermore, assume that if η, $u\in\Omega$ such $w(\eta,\eta) = w(u,u) = 0$ implies that if η and u are comparable, then the common fixed point of g and h is unique.

Proof. After proving *Theorem* 3.2 we create a a sequence $\{\xi_n\} \subseteq \Omega$ such that

$$\xi_n \to \nu \in \Omega \text{ with } w(\nu, \nu) = 0.$$

Since Ω is regular, $\xi_n \preceq \nu$ for all $n \in \mathbb{N}$.

Therefore, the elements ξ_n and ν are comparable for any $n \in \mathbb{N}$.

We now show that $w(\nu, h\nu) = 0$.

Suppose to the contrary, that is

$$w(\nu, h\nu) > 0.$$

By (3.1), we have

$$\begin{aligned}
\psi(w(\xi_{2n+1}, h\nu)) &= \psi(w(g\xi_{2n}, h\nu)) \\
&\leq \alpha(\max\{w(\xi_{2n}, \nu), w(\xi_{2n}, g\xi_{2n}), w(\nu, h\nu)\}) \\
&\quad \beta(\max\{w(\xi_{2n}, \nu), w(\xi_{2n}, g\xi_{2n}), w(\nu, h\nu)\}) \\
&= \alpha(\max\{w(\xi_{2n}, \nu), w(\xi_{2n}, \xi_{2n+1}), w(\nu, h\nu)\}) \\
&\quad \beta(\max\{w(\xi_{2n}, \nu), w(\xi_{2n}, \xi_{2n+1}), w(\nu, h\nu)\}).
\end{aligned}$$

Letting $n \to \infty$ in above inequalities, as a result, we say

$$\psi(w(\nu, h\nu)) \leq \alpha(w(\nu, h\nu))\beta(w(\nu, h\nu)).$$

Utilizing the properties of ψ, α and β,

$$\psi(w(\nu, h\nu)) < \psi(w(\nu, h\nu)),$$

a contradiction.

Thus, $w(\nu, h\nu) = 0$ that is ν is a fixed point of h.

We can prove that is ν is a fixed point of g by using arguments similar to those used above. Similar arguments to those used in the proof of *Theorem* 3.2 are used to establish the uniqueness of the common fixed point of g and h. □

Corollary 3.1. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g : \Omega \to \Omega$ be a mapping that meet the requirements listed below:*

(i) *There exist $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\begin{aligned}
\psi(w(g\mu, g\xi)) &\leq \alpha(\max\{w(\mu, \xi), w(\xi, g\xi), w(g\mu, \mu)\}) \\
&\quad \beta(\max\{w(\mu, \xi), w(\xi, g\xi), w(g\mu, \mu)\}),
\end{aligned}$$

holds for all comparable elements $\mu, \xi \in \Omega$,

(ii) *$g\xi \preceq g(g\xi)$ for all $\xi \in \Omega$,*

(iii) *g is continuous,*

(iv) *For all $\eta \in \Omega$, any iterative sequence $\{g^n \eta\}$ has convergent subsequences with respect to τ_δ.*

Then g has a fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if $\eta, u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the fixed point of g is unique.

Proof. Theorem 3.2 implies that by inserting $h = g$. □

Corollary 3.2. *Let we take that*

(iii) Ω *is regular,*

Instead of (iii) in Corollary 3.1, again we can have the same result.

Proof. Theorem 3.3 implies that by inserting $h = g$. □

Corollary 3.3. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g, h : \Omega \to \Omega$ be two mappings that meet the requirements listed below:*

(i) *There exist $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\psi(w(g\mu, h\xi)) \leq \alpha(w(\mu, \xi))\beta(w(\mu, \xi))$$

holds for all comparable elements $\mu, \xi \in \Omega$,

(ii) *The pair (g, h) is weakly increasing,*

(iii) *Either g or h is continuous,*

(iv) *For all $\eta \in \Omega$, any iterative sequences $\{g^n \eta\}$ and $\{h^n \eta\}$ have convergent subsequences with respect to τ_δ.*

Then g and h have a common fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if η, $u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the common fixed point of g and h is unique.

Corollary 3.4. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g, h : \Omega \to \Omega$ be two mappings that meet the requirements listed below:*

(i) *There exist $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\psi(w(g\mu, h\xi)) \leq \alpha(w(\mu, \xi))\beta(w(\mu, \xi))$$

holds for all comparable elements $\mu, \xi \in \Omega$,

(ii) *The pair (g, h) is weakly increasing,*

(iii) Ω *is regular,*

(iv) *For all $\eta \in \Omega$, any iterative sequences $\{g^n \eta\}$ and $\{h^n \eta\}$ have convergent subsequences with respect to τ_δ.*

Then g and h have a common fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if η, $u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the common fixed point of g and h is unique.

Corollary 3.5. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g : \Omega \to \Omega$ be a mapping that meet the requirements listed below:*

(i) *There exist $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\psi(w(g\mu, g\xi)) \leq \alpha(w(\mu, \xi))\beta(w(\mu, \xi))$$

holds for all comparable elements $\mu, \xi \in \Omega$,

(ii) *$g\xi \preceq g(g\xi)$ for all $\xi \in \Omega$,*

(iii) *g is continuous,*

(iv) *For all $\eta \in \Omega$, any iterative sequences $\{g^n\eta\}$ has convergent subsequences with respect to τ_δ.*

Then g has a fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if η, $u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the fixed point of g is unique.

Corollary 3.6. *Let $(\Omega, \preceq)$ be a partially ordered set, (Ω, δ) be a separated proximity space with w_0-distance w and $g : \Omega \to \Omega$ be a mapping that meet the requirements listed below:*

(i) *There exist $\alpha \in \Pi$, $\psi \in \Psi$ and a continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) \leq \psi(t)$ for all $t > 0$ such that*

$$\psi(w(g\mu, g\xi)) \leq \alpha(w(\mu, \xi))\beta(w(\mu, \xi))$$

holds for all comparable elements $\mu, \xi \in \Omega$,

(ii) *$g\xi \preceq g(g\xi)$ for all $\xi \in \Omega$,*

(iii) *Ω is regular,*

(iv) *For all $\eta \in \Omega$, any iterative sequence $\{g^n\eta\}$ has convergent subsequences with respect to τ_δ.*

Then g has a fixed point $\nu \in \Omega$ with $w(\nu, \nu) = 0$. Furthermore, assume that if η, $u \in \Omega$ such $w(\eta, \eta) = w(u, u) = 0$ implies that η and u are comparable then the fixed point of g is unique.

Example 3.1. Let $\Omega = \{0, 1, 2\}$ be equipped with the following partial order $\preceq$,

$$\preceq := \{(0, 0), (1, 1), (2, 2), (1, 0)\}.$$

Also, let Ω be endowed with the usual metric and the proximity δ on 2^Ω as

$$A\delta B \Leftrightarrow p(A, B) = 0, \text{where } p(A, B) = \min\{p(u, v) : u \in A, v \in B\}.$$

Define $w : \Omega \times \Omega \to [0, \infty)$ by

$$w(\xi, \mu) = \max\{|\xi|, |\mu|\},$$

$w(0, 0) = 0, w(1, 1) = 1, w(2, 2) = 2, w(0, 1) = w(1, 0) = 1, w(0, 2) = w(2, 0) = 2, w(1, 2) = w(2, 1) = 2.$

It is easy to see that (Ω, δ) be a separated proximity space with w_0 distance w.

Also define $g, h : \Omega \to \Omega$ with $g(0) = 0$, $g(1) = 0$, $g(2) = 1$ and $h(0) = 0$, $h(1) = 1$, $h(2) = 0$. It is simple to observe that g and h are continuous and that the pair (g, h) is weakly increasing with respect to $\preceq$.

Define $\alpha(t) = e^{\frac{-t}{16}}$, $\beta(t) = \frac{10}{11e}t$, $\psi(t) = \frac{1}{e}t$ if $t > 0$ and $\alpha(0) = 0$.

We next verify that the functions (g, h) satisfies the inequality

$$\psi(w(g\xi, h\mu)) \leq \alpha(K_{\xi,\mu})\beta(K_{\xi,\mu}).$$

For that, given $\xi, \mu \in \Omega$ with $\xi \preceq \mu$.

Then we have the following cases:

Case i $\xi = 0$ and $\mu = 0$. Then

$$\psi(w(g0, h0)) = \psi(w(0, 0)) = 0 \leq \alpha(K_{0,0})\beta(K_{0,0}).$$

Case ii $\xi = 1$ and $\mu = 1$. Then

$$\psi(w(g1, h1)) = \psi(w(0, 1)) = \psi(1) = \frac{1}{e}$$

and

$$K_{1,1} = \max\{w(1, 1), w(g1, 1), w(1, h1)\} = \max\{1, 1, 1\} = 1.$$

So,

$$\alpha(K_{1,1}) = \alpha(1) = e^{-\frac{1}{16}}$$

$$\beta(K_{1,1}) = \beta(1) = \frac{10}{11e}.$$

Hence

$$\psi(w(g1, h1)) \leq \alpha(K_{1,1})\beta(K_{1,1}).$$

Case iii $\xi = 2$ and $\mu = 2$. Then

$$\psi(w(g2, h2)) = \psi(w(1, 0)) = \psi(1) = \frac{1}{e}$$

and

$$K_{2,2} = \max\{w(2, 2), w(g2, 2), w(2, h2)\} = \max\{2, 2, 2\} = 2.$$

So,

$$\alpha(K_{2,2}) = \alpha(2) = e^{-\frac{2}{16}} = e^{-\frac{1}{8}}$$

$$\beta(K_{2,2}) = \beta(2) = \frac{20}{11e}.$$

Hence

$$\psi(w(g2, h2)) \leq \alpha(K_{2,2})\beta(K_{2,2}).$$

Case iv $1 \preceq 0$. Then we have two subcases:

Subcase i $\xi = 1$ and $\mu = 0$. Then

$$\psi(w(g1, h0)) = \psi(w(0, 0)) = 0$$

and

$$K_{1,0} = \max\{w(1, 0), w(g1, 0), w(1, h0)\} = \max\{1, 0, 1\} = 1.$$

So,

$$\alpha(K_{1,0}) = \alpha(1) = e^{-\frac{1}{16}}$$
$$\beta(K_{1,0}) = \beta(1) = \frac{10}{11e}.$$

Hence

$$\psi(w(g1, h0)) \leq \alpha(K_{1,0})\beta(K_{1,0}).$$

Subcase ii $\xi = 0$ and $\mu = 1$. Then

$$\psi(w(g0, h1)) = \psi(w(0, 1)) = \psi(1) = \frac{1}{e}$$

and

$$K_{0,1} = \max\{w(0, 1), w(g0, 1), w(0, h1)\} = \max\{1, 1, 1\} = 1.$$

So,

$$\alpha(K_{0,1}) = \alpha(1) = e^{-\frac{1}{16}}$$
$$\beta(K_{0,1}) = \beta(1) = \frac{10}{11e}.$$

Hence

$$\psi(w(g0, h1)) \leq \alpha(K_{0,1})\beta(K_{0,1}).$$

Therefore, requirements of *Theorem* 3.3 are all satisfied and so g and h have a common fixed point (0 is a common fixed point of g and h).

4. An application

Let (Ω, δ) be the proximity space, where $\Omega = C[0, 1]$ and δ is induced by the uniform metric $p_\infty(\xi, \mu) = \sup\{|\xi(t) - \mu(t)| : t \in [0, 1]\}$. In this case (Ω, δ) is separated proximity space. Consider the following w_0−distance w on Ω defined by

$$w(\xi, \mu) = \sup\{e^{-t} |\xi(t) - \mu(t)| : t \in [0, 1]\}.$$

Now, consider the integral equation

$$\xi(t) = G(t) + \int_0^1 S(t, s)F(s, \xi(s))ds, t \in [0, 1] \tag{4.1}$$

where $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $G : [0, 1] \to \mathbb{R}$, $S : [0, 1] \times [0, 1] \to [0, \infty)$. By utilizing the outcome from Corollary 3.1, the objective of this section is to provide an existence answer to (4.1). We give Ω the partial order "$\preceq$" provided by:

$$\xi \preceq \mu \Leftrightarrow \xi(t) \leq \mu(t)$$

for all $t \in [0, 1]$.

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(i) *There exists* $\alpha : [0,\infty) \to [0,1]$ *such that for all* $s \in [0,1]$ *and for all* $\xi, \mu \in \Omega$

$$0 \le |F(s,\xi(s)) - F(s,\mu(s))| \le \alpha(e^{-s}|\xi(s) - \mu(s)|)$$

and

$$\alpha(t_n) \to 1 \Rightarrow t_n \to 0,$$

(ii) $\int_0^1 S(t,s)ds \le |\xi(t) - \mu(t)|$.

Then the integral equation (4.1) *has a solution in* Ω.

Proof. Consider the mapping $g : \Omega \to \Omega$ defined by

$$g\xi(t) = \int_0^1 S(t,s)F(s,\xi(s))ds,$$

for all $\xi \in \Omega$ and $t \in [0,1]$. Then the (4.1) is equivalent to finding a fixed point of g.

Now, let $\xi, \mu \in \Omega$. We have:

$$\begin{aligned} |g\xi(t) - g\mu(t)| &= \left| \int_0^1 S(t,s)[F(s,\xi(s)) - F(s,\mu(s))]ds \right| \\ &\le \int_0^1 S(t,s)|F(s,\xi(s)) - F(s,\mu(s))|ds \\ &\le \int_0^1 S(t,s)\alpha(e^{-s}|\xi(s) - \mu(s)|)ds \\ &\le |\xi(t) - \mu(t)|\,\alpha(e^{-t}|\xi - \mu|) \\ &\le p_\infty(\xi,\mu)\alpha(e^{-t}|\xi - \mu|) \\ &\le p_\infty(\xi,\mu)\alpha(w(\xi,\mu)) \end{aligned}$$

and then we obtain

$$e^{-t}|g\xi(t) - g\mu(t)| \le w(\xi,\mu)\alpha(w(\xi,\mu))$$

i.e.,

$$w(g\xi, g\mu) \le w(\xi,\mu)\alpha(w(\xi,\mu))$$

for all $\xi, \mu \in \Omega$.

Now, let $\gamma \in C[0,1]$ be an arbitrary function. Define a sequence of functions $\{\xi_n\}$ as $g^n\xi = \xi_n$. Since $e^{-t}p_\infty(\xi,\mu) \le w(\xi,\mu) \le p_\infty(\xi,\mu)$ for all $\xi, \mu \in \Omega$, we have $p_\infty(\xi_n, \xi_m) \to 0$ as $m, n \to \infty$. That is the sequence $\{\xi_n\}$ is Cauchy and so has a convergent subsequence with respect to p_∞ since (Ω, p_∞) is complete. Consequently, there exists a unique $\xi \in \Omega$ which is a fixed point of the operator g, moreover $w(\xi,\xi) = 0$. Hence the integral equation (4.1) has a unique solution in Ω. □

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Conflict of interest

The author declares no conflict of interest.

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