Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14399–14425.
DOI: 10.3934/math.2023736
Received: 04 February 2023
Revised: 10 April 2023
Accepted: 12 April 2023
Published: 19 April 2023

Research article

Refined stability of the additive, quartic and sextic functional equations with counter-examples

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Abstract: In this study, we utilize the direct method (Hyers approach) to examine the refined stability of the additive, quartic, and sextic functional equations in modular spaces with and without the Δ_2-condition. We also use the direct approach to discuss the Ulam stability in 2-Banach spaces. Ultimately, we ensure that stability of above equations does not hold in a particular scenario by utilizing appropriate counter-examples.

Keywords: refined stability; modular space; 2-Banach space; additive; quartic and sextic functional equations
Mathematics Subject Classification: 39B52, 39B72, 39B82

1. Introduction and basic facts

In many different settings, functional equations are essential to the investigation of stability problems. The first in challenging the stability of group homomorphisms was Ulam [1]. His work laid the groundwork for subsequent research on stability phenomena. If an equation allows only one unique solution, we refer to that equation as being stable. Ulam [1] formulated the following Cauchy functional equation:

$$\Xi(s_1 + s_2) = \Xi(s_1) + \Xi(s_2).$$

In the context of a Banach space, Hyers [2] addressed Cauchy's functional equation in order to resolve this problems. Aoki [3] improved the work of Hyers by taking an unbounded Cauchy difference. Rassias [4] discussed additive mappings in his study, and Găvruţa [5] has already given identical results. For more details about the stability results, see [2, 6–13].

In 1950, Nakano [14] investigated the idea of modular linear spaces. Numerous writers have now extensively verified these hypotheses, e.g., Luxemburg [15], Amemiya [16], Musielak [17], Koshi [18], Mazur [19], Turpin [20] and Orlicz [21]. Both Orlicz spaces [22] and the concept of interpolation [17, 22] have several applications in the setting of modular spaces.

Several researchers examined stability in modular spaces via a fixed point approach of quasi-contractions without utilizing the Δ_2-condition, as suggested by Khamsi [23]. In recent years, Sadeghi [24] produced results on stability of some functional equations combining the Δ_2-condition with the Fatou property.

First, we review some terminology, notations, and common characteristics of the theory of given spaces.

Definition 1.1. *[24] Let Q be a linear space over $\Bbbk$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\varrho : Q \to [0, \infty)$ is said to be modular if the hypotheses below hold for all $\varpi, \rho \in Q$:*

(m_1) $\varrho(\varpi) = 0 \Leftrightarrow \varpi = 0$;
(m_2) $\varrho(a\varpi) = \varrho(\varpi)$ *for any scalar a with* $|a| = 1$;
(m_3) $\varrho(a_1\varpi + a_2\rho) \leq \varrho(\varpi) + \varrho(\rho)$ *for any scalar $a_1, a_2 \geq 0$ with* $a_1 + a_2 = 1$.

Also, ϱ is said to be convex modular, if the hypothesis (m_3) is replaced by

(m_3') $\varrho(a_1\varpi + a_2\rho) \leq a_1\varrho(\varpi) + a_2\varrho(\rho)$ *for any scalar $a_1, a_2 \geq 0$ with* $a_1 + a_2 = 1$.

Additionally, the vector space induced by a modular ϱ,

$$Q_\varrho = \{\varpi : \varrho(c\varpi) \to 0, \text{ as } c \to \infty\},$$

is a modular space (MS, for short). Denote by $\mathbb{N}$ the set of positive integers.

Definition 1.2. *[24] Let $\{\varpi_\mu\}$ be a sequence in an MS Q_ϱ.*

(i) If $\varrho(\varpi_j - \varpi) \to 0$ as $j \to \infty$, then $\{\varpi_j\}$ is called ϱ-convergent to a point ϖ and we write $\varpi_j \to \varpi$ as $j \to \infty$.
(ii) If $\varrho(\varpi_j - \varpi_\xi) < \epsilon$ for any $\epsilon > 0$ and for sufficiently large $j, \xi \in \mathbb{N}$, then $\{\varpi_j\}$ is called ϱ-Cauchy.
(iii) $K \subseteq Q_\varrho$ is called ρ-complete if any ϱ-Cauchy sequence is ϱ-convergent.
(iv) A modular ϱ is said to satisfy the Δ_2-condition if $\varrho(2\varpi_j) \to 0$ as $j \to \infty$, whenever $\varrho(\varpi_j) \to 0$ as $j \to \infty$.

If $\varrho(\varpi) \leq \liminf_{\mu\to\infty} \varrho(\varpi_\mu)$, the modular ϱ possesses the Fatou property, whereas the sequence $\{\varpi_\mu\}$ is ϱ-convergent to ϖ in the MS Q_ϱ and vice versa.

Proposition 1.1. *[25] In MSs,*

(1) if $\varpi_\alpha \to \varpi$ and λ is a constant vector, then $\varpi_\alpha + \lambda \to \varpi + \lambda$;
(2) if $\varpi_\alpha \to \varpi$ and $\rho_\alpha \to \rho$, then $a_1\varpi_\alpha + a_2\rho_\alpha \to a_1\varpi + a_2\rho$, where $a_1, a_2 \geq 0$ and $a_1 + a_2 \leq 1$.

Remark 1.1. *Suppose that ρ is convex and justifies the Δ_2-condition with Δ_2-constant $r > 0$. If $r < 2$, then $\varrho(\varpi) \leq r\varrho(\frac{\varpi}{2}) \leq \frac{r}{2}\varrho(\varpi)$, which suggests $\varrho = 0$. Therefore, if ρ is a convex modular, we ought to obtain the Δ_2-constant $r \geq 2$.*

It is clear that if μ is chosen from the analogous scalar field with $|\mu| > 1$ in MSs, then the convergence of a sequence $\{\varpi_\alpha\}$ to ϖ does not imply that $\{\mu\varpi_\alpha\}$ converges to $\mu\varpi$. This is due to the fact that in MSs, the multiples of the convergent sequence $\{\varpi_\alpha\}$ are convergent naturally.

In 1960, the idea of linear 2-normed spaces was created by Gahler [26] as follows:

Definition 1.3. *Assume that Λ over $\mathbb{R}$ is a linear space with* $\dim \Lambda > 1$ *and a function* $\|., .\| : \Lambda\times\Lambda \to \mathbb{R}$ *is such that for all $\varpi, \rho, \ell \in \Lambda$ and $\vartheta \in \mathbb{R}$,*

(i) $\|\varpi, \rho\| = 0$ *iff ϖ and ρ are linearly dependent;*
(ii) $\|\varpi, \rho\| = \|\rho, \varpi\|$;
(iii) $\|\vartheta\varpi, \rho\| = |\vartheta|\,\|\varpi, \rho\|$;
(iv) $\|\varpi, \ell + \rho\| \leq \|\varpi, \ell\| + \|\varpi, \rho\|$.

Then the function $\|., .\|$ is called a 2-norm on Λ, and $(\Lambda, \|., .\|)$ is called a linear 2-normed space (2-NS, for short).

For example of a 2-NS, consider $\mathbb{R}^2$ endowed with a 2-norm defined by $|\varpi - \rho|$ =the area of the triangle with vertices 0, ϖ and ρ.

It should be noted that, the assertion (iv) implies that

$$\|\varpi + \ell, \rho\| \leq \|\varpi, \rho\| + \|\ell, \rho\| \text{ and } |\|\varpi, \rho\| - \|\ell, \rho\|| \leq \|\varpi - \ell, \rho\|.$$

Hence, the mapping $\varpi \to \|\varpi, \rho\|$ is continuous from Λ onto $\mathbb{R}$, for any fixed $\rho \in \mathbb{R}$.

Definition 1.4. *Let Λ be a linear 2-NS and $\{\varpi_j\}_{j\geq 1}$ be a sequence in Λ.*

(1) A sequence $\{\varpi_j\}_{j\geq 1}$ is called convergent if there exists an element $\varpi \in \Lambda$ such that

$$\lim_{j\to\infty} \left\|\varpi_j - \varpi, \ell\right\| = 0 \textit{ for every } \ell \in \Lambda.$$

If $\{\varpi_j\}_{j\geq 1}$ converges to ϖ, then we can write $\varpi_j \to \varpi$ as $j \to \infty$ or $\lim_{j\to\infty} \varpi_j = \varpi$ and we say that ϖ is a limit point of $\{\varpi_j\}_{j\geq 1}$.

(2) Assume that $\ell, \rho \in \Lambda$ such that ℓ and ρ are linearly independent. Then $\{\varpi_j\}_{j\geq 1}$ is called a Cauchy sequence in Λ, if

$$\lim_{j,v\to\infty} \left\|\varpi_j - \varpi_v, \ell\right\| = 0,$$

and

$$\lim_{j,v\to\infty} \left\|\varpi_j - \varpi_v, \rho\right\| = 0.$$

Definition 1.5. *A linear 2-NS in which every Cauchy sequence is a convergent sequence is called a 2-Banach space (2-BS, for short).*

Lemma 1.1. *[27] Assume that* $(\Lambda, \|., .\|)$ *is a* 2*-NS. If* $\varpi \in \Lambda$ *and* $\|\varpi, \rho\| = 0$ *for each* $\rho \in \Lambda$*, then* $\rho = 0$.

Lemma 1.2. *[27] Let* $\{\varpi_j\}_{j\geq 1}$ *be a convergent sequence in a linear* 2*-NS* Λ*, then,*

$$\lim_{j\to\infty} \left\|\varpi_j, \ell\right\| = \left\|\lim_{j\to\infty} \varpi_j, \ell\right\| \text{ for all } \ell \in \Lambda.$$

Sadeghi [24] has confirmed the stability findings of functional equations utilizing the Fatou property and the Δ_2-condition in modular spaces. Our paper is aimed to discuss the refined stability of additive, quartic and sextic functional equations

$$\begin{aligned} & \Omega\left(\frac{(s_1 - s_2) + (s_3 - s_2)}{m} + s_4\right) + \Omega\left(\frac{(s_2 - s_3) + (s_4 - s_3)}{m} + s_1\right) \\ & +\Omega\left(\frac{(s_3 - s_4) + (s_1 - s_4)}{m} + s_2\right) + \Omega\left(\frac{(s_4 - s_1) + (s_2 - s_1)}{m} + s_3\right) \\ = \; & \Omega(s_1 + s_2 + s_3 + s_4), \end{aligned}$$

$$\begin{aligned} & \Omega(3s_1 + s_2) + \Omega(3s_1 - s_2) \\ = \; & 9\Omega(s_1 + s_2) + 9\Omega(s_1 - s_2) + 144\Omega(s_1) - 16\Omega(s_2), \end{aligned}$$

and

$$\begin{aligned} & \Omega(s_1 + 3s_2) - 6\Omega(s_1 + 2s_2) + 15\Omega(s_1 + s_2) - 20\Omega(s_1) \\ & +15\Omega(s_1 - s_2) - 6\Omega(s_1 - 2s_2) + \Omega(s_1 - 3s_2) \\ = \; & 720\Omega(s_2), \end{aligned}$$

respectively, in MSs with and without the Δ_2-condition and by the direct technique. Additionally, the Ulam stability in 2-BSs is examined. Finally, we show that the stability of these equations does not hold in a particular scenario using appropriate counter-examples.

2. Stability analysis

Here, we apply the direct technique to examine the stability theorems of the additive, quartic, and sextic functional equation. These results are considered as an improvement of forms due to Wongkum [28] and Sadeghi [24]. We assume here Λ is a linear space and Q_ϱ is a complete convex MS.

2.1. Additive functional equation and stability study

Kim [29] in 2013 studied the stability of the additive functional equation in fuzzy BSs. Inspired by the technique of Kim [29], we aim to study the stability of the additive functional equation:

$$\begin{aligned}
& \Omega\left(\frac{(s_1-s_2)+(s_3-s_2)}{m}+s_4\right)+\Omega\left(\frac{(s_2-s_3)+(s_4-s_3)}{m}+s_1\right) \\
& +\Omega\left(\frac{(s_3-s_4)+(s_1-s_4)}{m}+s_2\right)+\Omega\left(\frac{(s_4-s_1)+(s_2-s_1)}{m}+s_3\right) \\
= \; & \Omega\left(s_1+s_2+s_3+s_4\right) \qquad (2.1)
\end{aligned}$$

for any $m > 0$ in modular spaces by ignoring the conditions of Δ_2. For the convenience of notation, define the mapping $\Omega : \Lambda \to Q_\varrho$ as

$$\begin{aligned}
\Delta\Omega\left(s_1, s_2, s_3, s_4\right) = \; & \Omega\left(\frac{s_1-s_2}{m}+s_3+s_4\right)+\Omega\left(\frac{s_2-s_3}{m}+s_4+s_1\right)+\Omega\left(\frac{s_3-s_4}{m}+s_1+s_2\right) \\
& +\Omega\left(\frac{s_4-s_1}{m}+s_2+s_3\right)-\Omega\left(s_1+s_2+s_3+s_4\right),
\end{aligned}$$

where $s_1, s_2, s_3, s_4 \in \Lambda$, and m is a fixed nonzero integer.

Theorem 2.1. *Assume that there is a function $\Xi : \Lambda^4 \to [0, \infty)$ defined by*

$$\Xi\left(s_1, s_2, s_3, s_4\right) = \sum_{\mu=1}^{\infty} \frac{1}{4^\mu} E\left(4^{\mu-1} s_1, 4^{\mu-1} s_2, 4^{\mu-1} s_3, 4^{\mu-1} s_4\right) < \infty, \qquad (2.2)$$

such that a mapping $\Omega : \Lambda \to Q_\varrho$ satisfies $\Omega(0) = 0$ and for all $s_1, s_2, s_3, s_4 \in \Lambda$,

$$\varrho\left(\Delta\Omega\left(s_1, s_2, s_3, s_4\right)\right) \leq \Xi\left(s_1, s_2, s_3, s_4\right). \qquad (2.3)$$

Then there exists a unique additive mapping (AM) $W : \Lambda \to Q_\varrho$ fulfilling

$$\varrho\left(\Omega\left(s_1\right) - W(s_1)\right) \leq \Xi\left(s_1, s_1, s_1, s_1\right) \text{ for all } s_1 \in \Lambda. \qquad (2.4)$$

Proof. Putting $s_1 = s_2 = s_3 = s_4$ in (2.3), and setting $\Xi\left(s_1, s_1, s_1, s_1\right) = U(s_1)$, we have

$$\varrho\left(4\Omega\left(s_1\right) - \Omega\left(4s_1\right)\right) \leq \Xi\left(s_1, s_1, s_1, s_1\right) = U(s_1). \qquad (2.5)$$

Hence,

$$\varrho\left(\Omega\left(s_1\right) - \frac{1}{4}\Omega\left(4s_1\right)\right) \leq \frac{1}{4}U(s_1). \qquad (2.6)$$

Based on a mathematical induction, one can deduce that

$$\varrho\left(\Omega\left(s_1\right) - \frac{\Omega\left(4^\mu s_1\right)}{4^\mu}\right) \leq \sum_{j=1}^{\mu} \frac{1}{4^j} U(4^{j-1} s_1), \qquad (2.7)$$

for all $s_1 \in \Lambda$ and all natural numbers μ. Clearly, (2.6) follows immediately form (2.7) if we take $\mu = 1$. Assume that the inequality (2.7) is true for $\mu \in \mathbb{N}$, then we get

$$\begin{aligned}\varrho\left(\Omega\left(s_1\right)-\frac{\Omega\left(4^{\mu+1}s_1\right)}{4^{\mu+1}}\right) &= \varrho\left(\frac{1}{4}\left(\Omega(4s_1)-\frac{\Omega\left(4^{\mu}4s_1\right)}{4^{\mu}}\right)+\frac{1}{4}\left(4\Omega\left(s_1\right)-\Omega(4s_1)\right)\right)\\ &\leq \frac{1}{4}\varrho\left(\Omega(4s_1)-\frac{\Omega\left(4^{\mu}4s_1\right)}{4^{\mu}}\right)+\frac{1}{4}\varrho\left(4\Omega\left(s_1\right)-\Omega(4s_1)\right)\\ &\leq \frac{1}{4}\sum_{j=1}^{\mu}\frac{1}{4^j}U(4^js_1)+\frac{1}{4}U(s_1)\\ &= \sum_{j=1}^{\mu}\frac{1}{4^{j+1}}U(4^js_1)+\frac{1}{4}U(s_1)\\ &= \sum_{j=1}^{\mu+1}\frac{1}{4^j}U(4^{j-1}s_1).\end{aligned}$$

It follows that the inequality (2.7) is true for every $\mu \in \mathbb{N}$. Suppose that θ and η are natural numbers with $\theta < \eta$. Using (2.7), we can write

$$\begin{aligned}\varrho\left(\frac{\Omega\left(4^{\eta}s_1\right)}{4^{\eta}}-\frac{\Omega\left(4^{\theta}s_1\right)}{4^{\theta}}\right) &= \varrho\left(\frac{1}{4^{\theta}}\left(\frac{\Omega\left(4^{\eta-\theta}s_1\right)}{4^{\eta-\theta}}-\Omega\left(4^{\theta}s_1\right)\right)\right)\\ &\leq \frac{1}{4^{\theta}}\sum_{j=1}^{\eta-\theta}\frac{U(4^{j-1}4^{\theta}s_1)}{4^j}\\ &= \sum_{j=1}^{\eta-\theta}\frac{U(4^{\theta+j-1}s_1)}{4^{\theta+j}}\\ &= \sum_{\mu=\theta+1}^{\eta}\frac{U(4^{\mu-1}s_1)}{4^{\mu}}.\end{aligned}\tag{2.8}$$

Inequalities (2.2) and (2.8) illustrate that $\left\{\frac{\Omega(4^{\eta}s_1)}{4^{\eta}}\right\}$ is ϱ-Cauchy sequence in Q_{ϱ}. Since Q_{ϱ} is ϱ-complete, one can say $\left\{\frac{\Omega(4^{\eta}s_1)}{4^{\eta}}\right\}$ is ϱ-convergent. Now, describe the mapping $W:\Lambda\to Q_{\varrho}$ as

$$W(s_1)=\lim_{\eta\to\infty}\frac{\Omega\left(4^{\eta}s_1\right)}{4^{\eta}},\ \ s_1\in\Lambda.\tag{2.9}$$

Hence,

$$\begin{aligned}\varrho\left(\frac{4W\left(s_1\right)-W\left(4s_1\right)}{4^4}\right) &= \varrho\left(\frac{1}{4^4}\left(\frac{\Omega\left(4^{\eta+1}s_1\right)}{4^{\eta}}-W\left(4s_1\right)\right)+\frac{1}{4^2}\left(\frac{1}{4}W\left(s_1\right)-\frac{1}{4}\frac{\Omega\left(4^{\eta+1}s_1\right)}{4^{\eta+1}}\right)\right)\\ &\leq \frac{1}{4^4}\varrho\left(W\left(4s_1\right)-\frac{\Omega\left(4^{\eta+1}s_1\right)}{4^{\eta}}\right)+\frac{1}{4^3}\varrho\left(\frac{\Omega\left(4^{\eta+1}s_1\right)}{4^{\eta+1}}-W\left(s_1\right)\right),\end{aligned}\tag{2.10}$$

for all $s_1 \in \Lambda$. Applying (2.9) in (2.10) after taking the limit as $\eta \to \infty$, we find that the right-hand side of (2.10) tends to 0. Thus, one gets

$$4W\left(s_1\right)=W\left(4s_1\right),\ \text{for all}\ s_1\in\Lambda.\tag{2.11}$$

Also, for all $\eta \in \mathbb{N}$, by (2.11), we observe that

$$\begin{aligned}
&\varrho\left(\Omega\left(s_1\right)-W(s_1)\right)\\
= \;& \varrho\left(\sum_{\mu=1}^{\eta}\frac{4\Omega\left(4^{\mu-1}s_1\right)-\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}+\left(\frac{\Omega\left(4^{\eta}s_1\right)}{4^{\eta}}-W(s_1)\right)\right)\\
= \;& \varrho\left(\sum_{\mu=1}^{\eta}\frac{4\Omega\left(4^{\mu-1}s_1\right)-\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}+\frac{1}{4}\left(\frac{\Omega\left(4^{\eta-1}4s_1\right)}{4^{\eta-1}}-W(4s_1)\right)\right). \qquad (2.12)
\end{aligned}$$

Since $\sum\limits_{\mu=1}^{\eta}\frac{1}{4^{\mu}}+\frac{1}{4}<1$, by (2.5) and (2.12), one can write

$$\begin{aligned}
&\varrho\left(\Omega\left(s_1\right)-W(s_1)\right)\\
\leq \;& \sum_{\mu=1}^{\eta}\frac{1}{4^{\mu}}\varrho\left(4\Omega\left(4^{\mu-1}s_1\right)-\Omega\left(4^{\mu}s_1\right)\right)+\frac{1}{4}\varrho\left(\frac{\Omega\left(4^{\eta-1}4s_1\right)}{4^{\eta-1}}-W(4s_1)\right)\\
\leq \;& \sum_{\mu=1}^{\eta}\frac{1}{4^{\mu}}U\left(4^{\mu-1}s_1\right)+\frac{1}{4}\varrho\left(\frac{\Omega\left(4^{\eta-1}4s_1\right)}{4^{\eta-1}}-W(4s_1)\right)\\
= \;& \sum_{\mu=1}^{\eta}\frac{1}{4^{\mu}}\Xi\left(4^{\mu-1}s_1,4^{\mu-1}s_1,4^{\mu-1}s_1,4^{\mu-1}s_1\right)+\frac{1}{4}\varrho\left(\frac{\Omega\left(4^{\eta-1}4s_1\right)}{4^{\eta-1}}-W(4s_1)\right). \qquad (2.13)
\end{aligned}$$

Passing to the limit as $\eta\to\infty$ in (2.13), we have

$$\varrho\left(\Omega\left(s_1\right)-W(s_1)\right)\leq\Xi\left(s_1,s_1,s_1,s_1\right)\ \text{ for all } s_1\in\Lambda.$$

Therefore, the inequality (2.4) is true. Now, we shall prove that W is an AM. It is easy to observe that

$$\begin{aligned}
\varrho\left(\frac{1}{4^j}\Delta\Omega\left(4^js_1,4^js_2,4^js_3,4^js_4\right)\right) &\leq \frac{1}{4^j}\varrho\left(\Delta\Omega\left(4^js_1,4^js_2,4^js_3,4^js_4\right)\right)\\
&\leq \frac{1}{4^j}\Xi\left(4^js_1,4^js_2,4^js_3,4^js_4\right), \qquad (2.14)
\end{aligned}$$

for all $s_1,s_2,s_3,s_4\in\Lambda$. When $j\to\infty$ in (2.14), we get $\varrho\left(\Delta W\left(s_1,s_2,s_3,s_4\right)\right)\to 0$. Hence,

$$\Delta W\left(s_1,s_2,s_3,s_4\right)=0.$$

This implies that W is an additive mapping. For the uniqueness, assume that W_1 and W_2 are two AMs that satisfy (2.4). Then,

$$\begin{aligned}
\varrho\left(\frac{W_1(s_1)-W_2(s_1)}{2}\right) &= \varrho\left(\frac{1}{2}\left(\frac{W_1\left(4^{\mu}s_1\right)}{4^{\mu}}-\frac{\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}\right)+\frac{1}{2}\left(\frac{\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}-\frac{W_2\left(4^{\mu}s_1\right)}{4^{\mu}}\right)\right)\\
&\leq \frac{1}{2}\varrho\left(\frac{W_1\left(4^{\mu}s_1\right)}{4^{\mu}}-\frac{\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}\right)+\frac{1}{2}\varrho\left(\frac{\Omega\left(4^{\mu}s_1\right)}{4^{\mu}}-\frac{W_2\left(4^{\mu}s_1\right)}{4^{\mu}}\right)\\
&\leq \frac{1}{2}\frac{1}{4^{\mu}}\left[\varrho\left(W_1\left(4^{\mu}s_1\right)-\Omega\left(4^{\mu}s_1\right)\right)+\varrho\left(W_2\left(4^{\mu}s_1\right)-\Omega\left(4^{\mu}s_1\right)\right)\right]
\end{aligned}$$

$$\leq \frac{1}{4^{\mu}}\Xi\left(4^{\mu}s_1, 4^{\mu}s_1, 4^{\mu}s_1, 4^{\mu}s_1\right)$$
$$\leq \sum_{u=\mu+1}^{\infty}\frac{1}{4^{u}}E\left(4^{u-1}s_1, 4^{u-1}s_1, 4^{u-1}s_1, 4^{u-1}s_1\right) \to 0, \text{ as } u \to \infty,$$

which yields that $W_1 = W_2$. This finishes the proof. □

The corollaries below follow immediately from Theorem 2.1:

Corollary 2.1. *If there exists a mapping $\Omega : \Lambda \to Q_\varrho$ such that $\Omega(0) = 0$ and*

$$\varrho\left(\Delta\Omega\left(s_1, s_2, s_3, s_4\right)\right) \leq \varepsilon,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$, then there exists a unique AM $W : \Lambda \to Q_\varrho$ satisfying

$$\varrho\left(\Omega\left(s_1\right) - W(s_1)\right) \leq \frac{\varepsilon}{2} \textit{ for all } s_1 \in \Lambda.$$

Corollary 2.2. *If there exists a mapping $\Omega : \Lambda \to Q_\varrho$ such that $\Omega(0) = 0$ and*

$$\varrho\left(\Delta\Omega\left(s_1, s_2, s_3, s_4\right)\right) \leq \xi\left(\|s_1\|^p + \|s_2\|^p + \|s_3\|^p + \|s_4\|^p\right)$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$, $\xi > 0$ and $p \in (0, 1)$, then there exists a unique AM $W : \Lambda \to Q_\varrho$ fulfilling

$$\varrho\left(\Omega\left(s_1\right) - W(s_1)\right) \leq \frac{4\xi}{4 - 4^p}\|s_1\|^p \textit{ for all } s_1 \in \Lambda.$$

In the context of MSs, we present another stability result as in Theorem 2.1 with condition Δ_2 as follows:

Theorem 2.2. *Let Q be a linear space and Q_ϱ fulfill the Δ_2-condition with the mapping $\Omega : \Lambda \to Q_\varrho$ such that*

$$\varrho\left(\Delta\Omega\left(s_1, s_2, s_3, s_4\right)\right) \leq \Xi\left(s_1, s_2, s_3, s_4\right),$$

and

$$\lim_{\mu\to\infty} u^{\mu}\Xi\left(\frac{s_1}{4^{\mu}}, \frac{s_2}{4^{\mu}}, \frac{s_3}{4^{\mu}}, \frac{s_4}{4^{\mu}}\right) = 0 \textit{ and } \sum_{j=1}^{\infty}\left(\frac{u^2}{4}\right)^j \Xi\left(\frac{s_1}{4^{\mu}}, \frac{s_1}{4^{\mu}}, \frac{s_1}{4^{\mu}}, \frac{s_1}{4^{\mu}}\right) < \infty,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$. Then there exists a unique AM $W : \Lambda \to Q_\varrho$, described as

$$W(s_1) = \lim_{\mu\to\infty} 4^{\mu}\Omega\left(\frac{s_1}{4^{\mu}}\right),$$

and

$$\varrho\left(\Omega\left(s_1\right) - W(s_1)\right) \leq \frac{\alpha}{4u}\sum_{j=1}^{\infty}\left(\frac{u^2}{4}\right)^j \Xi\left(\frac{s_1}{2^j}, \frac{s_1}{2^j}, \frac{s_1}{2^j}, \frac{s_1}{2^j}\right),$$

for all $s_1 \in \Lambda$.

Proof. Since ϱ verifies the Δ_2-condition with α, Eq (2.3) implies that

$$\varrho\left(\Delta\Omega\left(s_1, s_2, s_3, s_4\right)\right) \leq \alpha\Xi\left(s_1, s_2, s_3, s_4\right) \text{ for all } s_1, s_2, s_3, s_4 \in \Lambda.$$

So, the proof of Theorem 2.1 directly leads to the conclusion. □

2.2. Quartic functional equation and stability study

In this part, without using the Fatou property, the refined and Ulam stability of the following quartic functional equation are investigated:

$$\Omega(3s_1+s_2)+\Omega(3s_1-s_2)=9\Omega(s_1+s_2)+9\Omega(s_1-s_2)+144\Omega(s_1)-16\Omega(s_2), \tag{2.15}$$

in modular spaces Q_ϱ. For ease of notations, we can define a mapping $\Omega : \Lambda \to Q_\varrho$ as

$$\Delta\Omega(s_1,s_2)=\Omega(3s_1+s_2)+\Omega(3s_1-s_2)+16\Omega(s_2)-144\Omega(s_1)-9\Omega(s_1+s_2)-9\Omega(s_1-s_2),$$

for all $s_1, s_2 \in \Lambda$.

Theorem 2.3. *Let Q be a linear space and Q_ϱ fulfill the Δ_2-condition with a mapping $\Xi : \Lambda\times\Lambda \to [0,\infty)$. Suppose also there exists a mapping $\Omega : \Lambda \to Q_\varrho$ such that*

$$\varrho(\Delta\Omega(s_1,s_2)) \le \Xi(s_1,s_2), \tag{2.16}$$

$$\lim_{\mu\to\infty} u^{4\mu}\Xi\left(\frac{s_1}{3^\mu},\frac{s_2}{3^\mu}\right)=0 \text{ and } \sum_{j=1}^{\infty}\left(\frac{u^4}{3}\right)^j \Xi\left(\frac{s_1}{3^j},0\right)<\infty,$$

for all $s_1, s_2 \in \Lambda$. Then there exists a unique quartic mapping (QM) $W : \Lambda \to Q_\varrho$ described as

$$W(s_1)=\lim_{\mu\to\infty} 3^{4\mu}\Omega\left(\frac{s_1}{3^\mu}\right),$$

and

$$\varrho(\Omega(s_1)-W(s_1)) \le \frac{1}{2u}\sum_{j=1}^{\infty}\left(\frac{u^4}{3}\right)^j \Xi\left(\frac{s_1}{3^j},0\right), \tag{2.17}$$

for all $s_1 \in \Lambda$.

Proof. Consider $\Omega(0)=0$ in view of $\Xi(0,0)=0$ along the convergence of

$$\sum_{j=1}^{\infty}\left(\frac{u^4}{3}\right)^j \Xi(0,0)<\infty.$$

Letting $s_2=0$ in (2.16), we have

$$\varrho\left(2\Omega(3s_1)-2\times 3^4\Omega(s_1)\right) \le \Xi(s_1,0) \text{ for all } s_1\in\Lambda.$$

Since $\sum_{j=1}^{\infty}\frac{1}{2^j}<1$, based on Δ_2-condition of ϱ, the subsequent functional inequality can be written as

$$\begin{aligned}\varrho\left(\Omega(s_1)-3^{4\mu}\Omega\left(\frac{s_1}{3^\mu}\right)\right) &= \varrho\left(\sum_{j=1}^{\mu}\frac{1}{3^j}\left(3^{4j-3}\Omega\left(\frac{s_1}{3^{j-1}}\right)-3^{5j}\Omega\left(\frac{s_1}{3^j}\right)\right)\right)\\ &\le \frac{1}{u^3}\sum_{j=1}^{\mu}\left(\frac{u^4}{3}\right)^j \Xi\left(\frac{s_1}{3^j},0\right) \text{ for all } s_1\in\Lambda. \end{aligned} \tag{2.18}$$

Now, in (2.18), replacing s_1 with $\frac{s_1}{3^\mu}$, we conclude that the series in (2.16) converges, and

$$\begin{aligned}
\varrho\left(3^{4\theta}\Omega\left(\frac{s_1}{3^\mu}\right)-3^{4(\theta+\mu)}\Omega\left(\frac{s_1}{3^{\theta+\mu}}\right)\right) &\leq u^{4\theta}\varrho\left(\Omega\left(\frac{s_1}{3^\mu}\right)-3^{4\mu}\Omega\left(\frac{s_1}{3^{\theta+\mu}}\right)\right)\\
&\leq u^{4\theta-3}\sum_{j=1}^{\mu}\left(\frac{u^4}{3}\right)^j\Xi\left(\frac{s_1}{3^{j+\theta}},0\right)\\
&\leq \frac{3^\theta}{u^{\theta+3}}\sum_{j=\theta+1}^{\mu+\beta}\left(\frac{u^4}{3}\right)^j\Xi\left(\frac{s_1}{3^j},0\right),
\end{aligned}$$

for all $s_1\in\Lambda$. Since $\frac{3^\theta}{u^{\theta+3}}\leq 1$, the right-hand side of the above inequality tends to 0. This proves that $\left\{3^{4\mu}\Omega\left(\frac{s_1}{3^\mu}\right)\right\}$ is a ϱ-Cauchy sequence in Q_ϱ. Since Q_ϱ is ϱ-complete, it is ϱ-convergent in Q_ϱ. Define the mapping $W:\Lambda\to Q_\varrho$ by

$$W(s_1)=\varrho\left(\lim_{\eta\to\infty}3^{4\mu}\Omega\left(\frac{s_1}{3^\mu}\right)\right)\text{ for all } s_1\in\Lambda,$$

that is,

$$\lim_{\eta\to\infty}\varrho\left(3^{4\mu}\Omega\left(\frac{s_1}{3^\mu}\right)-W(s_1)\right)=0\text{ for all } s_1\in\Lambda.$$

Now, consider

$$\begin{aligned}
\varrho\left(\Omega(s_1)-W(s_1)\right) &\leq \frac{1}{2}\varrho\left(2\Omega(s_1)-2\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)\right)+\frac{1}{2}\varrho\left(2\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)-2W(s_1)\right)\\
&\leq \frac{u}{2}\varrho\left(\Omega(s_1)-\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)\right)+\frac{u}{2}\varrho\left(\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)-W(s_1)\right)\\
&\leq \frac{1}{2u^2}\sum_{j=1}^{\mu}\left(\frac{u^4}{3}\right)^j\Xi\left(\frac{s_1}{3^j},0\right)+\frac{u}{2}\varrho\left(\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)-W(s_1)\right),
\end{aligned}$$

for all $s_1\in\Lambda$ and all $\mu>1$. Thus, the inequality is founded without utilizing the Fatou property. Letting $\mu\to\infty$, we have estimate of (2.17) of Ω as W. Replacing (s_1,s_2) with $\left(\frac{s_1}{3^\mu},\frac{s_2}{3^\mu}\right)$ in (2.16), one gets

$$\varrho\left(3^{4\mu}\Delta\Omega\left(\frac{s_1}{3^\mu},\frac{s_2}{3^\mu}\right)\right)\leq u^{4\mu}\Xi\left(\frac{s_1}{3^\mu},\frac{s_2}{3^\mu}\right)\to 0,\text{ as }\mu\to\infty.$$

It follows from the convexity of ϱ that

$$\begin{aligned}
&\varrho\left(\frac{1}{235}W(3s_1+s_2)+\frac{1}{235}W(3s_1-s_2)+\frac{16}{235}W(s_2)\right.\\
&\left.-\frac{9}{235}W(s_1+s_2)-\frac{9}{235}W(s_1-s_2)-\frac{144}{235}W(s_1)\right)\\
\leq\ & \frac{1}{235}\varrho\left(W(3s_1+s_2)-3^{4\mu}\Omega\left(\frac{3s_1+s_2}{3^\mu}\right)\right)+\frac{1}{235}\varrho\left(W(3s_1-s_2)-3^{4\mu}\Omega\left(\frac{3s_1-s_2}{3^\mu}\right)\right)\\
&+\frac{16}{235}\varrho\left(W(s_2)-16\left(3^{4\mu}\right)\Omega\left(\frac{s_2}{3^\mu}\right)\right)+\frac{9}{235}\varrho\left(W(s_1+s_2)-9\left(3^{4\mu}\right)\Omega\left(\frac{s_1+s_2}{3^\mu}\right)\right)\\
&+\frac{9}{235}\varrho\left(W(s_1-s_2)-9\left(3^{4\mu}\right)\Omega\left(\frac{s_1-s_2}{3^\mu}\right)\right)+\frac{144}{235}\varrho\left(W(s_1)-144\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)\right)
\end{aligned}$$

$$+\frac{1}{235}\varrho\left(3^{4\mu}\Omega\left(\frac{3s_1+s_2}{3^\mu}\right)+3^{4\mu}\Omega\left(\frac{3s_1-s_2}{3^\mu}\right)+16\left(3^{4\mu}\right)\Omega\left(\frac{s_2}{3^\mu}\right)\right.$$
$$\left.+9\left(3^{4\mu}\right)\Omega\left(\frac{s_1+s_2}{3^\mu}\right)+9\left(3^{4\mu}\right)\Omega\left(\frac{s_1-s_2}{3^\mu}\right)+144\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)\right),$$

for all $s_1, s_2 \in \Lambda$. Then the function W is a quartic (it is enough to let $\mu \to \infty$).

For the uniqueness, suppose that $W^* : \Lambda \to Q_\varrho$ is a QM such that

$$\varrho\left(\Omega\left(s_1\right)-W^*(s_1)\right) \leq \frac{1}{2u}\sum_{j=1}^{\infty}\left(\frac{u^4}{3}\right)^j \Xi\left(\frac{s_1}{3^j},0\right) \text{ for all } s_1 \in \Lambda.$$

Then, from the equations $W(3^{-\mu}s_1) = 3^{-4\mu}W(s_1)$ and $W^*(3^{-\mu}s_1) = 3^{-4\mu}W^*(s_1)$, one can write

$$\begin{aligned}
\varrho\left(W\left(s_1\right)-W^*(s_1)\right) &\leq \frac{1}{3}\varrho\left(3\left(3^{4\mu}\right)W\left(\frac{s_1}{3^\mu}\right)-3\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)\right)\\
&\quad+\frac{1}{3}\varrho\left(3\left(3^{4\mu}\right)\Omega\left(\frac{s_1}{3^\mu}\right)-3\left(3^{4\mu}\right)W^*\left(\frac{s_1}{3^\mu}\right)\right)\\
&\leq \frac{u^{4\mu+1}}{3}\varrho\left(W\left(\frac{s_1}{3^\mu}\right)-\Omega\left(\frac{s_1}{3^\mu}\right)\right)+\frac{u^{4\mu+1}}{3}\varrho\left(\Omega\left(\frac{s_1}{3^\mu}\right)-W^*\left(\frac{s_1}{3^\mu}\right)\right)\\
&\leq \frac{3^{\mu-1}}{2u^{\mu+3}}\sum_{j=1}^{\mu}\left(\frac{u^4}{3}\right)^j \Xi\left(\frac{s_1}{3^j},0\right), \text{ for all } s_1 \in \Lambda,
\end{aligned}$$

for all sufficiently large integers μ. Letting $\mu \to \infty$, we conclude that $W\left(s_1\right) = W^*(s_1)$, for all $s_1 \in \Lambda$. This completes the proof. □

Corollary 2.3. *Suppose that* $(\Lambda, \|.\|)$ *is a normed space and* Q_ϱ *fulfills* Δ_2*-condition. Assume also there are* $\xi > 0$, $p > \log_3 \frac{u^4}{3}$ *and the mapping* $\Omega : \Lambda \to Q_\varrho$ *such that*

$$\varrho\left(\Delta\Omega\left(s_1, s_2\right)\right) \leq \xi\left(\|s_1\|^p + \|s_2\|^p\right) \text{ for all } s_1, s_2 \in \Lambda.$$

Then there exists a unique QM $W : \Lambda \to Q_\varrho$ *fulfilling*

$$\Omega\left(s_1\right) - W(s_1) \leq \frac{u^3\xi}{3^{p+1}-u^4} \text{ for all } s_1 \in \Lambda.$$

Without utilizing the Δ_2-condition or the Fatou property, we provide another stability result in an MS.

Theorem 2.4. *Assume that there are a mapping* $\Omega : \Lambda \to Q_\varrho$ *that fulfills (2.16) and a function* $\Xi : \Lambda \times \Lambda \to [0, \infty)$ *such that*

$$\lim_{\mu\to\infty}\frac{\Xi\left(3^\mu s_1, 3^\mu s_2\right)}{3^{4\mu}} = 0 \text{ and } \sum_{j=1}^{\infty}\frac{\Xi\left(3^j s_1, 0\right)}{3^{4j}} < \infty \text{ for all } s_1, s_2 \in \Lambda.$$

Then there exists a unique QM $W : \Lambda \to Q_\varrho$ *fulfilling*

$$\varrho\left(\Omega\left(s_1\right)-\frac{1}{4}\Omega\left(0\right)-W(s_1)\right) \leq \frac{1}{3^4}\sum_{j=1}^{\infty}\frac{\Xi\left(3^j s_1, 0\right)}{3^{4j}} \text{ for all } s_1 \in \Lambda.$$

Proof. Setting $s_2 = 0$ in (2.16), one can write

$$\varrho\left(2\Omega(3s_1) - 2\times 3^4\Omega(s_1)\right) \le \Xi(s_1, 0), \text{ for all } s_1 \in \Lambda. \tag{2.19}$$

Taking $\Omega'(s_1) = \Omega(s_1) - \frac{1}{4}\Omega(0)$, then by the convexity of ϱ and using the fact $\sum_{j=0}^{\mu-1} \frac{1}{3^{4(j+1)}} < 1$, we get

$$\begin{aligned}
\varrho\left(\Omega'(s_1) - \frac{\Omega'(3^\mu s_1)}{3^{4\mu}}\right) &\le \varrho\left(\sum_{j=0}^{\mu-1}\left(\frac{3^4\Omega'\left(3^j s_1\right) - \Omega'\left(3^{j+1}s_1\right)}{3^{4(j+1)}}\right)\right) \\
&\le \sum_{j=0}^{\mu-1}\frac{\varrho\left(3^4\Omega'\left(3^j s_1\right) - \Omega'\left(3^{j+1}s_1\right)\right)}{3^{4(j+1)}} \\
&\le \frac{1}{3^4}\sum_{j=0}^{\mu-1}\frac{\Xi\left(3^j s_1, 0\right)}{3^{4j}} \quad \text{for all } s_1 \in \Lambda,\ \mu \in \mathbb{N}.
\end{aligned}$$

Then one gets $\left\{3^{4\mu}\Omega'\left(\frac{s_1}{3^\mu}\right)\right\}$ is a ϱ-Cauchy sequence and the mapping $W : \Lambda \to Q_\varrho$ is defined as

$$W(s_1) = \varrho\left(\lim_{\mu\to\infty}\frac{\Omega'(3^\mu s_1)}{3^{4\mu}}\right),$$

that is

$$\varrho\left(\lim_{\mu\to\infty}\frac{\Omega'(3^\mu s_1)}{3^{4\mu}} - W(s_1)\right) = 0 \text{ for all } s_1 \in \Lambda,$$

without utilizing the Δ_2-condition and the Fatou property. Furthermore, it is clear from the proof used in Theorem 2.3 that the mapping W satisfies the quartic functional equation.

Now, using the Fatou property and the Δ_2-condition, we demonstrate that (2.19) is true. According to the convexity of ϱ and using the fact $\sum_{j=0}^{\mu-1}\frac{1}{3^{4(j+1)}} + \frac{1}{3^4} < 1$, we have

$$\begin{aligned}
\varrho\left(\Omega'(s_1) - W(s_1)\right) &= \varrho\left(\sum_{j=1}^{\mu-1}\left(\frac{3^4\Omega'\left(3^j s_1\right) - \Omega'\left(3^{j+1}s_1\right)}{3^{4(j+1)}}\right) + \frac{\Omega'(3^\mu s_1)}{3^{4\mu}} - \frac{W(3s_1)}{3^4}\right) \\
&\le \sum_{j=0}^{\mu-1}\frac{1}{3^{4(j+1)}}\varrho\left(3^4\Omega'\left(3^j s_1\right) - \Omega'\left(3^{j+1}s_1\right)\right) + \frac{1}{3^4}\varrho\left(\frac{\Omega'\left(3^{\mu-1}3s_1\right)}{3^{4(\mu-1)}} - W(3s_1)\right) \\
&\le \frac{1}{3^4}\sum_{j=0}^{\mu-1}\frac{1}{3^{4j}}\Xi\left(3^j s_1, 0\right) + \frac{1}{3^4}\varrho\left(\frac{\Omega'\left(3^{\mu-1}3s_1\right)}{3^{4(\mu-1)}} - W(3s_1)\right),
\end{aligned}$$

for all $s_1 \in \Lambda$ and all natural number $\mu > 1$. Letting $\mu \to \infty$ in the above inequality, we get our desired result. □

Corollary 2.4. *Assume that there exists a function* $\Xi : \Lambda\times\Lambda \to [0, \infty)$ *such that*

$$\lim_{\mu\to\infty}\frac{\Xi(3^\mu s_1, 3^\mu s_2)}{3^{4\mu}} = 0 \ \textit{ and } \ \Xi(3s_1, 0) < 3^4 M\Xi(s_1, 0) \textit{ for all } s_1, s_2 \in \Lambda,$$

where $M \in (0,1)$. *If there exists a mapping* $\Omega : \Lambda \to Q_\varrho$ *fulfilling (2.16), then there exists a unique QM* $W : \Lambda \to Q_\varrho$ *fulfilling*

$$\varrho\left(\Omega(s_1) - \frac{1}{4}\Omega(0) - W(s_1)\right) \leq \frac{\Xi(s_1, 0)}{3^4(1-M)} \text{ for all } s_1 \in \Lambda.$$

Corollary 2.5. *Let* $(\Lambda, \|.\|)$ *be a normed linear space. If there are the real numbers* $\xi > 0$, $\varepsilon > 0$ *and a mapping* $\Omega : \Lambda \to Q_\varrho$ *such that*

$$\varrho(\Delta\Omega(s_1, s_2)) \leq \xi(\|s_1\|^p + \|s_2\|^p) + \varepsilon,$$

for all $s_1, s_2 \in \Lambda$, *then there exists a unique QM* $W : \Lambda \to Q_\varrho$ *fulfilling*

$$\varrho\left(\Omega(s_1) - \frac{1}{4}\Omega(0) - W(s_1)\right) \leq \frac{3\xi}{3^4 - 3^p}\|s_1\|^p + \frac{\varepsilon}{4} \text{ for all } s_1, s_2 \in \Lambda,$$

where $s_1 \neq 0$ *if* $p < 0$.

2.3. Sextic functional equation and stability study

Here, without using the Fatou property, the refined Ulam stability of the following sextic functional equation is introduced:

$$\begin{aligned} & \Omega(s_1 + 3s_2) - 6\Omega(s_1 + 2s_2) + 15\Omega(s_1 + s_2) - 20\Omega(s_1) \\ & +15\Omega(s_1 - s_2) - 6\Omega(s_1 - 2s_2) + \Omega(s_1 - 3s_2) \\ = \; & 720\Omega(s_2), \end{aligned}$$

in an MS Q_ϱ.

Theorem 2.5. *Let* Q *be a linear space and* Q_ϱ *fulfilling the* Δ_2*-condition with a mapping* $\Xi : \Lambda \times \Lambda \to [0, \infty)$. *Assume also there is a mapping* $\Omega : \Lambda \to Q_\varrho$ *such that*

$$\left.\begin{aligned} & \varrho(\Omega(s_1 + 3s_2) - 6\Omega(s_1 + 2s_2) + 15\Omega(s_1 + s_2) - 20\Omega(s_1) \\ & +15\Omega(s_1 - s_2) - 6\Omega(s_1 - 2s_2) + \Omega(s_1 - 3s_2) - 720\Omega(s_2)) \end{aligned}\right\} \leq \Xi(s_1, s_2), \tag{2.20}$$

and

$$\lim_{\mu\to\infty} u^{6\mu}\Xi\left(\frac{s_1}{2^\mu}, \frac{s_2}{2^\mu}\right) = 0, \text{ and } \sum_{j=1}^{\infty}\left(\frac{u^7}{2}\right)^j \Xi\left(\frac{s_1}{2^j}, 0\right) < \infty,$$

for all $s_1, s_2 \in \Lambda$. *Then there exists a unique sextic mapping (SM)* $W : \Lambda \to Q_\varrho$, *described as*

$$W(s_1) = \lim_{\mu\to\infty} 2^{6\mu}\Omega\left(\frac{s_1}{2^\mu}\right),$$

and

$$\varrho(\Omega(s_1) - W(s_1)) \leq \frac{1}{2u}\sum_{j=1}^{\infty}\left(\frac{u^7}{2}\right)^j \Xi\left(\frac{s_1}{2^j}, \frac{s_1}{2^j}\right), \tag{2.21}$$

for all $s_1 \in \Lambda$.

Proof. Firstly, let $\Omega(0) = 0$ in view of $\Xi(0,0) = 0$ along the convergence of

$$\sum_{j=1}^{\infty}\left(\frac{u^7}{2}\right)^j \Xi(0,0) < \infty.$$

Setting $s_1 = s_2$ in (2.20), we get

$$\varrho\left(\Omega(4s_1) - 6\Omega(3s_1) + 15\Omega(2s_1) - 20\Omega(s_1) - 6\Omega(-s_1) + \Omega(-2s_1) - 720\Omega(s_1)\right) \leq \Xi(s_1, s_1).$$

Since $\sum\limits_{j=1}^{\infty} \frac{1}{2^j} < 1$, and using Δ_2-condition of ϱ, the next functional inequality can be written as

$$\begin{aligned} \varrho\left(\Omega(s_1) - 2^{6\mu}\Omega\left(\frac{s_1}{3^\mu}\right)\right) &= \varrho\left(\sum_{j=1}^{\mu}\frac{1}{2^j}\left(3^{7j-6}\Omega\left(\frac{s_1}{3^{j-1}}\right) - 2^{6j}\Omega\left(\frac{s_1}{2^j}\right)\right)\right) \\ &\leq \frac{1}{u^6}\sum_{j=1}^{\mu}\left(\frac{u^6}{2}\right)^j \Xi\left(\frac{s_1}{2^j}, \frac{s_1}{2^j}\right), \text{ for all } s_1 \in \Lambda. \end{aligned} \tag{2.22}$$

Replacing s_1 with $\frac{s_1}{2^\mu}$ in (2.22), we see that the series in (2.16) converges, and

$$\begin{aligned} \varrho\left(2^{6\theta}\Omega\left(\frac{s_1}{2^\mu}\right) - 2^{6(\theta+\mu)}\Omega\left(\frac{s_1}{6^{\theta+\mu}}\right)\right) &\leq u^{6\theta}\varrho\left(\Omega\left(\frac{s_1}{2^\mu}\right) - 2^{6\mu}\Omega\left(\frac{s_1}{2^{\theta+\mu}}\right)\right) \\ &\leq u^{6\theta-6}\sum_{j=1}^{\mu}\left(\frac{u^7}{2}\right)^j \Xi\left(\frac{s_1}{2^{j+\theta}}, \frac{s_1}{2^{j+\theta}}\right) \\ &\leq \frac{2^\theta}{u^{\theta+5}}\sum_{j=\theta+1}^{\mu+\beta}\left(\frac{u^7}{2}\right)^j \Xi\left(\frac{s_1}{2^j}, \frac{s_1}{2^j}\right), \end{aligned}$$

for all $s_1 \in \Lambda$, which goes to 0 as $\theta \to \infty$ since $\frac{2}{u} \leq 1$, then the right-hand side of the above inequality tends to 0. This proves that $\left\{2^{6\mu}\Omega\left(\frac{s_1}{2^\mu}\right)\right\}$ is a ϱ-Cauchy sequence for all $s_1 \in \Lambda$ and it is ϱ-convergent in Q_ϱ. Hence, we can define the mapping $W : \Lambda \to Q_\varrho$ by

$$W(s_1) = \varrho\left(\lim_{\eta\to\infty} 2^{6\mu}\Omega\left(\frac{s_1}{2^\mu}\right)\right), \text{ that is, } \lim_{\eta\to\infty}\varrho\left(2^{6\mu}\Omega\left(\frac{s_1}{2^\mu}\right) - W(s_1)\right) = 0,$$

for all $s_1 \in \Lambda$. Consequently, without employing the Fatou property from the Δ_2-condition, the following inequality

$$\begin{aligned} \varrho\left(\Omega(s_1) - W(s_1)\right) &\leq \frac{1}{2}\varrho\left(2\Omega(s_1) - 2\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\right) + \frac{1}{2}\varrho\left(2\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right) - 2W(s_1)\right) \\ &\leq \frac{u}{2}\varrho\left(\Omega(s_1) - \left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\right) + \frac{u}{2}\varrho\left(\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right) - W(s_1)\right) \\ &\leq \frac{1}{2u}\sum_{j=1}^{\mu}\left(\frac{u^7}{2}\right)^j \Xi\left(\frac{s_1}{2^j}, \frac{s_1}{2^j}\right) + \frac{u}{2}\varrho\left(2^{6\mu}\Omega\left(\frac{s_1}{2^\mu}\right) - W(s_1)\right), \end{aligned}$$

is true for all $s_1 \in \Lambda$ and a nature number $\mu > 1$. Letting $\mu \to \infty$, we have estimate of (2.17) in Ω by W. Replacing (s_1, s_2) with $\left(\frac{s_1}{2^\mu}, \frac{s_2}{2^\mu}\right)$ in (2.21), we obtain that

$$\begin{aligned}
&\varrho\Bigg(2^{6\mu}\Omega\left(\frac{s_1+3s_2}{2^\mu}\right)-6\left(2^{6\mu}\right)\Omega\left(\frac{s_1+2s_2}{2^\mu}\right)+15\left(2^{6\mu}\right)\Omega\left(\frac{s_1+s_2}{2^\mu}\right)-20\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\\
&+15\left(2^{6\mu}\right)\Omega\left(\frac{s_1-s_2}{2^\mu}\right)-6\left(2^{6\mu}\right)\Omega\left(\frac{s_1-2s_2}{2^\mu}\right)+2^{6\mu}\Omega\left(\frac{s_1-3s_2}{2^\mu}\right)-720\Omega\left(\frac{s_2}{2^\mu}\right)\Bigg)\\
\le\ & u^{6\mu}\Xi\left(\frac{s_1}{2^\mu},\frac{s_2}{2^\mu}\right)\to 0,\ \text{as } \mu\to\infty,\ \text{for all } s_1,s_2\in\Lambda.
\end{aligned}$$

It follows from the convexity of ϱ that

$$\begin{aligned}
&\varrho\Bigg(\frac{1}{784}W(s_1+3s_2)-\frac{6}{784}W(s_1+2s_2)+\frac{15}{784}W(s_1+s_2)-\frac{20}{784}W(s_1)\\
&+\frac{15}{784}W(s_1-s_2)-\frac{6}{784}W(s_1-2s_2)+\frac{1}{784}W(s_1-3s_2)-\frac{720}{784}W(s_2)\Bigg)\\
\le\ & \frac{1}{784}\varrho\left(W(s_1+3s_2)-2^{6\mu}\Omega\left(\frac{s_1+3s_2}{3^\mu}\right)\right)+\frac{6}{784}\varrho\left(W(s_1+2s_2)-6\left(2^{6\mu}\right)\Omega\left(\frac{s_1+2s_2}{2^\mu}\right)\right)\\
&+\frac{15}{784}\varrho\left(W(s_1+s_2)-15\left(2^{6\mu}\right)\Omega\left(\frac{s_1+s_2}{2^\mu}\right)\right)+\frac{20}{784}\varrho\left(W(s_1)-20\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\right)\\
&+\frac{15}{784}\varrho\left(W(s_1-s_2)-15\left(2^{6\mu}\right)\Omega\left(\frac{s_1-s_2}{2^\mu}\right)\right)+\frac{6}{784}\varrho\left(W(s_1-2s_2)-6\left(2^{6\mu}\right)\Omega\left(\frac{s_1-2s_2}{2^\mu}\right)\right)\\
&+\frac{1}{784}\varrho\left(W(s_1-3s_2)-12^{6\mu}\Omega\left(\frac{s_1-3s_2}{2^\mu}\right)\right)+\frac{720}{784}\varrho\left(W(s_2)-720\left(2^{6\mu}\right)\Omega\left(\frac{s_2}{2^\mu}\right)\right)\\
&+\frac{1}{784}\varrho\Bigg(2^{6\mu}\Omega\left(\frac{s_1+3s_2}{3^\mu}\right)+6\left(2^{6\mu}\right)\Omega\left(\frac{s_1+2s_2}{2^\mu}\right)+15\left(2^{6\mu}\right)\Omega\left(\frac{s_1+s_2}{2^\mu}\right)+20\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\\
&+15\left(2^{6\mu}\right)\Omega\left(\frac{s_1-s_2}{2^\mu}\right)+6\left(2^{6\mu}\right)\Omega\left(\frac{s_1-2s_2}{2^\mu}\right)+12^{6\mu}\Omega\left(\frac{s_1-3s_2}{2^\mu}\right)+720\left(2^{6\mu}\right)\Omega\left(\frac{s_2}{2^\mu}\right)\Bigg),
\end{aligned}$$

for all $s_1, s_2 \in \Lambda$. Therefore, the mapping W is sextic (it is enough to let $\mu \to \infty$).

For the uniqueness, let $W'' : \Lambda \to Q_\varrho$ be another SM satisfying

$$\varrho\left(\Omega(s_1)-W''(s_1)\right)\le\frac{1}{2u}\sum_{j=1}^{\infty}\left(\frac{u^7}{2}\right)^j\Xi\left(\frac{s_1}{2^j},\frac{s_1}{2^j}\right),\ \text{for all } s_1\in\Lambda.$$

From the equations $W(2^{-\mu}s_1) = 2^{-6\mu}W(s_1)$ and $W''(2^{-\mu}s_1) = 2^{-6\mu}W''(s_1)$, we have

$$\begin{aligned}
\varrho\left(W(s_1)-W''(s_1)\right)\ &\le\ \frac{1}{2}\varrho\left(2\left(2^{6\mu}\right)W\left(\frac{s_1}{2^\mu}\right)-2\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)\right)+\frac{1}{2}\varrho\left(2\left(2^{6\mu}\right)\Omega\left(\frac{s_1}{2^\mu}\right)-2\left(2^{6\mu}\right)W''\left(\frac{s_1}{2^\mu}\right)\right)\\
&\le\ \frac{u^{6\mu+1}}{2}\varrho\left(W\left(\frac{s_1}{2^\mu}\right)-\Omega\left(\frac{s_1}{2^\mu}\right)\right)+\frac{u^{6\mu+1}}{2}\varrho\left(\Omega\left(\frac{s_1}{2^\mu}\right)-W''\left(\frac{s_1}{2^\mu}\right)\right)\\
&\le\ \frac{u^{6\mu}}{2}\sum_{j=1}^{\infty}\left(\frac{u^7}{2}\right)^j\Xi\left(\frac{s_1}{2^{j+\mu}},\frac{s_1}{2^{j+\mu}}\right)\le\frac{2^{\mu-1}}{u^{\mu+3}}\sum_{j=1}^{\mu}\left(\frac{u^7}{2}\right)^j\Xi\left(\frac{s_1}{2^j},\frac{s_1}{2^j}\right),
\end{aligned}$$

for all $s_1 \in \Lambda$ and for all sufficiently large natural numbers μ. Letting $\mu \to \infty$, we obtain that $W = W''$ and this completes the proof. □

Corollary 2.6. *Let $(\Lambda, \|.\|)$ be a normed space and Q_ϱ fulfill the Δ_2-condition. If there exist a real number $\xi > 0$, $p > \log_2 \frac{u^6}{2}$ and a mapping $\Omega : \Lambda \to Q_\varrho$ such that*

$$\left.\begin{array}{l} \varrho\left(\Omega\left(s_1+3s_2\right)-6\Omega\left(s_1+2s_2\right)+15\Omega\left(s_1+s_2\right)-20\Omega\left(s_1\right)\right. \\ \left.+15\Omega\left(s_1-s_2\right)-6\Omega\left(s_1-2s_2\right)+\Omega\left(s_1-3s_2\right)-720\Omega\left(s_2\right)\right) \end{array}\right\} \leq \xi\left(\|s_1\|^p+\|s_2\|^p\right),$$

for all $s_1, s_2 \in \Lambda$, then there exists a unique SM $W : \Lambda \to Q_\varrho$ fulfilling

$$\Omega\left(s_1\right)-W(s_1) \leq \frac{u^7 \xi}{2^{p+1}-u^8} \text{ for all } s_1 \in \Lambda.$$

Now, without employing the Δ_2-condition and the Fatou property, the following theorem provides an alternative stability result of Theorem 2.5 in an MS.

Theorem 2.6. *If there exist a mapping $\Omega : \Lambda \to Q_\varrho$, which fulfills (2.20) with the function $\Xi : \Lambda\times\Lambda \to [0, \infty)$ satisfying*

$$\lim_{\mu\to\infty} \frac{\Xi\left(2^\mu s_1, 2^\mu s_2\right)}{2^{6\mu}} = 0 \text{ and } \sum_{j=1}^{\infty} \frac{\Xi\left(2^j s_1, 2^j s_1\right)}{2^{6j}} < \infty \text{ for all } s_1, s_2 \in \Lambda,$$

then there exists a unique SM $W : \Lambda \to Q_\varrho$ such that

$$\varrho\left(\Omega\left(s_1\right)-\frac{1}{7}\Omega(0)-W(s_1)\right) \leq \frac{1}{2^6}\sum_{j=1}^{\infty} \frac{\Xi\left(2^j s_1, 2^j s_1\right)}{2^{6j}} \text{ for all } s_1 \in \Lambda. \tag{2.23}$$

Proof. Putting $s_1 = s_2$ in (2.20), we have

$$\begin{aligned} & \varrho\left(\Omega\left(4s_1\right)-6\Omega\left(3s_1\right)+15\Omega\left(2s_1\right)-20\Omega\left(s_1\right)-6\Omega\left(-s_1\right)+\Omega\left(-2s_1\right)-720\Omega\left(s_1\right)\right) \\ = & \varrho\left(\widehat{\Omega}\left(4s_1\right)-6\widehat{\Omega}\left(3s_1\right)+15\widehat{\Omega}\left(2s_1\right)-20\widehat{\Omega}\left(s_1\right)-6\widehat{\Omega}\left(-s_1\right)+\widehat{\Omega}\left(-2s_1\right)-720\widehat{\Omega}\left(s_1\right)\right) \\ \leq & \Xi\left(s_1, s_1\right), \end{aligned}$$

where $\widehat{\Omega}\left(s_1\right) = \Omega\left(s_1\right) - \frac{1}{7}\Omega(0)$. From the convexity of ϱ and using the fact $\sum_{j=0}^{\mu-1} \frac{1}{2^{6(j+1)}} < 1$, we get

$$\begin{aligned} \varrho\left(\widehat{\Omega}\left(s_1\right)-\frac{\widehat{\Omega}\left(2^\mu s_1\right)}{2^{6\mu}}\right) & \leq \varrho\left(\sum_{j=0}^{\mu-1}\left(\frac{2^6\widehat{\Omega}\left(2^j s_1\right)-\widehat{\Omega}\left(2^{j+1} s_1\right)}{2^{6(j+1)}}\right)\right) \\ & \leq \sum_{j=0}^{\mu-1} \frac{\varrho\left(3^4\widehat{\Omega}\left(3^j s_1\right)-\widehat{\Omega}\left(3^{j+1} s_1\right)\right)}{2^{6(j+1)}} \\ & \leq \frac{1}{2^6}\sum_{j=0}^{\mu-1} \frac{\Xi\left(2^j s_1, 2^j s_1\right)}{2^{6j}} \text{ for all } s_1 \in \Lambda,\ \mu \in \mathbb{N}. \end{aligned}$$

It follows that the sequence $\left\{\frac{\widehat{\Omega}\left(2^\mu s_1\right)}{2^{6\mu}}\right\}$ is ϱ-Cauchy and the mapping $W : \Lambda \to Q_\varrho$ is defined by

$$W(s_1) = \varrho\left(\lim_{\mu\to\infty} \frac{\widehat{\Omega}\left(2^\mu s_1\right)}{2^{6\mu}}\right),$$

that is,

$$\varrho\left(\lim_{\mu\to\infty}\frac{\widehat{\Omega}\left(2^{\mu}s_1\right)}{2^{6\mu}}-W(s_1)\right)=0 \text{ for all } s_1\in\Lambda,$$

without utilizing the Δ_2-condition and the Fatou property. Clearly, from the proof of Theorem 2.5, we conclude that the mapping W satisfies the sextic functional equation.

Now, using the Fatou property and the Δ_2-condition, we show that (2.23) holds. From the convexity property of ϱ and since $\sum_{j=0}^{\mu-1}\frac{1}{2^{6(j+1)}}+\frac{1}{2^6}<1$, we get

$$\begin{aligned}\varrho\left(\widehat{\Omega}(s_1)-W(s_1)\right) &= \varrho\left(\sum_{j=1}^{\mu-1}\left(\frac{2^6\widehat{\Omega}\left(3^j s_1\right)-\widehat{\Omega}\left(2^{j+1}s_1\right)}{2^{6(j+1)}}\right)+\frac{\widehat{\Omega}\left(2^{\mu}s_1\right)}{2^{6\mu}}-\frac{W(2s_1)}{2^6}\right)\\ &\le \sum_{j=0}^{\mu-1}\frac{1}{2^{6(j+1)}}\varrho\left(2^6\widehat{\Omega}\left(2^j s_1\right)-\widehat{\Omega}\left(2^{j+1}s_1\right)\right)+\frac{1}{2^6}\varrho\left(\frac{\widehat{\Omega}\left(2^{\mu-1}2s_1\right)}{2^{6(\mu-1)}}-W(2s_1)\right)\\ &\le \frac{1}{2^6}\sum_{j=0}^{\mu-1}\frac{1}{2^{6j}}\Xi\left(2^j s_1,2^j s_1\right)+\frac{1}{2^6}\varrho\left(\frac{\widehat{\Omega}\left(2^{\mu-1}2s_1\right)}{2^{6(\mu-1)}}-W(2s_1)\right),\end{aligned}$$

for all $s_1\in\Lambda$ and all natural number $\mu>1$. As $\mu\to\infty$ in the above inequality, we obtain our needed result. □

Corollary 2.7. *Assume that there exists a mapping $\Xi:\Lambda\times\Lambda\to[0,\infty)$ satisfying*

$$\lim_{\mu\to\infty}\frac{\Xi\left(2^{\mu}s_1,2^{\mu}s_2\right)}{2^{6\mu}}=0 \text{ and } \Xi\left(2s_1,2s_2\right)<2^6M^*\Xi\left(s_1,s_2\right),$$

for all $s_1,s_2\in\Lambda$ and for some $M^*\in(0,1)$. If there exists a mapping $\Omega:\Lambda\to Q_\varrho$ fulfilling (2.20), then there exists a unique SM $W:\Lambda\to Q_\varrho$ verifying

$$\varrho\left(\Omega(s_1)-\frac{1}{7}\Omega(0)-W(s_1)\right)\le\frac{\Xi\left(s_1,s_1\right)}{2^6(1-M^*)} \text{ for all } s_1\in\Lambda.$$

Corollary 2.8. *Let $(\Lambda,\|.\|)$ be a normed space. Assume that there are $\xi>0$, $\varepsilon>0$, $p\in(-\infty,2)$ and a mapping $\Omega:\Lambda\to Q_\varrho$ such that*

$$\left.\begin{array}{l}\varrho\left(\Omega(s_1+3s_2)-6\Omega(s_1+2s_2)+15\Omega(s_1+s_2)-20\Omega(s_1)\right.\\ \left.+15\Omega(s_1-s_2)-6\Omega(s_1-2s_2)+\Omega(s_1-3s_2)-720\Omega(s_2)\right)\end{array}\right\}\le\xi\left(\|s_1\|^p+\|s_2\|^p\right)+\varepsilon,$$

for all $s_1,s_2\in\Lambda$. Then there exists a unique SM $W:\Lambda\to Q_\varrho$ fulfilling

$$\varrho\left(\Omega(s_1)-\frac{1}{7}\Omega(0)-W(s_1)\right)\le\frac{2\xi}{2^6-2^p}\|s_1\|^p+\frac{\varepsilon}{3},$$

for all $s_1\in\Lambda$.

3. Stability analysis in 2-Banach spaces

In this section, we discuss the stability of the involved functional equations by considering Λ as a linear normed space and Q as a 2-BS.

3.1. Stability of Cauchy additive functional equation

For the convenience of notations, define the mapping $\Omega : \Lambda \to Q_\varrho$ as

$$\begin{aligned} R\Omega\left(s_1, s_2, s_3, s_4\right) &= \Omega\left(\frac{s_1 - s_2}{m} + s_3 + s_4\right) + \Omega\left(\frac{s_2 - s_3}{m} + s_4 + s_1\right) + \Omega\left(\frac{s_3 - s_4}{m} + s_1 + s_2\right) \\ &\quad + \Omega\left(\frac{s_4 - s_1}{m} + s_2 + s_3\right) - \Omega\left(s_1 + s_2 + s_3 + s_4\right), \end{aligned}$$

for each $s_1, s_2, s_3, s_4 \in \Lambda$.

Theorem 3.1. *Suppose that there exists a function* $\Xi : \Lambda^4 \times Q \to [0, \infty)$ *such that*

$$\lim_{j\to\infty} E\left(4^j s_1, 4^j s_2, 4^j s_3, 4^j s_4, \ell\right) = 0, \textit{ for all } s_1, s_2, s_3, s_4 \in \Lambda \textit{ and } \ell \in Q. \tag{3.1}$$

If there is a mapping $\Omega : \Lambda \to Q$ *with* $\Omega(0) = 0$ *such that*

$$\left\| R\Omega\left(s_1, s_2, s_3, s_4\right), \ell \right\| \le \Xi\left(s_1, s_2, s_3, s_4, \ell\right), \tag{3.2}$$

and

$$\widehat{\Xi}\left(s_1, \ell\right) = \sum_{j=1}^{\infty} \frac{1}{4^j} E\left(4^j s_1, 4^j s_1, 4^j s_1, 4^j s_1, \ell\right) < \infty,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$ *and* $\ell \in Q$*, then there is a unique AM* $V : \Lambda \to Q$ *fulfilling*

$$\left\| \Omega\left(s_1\right) - V(s_1), \ell \right\| \le \widehat{\Xi}\left(s_1, \ell\right) \textit{ for all } s_1 \in \Lambda \textit{ and all } \ell \in Q. \tag{3.3}$$

Proof. Setting $s_1 = s_2 = s_3 = s_4$ in (3.2), we get

$$\left\| 4\Omega\left(s_1\right) - \Omega\left(4s_1\right), \ell \right\| \le \Xi\left(s_1, s_2, s_3, s_4, \ell\right). \tag{3.4}$$

Replacing s_1 with $4^j s_1$ in (3.4), and using

$$\left\| \frac{1}{4^{j+1}} \Omega\left(4^{j+1} s_1\right) - \frac{1}{4^j} \Omega\left(4^j s_1\right), \ell \right\| \le \frac{1}{4^{j+1}} \Xi\left(4^j s_1, 4^j s_1, 4^j s_1, 4^j s_1, \ell\right),$$

for all $s_1 \in \Lambda$, $\ell \in Q$ and all $j > 0$, one writes

$$\begin{aligned} \left\| \frac{1}{4^{j+1}} \Omega\left(4^{j+1} s_1\right) - \frac{1}{4^r} \Omega\left(4^r s_1\right), \ell \right\| &\le \sum_{t=r}^{j} \left\| \frac{1}{4^{t+1}} \Omega\left(4^{t+1} s_1\right) - \frac{1}{4^t} \Omega\left(4^t s_1\right), \ell \right\| \\ &\le \frac{1}{4} \sum_{t=r}^{j} \frac{1}{4^t} \Xi\left(4^j s_1, 4^j s_1, 4^j s_1, 4^j s_1, \ell\right), \end{aligned} \tag{3.5}$$

for all $s_1 \in \Lambda$, $\ell \in Q$ and all integers $j > 0$ and $r > 0$ with $r \le j$. It follows from (3.4) and (3.5) that the sequence $\left\{\frac{\Omega(4^j s_1)}{4^j}\right\}$ is a Cauchy sequence in Q. The completeness of Q implies that the sequence $\left\{\frac{\Omega(4^j s_1)}{4^j}\right\}$ converges in Q for all $s_1 \in \Lambda$. Therefore, we can define that mapping $V : \Lambda \to Q$ as

$$V(s_1) = \lim_{j\to\infty} \frac{\Omega\left(4^j s_1\right)}{4^j}, \text{ for all } s_1 \in \Lambda. \tag{3.6}$$

Hence,

$$\lim_{j\to\infty}\left\|\frac{\Omega\left(4^j s_1\right)}{4^j}-V(s_1),\ell\right\|=0,\ \text{for all } s_1\in\Lambda \text{ and } \ell\in Q.$$

Putting $t=0$ and let $j\to\infty$ in (3.5), we have (3.3).

Now, we shall show that V is an AM. Using (3.1), (3.2), (3.6) and Lemma 1.2, one gets

$$\begin{aligned}\|R\Omega(s_1,s_2,s_3,s_4),\ell\| &= \lim_{j\to\infty}\left\|R\Omega\left(4^j s_1,4^j s_1,4^j s_1,4^j s_1\right),\ell\right\|\\ &\le \lim_{j\to\infty}\frac{1}{4^j}\Xi\left(4^j s_1,4^j s_1,4^j s_1,4^j s_1,\ell\right)=0.\end{aligned}$$

By Lemma 1.1,

$$\|RV(s_1,s_2,s_3,s_4),\ell\|=0.$$

Thus, V is an AM. For the uniqueness, consider another AM $V':\Lambda\to Q$ fulfilling (3.3). Then,

$$\begin{aligned}\|V(s_1)-V'(s_1),\ell\| &= \lim_{j\to\infty}\left\|V(4^j s_1)-\Omega(4^j s_1)+\Omega(4^j s_1)-V'(4^j s_1),\ell\right\|\\ &\le \lim_{j\to\infty}\frac{1}{4^j}\widehat{\Xi}\left(4^j s_1,\ell\right)=0 \text{ for all } s_1\in\Lambda \text{ and all } \ell\in Q.\end{aligned}$$

Based on Lemma 1.1, we have $V(s_1)-V'(s_1)=0$ for all $s_1\in\Lambda$, which implies that $V=V'$. □

Corollary 3.1. *Assume that $\mu:[0,\infty)\to[0,\infty)$ is a function such that $\mu(0)=0$ and the following assertions hold:*

(a_1) $\mu(\varkappa\omega)\le\mu(\varkappa)\mu(\omega)$;
(a_2) *For all $\varkappa>1$, $\mu(\varkappa)<\varkappa$.*

If there exists a mapping $\Omega:\Lambda\to Q_\varrho$ such that $\Omega(0)=0$ and

$$\|R\Omega(s_1,s_2,s_3,s_4),\ell\|\le\mu(\|s_1\|+\|s_2\|+\|s_3\|+\|s_4\|)+\mu(\ell),$$

for all $s_1,s_2,s_3,s_4\in\Lambda$ and $\ell\in Q$, then there is a unique AM $V:\Lambda\to Q$ fulfilling

$$\|\Omega(s_1)-V(s_1),\ell\|\le\left(\frac{4\mu(\|s_1\|)}{4-\mu(4)}+\mu(\ell)\right),\ \textit{for all } s_1\in\Lambda \textit{ and all } \ell\in Q. \tag{3.7}$$

Proof. Consider

$$\Xi(s_1,s_2,s_3,s_4,\ell)=\mu(\|s_1\|+\|s_2\|+\|s_3\|+\|s_4\|)+\mu(\ell),$$

for all $s_1,s_2,s_3,s_4\in\Lambda$ and $\ell\in Q$. Based on condition (a_1), we have

$$\mu\left(4^j\right)=(\mu(4))^j,$$

and

$$\Xi\left(4^j s_1,4^j s_2,4^j s_3,4^j s_4,\ell\right)\le(\mu(4))^j\left[\mu(\|s_1\|+\|s_2\|+\|s_3\|+\|s_4\|)\right]+\mu(\ell).$$

Using Theorem 3.1, we obtain (3.7). □

Corollary 3.2. *Assume that* $\eth : ([0,\infty))^4 \to [0,\infty)$ *is a homogeneous function with degree* q *and* $\Omega : \Lambda \to Q$ *is a mapping satisfying* $\Omega(0) = 0$ *and*

$$\|R\Omega\left(s_1, s_2, s_3, s_4\right), \ell\| \leq \eth\left(\|s_1\|, \|s_2\|, \|s_3\|, \|s_4\|\right)\|\ell\|,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$ *and* $\ell \in Q$*. Then there is a unique AM* $V : \Lambda \to Q$ *fulfilling*

$$\|\Omega\left(s_1\right) - V(s_1), \ell\| \leq \frac{\eth\left(\|s_1\|, \|s_1\|, \|s_1\|, \|s_1\|\right)\|\ell\|}{4 - 4^p},$$

for all $s_1 \in \Lambda$ *and all* $\ell \in Q$*, where* $p \in \mathbb{R}^+$ *with* $p < 1$*.*

Corollary 3.3. *Let* $c \in \mathbb{R}^+$ *with* $c < 1$ *and* $\eth : ([0,\infty))^4 \to [0,\infty)$ *be a homogeneous function with degree* a*. Assume that* $\Omega : \Lambda \to Q$ *is a mapping satisfying* $\Omega(0) = 0$ *and*

$$\|R\Omega\left(s_1, s_2, s_3, s_4\right), \ell\| \leq \eth\left(\|s_1\|, \|s_2\|, \|s_3\|, \|s_4\|\right) + \|\ell\|,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$ *and* $\ell \in Q$*. Then there is a unique AM* $V : \Lambda \to Q$ *fulfilling*

$$\|\Omega\left(s_1\right) - V(s_1), \ell\| \leq \frac{\eth\left(\|s_1\|, \|s_1\|, \|s_1\|, \|s_1\|\right) + \|\ell\|}{4 - c},$$

for all $s_1 \in \Lambda$ *and all* $\ell \in Q$*.*

Corollary 3.4. *Assume that a mapping* $\Omega : \Lambda \to Q$ *satisfies* $\Omega(0) = 0$ *and*

$$\|R\Omega\left(s_1, s_2, s_3, s_4\right), \ell\| \leq \|s_1\|^b + \|s_2\|^b + \|s_3\|^b + \|s_4\|^b + \|\ell\|,$$

for all $s_1, s_2, s_3, s_4 \in \Lambda$ *and* $\ell \in Q$*. Then there is a unique AM* $V : \Lambda \to Q$ *fulfilling*

$$\|\Omega\left(s_1\right) - V(s_1), \ell\| \leq \frac{2\|s_1\|^b + \|\ell\|}{4 - b},$$

for all $s_1 \in \Lambda$ *and all* $\ell \in Q$*, where* $b \in \mathbb{R}^+$ *with* $b < 1$*.*

3.2. Stability of the quartic functional equation

In this part, we assume that the mapping $\Omega : \Lambda \to Q$ is described as

$$\begin{aligned} R\Omega\left(s_1, s_2\right) &= \Omega\left(3s_1 + s_2\right) + \Omega\left(3s_1 - s_2\right) + 16\Omega\left(s_2\right) - 144\Omega\left(s_1\right) \\ &\quad -9\Omega\left(s_1 + s_2\right) - 9\Omega\left(s_1 - s_2\right), \end{aligned}$$

for all $s_1, s_2 \in \Lambda$.

Theorem 3.2. *Assume that* $\Xi : \Lambda^4 \times Q \to [0,\infty)$ *is a function such that*

$$\lim_{j\to\infty} \frac{1}{3^{4j}} \Xi\left(3^j s_1, 3^j s_1, \ell\right) = 0, \tag{3.8}$$

for all $s_1, s_2 \in \Lambda$ *and* $\ell \in Q$*. If there exists* $\Omega : \Lambda \to Q$ *with* $\Omega(0) = 0$ *such that*

$$\|R\Omega\left(s_1, s_2\right), \ell\| \leq 2\Xi\left(s_1, s_2, \ell\right), \tag{3.9}$$

and

$$\widehat{\Xi}(s_1, \ell) = \frac{1}{3} \sum_{j=1}^{\infty} \frac{1}{3^{4j}} E\left(3^j s_1, 0, \ell\right) < \infty,$$

for all $s_1 \in \Lambda$ and $\ell \in Q$, then there exists a unique QM $V_4 : \Lambda \to Q$ fulfilling

$$\|\Omega(s_1) - V_4(s_1), \ell\| \leq \widehat{\Xi}(s_1, \ell) \text{ for all } s_1 \in \Lambda \text{ and all } \ell \in Q. \tag{3.10}$$

Proof. Consider $s_2 = 0$ in (3.9). We have

$$\left\|2\Omega(3s_1) - 2 \times 3^4 \Omega(s_1), \ell\right\| \leq 2\Xi(s_1, 0, \ell),$$

which implies that

$$\left\|\frac{\Omega(3s_1)}{3^4} - \Omega(s_1), \ell\right\| \leq \frac{1}{3^4} \Xi(s_1, 0, \ell), \tag{3.11}$$

for all $s_1 \in \Lambda$ and $\ell \in Q$. In (3.11), replace s_1 with $3^j s_1$, to get

$$\left\|\frac{1}{3^{4(j+1)}} \Omega\left(3^{j+1} s_1\right) - \frac{1}{3^{4j}} \Omega\left(3^j s_1\right), \ell\right\| \leq \frac{1}{3^{4(j+1)}} \Xi(s_1, 0, \ell),$$

for all $s_1 \in \Lambda$, $\ell \in Q$ and all integer $j > 0$. Hence,

$$\begin{aligned}
\left\|\frac{1}{3^{4(j+1)}} \Omega\left(3^{j+1} s_1\right) - \frac{1}{3^{4m}} \Omega(3^m s_1), \ell\right\| &\leq \sum_{r=m}^{j} \left\|\frac{1}{3^{4(r+1)}} \Omega\left(3^{r+1} s_1\right) - \frac{1}{3^{4r}} \Omega(3^r s_1), \ell\right\| \\
&\leq \frac{1}{3} \sum_{r=m}^{j} \frac{1}{3^{4r}} E\left(3^j s_1, 0, \ell\right),
\end{aligned} \tag{3.12}$$

for all $s_1 \in \Lambda$, $\ell \in Q$ and all integers $j \geq m > 0$. Therefore, from (3.9) and (3.12), the sequence $\left\{\frac{\Omega(3^j s_1)}{3^{4j}}\right\}$ is a Cauchy sequence in Q. The completeness of Q implies that the sequence $\left\{\frac{\Omega(3^j s_1)}{3^j}\right\}$ converges in Q for all $s_1 \in \Lambda$. Therefore, we can describe the mapping $V_4 : \Lambda \to Q$ as

$$V(s_1) = \lim_{j \to \infty} \frac{\Omega\left(3^j s_1\right)}{3^j} \text{ for all } s_1 \in \Lambda. \tag{3.13}$$

Hence,

$$\lim_{j \to \infty} \left\|\frac{\Omega\left(3^j s_1\right)}{3^j} - V_4(s_1), \ell\right\| = 0 \text{ for all } s_1 \in \Lambda \text{ and } \ell \in Q.$$

Letting $m = 0$ and $j \to \infty$ in (3.12), we have (3.10).

Now, we shall show that V_4 is a QM. Using (3.8), (3.9), (3.13) and Lemma 1.2, one can write

$$\|R\Omega(s_1, s_2), \ell\| = \lim_{j \to \infty} \left\|R\Omega\left(3^j s_1, 3^j s_1\right), \ell\right\| \leq \lim_{j \to \infty} \frac{1}{3^{4j}} E\left(3^j s_1, 3^j s_2, \ell\right) = 0,$$

for all $s_1 \in \Lambda$, $\ell \in Q$. By Lemma 1.1, we get

$$\|RV_4(s_1, s_2, s_3, s_4), \ell\| = 0.$$

Thus, V is a QM. For the uniqueness, consider another QM $V_4' : \Lambda \to Q$ fulfilling (3.10). Then

$$\begin{aligned}\left\|V_4(s_1) - V_4'(s_1), \ell\right\| &= \lim_{j\to\infty} \frac{1}{3^{4j}} \left\|V_4(3^j s_1) - \Omega(3^j s_1) + \Omega(3^j s_1) - V_4'(3^j s_1), \ell\right\| \\ &\le \lim_{j\to\infty} \frac{1}{3^{4j}} \widehat{\Xi}\left(3^j s_1, \ell\right) = 0 \text{ for all } s_1 \in \Lambda \text{ and all } \ell \in Q.\end{aligned}$$

Based on Lemma 1.1, we have $V_4(s_1) - V_4'(s_1) = 0$ for all $s_1 \in \Lambda$, which implies that $V_4 = V_4'$ and this completes the proof. □

Corollary 3.5. *Let $\mu : [0, \infty) \to [0, \infty)$ be a given function with $\mu(0) = 0$ and*

(*i*) $\mu(\varkappa\omega) \le \mu(\varkappa)\mu(\omega)$,
(*ii*) *for all $\varkappa > 1$, $\mu(\varkappa) < \varkappa$.*

If there exists a mapping $\Omega : \Lambda \to Q_\varrho$ with $\Omega(0) = 0$ and

$$\|R\Omega\left(s_1, s_2\right), \ell\| \le \mu\left(\|s_1\| + \|s_2\|\right) + \mu\left(\ell\right),$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$, then there exists a unique QM $V_4 : \Lambda \to Q$ fulfilling

$$\|\Omega\left(s_1\right) - V_4(s_1), \ell\| \le \left(\frac{\mu\left(\|s_1\|\right)}{3 - \mu\left(3\right)} + \mu\left(\ell\right)\right), \text{ for all } s_1 \in \Lambda \text{ and all } \ell \in Q. \tag{3.14}$$

Proof. Assume that

$$\Xi\left(s_1, s_2, \ell\right) = \mu\left(\|s_1\| + \|s_2\|\right) + \mu\left(\ell\right),$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$. From the condition (*i*), we get

$$\mu\left(3^j\right) = \left(\mu\left(3\right)\right)^j,$$

and

$$\Xi\left(3^j s_1, 3^j s_2, \ell\right) \le \left(\mu\left(3\right)\right)^j \left[\mu\left(\|s_1\| + \|s_2\|\right)\right] + \mu\left(\ell\right).$$

By utilizing Theorem 3.1, we obtain (3.14). □

Corollary 3.6. *Assume that $\eth : ([0, \infty))^2 \to [0, \infty)$ is a homogeneous function with degree q and $\Omega : \Lambda \to Q$ is a mapping satisfying $\Omega(0) = 0$ and*

$$\|R\Omega\left(s_1, s_2\right), \ell\| \le \eth\left(\|s_1\|, \|s_2\|\right)\|\ell\|,$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$. Then there exists a unique QM $V_4 : \Lambda \to Q$ fulfilling

$$\|\Omega\left(s_1\right) - V_4(s_1), \ell\| \le \frac{\eth\left(\|s_1\|, \|s_1\|\right)\|\ell\|}{3 - 3^p},$$

for all $s_1 \in \Lambda$ and all $\ell \in Q$, where $p \in \mathbb{R}^+$ with $p < 1$.

Corollary 3.7. *Let $c \in \mathbb{R}^+$ with $c < 1$ and $\eth : ([0,\infty))^4 \to [0,\infty)$ be a homogeneous function with degree a. Assume that $\Omega : \Lambda \to Q$ is a mapping satisfying $\Omega(0) = 0$ and*

$$\|R\Omega(s_1, s_2), \ell\| \leq \eth(\|s_1\|, \|s_2\|) + \|\ell\|,$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$. Then there exists a unique AM $V_4 : \Lambda \to Q$ fulfilling

$$\|\Omega(s_1) - V_4(s_1), \ell\| \leq \frac{\eth(\|s_1\|, \|s_1\|) + \|\ell\|}{3 - c},$$

for all $s_1 \in \Lambda$ and all $\ell \in Q$.

Proof. We obtain the proof immediately, if we take in Theorem 3.2,

$$\Xi(s_1, s_2, \ell) = \eth(\|s_1\| + \|s_2\|) + \eth(\ell),$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$. □

Corollary 3.8. *Assume that a mapping $\Omega : \Lambda \to Q$ satisfies $\Omega(0) = 0$ and*

$$\|R\Omega(s_1, s_2), \ell\| \leq \|s_1\|^b + \|s_2\|^b + \|\ell\|,$$

for all $s_1, s_2 \in \Lambda$ and $\ell \in Q$. Then there exists a unique AM $V : \Lambda \to Q$ fulfilling

$$\|\Omega(s_1) - V_4(s_1), \ell\| \leq \frac{2\|s_1\|^b + \|\ell\|}{3 - b},$$

for all $s_1 \in \Lambda$ and all $\ell \in Q$, where $b \in \mathbb{R}^+$ with $b < 1$.

3.3. Stability results of the sextic functional equation

By the same methods used in sections 3.1 and 3.2, we can obtain the refined stability of the sextic functional equation by defining a mapping $\Omega : \Lambda \to Q$ as

$$\begin{aligned} R\Omega(s_1, s_2) &= \Omega(s_1 + 3s_2) - 6\Omega(s_1 + 2s_2) + 15\Omega(s_1 + s_2) - 20\Omega(s_1) + 15\Omega(s_1 - s_2) \\ &\quad -6\Omega(s_1 - 2s_2) + \Omega(s_1 - 3s_2) - 720\Omega(s_2), \end{aligned}$$

for all $s_1, s_2 \in \Lambda$, where Λ is a linear normed spaces and Q is a 2-BS.

4. Counter-examples

With the aid of a pertinent example, it is demonstrated that the functional equations (2.1) and (2.15) are unstable in the singular condition. To Gajda's outstanding example in [30], which demonstrates the instability in Corollaries 2.2 and 2.3 of equations (2.1) and (2.15), respectively, we propose the following examples as counter-examples via the assumptions $p \neq 1$ and $p \neq \log_3 \frac{u^4}{3}$, respectively.

Here, $\mathbb{R}$ stands for a real space, $\mathbb{Z}$ and $\mathbb{Q}$ refer to the sets of integer ad rational numbers. Our counter-examples can be demonstrated as in [31, 32].

Remark 4.1. *If a mapping $\Omega : \mathbb{R} \to \Lambda$ fulfills the functional equation (2.1), then the following assertions are true:*

(R_1) *For all* $s_1 \in \mathbb{R}$, $z \in \mathbb{Z}$ *and* $m \in \mathbb{Q}$, $\Omega(m^z s_1) = m^z \Omega(s_1)$;
(R_2) *For all* $s_1 \in \mathbb{R}$, *if the mapping* Ω *is continuous, then* $\Omega(s_1) = s_1 \Omega(1)$.

Example 4.1. *Assume that* $\Omega : \mathbb{R} \to \mathbb{R}$ *is a function described as*

$$\Omega(s_1) = \sum_{j=0}^{\infty} \frac{\varpi\left(4^j s_1\right)}{4^j},$$

where

$$\varpi(s_1) = \begin{cases} \delta s_1, & \text{if } s_1 \in (-1, 1), \\ \delta, & \text{otherwise.} \end{cases}$$

If we define a function $\Omega : \mathbb{R} \to \mathbb{R}$ *as in (2.1) such that*

$$|\Delta\Omega(s_1, s_2, s_3, s_4)| \leq 8\delta(|s_1| + |s_2| + |s_3| + |s_4|),$$

for all $s_1, s_2, s_3, s_4 \in \mathbb{R}$, *then we cannot found an AM* $W : \mathbb{R} \to \mathbb{R}$ *which satisfies*

$$|\Omega(s_1) - W(s_1)| \leq \varkappa |s_1|,$$

for all $s_1 \in \mathbb{R}$, *where* ξ *and* $\varkappa$ *are constants.*

Remark 4.2. *If a mapping* $\Omega : \mathbb{R} \to \Lambda$ *fulfills the functional equation (2.15), then the following hypotheses hold:*

(R_1) *For all* $s_1 \in \mathbb{R}$, $z \in \mathbb{Z}$ *and* $m \in \mathbb{Q}$, $\Omega\left(m^{\frac{z}{3}} s_1\right) = m^z \Omega(s_1)$;
(R_2) *For all* $s_1 \in \mathbb{R}$, *if a mapping* Ω *is continuous, then* $\Omega(s_1) = s_1^3 \Omega(1)$.

Example 4.2. *Assume that* $\Omega : \mathbb{R} \to \mathbb{R}$ *is a function described as*

$$\Omega(s_1) = \sum_{j=0}^{\infty} \frac{\varpi\left(3^j s_1\right)}{3^{4j}},$$

where

$$\varpi(s_1) = \begin{cases} \delta s_1^3, & \text{if } s_1 \in (-1, 1), \\ \delta, & \text{otherwise.} \end{cases}$$

If we define a function $\Omega : \mathbb{R} \to \mathbb{R}$ *as in (2.15) such that*

$$|\Delta\Omega(s_1, s_2)| \leq 234\delta\left(|s_1|^3 + |s_2|^3\right),$$

for all $s_1, s_2 \in \mathbb{R}$, *then we cannot found a QM* $W : \mathbb{R} \to \mathbb{R}$ *which satisfies*

$$|\Omega(s_1) - W(s_1)| \leq \varkappa |s_1|^3,$$

for all $s_1 \in \mathbb{R}$, *where* ξ *and* $\varkappa$ *are constants.*

5. Conclusions and future works

The concept of stability of a functional equation arises if we replace this functional equation by an inequality acting as a perturbation on the equation itself. Stability of the functional equation has been become an interesting subject over the last seventy years. Several results appeared in this direction. In our work, the direct method of Hyers has been utilized to study the refined stability of the additive, quartic, and sextic functional equations in modular spaces with and without the Δ_2-condition. Moreover, we used the direct approach to investigate the Ulam stability in 2-Banach spaces. At the end, some counter examples have been presented in order to ensure that the stability of these equations does not hold in a particular case. As future works, we look forward to study the stability of generalized additive, generalized quartic and generalized sextic functional equations. We will also study the effect of the multivalued mappings of these equations on D-metric spaces and generalized metric spaces.

Acknowledgments

This work was supported in part by the Basque Government under Grant IT1555-22.

Conflict of interest

The authors declare that they have no competing interests.

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