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Research article

A conserved Caginalp phase-field system with two temperatures and a nonlinear coupling term based on heat conduction

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Abstract: Our aim in this paper is to study generalizations of the Caginalp phase-field system based on a thermomechanical theory involving two temperatures and a nonlinear coupling. In particular, we prove well-posedness results. More precisely, the existence and uniqueness of solutions, the existence of the global attractor and the existence of an exponential attractor.

Keywords: conserved Caginalp system; two temperatures; well-posedness; dissipativity; global attractor; exponential attractor
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1. Introduction

The Caginalp phase-field system

$$\partial_t u + \Delta^2 u - \Delta f(u) = -\Delta\theta, \tag{1.1}$$

$$\partial_t \alpha - \Delta\theta = -\partial_t u, \tag{1.2}$$

has been proposed in [1] to model phase transition phenomena, e.g. melting-solidification phenomena, in certain classes of materials. In this context, u is the order parameter and θ the relative temperature (relative to the equilibrium melting temperature), f is a given function (precisely, the derivative of a doublewell potential F). This system has been studied, e.g., [2, 6–11, 13, 15] and [17].

Equations (1.1) and (1.2) are based on the total free energy

$$\Psi(u,\theta) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2)\mathrm{d}x, \tag{1.3}$$

where Ω is the domain occupied by the material (we assume that it is a bounded and smooth domain of $\mathbb{R}^n$, $n = 2$ or 3 with boundary Γ).

We then introduce the enthalpy H defined by

$$H = -\partial_\theta \psi, \tag{1.4}$$

where ∂ denotes a variational derivative, so that

$$H = u + \theta. \tag{1.5}$$

The gouverning equations for u and θ are finally given by

$$\partial_t u = \Delta \partial_u \psi, \tag{1.6}$$

where ∂u stands for the variational derivative with respect to u, which yields (1.1). Then, we have the energy equation

$$\partial_t H = -\mathrm{div} q, \tag{1.7}$$

where q is the thermal flux vector. Assuming the classical Fourier law

$$q = -\nabla \theta, \tag{1.8}$$

we obtain (1.1) and (1.2).

Now, one drawback of the Fourier law is that it predicts that thermal signals propagate with an infinite speed, which violates causality (the so-called "paradox of heat conduction", see, e.g. [5]). Therefore, several modifications of (1.8) have been proposed in the literature to correct this unrealistic feature, leading to a second order in time equation for the temperature.

A different approach to heat conduction was proposed in the Sixties (see, [14, 16]), where it was observed that two temperatures are involved in the definition of the entropy: the conductive temperature θ, influencing the heat conduction contribution, and the thermodynamic temperature, appearing in the heat supply part. For time-independent models, it appears that these two temperatures coincide in absence of heat supply. Actually, they are generally different in time for example, [8] and references therein for more discussion on the subject. In particular, this happens for non-simple materials. In that case, the two temperatures are related as follows (see [4, 5]).

$$\theta = \alpha - \triangle \alpha, \tag{1.9}$$

Our aim in this paper is to study a generalization of the Caginalp phase-field system based on these two temperatures theory and the usual Fourier law with a nonlinear coupling. In particular, we obtain the existence and the uniqueness of the solutions and we prove the existence of the exponential attractors and, thus, of finite-dimensional global attractors.

2. Setting of the problem

We consider the following initial and boundary value problem:

$$\partial_t u + \Delta^2 u - \Delta f(u) = -\Delta g(u)(\alpha - \triangle \alpha), \tag{2.1}$$

$$\partial_t \alpha - \Delta \partial_t \alpha + \Delta^2 \alpha - \Delta \alpha = -g(u)\partial_t u, \tag{2.2}$$

$$u = \Delta u = \alpha = \Delta\alpha = 0 \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \tag{2.4}$$

where Γ is the boundary of the spatial domain Ω.

We make the following assumptions on nonlinearities f and g:

$$f \text{ is of class } C^2(\mathbb{R}), \quad f(0) = 0, \; g \in C^2(\mathbb{R}), g(0) = 0, \tag{2.5}$$

$$| G(s) |< c_1 F(s) + c_2, \quad c_0, c_1, c_2 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.6}$$

$$| g(s)s |< c_3(| G(s) |^2 + 1), \quad c_3 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.7}$$

$$c_4 s^{k+2} - c_5 \leq F(s) \leq f(s)s + c_0 \leq c_6 s^{k+2} - c_7, \quad c_4, c_6 > 0, \quad c_5, c_7 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.8}$$

$$| g(s) |< c_8(| s | + 1), \quad | g'(s) | \leq c_9 \quad c_8, c_9 \geqslant 0, \quad s \in \mathbb{R}, \tag{2.9}$$

$$| f'(s) | \leq c_{10}(| s |^k + 1), \quad c_{10} \geqslant 0, \quad s \in \mathbb{R}, \tag{2.10}$$

where k is an integer, $G(s) = \displaystyle\int_0^s g(\tau)d\tau$, and , $F(s) = \displaystyle\int_0^s f(\tau)d(\tau)$.

We denote by $\|.\|$ the usual L^2-norm (with associated scalar product $((.,.))$) and set $\|.\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denotes the norm in the Banach space X. Throughout this paper, the same letters c, c' and c'' denotes (generally positive) constants which may change from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may change from line to line, or even in a same line.

Remark 2.1. *In our case, to obtain equations (2.1) and (2.2), the total free energy reads in terms of the conductive temperature θ*

$$\psi(u, \theta) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u) - G(u)\theta - \frac{1}{2}\theta^2\Big)\mathrm{d}x, \tag{2.11}$$

where $f = F'$ and $g = G'$, and (1.6) yields, in view of (1.9), the evolution equation for the order parameter (2.1). Furthermore, the enthalpy now reads

$$H = G(u) + \theta = G(u) + \alpha - \triangle\alpha,$$

which yields thanks to (1.7), the energy equation,

$$\frac{\partial\alpha}{\partial t} - \triangle\frac{\partial\alpha}{\partial t} + divq = -g(u)\frac{\partial u}{\partial t}.$$

Considering the usual Fourier law ($q = -\nabla\theta$), we have (2.2).

We can note that we still have an infinite speed of propagation here.

3. A priori estimates

The estimates derived in this section are formal, but they can easily be justified within a Galerkin scheme. In what follows, the Poincaré, Hölder and Young inequalities are extensively used, Without further referring to them.

We rewrite (2.1) in the equivalent form:

$$(-\Delta)^{-1}\partial_t u - \Delta u + f(u) = g(u)(\alpha - \Delta\alpha). \tag{3.1}$$

We multiply (3.1) by $\partial_t u$ and integrate over Ω, we have

$$((-\Delta)^{-1}\partial_t u, \partial_t u) + (-\Delta u, \partial_t u) + (f(u), \partial_t u) = (g(u)(\alpha - \Delta\alpha), \partial_t u),$$

which gives

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + 2\int_\Omega F(u)dx) + \|\partial_t u\|_{-1}^2 = \int_\Omega g(u)\partial_t u(\alpha - \Delta\alpha)dx. \tag{3.2}$$

We multiply (2.2) by $(\alpha - \Delta\alpha)$ and integrate over Ω, we have

$$(\partial_t\alpha, \alpha - \Delta\alpha) + (\Delta^2\alpha, \alpha - \Delta\alpha) + (-\Delta\partial_t\alpha, \alpha - \Delta\alpha) + (-\Delta\alpha, \alpha - \Delta\alpha) = -(g(u)\partial_t u, (\alpha - \Delta\alpha)),$$

which gives,

$$\frac{1}{2}\frac{d}{dt}\|\alpha - \Delta\alpha\|^2 + \|\nabla\alpha\|^2 + 2\|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 = -\int_\Omega g(u)\partial_t u(\alpha - \Delta\alpha)dx \tag{3.3}$$

(note that $\|\alpha - \Delta\alpha\|^2 = \|\alpha\|^2 + 2\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2$).

Summing (3.2) and (3.3), we find

$$\frac{d}{dt}(\|\nabla u\|^2 + \|\alpha - \Delta\alpha\|^2 + 2\int_\Omega F(u)dx) + 2\|\nabla\alpha\|^2 + 4\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2 + 2\|\partial_t u\|_{-1}^2 = 0, \tag{3.4}$$

which yields,

$$\frac{dE_1}{dt} + c(\|\nabla\alpha\|^2 + \|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2 + \|\partial_t u\|_{-1}^2) \leq c. \tag{3.5}$$

where

$$E_1 = \|\nabla u\|^2 + \|\alpha - \Delta\alpha\|^2 + 2\int_\Omega F(u)dx, \tag{3.6}$$

Owing to (2.8), we obtain

$$\begin{aligned} &c(\|u\|_{H^1(\Omega)}^2 + \|\alpha\|_{H^2(\Omega)}^2 + \|u\|_{L^{k+2}(\Omega)}^{k+2}) - c' \leq E_1 \\ &\leq c''(\|u\|_{H^1(\Omega)}^2 + \|\alpha\|_{H^2(\Omega)}^2 + \|u\|_{L^{k+2}(\Omega)}^{k+2}) - c''', \end{aligned} \tag{3.7}$$

We multiply (2.1) by u and integrate over Ω, we have

$$\frac{d}{dt}\|u\|_{-1}^2 + c(\|u\|_{H^1(\Omega)}^2 + \int_\Omega F(u)dx) \leq \frac{c}{2}\|\alpha\|_{H^2(\Omega)}^2 + 2c_0. \tag{3.8}$$

Summing (3.5) and δ(3.8), where $\delta > 0$ is small enough, we have

$$\frac{d}{dt}E_2 + c(E_2 + \|\nabla\Delta\alpha\|^2 + \|\partial_t u\|_{-1}^2) \leq c,\ c > 0, \tag{3.9}$$

where

$$E_2 = E_1 + \delta\|u\|_{-1}^2,$$

satifies

$$\begin{aligned} c(\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^{k+2}(\Omega)}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2) - c^{'} \leq \\ E_2 \leq c^{''}(\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^{k+2}(\Omega)}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2) - c^{'''}, \\ c, c^{''} > 0. \end{aligned}$$

In particular, we deduce from (3.9) and Gronwall's lemma the dissipative estimate

$$\begin{aligned} &\|u(t)\|_{H^1(\Omega)}^2 + \|u(t)\|_{L^{k+2}(\Omega)}^{k+2} + \|\alpha(t)\|_{H^2(\Omega)}^2 + \int_0^t e^{-c(t-s)}(\|\nabla\Delta\alpha(s)\|^2 + \|\partial_t u(s)\|_{-1}^2)ds \\ &\leq c(\|u_0\|_{H^1(\Omega)}^2 + \|u_0\|_{L^{k+2}(\Omega)}^{k+2} + \|\alpha_0\|_{H^2(\Omega)}^2)e^{-ct}, c > 0,\ t \geq 0. \end{aligned} \tag{3.10}$$

We multiply (2.1) by $\partial_t u$ and integrate over Ω, we obtain

$$\frac{d}{dt}\|\Delta u\|^2 + 2\|\partial_t u\|^2 = 2\int_\Omega \Delta f(u)\partial_t u dx - 2\int_\Omega \Delta g(u)(\alpha - \Delta\alpha)\partial_t u)dx. \tag{3.11}$$

We multiply (2.2) by $-\Delta(\alpha - \Delta\alpha)$ and integrate over Ω, we obtain

$$\frac{d}{dt}\|\nabla(\alpha - \Delta\alpha)\|^2 + 2\|\Delta\alpha\|^2 + 4\|\nabla\Delta\alpha\|^2 + 2\|\Delta^2\alpha\| = 2\int_\Omega g(u)\partial_t u.\Delta(\alpha - \Delta\alpha)dx. \tag{3.12}$$

(note that $\|\nabla(\alpha - \Delta\alpha)\|^2 = \|\nabla\alpha\|^2 + 2\|\Delta\alpha\|^2 + \|\nabla\Delta\alpha\|^2$)

Summing (3.11) and (3.12), we find

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|^2 + \|\nabla(\alpha - \Delta\alpha)\|^2) + 2\|\Delta\alpha\|^2 + 4\|\nabla\Delta\alpha\|^2 + 2\|\Delta^2\alpha\|^2 + 2\|\partial_t u\|^2 \\ &= 2(\Delta f(u), \partial_t u) - 2(\Delta g(u)(\alpha - \Delta\alpha), \partial_t u) + 2(g(u)\partial_t u, \Delta(\alpha - \Delta\alpha)). \end{aligned} \tag{3.13}$$

This, let find estimates of (3.13) right terms, using Hölder inequality, owing to $\alpha \in H^2(\Omega)$ with continuous injection the $H^2(\Omega) \subset L^\infty(\Omega)$, we have

$$2|(\Delta f(u), \partial_t u)| \leq c\|f(u)\|_{H^2(\Omega)}^2 + \frac{1}{3}\|\partial_t u\|^2. \tag{3.14}$$

Furthermore,

$$2|(\Delta g(u)(\alpha - \Delta\alpha), \partial_t u)| \ \ \leq \ \ 2\int_\Omega |\Delta g(u)||(\alpha - \Delta\alpha)|_{L^\infty(\Omega)}|\partial_t u|dx$$

$$\leq 2c\|\Delta g(u)\|\|\partial_t u\|$$
$$\leq c\|\Delta g(u)\|^2 + \frac{1}{3}\|\partial_t u\|^2, \quad (3.15)$$

and,

$$\begin{aligned} 2|(g(u)\partial_t u, \Delta(\alpha - \Delta\alpha))| &= 2|((\alpha - \Delta\alpha)\Delta g(u), \partial_t u)| \\ &\leq c_1\|\Delta g(u)\|^2 + c_3\|\partial_t u\|^2. \end{aligned} \quad (3.16)$$

Inserting (3.14), (3.15) and (3.16) into (3.13), we find

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|^2 + \|\nabla(\alpha - \Delta\alpha)\|^2) + 2\|\Delta\alpha\|^2 + 4\|\nabla\Delta\alpha\|^2 + 2\|\Delta^2\alpha\|^2 + c_5\|\partial_t u\|^2 \\ &\leq c\|f(u)\|^2_{H^2(\Omega)} + c\|\Delta g(u)\|^2. \end{aligned} \quad (3.17)$$

We recall that $H^2(\Omega) \subset C(\overline{\Omega})$ and owing to (2.5), we obtain

$$\|f(u)\|^2_{H^2(\Omega)} + \|g(u)\|^2_{H^2(\Omega)} \leq Q(\|u\|_{H^2(\Omega)}). \quad (3.18)$$

and inserting (3.18) into (3.17), we find

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|^2 + \|\nabla(\alpha - \Delta\alpha)\|^2) + 2\|\Delta\alpha\|^2 + 4\|\nabla\Delta\alpha\|^2 + \|\Delta^2\alpha\|^2 \\ &+c_5\|\partial_t u\|^2 \leq Q(\|u\|_{H^2(\Omega)}). \end{aligned} \quad (3.19)$$

In particular, we deduce

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\nabla(\alpha - \Delta\alpha)\|^2) \leq Q(\|u\|_{H^2(\Omega)}). \quad (3.20)$$

We set

$$y = \|\Delta u\|^2 + \|\nabla(\alpha - \Delta\alpha)\|^2, \quad (3.21)$$

we deduce from (3.20) an inequation of the form

$$y' \leq Q(y). \quad (3.22)$$

Let z be the solution to the ordinary differential equation

$$z' = Q(z), \quad z(0) = y(0). \quad (3.23)$$

It follows from the comparison principle, that there exists a time
$T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}) > 0$ belonging to, say $(0, \frac{1}{2})$ such that

$$y(t) \leq z(t), \quad \forall t \in [0, T_0], \quad (3.24)$$

hence

$$\|u(t)\|^2_{H^2(\Omega)} + \|\alpha(t)\|^2_{H^3(\Omega)} \leq Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}), \forall t \leq T_0. \quad (3.25)$$

We now differentiate (3.1) with respect to time, and have

$$(-\Delta)^{-1}\partial_t^2 u - \Delta\partial_t u + f'(u)\partial_t u = g'(u)\partial_t u(\alpha - \Delta\alpha) + g(u)(\partial_t\alpha - \Delta\partial_t\alpha). \tag{3.26}$$

Owing to (2.2), we have

$$(-\Delta)^{-1}\partial_t^2 u - \Delta\partial_t u + f'(u)\partial_t u = g'(u)\partial_t u(\alpha - \Delta\alpha) - g^2(u)\partial_t u + g(u)\Delta\alpha. \tag{3.27}$$

We multiply (3.27) by $t\partial_t u$ and integrate over Ω, we find for $t \le T_0$

$$\frac{d}{dt}(t\|\partial_t u\|_{-1}^2) + 2t\|\nabla\partial_t u\|^2 + 2(f'(u)\partial_t u, t\partial_t u) = 2\int_\Omega g'(u)\partial_t u(\alpha - \Delta\alpha).t\partial_t u dx$$
$$-2\int_\Omega g^2(u)\partial_t u.t\partial_t u dx + 2\int_\Omega g(u)\Delta\alpha.t\partial_t u dx. \tag{3.28}$$

Owing to (2.9), (3.18) and (3.25), we obtain $t \le T_0$

$$\begin{aligned} |2(f'(u)\partial_t u, t\partial_t u)| &\le 2tQ(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u\|^2 \\ &\le Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})(t\|\partial_t u\|_{-1}^2) + \frac{t}{2}\|\nabla\partial_t u\|^2. \end{aligned} \tag{3.29}$$

Using to the interpolation inequality note that, $\|\partial_t u\|^2 \le c\|\partial_t u\|_{-1}\|\nabla\partial_t u\|$.
Owing to (3.25) and that $-\Delta\alpha \in L^2(\Omega) \subset H^{-1}(\Omega)$, we have

$$\begin{aligned} |2(g(u)(\Delta\alpha, t\partial_t u)| &\le 2tQ(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\Delta\alpha\|\|\partial_t u\| \\ &\le Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})(t\|\Delta\alpha\|^2 + t\|\partial_t u\|^2) \\ &\le Q(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})(t\|\alpha\|_{H^2(\Omega)}^2 + t\|\nabla\partial_t u\|^2). \end{aligned} \tag{3.30}$$

Using the estimates (2.9) and owing to (3.25), we find

$$\begin{aligned} |2(g'(u)\partial_t u(\alpha - \Delta\alpha), t\partial_t u)| &\le 2tQ(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u\|^2 \\ &\le tQ(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u\|_{-1}^2 + \frac{t}{2}\|\nabla\partial_t u\|^2. \end{aligned} \tag{3.31}$$

Owing to (3.25), we have

$$|2(g^2(u)\partial_t u, t\partial_t u,)| \le tQ(\|u_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u\|_{-1}^2 + \frac{t}{2}\|\nabla\partial_t u\|^2. \tag{3.32}$$

In inserting (3.29), (3.30), (3.31), (3.32) into (3.28) and owing to (3.25), we find

$$\begin{aligned} &\frac{d}{dt}(t\|\partial_t u\|_{-1}^2) + ct\|\nabla\partial_t u\|^2 \\ &\le Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})(t\|\partial_t u\|_{-1}^2) + ct\|\alpha\|_{H^2(\Omega)}^2. \end{aligned} \tag{3.33}$$

In particular, owing to (3.5), (3.10), (3.25) and (3.33), Gronwall's lemma and, we find

$$\|\partial_t u\|_{-1}^2 \le \frac{1}{t}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}), \forall t \in (0, T_0]. \tag{3.34}$$

We multiply (3.27) by $\partial_t u$ and integrate over Ω, we have

$$\frac{d}{dt}\|\partial_t u\|_{-1}^2 + \|\nabla\partial_t u\|^2 \le c(\|\partial_t u\|_{-1}^2 + \|\alpha\|_{H^2(\Omega)}^2). \tag{3.35}$$

Owing to (3.10), (3.25) and Granwall's lemma, then the estimates (3.35) becomes

$$\|\partial_t u\|_{-1}^2 \le e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u(T_0)\|_{-1}^2,\ c \ge 0,\ t \ge T_0, \tag{3.36}$$

hence, owing to (3.34), we have

$$\|\partial_t u\|_{-1}^2 \le e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}),\ c \ge 0,\ t \ge T_0. \tag{3.37}$$

We now rewrite (3.1), for $t \ge T_0$ fixed, in the form

$$-\Delta u + f(u) = h_u(t), u = 0 \text{ on } \Gamma, \tag{3.38}$$

where

$$h_u(t) = -(-\Delta)^{-1}\partial_t u + g(u)(\alpha - \Delta\alpha). \tag{3.39}$$

We multiply (3.39) by $h_u(t)$ and integrate over Ω, we have

$$\|h_u(t)\|^2 \le c(\|\partial_t u\|_{-1}^2 + \|\alpha\|_{H^2(\Omega)}^2). \tag{3.40}$$

Owing to (3.34)-(3.37), we obtain

$$\|h_u(t)\|^2 \le e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}),\ c > 0,\ t \ge T_0. \tag{3.41}$$

We multiply (3.38) by u, owing to (2.8) and integrate over Ω, we find

$$\|\nabla u\|^2 + c\int_\Omega F(u)dx \le c\|h_u(t)\|^2 + c', c > 0. \tag{3.42}$$

We multiply (3.38) by $-\Delta u$, owing to (3.9)-(3.25) and integrate over Ω, we have

$$\|\Delta u\|^2 \le \|h_u(t)\|^2 + c\|\nabla u\|^2, \tag{3.43}$$

we deduce the (3.41)-(3.43), we obtain

$$\|u(t)\|_{H^2(\Omega)}^2 \le e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}) + c',\ c \ge 0,\ t \ge T_0. \tag{3.44}$$

Owing to (3.25), we find

$$\|u(t)\|_{H^2(\Omega)}^2 \le e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}),\ c \ge 0,\ t \ge 0. \tag{3.45}$$

Then the estimate (3.3) becomes

$$\frac{d}{dt}\|\alpha - \Delta\alpha\|^2 + 2\|\nabla\alpha\|^2 + 2\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2$$
$$\le Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})(\|\alpha - \Delta\alpha\|^2 + \|\partial_t u\|^2). \tag{3.46}$$

Owing to (3.25) and (3.36), we have

$$\|\alpha-\Delta\alpha\|^2+\|\partial_t u\|^2\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)}). \tag{3.47}$$

Owing to (3.35)-(3.37) and we integrate over T_0 to t, we deduce that

$$\begin{aligned}&\|\alpha(t)\|^2_{H^2(\Omega)}+\int_{T_0}^{t}(\|\nabla\alpha(s)\|^2+\|\Delta\alpha(s)\|^2+\|\nabla\Delta\alpha(s)\|^2)ds\\&\qquad\le e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)}),\ t\ge T_0,\end{aligned} \tag{3.48}$$

which implies

$$\|\alpha(t)\|^2_{H^2(\Omega)}\le e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)}),\ c>0\ ,t\ge T_0. \tag{3.49}$$

Combining (3.44) and (3.49), we have

$$\|u(t)\|^2_{H^2(\Omega)}+\|\alpha(t)\|^2_{H^2(\Omega)}\le e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})+c,c\ge 0,\ t\ge T_0. \tag{3.50}$$

Finally, we deduce (3.35) and (3.50) that

$$\|u(t)\|^2_{H^2(\Omega)}+\|\alpha(t)\|^2_{H^2(\Omega)}\le e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})+c,\ c>0,\ t\ge 0. \tag{3.51}$$

Integrating (3.51) between 0 to 1, we obtain

$$\int_0^1\|\alpha(t)\|^2_{H^2(\Omega)}dt\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})+c. \tag{3.52}$$

We multiply (2.1) by u and integrate over Ω, we have

$$\frac{d}{dt}\|u\|^2+c\|\Delta u\|^2\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})(\|\nabla u\|^2+\|\alpha\|^2_{H^3(\Omega)}). \tag{3.53}$$

Owing to (3.18) and (3.25), we find

$$\frac{d}{dt}\|u\|^2+c\|\Delta u\|^2\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)}). \tag{3.54}$$

We deduce the (3.54), we have

$$\int_0^1\|u(t)\|^2_{H^2(\Omega)}dt\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)}). \tag{3.55}$$

The estimates (3.52) and (3.55) conclude that there exists $T\in(0,1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)}+\|\alpha(T)\|^2\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})+c, \tag{3.56}$$

which implies

$$\|u(1)\|^2_{H^2(\Omega)}+\|\alpha(1)\|^2_{H^2(\Omega)}\le Q(\|u_0\|_{H^2(\Omega)},\|u_0\|_{L^{k+2}(\Omega)},\|\alpha_0\|_{H^3(\Omega)})+c. \tag{3.57}$$

Owing to (3.10), (3.51) and (3.57), we have the estimate dissipative following

$$\|u(t)\|^2_{H^2(\Omega)} + \|\alpha(t)\|^2_{H^2(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}) + c,\ c > 0,\ t \ge 0. \tag{3.58}$$

We multiply (2.2) by $\Delta\partial_t\alpha$ and integrate over Ω, we have

$$\frac{d}{dt}(\|\nabla\Delta\alpha\|^2 + \|\Delta\alpha\|^2) + \|\Delta\partial_t\alpha\|^2 + \|\nabla\partial_t\alpha\|^2 \le Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)})\|\partial_t u\|^2_{-1}. \tag{3.59}$$

Owing to (3.35)-(3.37) and integrate between T_0 to t, we deduce that

$$\int_{T_0}^t (\|\Delta\partial_t\alpha(s)\|^2 + \|\nabla\partial_t\alpha(s)\|^2)ds \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}),\ t \ge T_0, \tag{3.60}$$

Setting $y = \|\nabla\Delta\alpha\|^2, g = 0$ and $h = \|\partial_t u\|^2_{-1}$, we deduce from (3.60) that

$$y' \le gy + h, t \ge t_0, \tag{3.61}$$

where, owing to the above estimates, y, g and h satisfy the assumptions of the uniform Gronwall's lemme (for $t \ge t_0$), and for $t \ge t_0 + r$,

$$\int_t^{t+r} \|\nabla\Delta\alpha\|^2 ds \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, r) + c(r), c > 0,\ t \ge r, \tag{3.62}$$

which implies

$$\|\alpha(t)\|^2_{H^3(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, r) + c(r), c > 0,\ t \ge r. \tag{3.63}$$

We deduce owing to (3.58) and (3.63) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|\alpha(t)\|^2_{H^3(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|u_0\|_{L^{k+2}(\Omega)}, \|\alpha_0\|_{H^3(\Omega)}, r) + c(r), c(r) > 0,\ t \ge r. \tag{3.64}$$

4. The dissipative semigroup

Based on the a priori estimates, we have the

Theorem 4.1. *We assume that* $(u_0, \alpha_0) \in (H^2(\Omega) \cap H^1_0(\Omega) \cap L^{K+2}(\Omega)) \times (H^3(\Omega) \cap H^1_0(\Omega))$. *Then, the system (2.1)-(2.4) possesses at least solution* (u, α) *such that* $u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega) \cap L^{k+2}(\Omega))$, $\alpha \in L^\infty(0, T; H^3(\Omega) \cap H^1_0(\Omega))$ *and* $\partial_t u \in L^2(0, T; H^{-1}(\Omega))$, $\forall\ T > 0$.

Proof. The proof is based on the estimate (3.64) and, e.g., a standard Galerkin scheme. □

We have, concerning the uniqueness, the following.

Theorem 4.2. *We assume that the assumptions of Theorem 4.1 hold. Then, the solution obtained in Theorem 4.1 is unique.*

Proof. Let now (u_1, α_1) and (u_2, α_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0}, \alpha_{1,0})$ et $(u_{2,0}, \alpha_{2,0}) \in (H^2(\Omega) \cap H^1_0(\Omega) \cap L^{K+2}(\Omega)) \times (H^3(\Omega) \cap H^1_0(\Omega))$ respectively. We set $(u, \alpha) = (u_1, \alpha_1) - (u_2, \alpha_2)$ and $(u_0, \alpha_0) = (u_{1,0}, \alpha_{1,0}) - (u_{2,0}, \alpha_{2,0})$. Then (u, α) verifies the following problem. □

$$(-\Delta)^{-1}\partial_t u - \Delta u - (f(u_1) - f(u_2)) = g(u_1)(\alpha - \Delta\alpha) + (g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2), \tag{4.1}$$

$$\partial_t\alpha - \Delta\partial_t\alpha + \Delta^2\alpha - \Delta\alpha = -g(u_1)\partial_t u - (g(u_1) - g(u_2))\partial_t u_2, \tag{4.2}$$

$$u = \Delta u = \alpha = \Delta\alpha = 0 \quad \text{on} \quad \Gamma, \tag{4.3}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0. \tag{4.4}$$

We multiply (4.1) by $\partial_t u$ and integrate over Ω, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\partial_t u\|_{-1}^2 + \int_\Omega (f(u_1) - f(u_2))\partial_t u dx \\ = & \int_\Omega g(u_1)(\alpha - \Delta\alpha)\partial_t u dx - \int_\Omega (g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2)\partial_t u dx. \end{aligned} \tag{4.5}$$

We multiply (4.2) by $(\alpha - \Delta\alpha)$ and integrate Ω, we obtain

$$\begin{aligned} & \frac{d}{dt}\|\alpha - \Delta\alpha\|^2 + 2\|\nabla\alpha\|^2 + 4\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2 \\ = & -2\int_\Omega g(u_1)\partial_t u(\alpha - \Delta\alpha)dx - 2\int_\Omega (g(u_1) - g(u_2))\partial_t u_2(\alpha - \Delta\alpha)dx \\ \leq & \ 2\int_\Omega |\nabla g(u_1)||(-\Delta)^{-1}\partial_t u||\alpha - \Delta\alpha|dx + 2\int_\Omega |g(u_1) - g(u_2)||\partial_t u_2||\alpha - \Delta\alpha|dx \\ \leq & \ \frac{c}{4}\|\partial_t u\|_{-1}^2 + c\|\partial_t u_2\|^2 + c\|\alpha - \Delta\alpha\|^2. \end{aligned} \tag{4.6}$$

Summing (4.5) and (4.6) and integrate over Ω, we find

$$\begin{aligned} & \frac{d}{dt}(\|\nabla u\|^2 + \|\alpha - \Delta\alpha\|^2) + 2\|\nabla\alpha\|^2 + 4\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2 + 2\|\partial_t u\|_{-1}^2 \\ & +2\int_\Omega (f(u_1) - f(u_2))\partial_t u dx \\ = & \ c\|\alpha - \Delta\alpha\|^2 + c\|\partial_t u_2\|^2 + \frac{c}{4}\|\partial_t u\|_{-1}^2 - 2\int_\Omega (g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2)\partial_t u dx \\ & +2\int_\Omega g(u_1)(\alpha - \Delta\alpha)\partial_t u dx, \end{aligned} \tag{4.7}$$

which implies

$$\begin{aligned} & \frac{d}{dt}E_4 + 2\|\nabla\alpha\|^2 + 4\|\Delta\alpha\|^2 + 2\|\nabla\Delta\alpha\|^2 + c'\|\partial_t u\|_{-1}^2 + 2\int_\Omega (f(u_1) - f(u_2))\partial_t u dx \\ = & \ +c\|\alpha - \Delta\alpha\|^2 + c\|\partial_t u_2\|^2 - 2\int_\Omega (g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2)\partial_t u dx \end{aligned}$$

$$+2\int_\Omega g(u_1)(\alpha-\Delta\alpha)\partial_t u dx, \tag{4.8}$$

where

$$E_4 = \|\nabla u\|^2 + \|\alpha-\Delta\alpha\|^2,$$

satisfies

$$E_4 \geq c(\|u\|^2_{H^1(\Omega)} + \|\alpha\|^2_{H^2(\Omega)}). \tag{4.9}$$

Find the estimates for (4.8),

$$\begin{aligned} 2|((f(u_1)-f(u_2),\partial_t u)| &\leq 2|\nabla(fu_1)-f(u_2))||(-\Delta)^{-\frac{1}{2}}\partial_t u| \\ &\leq 2\|\nabla(fu_1)-f(u_2))\|^2 + \frac{1}{2}\|\partial_t u\|^2_{-1}. \end{aligned} \tag{4.10}$$

Besides

$$\begin{aligned} 2\|\nabla(fu_1)-f(u_2))\|^2 &= 2 \int_\Omega |\nabla(f'((u_1 s+(1-s)u_2)u|^2 dx \\ &\leq 2\int_\Omega |(\int_0^1 f'((u_1 s+(1-s)u_2)ds\nabla u \\ &\quad + \int_0^1 f''((u_1 s+(1-s)u_2)(|u||\nabla u_1|+|u||\nabla u_2|)ds|^2 dx \\ &\leq Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\|\nabla u\|^2 \\ &\quad +Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\|u\|^2 \\ &\leq Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})(\|\nabla u\|^2+\|u\|^2) \\ &\leq Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\|\nabla u\|^2. \end{aligned} \tag{4.11}$$

Inserting (4.11) into the estimates (4.10), we find

$$\begin{aligned} &2|((f(u_1)-f(u_2),\partial_t u)| \\ \leq\ & Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\|\nabla u\|^2 + \frac{1}{2}\|\partial_t u\|^2_{-1}. \end{aligned} \tag{4.12}$$

Furthermore

$$\begin{aligned} &2|(g(u_1)-g(u_2)(\alpha_2-\Delta\alpha_2),\partial_t u_2)| \\ \leq\ & 2\int_\Omega |\nabla(g(u_1)-g(u_2)||\alpha_2-\Delta\alpha_2||(\Delta)^{-\frac{1}{2}}\partial_t u_2|dx \\ \leq\ & |\alpha_2-\Delta\alpha_2|_{L^\infty(\Omega)}(\int_\Omega |u||(-\Delta)^{-\frac{1}{2}}\partial_t u|| \times \int_0^1 |g'(su_1 2+(1-s)u_2)|dsdx) \\ \leq\ & Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\int_\Omega |u||(-\Delta)^{-\frac{1}{2}}\partial_t u_2|dx \\ \leq\ & Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})\times\|\nabla u\|^2 + \frac{1}{2}\|\partial_t u\|^2_{-1}. \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
2|(g(u_1)(\alpha-\Delta\alpha),\partial_t u)| &\leq 2|((-\Delta)^{\frac{1}{2}}(g(u_1))(\alpha-\Delta\alpha),(-\Delta)^{-\frac{1}{2}}\partial_t u)| \\
&\leq 2\int_\Omega |\nabla g(u_1)||\alpha-\Delta\alpha||(-\Delta)^{-1}\partial_t u|dx \\
&\leq \int_\Omega |\nabla g(u_1)|_{L^\infty}|\alpha-\Delta\alpha||\partial_t u|_{-1}dx \\
&\leq 2c\|\alpha-\Delta\alpha\|\|\partial_t u\|_{-1} \\
&\leq c\|\alpha-\Delta\alpha\|+\frac{c}{2}\|\partial_t u\|^2. \qquad (4.14)
\end{aligned}$$

Inserting (4.12), (4.13) and (4.14) into (4.8), owing (3.37), we find

$$\begin{aligned}
&\frac{d}{dt}E_4+2\|\nabla\alpha\|^2+c\|\alpha\|^2_{H^2(\Omega)}+2\|\nabla\Delta\alpha\|^2+c\|\partial_t u\|^2_{-1} \\
\leq\ & Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega))})(\|\nabla u\|^2), \qquad (4.15)
\end{aligned}$$

which gives

$$\begin{aligned}
&\frac{d}{dt}E_4+2\|\nabla\alpha\|^2+c\|\alpha\|^2_{H^2(\Omega)}+2\|\nabla\Delta\alpha\|^2+c\|\partial_t u\|^2_{-1} \\
\leq\ & Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})E_4. \qquad (4.16)
\end{aligned}$$

Applying Granwall's lemme, into (4.16), we find

$$\begin{aligned}
&\|u(t)\|^2_{H^1(\Omega)}+\|\alpha(t)\|^2_{H^2(\Omega)} \\
\leq\ & e^{ct}Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)},\|\alpha_{1,0}\|_{H^3(\Omega)},\|\alpha_{2,0}\|_{H^3(\Omega)})(\|\alpha_0\|^2_{H^2(\Omega)}+\|u_0\|^2_{H^1(\Omega)}), \qquad (4.17)
\end{aligned}$$

hence the uniqueness, as well as continuous depending with respect to the initial data.

We set $\Psi=(H^2(\Omega)\cap H^1_0(\Omega)\cap L^{K+2}(\Omega))\times(H^3(\Omega)\cap H^1_0(\Omega))$. It follows from Theorem 4.2, that we have the continuous (with respect to the $H^1(\Omega)\times H^2(\Omega)$-norm) of the following semigroup

$$S(t):\Psi\longrightarrow\Psi,(u_0,\alpha_0)\longrightarrow(u(t),\alpha(t)),$$

$(i.e, S(0)=I, S(t)oS(s)=S(t+s),\ t,s\geq 0)$. We then deduce from (3.47) the following theorem.

Theorem 4.3. *The semigroup S(t) is dissipative in* Ψ*, i.e., there exists a bounded set* $B\in\Psi$*(called absorbing set) such that, for every bounded* $B\in\Psi$*, there exists* $t_0=t_0(B)\geq 0$ *such that* $t\geq t_0$ *implies* $S(t)B\subset B_0$.

Remark 4.1. *It is easy to see that we can assume, without loss of generality, that* B_0 *is positively invariant by* $S(t)$*, i.e.,* $S(t)B_0\subset B_0, \forall t\geq 0$*. Furthermore, it follows from (3.64) that* $S(t)$ *is dissipative in* $H^2(\Omega)\times H^3(\Omega)$ *and it follows from (3.63) that we can take* B_0 *in* $H^2(\Omega)\times H^3(\Omega)$.

Corollary 4.1. *The semigroup* $S(t)$ *possesses the global attractor* $\mathcal{A}$ *who is bounded in* $H^2(\Omega)\times H^3(\Omega)$ *and compact in* Ψ.

5. Existence of exponential attractors

The aim of this section is to prove the existence of exponential attractors for the semigroup $S(t), t \geq 0$, associated to the problem (2.1)-(2.4). To do so, we need the semigroup that has to be Lipschitz continuous, satisfying the smoothing property and checking a Hölder continuous with respect to time. This is enough to conclude on the existence of exponential attractors.

Lemma 5.1. *Let (u_1, α_1) and (u_2, α_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0}, \alpha_{1,0})$ and $(u_{2,0}, \alpha_{2,0})$, respectively, belonging to B_0. Then, the corresponding solutions of the problem (2.1)-(2.4) satisfy the following estimate*

$$\begin{aligned} & \|u_1(t) - u_2(t)\|^2_{H^2(\Omega)} + \|\alpha_1(t) - \alpha_2(t)\|^2_{H^3(\Omega)} \\ \leq\ & ce^{c't}(\|u_{1,0} - u_{2,0}\|^2_{H^1(\Omega)} + \|\alpha_{1,0} - \alpha_{2,0}\|^2_{H^2(\Omega)}),\ t \geq 1, \end{aligned} \tag{5.1}$$

where the constants only depend on B_0.

Proof. We set $(u, \alpha) = (u_1, \alpha_1) - (u_2, \alpha_2)$ and $(u_0, \alpha_0) = (u_{1,0}, \alpha_{1,0}) - (u_{2,0}, \alpha_{2,0})$, then (u, α) satisfies □

$$\begin{aligned} & (-\Delta)^{-1}\partial_t u - \Delta u - (f(u_1) - f(u_2)) \\ & = g(u_1)(\alpha - \Delta\alpha) - (g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2), \end{aligned} \tag{5.2}$$

$$\partial_t\alpha - \Delta\partial_t\alpha + \Delta^2\alpha - \Delta\alpha = -g(u_1)\partial_t u - (g(u_1) - g(u_2))\partial_t u_2, \tag{5.3}$$

$$u = \Delta u = \alpha = \Delta\alpha = 0 \quad \text{on} \quad \Gamma \tag{5.4}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0. \tag{5.5}$$

We first deduce from (4.16) that

$$\|\nabla u(t)\|^2 + \|\alpha(t)\|^2_{H^2(\Omega)} \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)}), c' > 0, t \geq 0, \tag{5.6}$$

and

$$\int_0^t (\|\nabla\alpha(s)\|^2 + \|\nabla\Delta\alpha(s)\|^2 + \|\partial_t u(s)\|^2_{-1})ds \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)}), c' > 0, t \geq 0, \tag{5.7}$$

where the constants only depend on B_0.
We differentiate (5.2) with respect to time and have, owing to (5.3), we obtain

$$\begin{aligned} & (-\Delta)^{-1}\partial_t\theta + \Delta\theta - f'(u_1)\theta + (f'(u_1) - f'(u_2))\partial_t u_2 \\ & = g'(u_1)\partial_t u_1(\alpha - \Delta\alpha) + g^2(u_1)\theta - g(u_1)(g(u_1) - g(u_2))\partial_t u_2 + g(u_1)\Delta\alpha \\ & + g'(u_1)\theta(\alpha_2 - \Delta\alpha_2) + (g'(u_1) - g'(u_2))\partial_t u_2(\alpha_2 - \Delta\alpha_2) \\ & + (g(u_1) - g(u_2))(\partial_t\alpha_2 - \Delta\partial_t\alpha_2), \end{aligned} \tag{5.8}$$

where $\theta = \partial_t u$ and $u_1 = u + u_2$.
We multiply (5.8) by $(t - T_0)\theta$ and integrate over Ω, where T_0 is same as in one of previous section, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}((t-T_0)\|\theta\|_{-1}^2) + (t-T_0)\|\nabla\theta\|^2 \\
&\leq |(g^2(u_1)\theta, (t-T_0)\theta)| + |((f'(u_1) - f'(u_2))\partial_t u_2, (t-T_0)\theta)| \\
&+|(g'(u_1)\partial_t u_1(\alpha - \Delta\alpha), (t-T_0)\theta| + |(g(u_1)(g(u_1) - g(u_2))\partial_t u_2, (t-T_0)\theta)| \\
&+|(g(u_1)\Delta\alpha, (t-T_0)\theta| + |(g'(u_1)\theta(\alpha_2 - \Delta\alpha_2), (t-T_0)\theta)| \\
&+|((g'(u_1) - g'(u_2))\partial_t u_2(\alpha_2 - \Delta\alpha_2), (t-T_0)\theta)| \\
&+|((g(u_1) - g(u_2))(\partial_t\alpha_2 - \Delta\partial_t\alpha_2), (t-T_0)\theta)| + |(f'(u_1)\theta, (t-T_0)\theta)|. \qquad (5.9)
\end{aligned}$$

We have,

$$\begin{aligned}
|(g^2(u)\theta, (t-T_0)\theta)| &\leq (t-T_0)\int_\Omega |g^2(u)||\theta|^2 dx \\
&\leq c(t-T_0)\|\theta\|^2 \text{ (owing (2.9) and } H^2(\Omega) \subset L^\infty(\Omega)), \qquad (5.10)
\end{aligned}$$

Noting that $u_1, u_2 \in H^2(\Omega) \subset L^\infty(\Omega)$, we have

$$\begin{aligned}
|((f'(u_1) - f'(u_2))\partial_t u_2, (t-T_0)\theta)| &\leq (t-T_0)\int_\Omega |f'(u_1) - f'(u_2)||\theta||\partial_t u_2| dx \\
&\leq (t-T_0)\int_\Omega |3u_1^2 - 3u_2^2||\theta||\partial_t u_2| dx \\
&\leq c(t-T_0)(\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty})\int_\Omega |u||\theta||\partial_t u_2| dx \\
&\leq c(t-T_0)\int_\Omega |u|_{L^4}|\theta|_{L^4}|\partial_t u_2| dx \\
&\leq c(t-T_0)\|u\|_{L^4}\|\theta\|_{L^4}\|\partial_t u_2\| \\
&\leq c(t-T_0)\|\nabla u\|\|\nabla\theta\|\|\partial_t u_2\|, \qquad (5.11)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|((g'(u_1) - g'(u_2))\partial_t u_2(\alpha_2 - \Delta\alpha_2), (t-T_0)\theta)| &\leq (t-T_0)\int_\Omega |g'(u_1) - g'(u_2)||\partial_t u_2||\alpha_2 - \Delta\alpha_2||\theta| dx \\
&\leq c(t-T_0)\int_\Omega |\partial_t u_2||\alpha_2 - \Delta\alpha_2||\theta| dx \\
&\leq c(t-T_0)\|\alpha_2 - \Delta\alpha_2\|_{L^4}\|\theta\|_{L^4}\|\partial_t u_2\| \\
&\leq c(t-T_0)\|\nabla(\alpha_2 - \Delta\alpha_2)\|\|\nabla\theta\|\|\partial_t u_2\| \\
&\leq c(t-T_0)\|\nabla\theta\|\|\partial_t u_2\|, \qquad (5.12)
\end{aligned}$$

Using (2.9) and (4.17), we find

$$|(g'(u_1)\partial_t u_1(\alpha - \Delta\alpha), (t-T_0)\theta| \leq (t-T_0)\int_\Omega |g'(u_1)||\partial_t u_1||\alpha - \Delta\alpha||\theta| dx$$

$$
\begin{aligned}
&\le c(t-T_0)\int_\Omega |\partial_t u_1||\alpha-\Delta\alpha||\theta|dx \\
&\le c(t-T_0)\int_\Omega |\partial_t u_1||\theta|dx \\
&\le c(t-T_0)\|\partial_t u_1\|\|\theta\|, \qquad (5.13)
\end{aligned}
$$

noting that $u_1 \in H^2(\Omega)$ and $\alpha \in H^3(\Omega)$, then

$$
\begin{aligned}
|(g'(u_1)\theta(\alpha_2-\Delta\alpha_2),(t-T_0)\theta)| &\le (t-T_0)\int_\Omega |g'(u_1)||\alpha_2-\Delta\alpha_2||\theta|^2dx \\
&\le c(t-T_0)\|\theta\|^2, \qquad (5.14)
\end{aligned}
$$

after Green, we have

$$
\begin{aligned}
|(g(u)\Delta\alpha,(t-T_0)\theta| &\le (t-T_0)\int_\Omega |\nabla\alpha||g'(u)\nabla u||\theta|dx + (t-T_0)\int_\Omega |\nabla\alpha||g(u)||\nabla\theta|dx \\
&\le c(t-T_0)\|\nabla\alpha\|\|\theta\| + c(t-T_0)\|\nabla\alpha\|\|\nabla\theta\|, \qquad (5.15)
\end{aligned}
$$

we have that $\alpha_2 \in H^2(\Omega)$, then

$$
\begin{aligned}
|((g(u_1)-g(u_2))(\partial_t\alpha_2-\Delta\partial_t\alpha_2),(t-T_0)\theta)| &\le (t-T_0)\int_\Omega |g(u_1)-g(u_2)||\partial_t\alpha_2-\Delta\partial_t\alpha_2||\theta|dx \\
&\le (t-T_0)\int_\Omega |u|_{L^4}|\partial_t\alpha_2-\Delta\partial_t\alpha_2||\theta|_{L^4}dx \\
&\le (t-T_0)\|u\|_{L^4}\|\partial_t\alpha_2-\Delta\partial_t\alpha_2\|\|\theta\|_{L^4} \\
&\le (t-T_0)\|\nabla u\|\|\partial_t\alpha_2-\Delta\partial_t\alpha_2\|\|\nabla\theta\| \\
&\le c(t-T_0)\|\nabla u\|\|\nabla\theta\|, \qquad (5.16)
\end{aligned}
$$

moreover

$$
\begin{aligned}
|(g(u_1)(g(u_1)-g(u_2))\partial_t u_2,(t-T_0)\theta)| &\le (t-T_0)\int_\Omega |g(u_1)||g(u_1)-g(u_2)||\partial_t u_2||\theta|dx \\
&\le c(t-T_0)\int_\Omega |g(u_1)||\partial_t u_2||\theta|dx \\
&\le c(t-T_0)\int_\Omega (|u_1|_{L^4}+1)|\partial_t u_2||\theta|_{L^4}dx \\
&\le c(t-T_0)(\|u_1\|_{L^4}+1)\|\partial_t u_2\|\|\theta\|_{L^4} \\
&\le c(t-T_0)\|\partial_t u_2\|\|\theta\|_{L^4}, \qquad (5.17)
\end{aligned}
$$

and

$$
|(f'(u_1)\theta,(t-T_0)\theta)| \le c(t-T_0)\|\theta\|^2, \qquad (5.18)
$$

where the constants only depend on B_0.
By substituting (5.10), (5.11), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17) and (5.18) into (5.9), we have, owing to the interpolation inequality,

$$
\frac{d}{dt}((t-T_0)\|\theta\|_{-1}^2) + \frac{3}{4}(t-T_0)\|\nabla\theta\|^2
$$

$$\leq c(t-T_0)\|\theta\|_{-1}^2 + c(t-T_0)\|\theta\|\|\partial_t u_2\| + 2c(t-T_0)\|\theta\|\|\partial_t u_1\|$$
$$+\frac{1}{4}c(t-T_0)\|\nabla\alpha\|. \tag{5.19}$$

We now multiply (5.3) by $-(t-T_0)\alpha$ and intégrate over Ω, we obtain

$$(\partial_t\alpha, -(t-T_0)\alpha) + (-\Delta\partial_t\alpha, -(t-T_0)\alpha) + (-\Delta\alpha, -(t-T_0)\alpha) + (\Delta^2\alpha, (t-T_0)\alpha)$$
$$= (-g(u_1)\partial_t u, -(t-T_0)\alpha) + ((g(u_1)-g(u_2))\partial_t u_2, -(t-T_0)\alpha),$$

which implies

$$\frac{1}{2}\frac{d}{dt}((t-T_0)\|\alpha\|^2 + (t-T_0)\|\nabla\alpha\|^2) + (t-T_0)\|\nabla\alpha\|^2 + (t-T_0)\|\Delta\alpha\|^2$$
$$\leq |(-g(u_1)\partial_t u, -(t-T_0)\alpha)| + |((g(u_1)-g(u_2))\partial_t u_2, -(t-T_0)\alpha)|. \tag{5.20}$$

For that, let find the estimates of (5.20) right terms, using Hölder inequality, we have

$$\begin{aligned}|(-g(u_1)\partial_t u, -(t-T_0)\alpha)| &\leq (t-T_0)\int_\Omega |g(u_1)||\partial_t u||\alpha|dx\\ &\leq c(t-T_0)\int_\Omega (|u_1|_{L^4}+1)|\partial_t u||\alpha|_{L^4}dx\\ &\leq c(t-T_0)(\|\nabla u_1\|_{L^4}+1)\|\partial_t u\|\|\alpha\|_{L^4}\\ &\leq c(t-T_0)\|\nabla\alpha\|\|\partial_t u\|,\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}|((g(u_1)-g(u_2))\partial_t u_2, -(t-T_0)\alpha)| &\leq (t-T_0)\int_\Omega |g(u_1)-g(u_2)||\alpha||\partial_t u_2|dx\\ &\leq c(t-T_0)\int_\Omega |u|_{L^4}|\alpha|_{L^4}|\partial_t u_2|dx\\ &\leq c(t-T_0)\|\nabla u\|\|\nabla\alpha\|\|\partial_t u_2\|\\ &\leq c(t-T_0)\|\nabla\alpha\|\|\partial_t u_2\|.\end{aligned} \tag{5.22}$$

Inserting (5.21) and (5.22) into (5.20), we find

$$\frac{d}{dt}[(t-T_0)(\|\alpha\|^2+\|\nabla\alpha\|^2)] + 2(t-T_0)\|\nabla\alpha\|^2 + 2(t-T_0)\|\Delta\alpha\|^2$$
$$\leq 2c(t-T_0)\|\nabla\alpha\|\|\partial_t u\| + 2c(t-T_0)\|\nabla\alpha\|\|\partial_t u_2\|. \tag{5.23}$$

Noting that $(u,\alpha) = (u_2,\alpha_2) = (u_1,\alpha_1)$, then

$$\int_{T_0}^t \|\partial_t u_2(s)\|^2 dx \leq ce^{c't}, t \geq T_0, \tag{5.24}$$

where the constants only depend on B_0.
Combining (5.19) and (5.23), we find

$$\frac{d}{dt}E_5 + +2(t-T_0)\|\Delta\alpha\|^2 + c(t-T_0)\|\nabla\theta\|^2 + 2(t-T_0)\|\nabla\alpha\|^2$$

$$\leq \quad c(t-T_0)(\|\theta\|_{-1}^2 + \|\alpha\|_{H^1(\Omega)}^2) + c(t-T_0)(\|\partial_t u_1\|^2 + \|\partial_t u_2\|^2), \tag{5.25}$$

where

$$E_5 = (t-T_0)(\|\theta\|_{-1}^2 + \|\nabla\alpha\|^2 + \|\alpha\|^2). \tag{5.26}$$

Applying Gronwall's lemma to (5.25) over $[T_0, t]$, we have

$$\begin{aligned} & \|\theta(t)\|_{-1}^2 + \|\alpha(t)\|_{H^1(\Omega)}^2 + \int_{T_0}^t (\|\nabla\alpha(s)\|^2 + \|\Delta\alpha(s)\|^2 + \|\nabla\theta(s)\|^2) e^{-c(s-t)} ds \\ \leq & \int_{T^0}^t (\|\partial_t u_1(s)\|^2 + \|\partial_t u_2(s)\|^2) e^{-c(s-t)} ds + E(0)e^{ct}, \end{aligned} \tag{5.27}$$

which implies

$$\|\theta(t)\|_{-1}^2 + \|\alpha(t)\|_{H^1(\Omega)}^2 \leq ce^{c't}(\|u_0\|_{H^1(\Omega)}^2 + \|\alpha_0\|_1^2), c > 0, \tag{5.28}$$

finally, we obtain

$$\|\partial_t u(t)\|_{-1}^2 + \|\alpha(t)\|_{H^1(\Omega)}^2 \leq ce^{c't}(\|u_0\|_{H^1(\Omega)}^2 + \|\alpha_0\|_{H^2(\Omega)}^2), c > 0, t \geq 1, \tag{5.29}$$

where the constants only depend on B_0.

We rewrite (5.8) in the form

$$-\Delta u = \tilde{h}_u(t), \quad u = 0 \ on \ \Gamma. \tag{5.30}$$

for $t \geq 1$ fixed, where

$$\begin{aligned} \tilde{h}_u(t) \;\; = \;\; & -(-\Delta)^{-1}\partial_t u - (f(u_1) - f(u_2)) + g(u_1)(\alpha - \Delta\alpha) \\ & +(g(u_1) - g(u_2))(\alpha_2 - \Delta\alpha_2). \end{aligned} \tag{5.31}$$

We multiply (5.31) by $\tilde{h}_u(t)$ and integrate over Ω, we have

$$\begin{aligned} & (\tilde{h}_u(t), \tilde{h}_u(t)) \\ & = -((-\Delta)^{-1}\partial_t u, \tilde{h}_u(t)) - (f(u_1) - f(u_2), \tilde{h}_u(t)) - (g(u_1)(\alpha - \Delta\alpha), \tilde{h}_u(t)) \\ & +((-\Delta)^{-1}[(\Delta g(u_1) - \Delta g(u_2))(\alpha_2 - \Delta\alpha_2)], \tilde{h}_u(t)), \end{aligned}$$

which implies

$$\begin{aligned} \|\tilde{h}_u(t)\|^2 \;\; \leq \;\; & c\|\tilde{h}_u(t)\|\|\partial_t u\|_{-1} + |((f(u_1) - f(u_2), \tilde{h}_u(t))| + |(g(u_1)(\alpha - \Delta\alpha), \tilde{h}_u(t))| \\ & +|((-\Delta)^{-1}[(\Delta g(u_1) - \Delta g(u_2))(\alpha_2 - \Delta\alpha_2)], \tilde{h}_u(t))|. \end{aligned} \tag{5.32}$$

Here

$$\begin{aligned} |((f(u_1) - f(u_2), \tilde{h}_u(t))| \;\; \leq \;\; & \int_\Omega |f(u_1) - f(u_2)||\tilde{h}_u(t)| dx \\ \leq \;\; & \|f(u_1) - f(u_2)\|\|\tilde{h}_u(t)\| \end{aligned}$$

$$\leq \quad c\|f(u_1)-f(u_2)\|^2+\frac{1}{6}\|\tilde{h}_u(t)\|^2, \tag{5.33}$$

furthermore $u_1,u_2\in H^2(\Omega)\subset L^\infty(\Omega)$, then

$$\begin{aligned}
\|f(u_1)-f(u_2)\|^2 &\leq \int_\Omega |f(u_1)-f(u_2)|^2dx\\
&\leq \int_\Omega\int_0^1 |f'(u_1s+(1-s)u_2|^2|u|^2dsdx\\
&\leq \int_0^1 |f'(u_1s+(1-s)u_2|^2ds\int_\Omega |u|^2dx\\
&\leq c\int_0^1 (\|su_1+(1-s)u_2\|_{L^\infty}^{2k}+1)ds\int_\Omega |u|^2dx\\
&\leq c(\|u_1+u_2\|_{L^\infty}^{2k}+1)\|u\|^2,
\end{aligned} \tag{5.34}$$

if $n=2$ where $n=3$, for $k\leq 1$(in particular $k=1$), preceding estimate give

$$\|f(u_1)-f(u_2)\|^2 \quad\leq\quad c(\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2+1)\|u\|_{H_0^1}^2, \tag{5.35}$$

if $n=2$ where $n=3$ with $k>1$, owing we have

$$\|f(u_1)-f(u_2)\|^2 \quad\leq\quad c((\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2)^k+1)\|u\|_{H_0^1}^2, \tag{5.36}$$

on the one hand,

$$\begin{aligned}
|(g(u_1)(\alpha-\Delta\alpha),\tilde{h}_u(t))| &\leq \int_\Omega |g(u_1)||\alpha-\Delta\alpha||\tilde{h}_u(t)|dx\\
&\leq \int_\Omega |g(u_1)|_{L^\infty}|\alpha-\Delta\alpha||\tilde{h}_u(t)|dx\\
&\leq c\|\alpha-\Delta\alpha\|\|\tilde{h}_u(t)\|\\
&\leq c\|\alpha-\Delta\alpha\|^2+\frac{1}{6}\|\tilde{h}_u(t)\|^2\\
&\leq c\|\alpha\|_{H^2(\Omega)}^2+\frac{1}{6}\|\tilde{h}_u(t)\|^2,
\end{aligned} \tag{5.37}$$

on the other hand,

$$\begin{aligned}
&|((-\Delta)^{-1}[(\Delta g(u_1)-\Delta g(u_2))(\alpha_2-\Delta\alpha_2)],\tilde{h}_u(t))|\\
&\quad= |((\Delta g(u_1)-\Delta g(u_2))(\alpha_2-\Delta\alpha_2),\tilde{h}_u(t))|\\
&\quad\leq \int_\Omega |\Delta g(u_1)-\Delta g(u_2)||\alpha_2-\Delta\alpha_2||(-\Delta)^{-1}\tilde{h}_u(t)|dx\\
&\quad\leq \int_\Omega |\Delta u_1-\Delta u_2||\alpha_2-\Delta\alpha_2||(-\Delta)^{-1}\tilde{h}_u(t)|dx\\
&\quad\leq \int_\Omega |\Delta u_1-\Delta u_2|_{L^\infty}|\alpha_2-\Delta\alpha_2||(-\Delta)^{-1}\tilde{h}_u(t)|dx
\end{aligned}$$

$$\leq \quad c\int_{\Omega}|\alpha_2-\Delta\alpha_2||(-\Delta)^{-1}\tilde{h}_u(t)|dx$$
$$\leq \quad c\|\alpha_2-\Delta\alpha_2\|^2+\frac{1}{6}\|\tilde{h}_u(t)\|^2, \tag{5.38}$$

combining (5.33), (5.36), (5.37) and (5.38), we find

$$\|\tilde{h}_u(t)\|^2 \leq c(\|\partial_t u\|_{-1}^2+\|\alpha\|_{H^2(\Omega)}^2)+c\|\nabla u\|^2+c\|\alpha_2-\Delta\alpha_2\|^2. \tag{5.39}$$

Using (5.29) and (5.6), we obtain

$$\|\tilde{h}_u(t)\|^2 \leq ce^{c't}(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{H^2(\Omega)}^2), t\geq 1, \tag{5.40}$$

where the constants only depend on B_0
We multiply (5.30) by and intergrate over Ω, we find

$$\|\Delta u\|^2 \leq \|\tilde{h}_u(t)\|^2, \tag{5.41}$$

hence, owing to (5.40), we have

$$\|\Delta u\|^2 \leq ce^{c't}(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|_{H^2(\Omega)}^2),\ t\geq 1, \tag{5.42}$$

we finally deduce from (5.29) and (5.42), the estimate (5.1) which concludes the proof

Lemma 5.2. *Let (u_1,α_1) and (u_2,α_2) be two solutions to (2.1)-(2.4) with initial data $(u_{1,0},\alpha_{1,0})$ and $(u_{2,0},\alpha_{2,0})$, respectively, belonging to B_0. Then, the semigroup $\{S(t))\}_{t\geq 0}$ is Lipschitz continuity with respect to space, i.e, there exists the constant $c>0$ such that*

$$\begin{aligned}&\|u_1(t)-u_2(t)\|_{H^1(\Omega)}^2+\|\alpha_1(t)-\alpha_2(t)\|_{H^2(\Omega)}^2\\ \leq\quad & ce^{c't}(\|u_{1,0}-u_{2,0}\|_{H^1(\Omega)}^2+\|\alpha_{1,0}-\alpha_{2,0}\|_{H^2(\Omega)}^2), t\geq 1,\end{aligned} \tag{5.43}$$

where the constants only depend on B_0.

Proof. The proof of the Lemma 5.2 is a direct consequence of the estimate (5.6). □

It just remains to prove the Hölder continuity with respect to time.

Lemma 5.3. *Let (u,α) be the solution of (5.2)-(5.5) with intial data (u_0,α_0) in B_0. Then, the semigroup $\{S(t))\}_{t\geq 0}$ is Hölder continuous with respect to time, i.e, there exists the constant $c>0$ such that $\forall t_1,t_2\in[0,T]$*

$$\|S(t_1)(u_0,\alpha_0)-S(t_2)(u_0,\alpha_0)\|_{\Psi}\leq c|t_1-t_2|^{\frac{1}{2}}, \tag{5.44}$$

where the constants only depend on B_0 and Γ.

Proof. □

$$
\begin{aligned}
\|S(t_1)(u_0,\alpha_0)-S(t_2)(u_0,\alpha_0)\|_\Psi &= \|(u(t_1)-u(t_2),\alpha(t_1)-\alpha_2))\|_\Psi\\
&\le \|u(t_1)-u(t_2)\|_{H^1(\Omega)}+\|\alpha(t_1)-\alpha(t_2)\|_{H^2(\Omega)}\\
&\le c(\|\nabla(u(t_1)-u(t_2))\|+\|\alpha(t_1)-\alpha(t_2)\|_{H^2(\Omega)})\\
&\le (\|\int_{t_1}^{t_2}\nabla\partial_t u ds\|+\|\int_{t_1}^{t_2}\partial_t\alpha\|_{H^2})\\
&\le c|t_1-t_2)|^{\frac12}|\int_{t_1}^{t^2}(\|\nabla\partial_t u\|^2+\|\partial_t\alpha\|_{H^2}^2)ds|^{\frac12}. \qquad (5.45)
\end{aligned}
$$

Noting that, thanks to (3.5) and (3.37), we have

$$
|\int_{t_1}^{t_2}\|\nabla\partial_t u\|^2 ds|\le c, \qquad (5.46)
$$

where the constant c depends only on B_0 and $T\ge T_0$ such that $t_1,t_2\in[0,T]$.
Furthermore, multiplying (5.3) by $(-\Delta)^{-1}\partial_t\alpha$ and integrate over Ω, we obtain

$$
\frac{d}{dt}\|\alpha\|^2+c\|\partial_t\alpha\|_{-1}^2+2\|\nabla\partial_t\alpha\|^2+2\|\partial_t\alpha\|^2\le c(\|\partial_t u_2\|^2+\|\partial_t u\|^2), \qquad (5.47)
$$

and it follows from (3.60), (5.24), (5.46) and (5.47) that

$$
|\int_{t_1}^{t_2}\|\partial_t\alpha\|_{H^2(\Omega)}^2 ds|\le c, \qquad (5.48)
$$

$$
\|S(t_1)(u_0,\alpha_0)-S(t_2)(u_0,\alpha_0)\|_\Psi\le c|t_1-t_2|^{\frac12}, \qquad (5.49)
$$

where c only depends on B_0 and T such that $t_1,t_2\in[0,T]$.
Finally, we obtain thanks to (5.46) and (5.48), the estimate (5.44). Thus, the Lemma is proved. We finally deduce from Lemma 5.1, Lemma 5.2 and Lemma 5.3 the following result (see, e.g. [12]).

Theorem 5.1. *The semigroup $S(t)$ possesses an exponential attractor $M\subset B_0$, i.e.,*
(i) M is compact in $H^1(\Omega)\times H^2(\Omega)$;
(ii) M is positively invariant, $S(t)M\subset M, t\ge 0$;
(iii) M has finite fractal dimension in $H^1(\Omega)\times H^2(\Omega)$;
(iv) M attracts exponentially fast the bounded subsets of Ψ

$$
\forall B\in\Psi \ bounded, dist_{H^1(\Omega)\times H^2(\Omega)}(S(t)B,M)\le Q(\|B\|_\Psi)e^{-ct},\ c>0, t\ge 0,
$$

where the constant c is independent of B and $dist_{H^1(\Omega)\times H^2(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$
dist_{H^1(\Omega)\times H^2(\Omega)}(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_{H^1(\Omega)\times H^2(\Omega)}.
$$

Remark 5.1. *Setting $\tilde{M} = S(1)M$, we can prove that $\tilde{M}$ is an exponential attractor for $S(t)$, but now in the topology of Ψ.*
Since M (or $\tilde{M}$) is a compact attracting set, we deduce from Theorem 5.1 and standard results (see, e.g, [3, 12]) the

Corollary 5.1. *The semigroup $S(t)$ possesses the finite-dimensional global attractor $A \subset B_0$.*

Remark 5.2. *We note that the global attractor A is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e. $S(t)A = A,\ \forall t \geq 0$) and attractors all bounded sets of initial data as time goes to infinity; thus, it appears as a suitable object in view of the study of asymptotic behaviour of the system. Furthermore, the finite dimensionality means, roughly speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics is, in some proper sense, finite dimensional and can be described by a finite number of parameters.*

The existence of the global attractor being established, one question is to know whether this attractor has a finite dimension in terms of the fractal or Hausdorff dimension. This is the aim of the final section.

Remark 5.3. *Comparing to the global attractor, an exponentiel attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations.*

6. Conclusions

This manuscript explains in a clear way, the context of dynamic system with two temperatures, when the relative solution exists. The existence of exponential attractor, associated to the problem (2.1)-(2.4) that we have proved, allow to assert that the existing solution of the problem (2.1)-(2.4) that we have shown in this work, belongs to the finite-dimensional subset called global attractor, from a certain time.

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Conflict of interest

The authors declare that there is no conflict of interests in this paper.

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