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Research article

Attractor of a nonlinear hybrid reaction–diffusion model of neuroendocrine transdifferentiation of human prostate cancer cells with time-lags

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Abstract: Prostate cancer is a serious disease that endangers men's health. The genetic mechanism and treatment of prostate cancer have attracted the attention of scientists. In this paper, we focus on the nonlinear mixed reaction diffusion dynamics model of neuroendocrine transdifferentiation of prostate cancer cells with time delays, and reveal the evolutionary mechanism of cancer cells mathematically. By applying operator semigroup theory and the comparison principle of parabolic equation, we study the global existence, uniqueness and boundedness of the positive solution for the model. Additionally, the global invariant set and compact attractor of the positive solution are obtained by Kuratowski's measure of noncompactness. Finally, we use the Pdepe toolbox of MATLAB to carry out numerical calculations and simulations on an example to check the correctness and effectiveness of our main results. Our results show that the delay has no effect on the existence, uniqueness, boundedness and invariant set of the solution, but will affect the attractor.

Keywords: human prostate cancer cells; nonlinear hybrid reaction–diffusion model; time–lags; well-posedness; invariant set and compact attractor
Mathematics Subject Classification: 35K57, 35K61, 35B40, 92C60

1. Introduction

In recent decades, many models have been established to study prostate cancer. For example, some biological models, such as TRAP, LADY and LNCaP models [1], use genetically engineered mice with human prostate cancer cells to simulate the growth of prostate cancer cells. These biological models are based on extracorporeal experiments, and cannot truly reflect the growth of cancer cells in the human body. Swanson et al. [2] and Vollmer et al. [3, 4] were early researchers who applied mathematical models to study prostate cancer. In their models, the relationship between serum prostate specific antigen concentration and tumour volume was mainly discussed. Subsequently,

Kuang et al. [5] applied the idea of ecostoichiometry to establish the KNE model to study tumour growth under various physical conditions. Some scholars [6–8] have revealed that androgens are significantly associated with prostate cancer. Later mathematical models focused on the effect of androgen concentration on prostate tumour growth. One of the topics most worthy of separate discussion is the mathematical models [9, 10] that Jackson put forward in 2004. The main contribution of Jackson's models is to show that there are two types of tumours, androgen dependent and androgen independent. The subsequent mathematical models of prostate cancer were established and studied under this framework. For example, Ideta et al. [11] found that intermittent ADT can shorten the time of cancer recurrence by establishing a mathematical model. In [12], Eikenberry et al. built a mathematical model for the intracellular dynamics of androgens and their receptors. Their research showed that lowering the androgen level would increase the PCa cell mutation rate, and lead to more heterogeneous populations.

Subsequently, research has focused on the role of neuroendocrine cells in the recurrence of prostate cancer. Neuroendocrine cells throughout the human body are special secretory cells with a cell structure similar to that of neurons. Therefore, they usually help to stabilize the surrounding tissues [13]. Based on an extracorporeal experiment with androgen deprivation conditions of LNCaP cells grown in petri dishes, Cerasolo et al. [14] proposed a discrete delay kinetic model to study the theory of neuroendocrine transdifferentiation in PCa. This model considered androgen-dependent cells, neuroendocrine androgen-dependent cells and androgen concentration. In the model of Morken et al. [15], the cancer cell population is divided into androgen-dependent cells and androgen-independent cells, which can proliferate in vivo. In 2021, Turner et al. [16] proposed a delayed nonlinear ODE model to investigate the neuroendocrine transdifferentiation of prostate cancer cells. Combined with biological significance, the author carefully studied the existence and global asymptotic stability of the equilibrium point of the model, as well as Hopf bifurcation.

Inspired by modelling ideas and methods in [16], we put forward a nonlinear reaction-diffusion model to describe the diffusion distribution of androgens in human prostate tumours. Our main contributions are as follows. (a) We are the first to establish a diffusion PDE model to reveal the dynamic mechanism of human prostate cancer. (b) We obtain some sufficient conditions for the global existence, uniqueness, boundedness and compact attraction of the positive solution for our model. (c) Our findings are helpful for the prevention and treatment of human prostate cancer.

The remaining structure of the paper is as follows. Section 2 gives the process of mathematical modelling. In Section 3, we apply operator semigroup theory and the comparison principle of parabolic equations to prove that our model has a global bounded unique positive solution. In Section 4, we obtain the global invariant set and compact attractor of our model by utilizing Kuratowski's measure of noncompactness. In Section 5, we provide an example and carry out a numerical simulation to examine the validity of our results. Section 6 gives a brief summary and outlook.

2. Mathematical modelling

This section describes the process of mathematical modelling in detail. Since we benefit from the ODE model [16] in terms of thinking methods, let us review the establishment course of this ODE model first.

Human prostate cancer cells are divided into transdifferentiated nonmalignant neuroendocrine androgen-independent cells (denoted by $Z(t)$) and LNCaP androgen-dependent cells (denoted by $Y(t)$). During the growth of $Y(t)$ cells, their proliferation ability is affected by the change in androgen concentration (denoted by $X(t)$) in the environment. In the experiment, androgen serum stripped by charcoal is introduced into the culture dish. Since this is the only external source of androgen, it can be considered that the concentration of serum is equivalent to the initial androgen concentration. $Z(t)$ cells are considered to be androgen-independent and mitotic products. They have no proliferative ability and can only be produced by transdifferentiation of $Y(t)$ cells. $Y(t)$ cells undergo asymmetric cell division, which means that a certain proportion of daughter cells will differentiate when undergoing proliferation. $Y(t)$ cells are divided into three groups: mature/resting cells (still denoted by $Y(t)$), proliferating cells (denoted by $U(t)$) and transdifferentiated cells (denoted by $V(t)$). It is worth noting that $U(t)$ and $V(t)$ represent different stages of the life cycle of $Y(t)$ cells, so they do not represent new cell types. The average cell cycle duration of $Y(t)$ is denoted as τ_1, and the cell transdifferentiation duration from $Y(t)$ to $Z(t)$ is denoted as τ_2.

Next, the biological transformation relationship and its quantification in terms of $X(t)$, $Y(t)$, $Z(t)$, $U(t)$ and $V(t)$ are given as follows. Androgen $X(t)$ is depleted at a constant rate d_1 and is secreted by $Z(t)$ cells at a secretion rate r. $Y(t)$ and $U(t)$ die according to the ratio d_2. The proportions of death of $Z(t)$ and $V(t)$ cells are d_3 and d_4, respectively. The function $af(X)$ is the rate of $Y(t)$ cells becoming $U(t)$ cells. The function $g(X, Y)$ is the rate of $Y(t)$ cells becoming $V(t)$ cells. $2[1 - bf(X(t - \tau_2))]e^{-d_4\tau_2}g(X(t - \tau_2), Y(t - \tau_2))Y(t - \tau_2)$ is the measure of $U(t)$ cells asymmetrically dividing into $Y(t)$ cells. $2bf(X(t - \tau_1))e^{-d_4\tau_1}g(X(t - \tau_1), X(t - \tau_1))Y(t - \tau_1)$ is the measure of $V(t)$ cells asymmetrically dividing into $Z(t)$ cells. $e^{-d_2\tau_2}af(X(t - \tau_2))Y(t - \tau_2)$ is the measure of $U(t)$ cells becoming $Z(t)$ cells.

Based on the above analysis, the following delayed nonlinear ODE model describing the neuroendocrine transdifferentiation of prostate cancer cells was proposed in [16]

$$\begin{cases} \frac{dX}{dt} = -d_1X(t) + rZ(t), \\ \frac{dY}{dt} = -d_2Y(t) - af(X(t))Y(t) - g(X(t), Y(t))Y(t) + 2[1 - bf(X(t - \tau_1))]e^{-d_4\tau_1} \\ \qquad \times g(X(t - \tau_1), Y(t - \tau_1))Y(t - \tau_1), \\ \frac{dZ}{dt} = -d_3Z(t) + ae^{-d_2\tau_2}f(X(t - \tau_2))Y(t - \tau_2) + 2bf(X(t - \tau_1))e^{-d_4\tau_1} \\ \qquad \times g(X(t - \tau_1), Y(t - \tau_1))Y(t - \tau_1), \\ \frac{dU}{dt} = -d_2U(t) + af(X(t))Y(t) - e^{-d_2\tau_2}af(X(t - \tau_2))Y(t - \tau_2), \\ \frac{dV}{dt} = -d_4V(t) + g(X(t), Y(t))Y(t) - e^{d_4\tau_1}g(X(t - \tau_1), Y(t - \tau_1))Y(t - \tau_1). \end{cases} \tag{2.1}$$

In practice, however, the concentration of cancer cells and androgen also depends on spatial variables. In particular, androgen concentration will produce a diffusion phenomenon during concentrated consumption and synthesis. In fact, the movement of particles in liquid medium is mainly in the form of diffusion. The PDE model describing the diffusion phenomenon has better accuracy than the ODE model. In recent years, some scholars have applied the diffusion PDE model to study practical problems such as virus transmission, cancer prevention and treatment, and online game addiction, and some good results have been achieved (see [17–20]). Therefore, we introduce spatial variables and androgen diffusion to generalize Model (2.1). In addition, for better alignment with biology and experimental results, we take $f(Z)$ as the Ricker function regulated by $f(Z) = \beta Ze^{-\xi Z}$, which is suggested by experimental evidence [14], where β is the gradient of the differentiation increase, and ξ is the inverse of the maximum differentiation rate. $g(X, Y) = \gamma_0 \frac{X}{b+X} \frac{\theta^n}{\theta^n+Y^n}$

is a Hill-type function used by Adimy et al. in [21,22], where γ_0 is the maximum rate of transfer from $Y(t)$ cells to $V(t)$ cells, b is the half-saturation constant for androgen concentration, and θ and n have similar roles to the Hill coefficients and represent the response to human prostate cancer cell population changes. Based on the above discussion, we focus on the following model in this paper

$$\begin{cases} \frac{\partial X}{\partial t} = D\Delta X - d_1(\cdot)X + r(\cdot)Z, \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial Y}{\partial t} = -d_2(\cdot)Y - a(\cdot)\beta(\cdot)XYe^{-\xi(\cdot)X} - \gamma_0(\cdot)\frac{X}{b(\cdot)+X}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+Y^{n(\cdot)}}Y \\ \qquad +2[1-b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}]e^{-d_4(\cdot)\tau_1} \\ \qquad \times\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1), \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial Z}{\partial t} = -d_3(\cdot)Z + a(\cdot)e^{-d_2(\cdot)\tau_2}\beta(\cdot)X(\cdot,t-\tau_2)Y(\cdot,t-\tau_2)e^{-\xi(\cdot)X(\cdot,t-\tau_2)} \\ \qquad +2b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}e^{-d_4(\cdot)\tau_1} \\ \qquad \times\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1), \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial U}{\partial t} = -d_2(\cdot)U + a(\cdot)\beta(\cdot)XYe^{-\xi(\cdot)X} - e^{-d_2(\cdot)\tau_2}a(\cdot)\beta(\cdot) \\ \qquad \times X(\cdot,t-\tau_2)Y(\cdot,t-\tau_2)e^{-\xi(\cdot)X(\cdot,t-\tau_2)}, \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial V}{\partial t} = -d_4(\cdot)V + \gamma_0(\cdot)\frac{X}{b(\cdot)+X}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+Y^{n(\cdot)}}Y - e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot) \\ \qquad \times\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1), \ (x,t) \in \Omega \times \mathbb{R}_+, \\ \frac{\partial X}{\partial \nu} = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ (X,Y,Z,U,V) = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5), \ (x,t) \in \Omega \times [-\tau,0], \end{cases} \tag{2.2}$$

where $\mathbb{R}_+ = (0,+\infty)$, and the lags $\tau_1, \tau_2 > 0$ are some constants, $\tau = \max\{\tau_1,\tau_2\}$. $\Omega \subset \mathbb{R}^n$ is a spatially bounded domain with smooth boundary $\partial\Omega$ associated with the non flux boundary condition $\frac{\partial X}{\partial \nu} = 0$. ν is the outer normal vector. $D > 0$ is a diffusion coefficient. Δ is a Laplace operator. $\phi_k = \phi_k(\cdot,t)$ $(k = 1,2,3,4,5)$ is the initial delay function. The biological interpretation of the remaining spatially dependent parameters is similar to system (2.1). The state changes of human prostate cancer cells and androgens in tumours are shown in Figure 1.

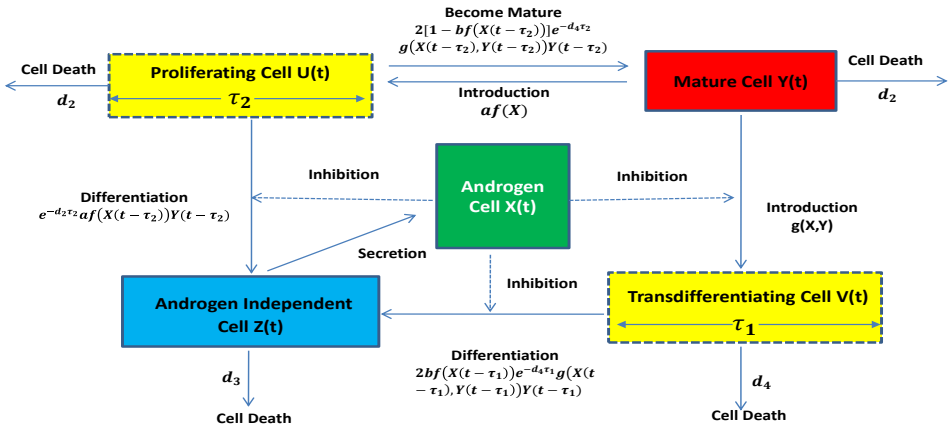

Figure 1. General scheme of the state transition in the models (2.1) and (2.2).

The parameter symbols and their meanings in the model are listed in the following table.

t	time variable
x	spatial variable
$X(x,t)$	the concentration of androgen cells in human prostate tumors
$Y(x,t)$	the concentration of mature cells in human prostate tumors
$Z(x,t)$	the concentration of Androgen-independent cells in human prostate tumors
$U(x,t)$	the concentration of proliferating cells in human prostate tumors
$V(x,t)$	the concentration of transdifferentiating cells in human prostate tumors
$r(x)$	the rate of increase of $X(x,t)$ secreted by $Z(x,t)$
$d_1(x)$	the consumption rate of $X(x,t)$
$d_2(x)$	the rate of $Y(x,t)$ and $U(x,t)$
$d_3(x)$	the rate of $Z(x,t)$
$d_4(x)$	the rate of $V(x,t)$
$a(x)$	the differentiation ratio from $U(x,t)$ to $Z(x,t)$
$b(x)$	the half-saturation constant of $X(x,t)$
$\beta(x)$	the gradient of the differentiation increase
$\xi(x)$	the inverse of the maximum differentiation rate
$\theta(x)$,$n(x)$	the Hill coefficients
$\gamma_0(x)$	the maximum rate of transfer from the $Y(x,t)$ to $V(x,t)$
τ_1	the average cell cycle duration of $Y(x,t)$
τ_2	the cell transdifferentiation duration from $Y(x,t)$ to $Z(x,t)$
D	the diffusion coefficient
ν	the outer normal vector
Δ	the Laplace operator
$\phi_i(i=1,2,3,4,5)$	the delay initial function
τ	$\tau=\max\|\tau_1,\tau_2\|$

To ensure that $V(x,t)$ cells asymmetrically divide into $Y(x,t)$ cells, it is necessary that

$$2[1-b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}]e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}>0. \tag{2.3}$$

To do so, let $h(u)=1-b(\cdot)\beta(\cdot)ue^{-\xi(\cdot)u}$, then

$$h'(u)=b(\cdot)\beta(\cdot)e^{-\xi(\cdot)u}(u\xi(\cdot)-1),\ \ h''(u)=b(\cdot)\beta(\cdot)e^{-\xi(\cdot)u}\xi(\cdot)(2-u\xi(\cdot)).$$

It follows from $h'(u)$ and $h''(u)$ that $h(u)$ has only one minimum value point $u_0=\frac{1}{\xi(\cdot)}\in(0,+\infty)$. Thereby, $\min_{u>0}h(u)=h(u_0)=1-\frac{b(\cdot)\beta(\cdot)}{e\xi(\cdot)}$. If (2.3) holds, it suffices that $\min_{u>0}h(u)>0$, and $0<\frac{b(\cdot)\beta(\cdot)}{\xi(\cdot)}<e$. Therefore, this paper requires the following underlying assumptions.

(H_1) The functions $0<d_1(\cdot), d_2(\cdot), d_3(\cdot), d_4(\cdot), r(\cdot), a(\cdot), b(\cdot), \beta(\cdot), \xi(\cdot), \gamma_0(\cdot), n(\cdot)\in C(\Omega,\mathbb{R})$. $\tau_1,\tau_2>0$ are two constants, $\tau=\max\{\tau_1,\tau_2\}$.

(H_2) For all $x\in\Omega$, $0<\frac{b(\cdot)\beta(\cdot)}{\xi(\cdot)}<e$, which ensures that $V(\cdot,t)$ cells asymmetrically divide into $Y(\cdot,t)$ cells.

(H_3) The initial delay function $\phi_k=\phi_k(\cdot,t)(k=1,2,3,4,5)$ satisfies $0<\phi_k\in C(\Omega\times[-\tau,0],\mathbb{R})$.

(H_4) For all $x\in\Omega$, $d_2(\cdot)<2\gamma_0(\cdot)$, which ensures that $Y(\cdot,t)$ cells do not vanish because of natural death.

3. Global existence, uniqueness and boundedness of the solution

This section focuses on the global existence, uniqueness and boundedness of the solution to (2.2). First, we need to review the relevant knowledge of operator semigroup theory and the comparison principle of parabolic equations.

Let $\mathscr{X} = C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^5)$, $\mathscr{X}^+ = C(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}^5_+)$, $\mathscr{X}^+_\tau = C(\overline{\Omega} \times [-\tau, 0], \mathbb{R}^5_+)$. For any $W = (W_1, W_2, W_3, W_4, W_5)^T \in \mathscr{X}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T \in \mathscr{X}^+_\tau$, two norms are defined by

$$\|W\| = \sup_{t\in\mathbb{R}_+}\left(\int_\Omega \sum_{j=1}^{5}[W_j(x,t)]^2 dx\right)^{\frac{1}{2}},$$

and

$$\|\phi\|_\tau = \sup_{t\in[-\tau,0]}\left(\int_\Omega \sum_{j=1}^{5}[\phi_j(x,t)]^2 dx\right)^{\frac{1}{2}},$$

then $(\mathscr{X}, \|\cdot\|)$ and $(\mathscr{X}^+_\tau, \|\cdot\|_\tau)$ are two Banach spaces. $\mathscr{X}^+ = C(\overline{\Omega}\times\mathbb{R}_+, \mathbb{R}^5_+)$ is the closed positive cone of $\mathscr{X}$. For any $f(x) \in C(\overline{\Omega}, \mathbb{R})$, denote

$$f^+ = \sup_{x\in\overline{\Omega}} f(x), \quad f^- = \inf_{x\in\overline{\Omega}} f(x).$$

Set $W(t) = (X(\cdot,t), Y(\cdot,t), Z(\cdot,t), U(\cdot,t), V(\cdot,t))^T$ and $\phi(t) = (\phi_1(\cdot,t), \phi_2(\cdot,t), \phi_3(\cdot,t), \phi_4(\cdot,t), \phi_5(\cdot,t))^T$, then the model (2.2) is rewritten by

$$\begin{cases} \frac{dW(t)}{dt} = \mathcal{L}(W(t)) + \mathcal{F}(W(t)), \ t \in \mathbb{R}_+ \\ W(t) = \phi(t), \ t \in [-\tau, 0], \end{cases} \tag{3.1}$$

where the mappings $\mathcal{L}$ and $\mathcal{F}$ are defined by

$$\mathcal{L}(W(t)) = \mathcal{L}\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} D\Delta X - d_1(\cdot)X \\ -d_2(\cdot)Y \\ -d_3(\cdot)Z \\ -d_2(\cdot)U \\ -d_4(\cdot)V \end{pmatrix}, \tag{3.2}$$

and

$$\mathcal{F}(W(t)) = \mathcal{F}\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} F_1(W) \\ F_2(W) \\ F_3(W) \\ F_4(W) \\ F_5(W) \end{pmatrix}, \tag{3.3}$$

here

$$F_1(W) = r(\cdot)Z,$$

$$
\begin{aligned}
F_2(W) = &- a(\cdot)\beta(\cdot)XYe^{-\xi(\cdot)X} - \gamma_0(\cdot)\frac{X}{b(\cdot)+X}\frac{\theta^{n(\cdot)}(\cdot)}{[\theta(\cdot)]^{n(\cdot)}+Y^{n(\cdot)}}Y \\
&+ 2[1 - b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}]e^{-d_4(\cdot)\tau_1} \\
&\times \gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1),
\end{aligned}
$$

$$
\begin{aligned}
F_3(W) =&a(\cdot)e^{-d_2(\cdot)\tau_2}\beta(\cdot)X(\cdot,t-\tau_2)Y(\cdot,t-\tau_2)e^{-\xi(\cdot)X(\cdot,t-\tau_2)} \\
&+ 2b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}e^{-d_4(\cdot)\tau_1} \\
&\times \gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1),
\end{aligned}
$$

$$
F_4(W) = a(\cdot)\beta(\cdot)XYe^{-\xi(\cdot)X} - e^{-d_2(\cdot)\tau_2}a(\cdot)\beta(\cdot)X(\cdot,t-\tau_2)Y(\cdot,t-\tau_2)e^{-\xi(\cdot)X(\cdot,t-\tau_2)},
$$

and

$$
\begin{aligned}
F_5(W) =&\gamma_0(\cdot)\frac{X}{b(\cdot)+X}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+Y^{n(\cdot)}}Y - e^{d_4(\cdot)\tau_1}\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)} \\
&\times \frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1).
\end{aligned}
$$

For all $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)^T \in \mathscr{X}^+$, define

$$
\begin{cases}
\mathcal{T}_1(t)\varphi_1 = e^{-d_1(\cdot)t}\int_\Omega \Lambda(\cdot,s,\varphi_1(s))ds,\ \mathcal{T}_2(t)\varphi_2 = e^{-d_2(\cdot)t}\varphi_2, \\
\mathcal{T}_3(t)\varphi_3 = e^{-d_3(\cdot)t}\varphi_3,\ \mathcal{T}_4(t)\varphi_4 = e^{-d_2(\cdot)t}\varphi_4,\ \mathcal{T}_5(t)\varphi_5 = e^{-d_4(\cdot)t}\varphi_5,
\end{cases} \tag{3.4}
$$

where $\mathcal{T}_1(t)$ is a compact and strongly positive C_0-semigroup [23] induced by $D\Delta - d_1(\cdot)$ with the Neumann boundary $\frac{\partial X}{\partial \nu} = 0$. Λ is Green's function. C_0-semigroup $\{e^{\mathcal{L}t}\}_{t\geq 0} = (\mathcal{T}_1(t), \mathcal{T}_1(t), \mathcal{T}_3(t), \mathcal{T}_4(t), \mathcal{T}_5(t))$ on $\mathscr{X}^+$ is generated by the mapping $\mathcal{L}$ corresponding to the linear part of Eq (2.2), and satisfies $e^{\mathcal{L}t}\mathscr{X}^+ \subset \mathscr{X}^+$. In this way, the differential equation (3.1) is transformed into the following integral equation form:

$$
\begin{cases}
W(t) = e^{\mathcal{L}t}\varphi + \int_0^t e^{\mathcal{L}(t-s)}\mathcal{F}(W(s))ds, & t > 0, \\
W(t) = \phi(t), & -\tau \leq t \leq 0.
\end{cases} \tag{3.5}
$$

For any $\varphi \in \mathscr{X}^+$, we have

$$
\varphi + \epsilon\mathcal{F}(\varphi) = \begin{pmatrix} \varphi_1 + \epsilon F_1(\varphi) \\ \varphi_2 + \epsilon F_2(\varphi) \\ \varphi_3 + \epsilon F_3(\varphi) \\ \varphi_4 + \epsilon F_4(\varphi) \\ \varphi_5 + \epsilon F_5(\varphi) \end{pmatrix} \geq \begin{pmatrix} \varphi_1 + \epsilon r^- \varphi_3 \\ \varphi_2[1 - \epsilon(a^+\beta^+\varphi_1 + \gamma_0^+)] \\ \varphi_3 \\ \varphi_4 - \epsilon a^+\beta^+\varphi_1\varphi_2 \\ \varphi_5 - \epsilon\gamma_0^+\varphi_2 \end{pmatrix}.
$$

Taking $\epsilon > 0$ small enough, one has $\varphi + \epsilon\varphi \in \mathscr{X}^+$, and $\lim_{\epsilon\to 0^+} \mathrm{dist}(\varphi + \epsilon\varphi, \mathscr{X}^+) = 0$. By Corollary 4 in [24] and noting that $\phi \in \mathscr{X}_\tau^+$, we derive the following conclusion.

Lemma 3.1. *For each $\varphi \in \mathscr{X}^+$, Eq* (2.2) *has a unique discontinuous mild solution*

$$W = \begin{cases} \varpi(\cdot, t, \varphi), & 0 < t < t_\varphi, \\ \phi(\cdot, t), & -\tau \le t \le 0, \end{cases}$$

such that $\varpi(\cdot, t, \varphi) \in C([0, t_\varphi], \mathscr{X}) \cap C^1((0, t_\varphi), \mathscr{X})$ with $\varpi(\cdot, 0^+, \varphi) = \varphi$. Furthermore, when $t \in [-\tau, t_\varphi)$, W is the classical solution of (2.2)*, and $\limsup_{t \to t_\varphi} \|\varpi(\cdot, t, \varphi)\| = +\infty$ provided that $t_\varphi < +\infty$.*

The proof of Lemma 3.1 is the same as that of Corollary 4 in [24], so we omit it.

Lemma 3.2. *[25] Consider*

$$\begin{cases} \frac{\partial w}{\partial t} = d\Delta w + g(\cdot) - h(\cdot)w, & (x, t) \in \Omega \times (0, +\infty), \\ \frac{\partial w}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, +\infty), \end{cases} \tag{3.6}$$

where $d > 0$, $g(\cdot), h(\cdot) \in C(\overline{\Omega}, \mathbb{R}_+)$. Then Eq (3.6) *has a unique globally asymptotically stable steady-state solution $w^*(\cdot) \in C(\overline{\Omega}, \mathbb{R}_+)$. Furthermore, for all $x \in \Omega$, $w^*(\cdot) = \frac{g}{h}$ provided that $g(\cdot) \equiv g$ and $h(\cdot) \equiv h$, where $h, g > 0$ are some constants.*

Theorem 3.1. *For each $\varphi \in \mathscr{X}^+$, Eq* (2.2) *has a unique solution*

$$W = \begin{cases} \varpi(\cdot, t, \varphi), & 0 < t < +\infty, \\ \phi(\cdot, t), & -\tau \le t \le 0, \end{cases} \tag{3.7}$$

such that $\varpi(\cdot, t, \varphi) \in \mathscr{X}^+$ with $\varpi(\cdot, 0^+, \varphi) = \varphi$. And there has a constant $\mathcal{M} > 0$ independent of φ such that $\limsup_{t \to +\infty} \varpi(\cdot, t, \varphi) \le \mathcal{M}$, $\forall\, x \in \Omega$.

Proof. For each $\varphi \in \mathscr{X}^+$, according to Lemma 3.1, we know that Eq (2.2) has a unique classical solution

$$W = \begin{cases} \varpi(\cdot, t, \varphi), & 0 < t < t_\varphi, \\ \phi(\cdot, t), & -\tau \le t \le 0, \end{cases}$$

such that $\varpi(\cdot, t, \varphi) \in C([0, t_\varphi], \mathscr{X}) \cap C^1((0, t_\varphi), \mathscr{X})$ with $\varpi(\cdot, 0^+, \varphi) = \varphi$. Next, it suffices to prove that $t_\varphi = +\infty$, and there has a constant $\mathcal{M} > 0$ independent of φ such that $\limsup_{t \to +\infty} \varpi(\cdot, t, \varphi) \le \mathcal{M}$, $\forall\, x \in \Omega$. In fact, from the second equation of (2.2) and (H_1)–(H_4), we have

$$\begin{aligned} \frac{\partial Y}{\partial t} &\le -d_2(\cdot)Y + 2e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot)\frac{X(\cdot, t-\tau_1)}{b(\cdot) + X(\cdot, t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)} + [Y(\cdot, t-\tau_1)]^{n(\cdot)}}Y(\cdot, t-\tau_1) \\ &\le -d_2^- Y(\cdot, t) + 2\gamma_0^+ Y(\cdot, t-\tau_1). \end{aligned} \tag{3.8}$$

Consider the following delayed differential equation

$$\frac{\partial Y}{\partial t} = -d_2^- Y(\cdot, t) + 2\gamma_0^+ Y(\cdot, t-\tau_1), \tag{3.9}$$

let $Y = e^{\lambda t}$, then the characteristic equation of (3.9) is read as

$$\lambda + d_2^- = 2\gamma_0^+ e^{-\lambda\tau_1}. \tag{3.10}$$

Set $Q(\lambda) = \lambda + d_2^- - 2\gamma_0^+ e^{-\lambda\tau_1}$, together with (H_1) and (H_4), then $Q'(\lambda) = 1 + 2\gamma_0^+\tau_1 e^{-\lambda\tau_1} > 0$, $Q(0) = d_2^- - 2\gamma_0^+ < 0$, $Q(+\infty) = +\infty > 0$. Therefore, $Q(\lambda)$ is strictly monotonically increasing in $\mathbb{R}$. According to the existence theorem of zero point of continuous function, there has a unique constant $\lambda_0 \in (0, \infty)$ such that $Q(\lambda_0) = 0$, that is, the characteristic Eq (3.10) has a unique characteristic root $\lambda = \lambda_0 > 0$. Therefore, the general solution of Eq (3.9) is $Y = Ce^{\lambda_0 t}$. Thus, (3.8) becomes

$$\frac{\partial[e^{\lambda_0 t}Y(\cdot,t)]}{\partial t} = [-d_2^- Y(\cdot,t) + 2\gamma_0^+ Y(\cdot,t-\tau_1)]e^{\lambda_0 t} \le 0. \tag{3.11}$$

Integrating from 0 to t on both side of (3.11), we get

$$e^{\lambda_0 t}Y(\cdot,t) - Y(\cdot,0) = e^{\lambda_0 t}Y - \phi_2(\cdot,0) \le 0,$$

which implies that

$$Y(\cdot,t) \le \phi_2^+(0)e^{-\lambda_0 t} \le \phi_2^+(0),\ 0 < t \le t_\varphi. \tag{3.12}$$

One derives from (3.12) and (H_3) that there has a constant $\mathcal{M}_0$ such that

$$\mathcal{M}_0 \ge \max\left\{\sup_{x\in\overline{\Omega},-\tau\le t\le 0}\phi_2(\cdot,t),\ \phi_2^+(0)\right\} > 0,\quad Y(\cdot,t) \le \mathcal{M}_0,\quad \forall\ -\tau \le t < t_\varphi. \tag{3.13}$$

Let

$$\Theta(\cdot,t) = Y(\cdot,t) + Z(\cdot,t) + U(\cdot,t) + V(\cdot,t),$$

then

$$\Theta(\cdot,0) = Y(\cdot,0) + Z(\cdot,0) + U(\cdot,0) + V(\cdot,0) = \sum_{k=2}^{5}\phi_k(\cdot,0).$$

Denote

$$\underline{d} = \min\{d_2^-, d_3^-, d_4^-\},\quad \Theta^+(0) = \sup_{x\in\overline{\Omega}}\Theta(\cdot,0) = \sum_{k=2}^{5}\sup_{x\in\overline{\Omega}}\phi_k(\cdot,0).$$

In view of (H_1)–(H_4), (2.2) and (3.13), we obtain

$$\begin{aligned}\frac{\partial\Theta}{\partial t} =& -d_2(\cdot)Y - d_3(\cdot)Z - d_2(\cdot)U - d_4(\cdot)V\\ &+ e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[Y(\cdot,t-\tau_1)]^{n(\cdot)}}Y(\cdot,t-\tau_1)\\ \le& -\underline{d}\Theta(\cdot,t) + \gamma_0^+ Y(\cdot,t-\tau_1) \le -\underline{d}\Theta(\cdot,t) + \gamma_0^+\mathcal{M}_0.\end{aligned} \tag{3.14}$$

By (3.14), we have

$$\frac{\partial[e^{\underline{d}t}\Theta(\cdot,t)]}{\partial t} = e^{\underline{d}t}\left[\frac{\partial\Theta(\cdot,t)}{\partial t} + \underline{d}\Theta(\cdot,t)\right] \le \gamma_0^+\mathcal{M}_0 e^{\underline{d}t}. \tag{3.15}$$

Integrating from 0 to t on both side of (3.15), we obtain

$$e^{\underline{d}t}\Theta(\cdot,t) \le \Theta(\cdot,0) + \gamma_0^+\mathcal{M}_0\int_0^t e^{\underline{d}t}dt = \Theta(\cdot,0) + \frac{\gamma_0^+\mathcal{M}_0}{\underline{d}}(e^{\underline{d}t} - 1). \tag{3.16}$$

From (3.16) and noting that $\underline{d} > 0$, we yield

$$\Theta(\cdot,t) \le \Theta^+(0)e^{-\underline{d}t} + \frac{\gamma_0^+\mathcal{M}_0}{\underline{d}}(1-e^{-\underline{d}t}) \le \Theta^+(0) + \frac{\gamma_0^+\mathcal{M}_0}{\underline{d}},\ 0<t<t_\varphi. \tag{3.17}$$

From (H_3) and (3.17), we take

$$\mathcal{M}_1 \ge \max\left\{\sup_{x\in\overline{\Omega},\,-\tau\le t\le 0}\Theta(\cdot,t),\ \Theta^+(0)+\frac{\gamma_0^+\mathcal{M}_0}{\underline{d}}\right\} > 0,$$

then

$$\Theta(\cdot,t) = Y(\cdot,t)+Z(\cdot,t)+U(\cdot,t)+V(\cdot,t) \le \mathcal{M}_1,\ \forall\, x\in\Omega,\ t\in[-\tau,t_\varphi). \tag{3.18}$$

From the first equation of (2.2) and (3.18), we have

$$\begin{cases} \frac{\partial X}{\partial t} = D\Delta X - d_1(\cdot)X + r(\cdot)Z \le D\Delta X + r^+\mathcal{M}_1 - d_1^-X, & (x,t)\in\Omega\times(0,t_\varphi),\\ \frac{\partial X}{\partial \nu} = 0, & (x,t)\in\partial\Omega\times(0,t_\varphi). \end{cases} \tag{3.19}$$

By Lemma 3.2 and the comparison principle, there has a constants $\mathcal{M}_2 \ge \frac{r^+\mathcal{M}_1}{d_1^-} > 0$ such that $X(\cdot,t) \le \mathcal{M}_2$, $\forall\,(x,t)\in\Omega\times(0,t_\varphi)$. Thus, taking $\mathcal{M} = \max\{\mathcal{M}_1,\mathcal{M}_2,\sup_{x\in\Omega,-\tau\le t\le 0}\phi_1(\cdot,t)\} > 0$ independent of φ, one has

$$\varpi(\cdot,t,\varphi) \le \mathcal{M},\ \forall\,(x,t)\in\Omega\times(0,t_\varphi);\quad \phi(\cdot,t)\le\mathcal{M},\ \forall\,(x,t)\in\Omega\times[-\tau,0]. \tag{3.20}$$

In the light of (3.20) and Lemma 3.1, we conclude that $t_\varphi = +\infty$ and $\limsup_{t\to+\infty}\varpi(\cdot,t,\varphi) \le \mathcal{M}$, $\forall\, x\in\Omega$. The proof is completed. □

4. Global compact attractor

Let

$$\mathcal{M}_0^* = \max\left\{\sup_{x\in\overline{\Omega},-\tau\le t\le 0}\phi_2(\cdot,t),\ \phi_2^+(0)\right\},\ \mathcal{M}_1^* = \max\left\{\sup_{x\in\overline{\Omega},\,-\tau\le t\le 0}\Theta(\cdot,t),\ \Theta^+(0)+\frac{\gamma_0^+\mathcal{M}_0^*}{\underline{d}}\right\},$$

$$\mathcal{M}_2^* = \frac{r^+\mathcal{M}_1^*}{d_1^-},\quad \mathcal{M}^* = \max\left\{\mathcal{M}_1^*,\mathcal{M}_2^*,\sup_{x\in\Omega,-\tau\le t\le 0}\phi_1(\cdot,t)\right\}.$$

Define a set $\mathscr{D}$ and a solution semiflow $\mathscr{F}_t:\ \mathscr{X}^+\to\mathscr{X}^+$ of (2.2) as

$$\mathscr{D} = \left\{(X,Y,Z,U,V)^T\in\mathscr{X}^+:\ 0\le X\le\mathcal{M}^*, 0\le Y+Z+U+V\le\mathcal{M}_1^*\right\}, \tag{4.1}$$

and

$$\mathscr{F}_t(\varphi) = W(x,t,\varphi) = (X(\cdot,t,\varphi),Y(\cdot,t,\varphi),Z(\cdot,t,\varphi),U(\cdot,t,\varphi),V(\cdot,t,\varphi)),\ t\ge 0,\ \varphi\in\mathscr{X}^+. \tag{4.2}$$

To ensure the existence of a global compact attractor for the solution semiflow $\mathscr{F}_t(\varphi)$, it needs to be further assumed that the condition below is true.

(H$_5$) $\kappa_1 = -d_2^- + \gamma_0^+ + 2\gamma_0^+ e^{-d_4^-\tau_1} + \frac{1}{2}[\frac{a^+\beta^+ e^{-d_2^-\tau_2}}{\xi^- e} + \frac{2b^+\beta^+\gamma_0^+ e^{-d_4^-\tau_1}}{\xi^- e}(1+n^+)] + \frac{1}{2}[\frac{a^+\beta^+}{\xi^- e} + \frac{a^+\beta^+}{\xi^- e}e^{-d_2^-\tau_2}] + \frac{1}{2}[\gamma_0^+(1+n^+) + \gamma_0^+ e^{-d_4^-\tau_1}(1+n^+)] < 0$, $\kappa_2 = -d_3^- + \frac{1}{2}[\frac{a^+\beta^+ e^{-d_2^-\tau_2}}{\xi^- e} + \frac{2b^+\beta^+\gamma_0^+ e^{-d_4^-\tau_1}}{\xi^- e}(1+n^+)] < 0$ and $\kappa_3 = -d_4^- + \frac{1}{2}[\gamma_0^+(1+n^+) + \gamma_0^+ e^{-d_4^-\tau_1}(1+n^+)] < 0$.

Theorem 4.1. *If* (H$_1$)–(H$_5$) *hold, then the solution semiflow* $\mathscr{F}_t(\varphi)$ *defined by* (4.2) *has a global compact attractor* $\mathscr{A}$ *on* $\mathscr{X}^+$.

Next, we shall fulfil the proof of Theorem 4.1 from the under lemmas.

Lemma 4.1. $\mathscr{D}$ *is a positive invariant set of* $\mathscr{F}_t$ *defined by* (4.2) *in the sense of* $\mathscr{F}_t(\varphi) \in \mathscr{D}$, $\forall\, \varphi \in \mathscr{D}$, $t \geq 0$.

Proof. From the proof of Theorem 3.1, we know that the constants $\mathcal{M}_1^*$ and $\mathcal{M}^*$ can be regarded as the upper solution of Eqs (3.14) and (3.19), respectively. That is, for all $\varphi \in \mathscr{D}$, $t \geq 0$, noting that $\mathscr{F}_t(\varphi)$ is a solution of Eq (2.2), we have $0 \leq X(\cdot, t, \varphi) \leq \mathcal{M}^*$ and $0 \leq Y(\cdot, t, \varphi)+Z(\cdot, t, \varphi)+U(\cdot, t, \varphi)+V(\cdot, t, \varphi) \leq \mathcal{M}_1^*$, namely, $\mathscr{F}_t(\varphi) \in \mathscr{D}$. The proof is completed. □

Since there is a lack of diffusion terms in the last four equations of system (2.2), the semiflow $\mathscr{F}_t$ is no longer compact. Therefore, we adopt Kuratowski's measure of noncompactness (see [26]) $\mathcal{K}$ defined by

$$\mathcal{K}(\mathbb{E}) = \inf\{\sigma > 0 : \mathbb{E} \textit{ has a finite cover by subsets of } \mathbb{E}_i \subset \mathbb{E} \textit{ such that } diam(\mathbb{E}_i) \leq \sigma\}, \quad (4.3)$$

where $\mathbb{E}$ is any bounded set and $diam(\mathbb{E}_i)$ is the diameter of the set $\mathbb{E}_i$. If $\mathbb{E}$ is unbounded, then $\mathcal{K}(\mathbb{E}) = +\infty$. Moreover, $\mathcal{K}(\mathbb{E}) = 0$ if and only if $\mathbb{E}$ is precompact (i.e. $\overline{\mathbb{E}}$ is compact). For the sake of unity and convenience, let $\mathbf{w} = (w_1, w_2, w_3, w_4) = (Y, Z, U, V)$, then the last four equations of system (2.2) are rewritten as

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{h}(x, X, \mathbf{w}), \quad x \in \Omega,\ t > 0. \quad (4.4)$$

For Eq (4.4), we have the following conclusion.

Lemma 4.2. *If* (H$_5$) *holds, then there has a constant* $\kappa > 0$ *such that*

$$\zeta\left[\frac{\partial \boldsymbol{h}(x, X, \boldsymbol{w})}{\partial \boldsymbol{w}}\right]\zeta^T \leq -\kappa\zeta\zeta^T, \quad \forall\, \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{R}^4, \quad (4.5)$$

where $\left[\frac{\partial \boldsymbol{h}(x,t,X,\boldsymbol{w})}{\partial \boldsymbol{w}}\right]$ *is the Jacobian matrix of* $\boldsymbol{h}(x, t, X, \boldsymbol{w})$ *with respect to* $\boldsymbol{w}$.

Proof. A direct calculation gives

$$\begin{aligned}\frac{\partial h_1}{\partial w_1} =& -d_2(\cdot) - a(\cdot)\beta(\cdot)Xe^{-\xi(\cdot)X} - \gamma_0(\cdot)\frac{X}{b(\cdot)+X}\Big[\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)} + w_1^{n(\cdot)}} - \frac{n(\cdot)[\theta(\cdot)]^{n(\cdot)}w_1^{n(\cdot)}}{([\theta(\cdot)]^{n(\cdot)} + w_1^{n(\cdot)})^2}\Big] \\ &+ 2[1 - b(\cdot)\beta(\cdot)X(\cdot, t-\tau_1)e^{-\xi(\cdot)X(\cdot,t-\tau_1)}]e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot)\frac{X(\cdot, t-\tau_1)}{b(\cdot)+X(\cdot, t-\tau_1)} \\ &\times\Big[\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)} + [w_1(\cdot, t-\tau_1)]^{n(\cdot)}} - \frac{n(\cdot)[\theta(\cdot)]^{n(\cdot)}[w_1(\cdot, t-\tau_1)]^{n(\cdot)}}{([\theta(\cdot)]^{n(\cdot)} + [w_1(\cdot, t-\tau_1)]^{n(\cdot)})^2}\Big], \end{aligned} \quad (4.6)$$

$$\frac{\partial h_1}{\partial w_2} = \frac{\partial h_1}{\partial w_3} = \frac{\partial h_1}{\partial w_4} = 0, \tag{4.7}$$

$$\begin{aligned}\frac{\partial h_2}{\partial w_1} =& a(\cdot)e^{-d_2(\cdot)\tau_2}\beta(\cdot)X(\cdot,t-\tau_2)e^{-\xi(\cdot)X(\cdot,t-\tau_2)} + 2b(\cdot)\beta(\cdot)X(\cdot,t-\tau_1)e^{-d_4(\cdot)\tau_1}\\ &\times e^{-\xi(\cdot)X(\cdot,t-\tau_1)}\gamma_0(\cdot)\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\Big[\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[w_1(\cdot,t-\tau_1)]^{n(\cdot)}}\\ &-\frac{n(\cdot)[\theta(\cdot)]^{n(\cdot)}[w_1(\cdot,t-\tau_1)]^{n(\cdot)}}{([\theta(\cdot)]^{n(\cdot)}+[w_1(\cdot,t-\tau_1)]^{n(\cdot)})^2}\Big],\end{aligned} \tag{4.8}$$

$$\frac{\partial h_2}{\partial w_2} = -d_3(\cdot),\ \frac{\partial h_2}{\partial w_3} = \frac{\partial h_2}{\partial w_4} = 0, \tag{4.9}$$

$$\frac{\partial h_3}{\partial w_1} = a(\cdot)\beta(\cdot)Xe^{-\xi(\cdot)X} - e^{-d_2(\cdot)\tau_2}a(\cdot)\beta(\cdot)X(\cdot,t-\tau_2), e^{-\xi(\cdot)X(\cdot,t-\tau_2)} \tag{4.10}$$

$$\frac{\partial h_3}{\partial w_3} = -d_2(\cdot),\ \frac{\partial h_3}{\partial w_2} = \frac{\partial h_3}{\partial w_4} = 0, \tag{4.11}$$

$$\begin{aligned}\frac{\partial h_4}{\partial w_1} =& \gamma_0(\cdot)\frac{X}{b(\cdot)+X}\Big[\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+w_1^{n(\cdot)}} - \frac{n(\cdot)[\theta(\cdot)]^{n(\cdot)}w_1^{n(\cdot)}}{([\theta(\cdot)]^{n(\cdot)}+w_1^{n(\cdot)})^2}\Big] - e^{-d_4(\cdot)\tau_1}\gamma_0(\cdot)\\ &\times\frac{X(\cdot,t-\tau_1)}{b(\cdot)+X(\cdot,t-\tau_1)}\Big[\frac{[\theta(\cdot)]^{n(\cdot)}}{[\theta(\cdot)]^{n(\cdot)}+[w_1(\cdot,t-\tau_1)]^{n(\cdot)}} - \frac{n(\cdot)[\theta(\cdot)]^{n(\cdot)}[w_1(\cdot,t-\tau_1)]^{n(\cdot)}}{([\theta(\cdot)]^{n(\cdot)}+[w_1(\cdot,t-\tau_1)]^{n(\cdot)})^2}\Big],\end{aligned} \tag{4.12}$$

and

$$\frac{\partial h_4}{\partial w_4} = -d_4(\cdot),\ \frac{\partial h_4}{\partial w_2} = \frac{\partial h_4}{\partial w_3} = 0. \tag{4.13}$$

From (4.6)–(4.13) and Schwartz's inequality, we have

$$\begin{aligned}&\zeta\left[\frac{\partial \mathbf{h}(x,X,\mathbf{w})}{\partial \mathbf{w}}\right]\zeta^T = \frac{\partial h_1}{\partial w_1}\zeta_1^2 + \frac{\partial h_2}{\partial w_2}\zeta_2^2 + \frac{\partial h_3}{\partial w_3}\zeta_3^2 + \frac{\partial h_4}{\partial w_4}\zeta_4^2 + \frac{\partial h_2}{\partial w_1}\zeta_1\zeta_2 + \frac{\partial h_3}{\partial w_1}\zeta_1\zeta_3 + \frac{\partial h_4}{\partial w_1}\zeta_1\zeta_4\\ \le&\frac{\partial h_1}{\partial w_1}\zeta_1^2 + \frac{\partial h_2}{\partial w_2}\zeta_2^2 + \frac{\partial h_3}{\partial w_3}\zeta_3^2 + \frac{\partial h_4}{\partial w_4}\zeta_4^2 + \frac{1}{2}\Big|\frac{\partial h_2}{\partial w_1}\Big|(\zeta_1^2+\zeta_2^2) + \frac{1}{2}\Big|\frac{\partial h_3}{\partial w_1}\Big|(\zeta_1^2+\zeta_3^2) + \frac{1}{2}\Big|\frac{\partial h_4}{\partial w_1}\Big|(\zeta_1^2+\zeta_4^2)\\ \le&\Big\{-d_2(\cdot)+\gamma_0(\cdot)+2\gamma_0(\cdot)e^{-d_4(\cdot)\tau_1} + \frac{1}{2}\Big[\frac{a(\cdot)\beta(\cdot)e^{-d_2(\cdot)\tau_2}}{\xi(\cdot)e} + \frac{2b(\cdot)\beta(\cdot)\gamma_0(\cdot)e^{-d_4(\cdot)\tau_1}}{\xi(\cdot)e}(1+n(\cdot))\Big]\\ &+\frac{1}{2}\Big[\frac{a(\cdot)\beta(\cdot)}{\xi(\cdot)e} + \frac{a(\cdot)\beta(\cdot)}{\xi(\cdot)e}e^{-d_2(\cdot)\tau_2}\Big] + \frac{1}{2}\Big[\gamma_0(\cdot)(1+n(\cdot)) + \gamma_0(\cdot)e^{-d_4(\cdot)\tau_1}(1+n(\cdot))\Big]\Big\}\zeta_1^2\\ &+\Big\{-d_3(\cdot)+\frac{1}{2}\Big[\frac{a(\cdot)\beta(\cdot)e^{-d_2(\cdot)\tau_2}}{\xi(\cdot)e} + \frac{2b(\cdot)\beta(\cdot)\gamma_0(\cdot)e^{-d_4(\cdot)\tau_1}}{\xi(\cdot)e}(1+n(\cdot))\Big]\Big\}\zeta_2^2\\ &+\Big\{-d_2(\cdot)+\frac{1}{2}\Big[\frac{a(\cdot)\beta(\cdot)}{\xi(\cdot)e} + \frac{a(\cdot)\beta(\cdot)}{\xi(\cdot)e}e^{-d_2(\cdot)\tau_2}\Big]\Big\}\zeta_3^2\end{aligned}$$

$$
\begin{aligned}
&+\Big\{-d_4(\cdot)+\frac{1}{2}\Big[\gamma_0(\cdot)(1+n(\cdot))+\gamma_0(\cdot)e^{-d_4(\cdot)\tau_1}(1+n(\cdot))\Big]\Big\}\zeta_4^2\\
\le&\Big\{-d_2^-+\gamma_0^++2\gamma_0^+e^{-d_4^-\tau_1}+\frac{1}{2}\Big[\frac{a^+\beta^+e^{-d_2^-\tau_2}}{\xi^-e}+\frac{2b^+\beta^+\gamma_0^+e^{-d_4^-\tau_1}}{\xi^-e}(1+n^+)\Big]\\
&+\frac{1}{2}\Big[\frac{a^+\beta^+}{\xi^-e}+\frac{a^+\beta^+}{\xi^-e}e^{-d_2^-\tau_2}\Big]+\frac{1}{2}\Big[\gamma_0^+(1+n^+)+\gamma_0^+e^{-d_4^-\tau_1}(1+n^+)\Big]\Big\}\zeta_1^2\\
&+\Big\{-d_3^-+\frac{1}{2}\Big[\frac{a^+\beta^+e^{-d_2^-\tau_2}}{\xi^-e}+\frac{2b^+\beta^+\gamma_0^+e^{-d_4^-\tau_1}}{\xi^-e}(1+n^+)\Big]\Big\}\zeta_2^2\\
&+\Big\{-d_2^-+\frac{1}{2}\Big[\frac{a^+\beta^+}{\xi^-e}+\frac{a^+\beta^+}{\xi^-e}e^{-d_2^-\tau_2}\Big]\Big\}\zeta_3^2\\
&+\Big\{-d_4^-+\frac{1}{2}\Big[\gamma_0^+(1+n^+)+\gamma_0^+e^{-d_4^-\tau_1}(1+n^+)\Big]\Big\}\zeta_4^2. \qquad (4.14)
\end{aligned}
$$

Take $0<\kappa\le-\min\{\kappa_1,\kappa_2,\kappa_3\}$, then it follows from (H_5) and (4.14) that the formula (4.5) holds. The proof is completed. □

Lemma 4.3. *If* (H_5) *is true, then* $\mathscr{F}_t$ *defined as* (4.3) *is asymptotically compact and* $\mathcal{K}$*-contract in sense of*

$$\lim_{t\to+\infty}\mathcal{K}(\mathscr{F}_t(\mathbb{E}))=0,\ \ for\ any\ bounded\ set\ \mathbb{E}\in\mathscr{X}^+.$$

Proof. For any bounded subset $\mathbb{E}\subset\mathscr{D}$, $\mathscr{F}_t$ is asymptotically compact on $\mathbb{E}\Longleftrightarrow$ For any sequences $t_n\to+\infty$ and $\varphi_n\in\mathscr{D}$, there have common subsequences t_{n_k} and φ_{n_k} such that $\mathscr{F}_{t_{n_k}}(\varphi_{n_k})$ converges in $\mathscr{X}$ as $k\to\infty$. We derive from Theorem 3.1 that $\mathscr{F}_{t_n}(\varphi_n)$ is uniformly bounded on $\overline{\Omega}$. According to Ascoli-Arezlà theorem, it suffices to prove that $\mathscr{F}_{t_n}(\varphi_n)$ is equicontinuous on $\overline{\Omega}$. To this end, let

$$\mathscr{F}_t(\varphi_n)=(X_n(x,t),\mathbf{w}_n(x,t)),\ \ \forall\,\varphi_n\in\mathscr{D},\ \ t\ge0,\ \ x\in\Omega,$$

and

$$(\overline{X}_n(x,t),\overline{\mathbf{w}}_n(x,t))=(X_n(x,t+t_n),\mathbf{w}_n(x,t+t_n)),\ \ \forall\,t\ge t_n,\ \ x\in\Omega.$$

Obviously, $(\overline{X}_n(x,0),\overline{\mathbf{w}}_n(x,0))=\mathscr{F}_{t_n}(\varphi_n)(x)$, for all $n\ge1$, $x\in\Omega$. One knows from Theorem 3.1 that $(\overline{X}_n(x,t),\overline{\mathbf{w}}_n(x,t))$ is also uniformly bounded, for all $n\ge1$, $x\in\Omega$ and $t\ge0$. For arbitrary $x_1,x_2\in\overline{\Omega}$ satisfying $|x_1-x_2|<\delta$, we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2\\
=&2(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))\cdot(\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_1,t))-\mathbf{h}(x_2,\overline{X}_n(x_2,t),\overline{\mathbf{w}}_n(x_2,t)))^T\\
=&2(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))\cdot(\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_1,t))-\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_2,t)))^T\\
&+2(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))\cdot(\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_2,t))-\mathbf{h}(x_2,\overline{X}_n(x_2,t),\overline{\mathbf{w}}_n(x_2,t)))^T. \qquad (4.15)
\end{aligned}
$$

For all $t \geq t_n$, from Lemma 4.2 and (4.15), we have

$$\begin{aligned}
&(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))\cdot(\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_1,t))-\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_2,t)))^T\\
=&(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))\cdot\Big[\int_0^1\frac{\partial\mathbf{h}}{\partial\mathbf{w}}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_1,t)+\mu(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))d\mu\Big]\\
&\cdot(\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t))^T\\
\leq&-\kappa\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2. \qquad (4.16)
\end{aligned}$$

Denote

$$\mathcal{H}_n(x_1,x_2,t)=\|\mathbf{h}(x_1,\overline{X}_n(x_1,t),\overline{\mathbf{w}}_n(x_2,t))-\mathbf{h}(x_2,\overline{X}_n(x_2,t),\overline{\mathbf{w}}_n(x_2,t))\|, \qquad (4.17)$$

then we derive from (4.1) and (4.15)–(4.17) that

$$\frac{\partial}{\partial t}\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2\leq-2\kappa\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2+8\mathcal{M}_1^*\mathcal{H}_n(x_1,x_2,t). \qquad (4.18)$$

By (4.18), we get

$$\frac{\partial}{\partial t}\big[e^{2\kappa t}\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2\big]\leq 8\mathcal{M}_1^*\mathcal{H}_n(x_1,x_2,t)e^{2\kappa t}. \qquad (4.19)$$

For all $t\geq s\geq -t_n$, $x_1,x_2\in\Omega$, integrating from s to t on both side of (4.19), we obtain

$$\begin{aligned}
\|\overline{\mathbf{w}}_n(x_1,t)-\overline{\mathbf{w}}_n(x_2,t)\|^2\leq&\, e^{-2\kappa(t-s)}\|\overline{\mathbf{w}}_n(x_1,s)-\overline{\mathbf{w}}_n(x_2,s)\|^2\\
&+8\mathcal{M}_1^*\int_s^t e^{-2\kappa(t-\eta)}\mathcal{H}_n(x_1,x_2,\eta)d\eta. \qquad (4.20)
\end{aligned}$$

Fixing $t=0$ and $s=-t_n$ in (4.20), we have

$$\begin{aligned}
\|\mathbf{w}_n(x_1,t_n)-\mathbf{w}_n(x_2,t_n)\|^2\leq&\, e^{-2\kappa t_n}\|\mathbf{w}_n(x_1,0)-\mathbf{w}_n(x_2,0)\|^2\\
&+8\mathcal{M}_1^*\int_{t_n}^0 e^{2\kappa\eta}\mathcal{H}_n(x_1,x_2,\eta)d\eta,\ \forall\, n\geq 1,\ x_1,x_2\in\Omega. \qquad (4.21)
\end{aligned}$$

According to Sobolev imbedding theorem and the L^p estimate of parabolic equations, and noting that $(X_n(x,t),\mathbf{w}_n(x,t))=\mathscr{F}_t(\varphi_n)$, $\varphi_n\in\mathbb{E}$, $\forall\, n\geq 1$, $t\geq 0$, $x\in\Omega$, we conclude that $\overline{X}_n(x,0)$ is equicontinuous with respect to $x\in\Omega$. Furthermore, for any given $t^*>0$, when $t\in[-t^*,0]$ with $t_n>t^*$, $\overline{X}_n(x,t)$ is also equicontinuous with respect to $x\in\Omega$. Therefore, we need to show that $\{\mathbf{w}_n(x,t_n)\}_{n=1}^{\infty}$ is equicontinuous, that is, $\forall\,\varepsilon>0$, $\forall\, x_1,x_2\in\Omega$ and $n\geq 1$ (large enough), $\exists\,\delta>0$, when $|x_1-x_2|<\delta$, we have

$$\|\mathbf{w}_n(x_1,t_n)-\mathbf{w}_n(x_2,t_n)\|<\varepsilon. \qquad (4.22)$$

Indeed, $\mathbf{w}_n(x,0)$ is bounded on Ω, there has an integer $n_0\geq 1$ such that

$$e^{-2\kappa t_n}\|\mathbf{w}_n(x_1,0)-\mathbf{w}_n(x_2,0)\|^2<\frac{\varepsilon}{2},\quad n>n_0. \qquad (4.23)$$

In addition, X and $\mathbf{w}$ are bounded $\Rightarrow \mathcal{H}_n$ is also bounded on $\Omega \times \Omega \times [-t_n, 0]$. Therefore, there has $\overline{T} > 0$ such that

$$8\mathcal{M}_1^* \int_{-t_n}^{-\overline{T}} e^{2\kappa\eta}\mathcal{H}_n(x_1, x_2, \eta)d\eta < \frac{\varepsilon}{4}, \quad \forall\, t_n > \overline{T}. \tag{4.24}$$

Moreover, since $\overline{X}_n(x, 0)$ is equicontinuous with respect to $x \in \Omega$, there has $\delta > 0$ satisfying $|x_1 - x_2| < \delta$ such that

$$\mathcal{H}_n(x_1, x_2, \eta) < \frac{\kappa\varepsilon}{16\mathcal{M}_1^*},$$

which implies that

$$8\mathcal{M}_1^* \int_{-\overline{T}}^{0} e^{2\kappa\eta}\mathcal{H}_n(x_1, x_2, \eta)d\eta < \frac{\varepsilon}{4}. \tag{4.25}$$

Thus, it follows from (4.21) and (4.23)–(4.25) that (4.22) holds. As a result, $\mathscr{F}_t$ is asymptotically compact on $\mathbb{E}$. Based on Lemma 23 in [27], we know that the omega limit set $\omega(\mathbb{E})$ is nonempty, compact and invariant, and attracts $\mathbb{E}$, which shows that $\mathscr{F}_t$ is asymptotically smooth. Thank to Lemma 2.1 in [28], we have

$$\mathcal{K}(\mathscr{F}_t(\mathbb{E})) \leq \mathcal{K}(\omega(\mathbb{E})) + \rho(\mathscr{F}_t(\mathbb{E}), \omega(\mathbb{E})) = \rho(\mathscr{F}_t(\mathbb{E}), \omega(\mathbb{E})), \tag{4.26}$$

where $\rho(\mathscr{F}_t(\mathbb{E}), \omega(\mathbb{E}))$ stands for the distance between $\mathscr{F}_t(\mathbb{E})$ and $\omega(\mathbb{E})$. Taking the limit $t \to +\infty$ on the both side of (4.26), we obtain $\lim_{t\to+\infty} \mathcal{K}(\mathscr{F}_t(\mathbb{E})) = 0$. Therefore, $\mathscr{F}_t$ is $\mathcal{K}$-contract. The proof is completed. □

By applying Theorem 3.1 and Lemma 4.3, we conclude that Theorem 4.1 holds.

5. An example and numerical simulation

It is worth noting that some parameters such as r, a and b in system (2.1) cannot be estimated by experimental design, which is shown by the authors of [14]. Therefore, the following numerical simulation only reveals the global dynamic behaviour of model (2.2) mathematically.

Let $\Omega = (0, 1) \subset \mathbb{R}$, $D = 2$, $d_1(\cdot) \equiv 0.2$, $r(\cdot) \equiv 0.5$, $d_2(\cdot) \equiv 0.6$, $d_3(\cdot) \equiv 0.3$, $d_4(\cdot) \equiv 0.5$, $a(\cdot) \equiv 1$, $\beta(\cdot) \equiv 2$, $\xi(\cdot) \equiv 4$, $\gamma_0(\cdot) \equiv 0.1$, $b(\cdot) \equiv 5$, $\theta(\cdot) \equiv 0.2$, $n(\cdot) \equiv 0.3$, $\tau_1 = \tau_2 = 1$, $\phi_k = 0.1, k = 1, 2, 3, 4, 5$. Obviously, the conditions (H_1)–(H_4) hold. From Theorem 3.1, we conclude that system (2.2) has a unique bounded solution.

Next, we verify the condition (H_5) is true. Indeed, a direct calculation gives

$$\kappa_3 = -d_4^- + \frac{1}{2}[\gamma_0^+(1 + n^+) + \gamma_0^+ e^{-d_4^-\tau_1}(1 + n^+)] \approx -0.3956 < 0,$$

$$\kappa_2 = -d_3^- + \frac{1}{2}\Big[\frac{a^+\beta^+ e^{-d_2^-\tau_2}}{\xi^- e} + \frac{2b^+\beta^+\gamma_0^+ e^{-d_4^-\tau_1}}{\xi^- e}(1 + n^+)\Big] \approx -0.1770 < 0,$$

and

$$\kappa_1 = -d_2^- + \gamma_0^+ + 2\gamma_0^+ e^{-d_4^-\tau_1} + \frac{1}{2}\left[\frac{a^+\beta^+ e^{-d_2^-\tau_2}}{\xi^- e} + \frac{2b^+\beta^+\gamma_0^+ e^{-d_4^-\tau_1}}{\xi^- e}(1+n^+)\right]$$
$$+ \frac{1}{2}\left[\frac{a^+\beta^+}{\xi^- e} + \frac{a^+\beta^+}{\xi^- e}e^{-d_2^-\tau_2}\right] + \frac{1}{2}\left[\gamma_0^+(1+n^+) + \gamma_0^+ e^{-d_4^-\tau_1}(1+n^+)\right] \approx -0.0088 < 0.$$

Thus, it follows from Theorem 4.1 that system (2.2) has a global compact attractor.

With the help of the Pdepe toolbox in MATLAB 2018b, we now give the numerical solutions and simulations of the example. Taking the initial values $X_0 = 1$, $Y_0 = 0.8$, $Z_0 = 0.9$, $U_0 = 0.8$, and $V_0 = 0.8$, when $x = 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95$, and $t = 1.55, 1.60, 1.65, 1.70, 1.75, 1.80, 1.85, 1.90, 1.95$, the numerical solutions of $X(x,t)$, $Y(x,t)$, $Z(x,t)$, $U(x,t)$ and $V(x,t)$ are as given in Tables 1–5, respectively. Figures 2–6 are their respective numerical simulations. It can be seen from the numerical solutions and simulation diagrams that this example has a positive solution of global attraction.

Table 1. The numerical solution $X(x,t)$.

t \ x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1.55	1.1259	1.1007	1.0753	1.0500	1.0249	0.9997	0.9745	0.9493	0.9239
1.60	1.1409	1.1154	1.0892	1.0632	1.0371	1.0111	0.9851	0.9590	0.9329
1.65	1.1547	1.1289	1.1021	1.0755	1.0489	1.0222	0.9956	0.9689	0.9422
1.70	1.1681	1.1419	1.1146	1.0874	1.0601	1.0329	1.0057	0.9785	0.9511
1.75	1.1811	1.1546	1.1267	1.0988	1.0710	1.0432	1.0154	0.9876	0.9597
1.80	1.1936	1.1668	1.1383	1.1099	1.0815	1.0531	1.0247	0.9963	0.9678
1.85	1.2056	1.1786	1.1495	1.1205	1.0915	1.0626	1.0336	1.0046	0.9756
1.90	1.2173	1.1899	1.1603	1.1307	1.1012	1.0717	1.0422	1.0126	0.9830
1.95	1.2285	1.2008	1.1706	1.1405	1.1105	1.0804	1.0503	1.0202	0.9900

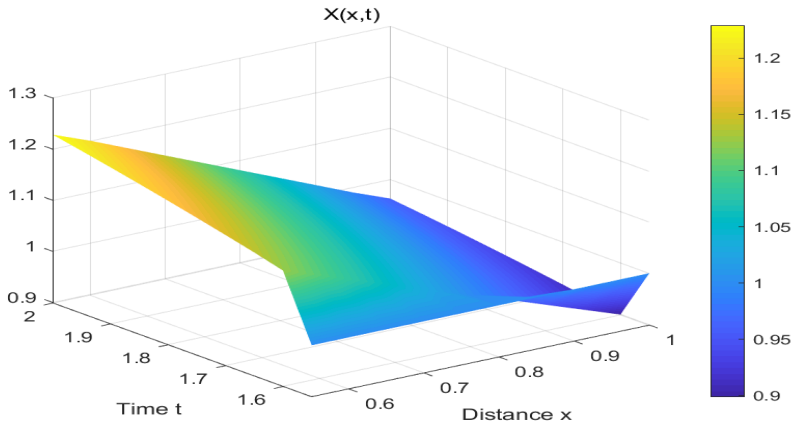

Figure 2. Simulation of solution of $X(x,t)$

Table 2. The numerical solution $Y(x,t)$.

t \ x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1.55	2.7597	0.7675	0.7752	0.7751	0.7750	0.7749	0.7748	0.7747	0.7826
1.60	4.6898	0.7207	0.7513	0.7510	0.7508	0.7507	0.7505	0.7501	0.7817
1.65	6.5911	0.6601	0.7285	0.7277	0.7275	0.7272	0.7269	0.7260	0.7966
1.70	8.4642	0.5862	0.7068	0.7052	0.7049	0.7045	0.7041	0.7024	0.8268
1.75	10.3098	0.4994	0.6863	0.6834	0.6830	0.6826	0.6821	0.6790	0.8719
1.80	12.1286	0.4000	0.6670	0.6623	0.6619	0.6614	0.6608	0.6558	0.9312
1.85	13.9211	0.2885	0.6490	0.6419	0.6414	0.6408	0.6403	0.6327	1.0045
1.90	15.6882	0.1653	0.6323	0.6221	0.6216	0.6210	0.6204	0.6095	1.0910
1.95	17.4303	0.0308	0.6171	0.6029	0.6025	0.6018	0.6012	0.5861	1.1905

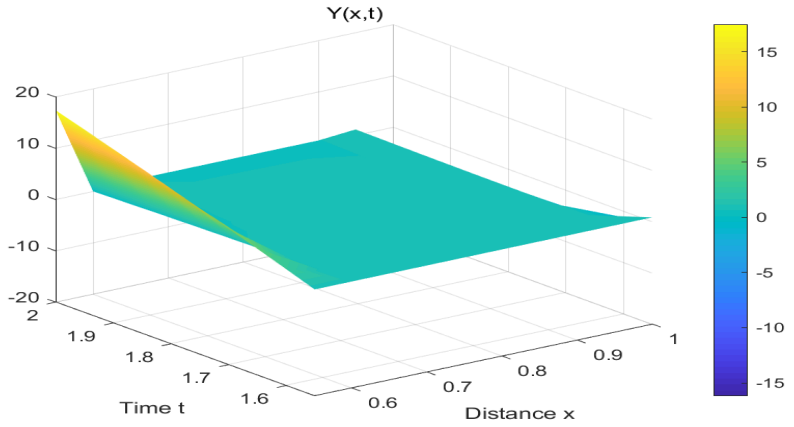

Figure 3. Simulation of solution of $Y(x,t)$.

Table 3. The numerical solution $Z(x,t)$.

t \ x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1.55	2.8801	0.8837	0.8873	0.8874	0.8874	0.8875	0.8875	0.8876	0.8910
1.60	4.8462	0.8605	0.8748	0.8748	0.8749	0.8750	0.8752	0.8753	0.8890
1.65	6.7984	0.8304	0.8624	0.8625	0.8626	0.8628	0.8629	0.8630	0.8939
1.70	8.7367	0.7936	0.8503	0.8502	0.8504	0.8506	0.8509	0.8509	0.9055
1.75	10.6612	0.7500	0.8384	0.8381	0.8384	0.8386	0.8389	0.8388	0.9239
1.80	12.5721	0.6998	0.8268	0.8262	0.8265	0.8268	0.8271	0.8267	0.9490
1.85	14.4695	0.6431	0.8155	0.8144	0.8147	0.8151	0.8155	0.8147	0.9807
1.90	16.3535	0.5799	0.8044	0.8027	0.8031	0.8035	0.8039	0.8026	1.0190
1.95	18.2242	0.5103	0.7937	0.7912	0.7916	0.7921	0.7926	0.7905	1.0638

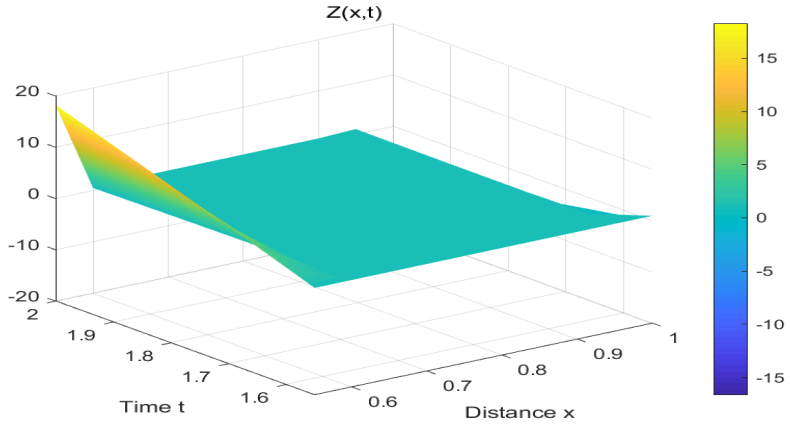

Figure 4. Simulation of solution of $Y(x,t)$.

Table 4. The numerical solution $U(x,t)$.

t \ x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1.55	2.7623	0.7696	0.7769	0.7769	0.7770	0.7770	0.7770	0.7771	0.7843
1.60	4.6963	0.7255	0.7546	0.7545	0.7546	0.7547	0.7547	0.7547	0.7833
1.65	6.6029	0.6682	0.7331	0.7327	0.7328	0.7329	0.7331	0.7327	0.7966
1.70	8.4824	0.5980	0.7125	0.7115	0.7116	0.7118	0.7120	0.7111	0.8238
1.75	10.3354	0.5152	0.6929	0.6909	0.6911	0.6913	0.6915	0.6896	0.8646
1.80	12.1625	0.4203	0.6744	0.6708	0.6711	0.6713	0.6716	0.6682	0.9185
1.85	13.9642	0.3135	0.6570	0.6513	0.6516	0.6519	0.6523	0.6469	0.9852
1.90	15.7409	0.1952	0.6408	0.6324	0.6328	0.6331	0.6335	0.6254	1.0645
1.95	17.4933	0.0657	0.6259	0.6139	0.6144	0.6147	0.6153	0.6038	1.1558

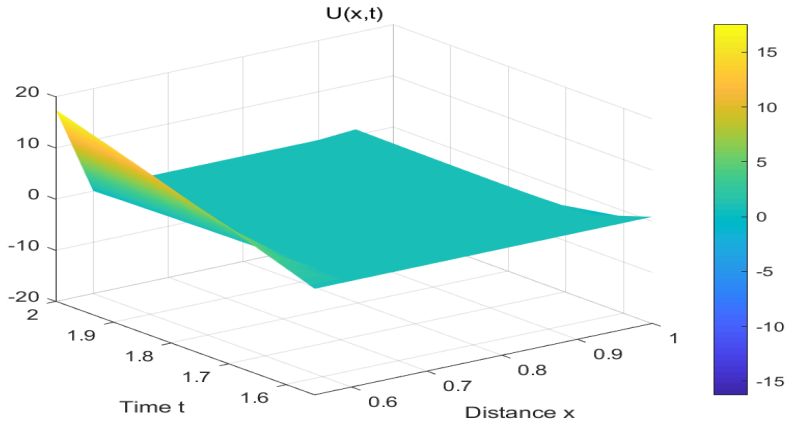

Figure 5. Simulation of solution of $Y(x,t)$.

Table 5. The numerical solution $V(x,t)$.

t \ x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1.55	2.7679	0.7742	0.7804	0.7804	0.7804	0.7804	0.7803	0.7803	0.7865
1.60	4.7119	0.7366	0.7613	0.7612	0.7612	0.7612	0.7612	0.7611	0.7858
1.65	6.6322	0.6877	0.7429	0.7425	0.7425	0.7425	0.7425	0.7421	0.7974
1.70	8.5294	0.6277	0.7251	0.7243	0.7243	0.7243	0.7243	0.7234	0.8210
1.75	10.4038	0.5568	0.7081	0.7065	0.7065	0.7065	0.7065	0.7049	0.8564
1.80	12.2559	0.4753	0.6919	0.6891	0.6892	0.6891	0.6892	0.6864	0.9032
1.85	14.0860	0.3834	0.6765	0.6722	0.6722	0.6722	0.6723	0.6680	0.9612
1.90	15.8946	0.2815	0.6620	0.6557	0.6558	0.6557	0.6558	0.6495	1.0302
1.95	17.6821	0.1698	0.6484	0.6396	0.6397	0.6397	0.6398	0.6309	1.1098

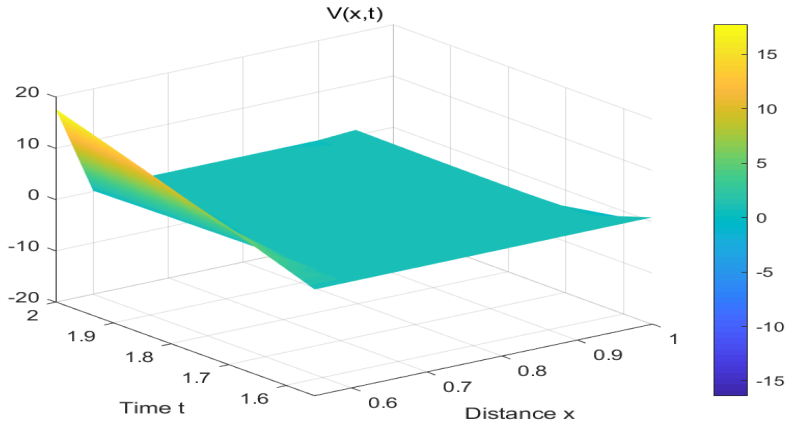

Figure 6. Simulation of solution of $Y(x,t)$.

6. Conclusions and outlooks

Prostate cancer is a common type of malignant tumour and one of the main diseases causing death in males. Many researchers have conducted extensive and in-depth experimental and theoretical research on prostate cancer. In this manuscript, we applied the PDE model (2.2) to explore the spatial distribution and evolution of prostate cancer cells and androgen in tumours. Applying C_0-semigroup theory and the comparison principle of PDEs, we obtain the global existence, uniqueness and boundedness of the solution of our model (2.2). At the same time, we obtain some sufficient conditions for the existence of global invariant sets and global attractors by the solution semiflow and Kuratowski's measure of noncompactness. Our theoretical results show that the time delay has no effect on the global existence, uniqueness and boundedness of the solution. However, the time delay and natural cell death rate will have a great impact on global attraction. The numerical solution and simulation of an example verify the correctness and effectiveness of our theoretical outcomes. Our results show that the evolution of prostate cancer cells and androgens over time is not only attractive, but also asymptotically stable. Our research provides certain theoretical support for the prevention

and treatment of human prostate cancer. In addition, enlightened by some recent research [29–42], we can further apply fractional calculus theory, fixed point theory, coincidence theory and Lyapunov stability theory to study human diseases in the future, reveal the pathogenesis of these diseases through mathematical methods, and make certain contributions to the prevention and treatment of human diseases.

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Conflict of interest

The author declares that there is no competing interest.

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