



Research article

Threshold dynamics of a nonlocal diffusion West Nile virus model with spatial heterogeneity

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Abstract: In this study, we investigated the threshold dynamics of a spatially heterogeneous nonlocal diffusion West Nile virus model. By employing semigroup theory and continuous Fréchet-differentiable, we established the well-posedness of the solution. The expression for the basic reproduction number derived using the next-generation matrix method. The authors demonstrated the threshold dynamics of the system by constructing a Lyapunov function and applying the comparison principle. Finally, numerical simulations were used to validate the theorem results. It can be suggested that to control disease development rapidly, measures should be taken to reduce the spread of mosquitoes and birds.

Keywords: threshold dynamics; nonlocal diffusion; West Nile virus model; spatial heterogeneity

Mathematics Subject Classification: 35B40, 37N25, 92B05

1. Introduction

West Nile virus disease is zoonotic and is becoming increasingly prevalent in Africa, Europe, the Americas, Australia, the Middle East, the Indian subcontinent and other regions. This poses a significant risk to human and animal health, with severe cases resulting in encephalitis, and even death. In nature, birds act as reservoirs for the West Nile virus, which circulates through a mosquito-bird cycle. Mosquitoes carrying the virus bite susceptible birds, leading to the development of toxemia in the birds. Subsequently, susceptible mosquitoes bite infected birds, contributing to the spread of the virus. Over 60 mosquito species and hundreds of bird species have been implicated in the transmission of the virus in the United States of America [1]. There is no specific treatment for the disease, and mathematical models can help us understand the pathogenesis of disease.

There is currently no specific treatment for the disease, but mathematical models can help us understand its pathogenesis. Numerous models have been developed to study the dynamic behavior

of the West Nile virus [2–13]. Bai and Zhang [2] examined the dynamics of a periodic West Nile virus model, and concluded that using time-averaged extrinsic incubation period might underestimate the risk of disease outbreaks. Ge et al. [6] investigated the spatial transmission of reaction-diffusion West Nile virus with free boundaries in a heterogeneous environment, providing sufficient conditions for virus persistence and extinction based on the spatial risk index. Cheng and Zheng [9] explored the spatiotemporal spread and asymptotic behavior of the West Nile virus through a reaction-advection-diffusion system with free boundaries. They considered the impact of advection terms on the virus's extinction and persistence. Tarboush and Zhang [13] analyzed the effects of a periodically evolving domain on the dynamics of the West Nile virus spread. They introduced a spatiotemporal basic reproduction number that varies with the rate of periodic evolution to study the virus long-term behavior.

The existing literature West Nile virus models predominantly focus on reaction-diffusion approaches, in which the Laplacian operator accounts for the local behavior of random diffusion at adjacent spatial locations. However, the reaction-diffusion model represented by the Laplacian operator may not accurately capture the long-distance effects of disease transmission [14]. Consequently, to better represent the spread of the disease, integral operators are employed to describe the diffusion process, which accounts for the movement between non-adjacent spatial locations. Generally, $\int_{\Omega} \mathfrak{I}(x-y)\varphi(y)dy - \varphi(x)$ (convolution operator) is utilized to model the nonlocal diffusion [15]. For instance, Du and Nie [16] explored the long-term dynamic behavior of a West Nile virus model with non-local spread and free boundaries. Additionally, Pu et al. [17] incorporated the influence of seasonal variation into the model based on Reference [16]. Some studies, e.g., [16] and [17] only considered the impact of non-local diffusion on the infected population; thus, we introduce a model [18] to facilitate the analysis of non-local diffusion's influence on both susceptible and infected populations.

$$\begin{cases} \frac{dS_m}{dt} = \Lambda_m - \mu_m S_m - \frac{b_1 \beta_m}{N_b} S_m I_b, \\ \frac{dI_m}{dt} = \frac{b_1 \beta_m}{N_b} S_m I_b - \mu_m I_m, \\ \frac{dS_b}{dt} = \Lambda_b - \mu_b S_b - \frac{b_1 \beta_b}{N_b} S_b I_m, \\ \frac{dI_b}{dt} = \frac{b_1 \beta_b}{N_b} S_b I_m - \mu_b I_b - d_b I_b, \end{cases} \quad (1.1)$$

where S_m and I_m represent the populations of uninfected and infected female mosquitoes, respectively. S_b and I_b denote the population of susceptible and infected birds, respectively. Λ_m and Λ_b are the recruitment rates of susceptible mosquitoes and birds, respectively. b_1 is the per capita biting rate of mosquitoes on the bird. β_m and β_b are the West Nile virus transmission probabilities from infected birds to uninfected mosquitoes and from mosquitoes to birds, respectively. μ_m and μ_b are the natural death rates of mosquitoes and birds, respectively; d_b denotes the disease-induced death rate of infected birds. The parameters in this model are all positive constants.

It is worth noting that the activity range of mosquitoes is generally limited, in general, with most only flying tens to hundreds of meters, and the longest flight distance being one to two kilometers.

Therefore, we disregard the non-local spread of mosquitoes and we consider the following model.

$$\begin{cases} \frac{\partial S_m}{\partial t} = \Lambda_m(x) - \mu_m(x)S_m - \frac{b_1(x)\beta_m(x)}{N_b}S_m(x,t)I_b(x,t), \\ \frac{\partial I_m}{\partial t} = \frac{b_1(x)\beta_m(x)}{N_b}S_m(x,t)I_b(x,t) - \mu_m(x)I_m(x,t), \\ \frac{\partial S_b}{\partial t} = d_1 \int_{\Omega} \mathfrak{I}(x-y)S_b(y,t)dy - d_1S_b(x,t) + \Lambda_b(x) - \mu_b(x)S_b(x,t) - \frac{b_1(x)\beta_b(x)}{N_b}S_b(x,t)I_m(x,t), \\ \frac{\partial I_b}{\partial t} = d_2 \int_{\Omega} \mathfrak{I}(x-y)I_b(y,t)dy - d_2I_b(x,t) + \frac{b_1(x)\beta_b(x)}{N_b}S_b(x,t)I_m(x,t) - \mu_b(x)I_b(x,t) - d_bI_b(x,t), \\ S_m(x,0) = S_{m,0}(x), I_m(x,0) = I_{m,0}(x), S_b(x,0) = S_{b,0}(x)I_b(x,0) = I_{b,0}(x), \\ x \in \Omega, t > 0. \end{cases} \quad (1.2)$$

with the boundary condition

$$\frac{\partial S_m}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = \frac{\partial S_b}{\partial \nu} = \frac{\partial I_b}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \quad (1.3)$$

and initial condition

$$S_m(x,0) = S_{m,0}(x), I_m(x,0) = I_{m,0}(x), S_b(x,0) = S_{b,0}(x), I_b(x,0) = I_{b,0}(x), x \in \Omega. \quad (1.4)$$

$d_1 > 0$ and $d_2 > 0$ represent the diffusion coefficient of susceptible and infectious for birds, respectively. $\mu_h(x)$, $\mu(x)$, $\beta_H(x)$, $b(x)$, $\gamma_H(x)$, $\beta_v(x)$ and $\nu(x)$ are positive continuous functions on $\bar{\Omega}$. The dispersal kernel function \mathfrak{I} is continuous and satisfies the following properties

$$\mathfrak{I}(0) > 0, \quad \int_{\mathbb{R}} \mathfrak{I}(x)dx = 1, \quad \mathfrak{I}(x) > 0 \text{ on } \bar{\Omega}, \quad \mathfrak{I}(x) = \mathfrak{I}(-x) \geq 0 \text{ on } \mathbb{R}. \quad (1.5)$$

This paper is organized as follows. In Section 2, we prove the existence, uniqueness, positivity, and boundedness of solutions, and derive the expression for the basic reproduction number using the next-generation matrix method. In Section 3, we establish the global asymptotic stability and uniform persistence of the system by constructing Lyapunov functions and applying the comparison principle. Section 4 presents numerical simulations, and Section 5 concludes the paper with some final remarks.

2. Well-posedness of the solution

Let us consider the following function spaces and positive cones.

$$\mathbb{X} := \mathbb{C}(\bar{\Omega}), \quad \mathbb{X}_+ := \mathbb{C}_+(\bar{\Omega}), \quad \mathcal{Y} := \mathbb{C}(\bar{\Omega}) \times \mathbb{C}(\bar{\Omega}) \times \mathbb{C}(\bar{\Omega}) \times \mathbb{C}(\bar{\Omega}), \quad \mathcal{Y}_+ := \mathbb{C}_+(\bar{\Omega}) \times \mathbb{C}_+(\bar{\Omega}) \times \mathbb{C}_+(\bar{\Omega}) \times \mathbb{C}_+(\bar{\Omega}).$$

The norms in \mathbb{X} and \mathcal{Y} are defined as follows, respectively: $\|\vartheta\|_{\mathbb{X}} := \sup_{x \in \bar{\Omega}} |\vartheta(x)|$, $\vartheta \in \mathbb{X}$, and

$$\|(z_1, z_2, z_3, z_4)\|_{\mathcal{Y}} := \sup_{x \in \bar{\Omega}} \sqrt{|z_1(x)|^2 + |z_2(x)|^2 + |z_3(x)|^2 + |z_4(x)|^2}, \quad (z_1, z_2, z_3, z_4) \in \mathcal{Y}.$$

Next, we define the linear operators on \mathbb{X} .

$$\begin{aligned} \mathbb{A}_1 \vartheta_1(x) &:= -\mu_m(x)\vartheta_1(x), \\ \mathbb{A}_2 \vartheta_2(x) &:= -\mu_m(x)\vartheta_2(x), \\ \mathbb{A}_3 \vartheta_3(x) &:= d_1 \int_{\Omega} \mathfrak{I}(x-y)\vartheta_3(y)dy - d_1\vartheta_3(x) - \mu_b(x)\vartheta_3(x), \\ \mathbb{A}_4 \vartheta_4(x) &:= d_2 \int_{\Omega} \mathfrak{I}(x-y)\vartheta_4(y)dy - d_2\vartheta_4(x) - \mu_b(x)\vartheta_4(x) - d_b(x)\vartheta_4(x). \end{aligned} \quad (2.1)$$

From the above, we know that \mathbb{A}_1 , \mathbb{A}_2 , \mathbb{A}_3 and \mathbb{A}_4 are bounded linear operators; by virtue of [19, Theorem 1.2], we obtain that $\mathbb{A}_1(t)_{t \geq 0}$, $\mathbb{A}_2(t)_{t \geq 0}$, $\mathbb{A}_3(t)_{t \geq 0}$ and $\mathbb{A}_4(t)_{t \geq 0}$ are uniformly continuous semigroups on \mathbb{X} . Furthermore, according to [20, Section 2.1.1], the semigroups $\mathbb{A}_1(t)_{t \geq 0}$, $\mathbb{A}_2(t)_{t \geq 0}$, $\mathbb{A}_3(t)_{t \geq 0}$ and $\mathbb{A}_4(t)_{t \geq 0}$ are positive.

Next, we will prove the existence and uniqueness of the solution for system (1.2).

Lemma 2.1. *For any initial data $(S_{m,0}, I_{m,0}, S_{b,0}, I_{b,0})$ and $t \in [0, t_0)$, the solution $(S_m(x, t), I_m(x, t), S_b(x, t), I_b(x, t))$ of system (1.2) satisfies that*

$$\limsup_{t \rightarrow \infty} \int_{\Omega} [S_m(x, t) + I_m(x, t) + S_b(x, t) + I_b(x, t)] dx < \infty.$$

Here $|\Omega|$ denotes the volume of Ω .

Proof. By (1.2), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} [S_m(x, t) + I_m(x, t) + S_b(x, t) + I_b(x, t)] dx \\ &= \int_{\Omega} \Lambda_m(x) dx - \int_{\Omega} \mu_m(x) S_m dx - \int_{\Omega} \frac{b_1(x) \beta_m(x)}{N_b} S_m(x, t) I_b(x, t) dx - \int_{\Omega} \frac{b_1(x) \beta_m(x)}{N_b} S_m(x, t) I_b(x, t) dx \\ & \quad - \int_{\Omega} \mu_m(x) I_m(x, t) dx + d_1 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) S_b(y, t) dy dx - d_1 \int_{\Omega} S_b(x, t) dx + \int_{\Omega} \Lambda_b(x) dx \\ & \quad - \int_{\Omega} \mu_b(x) S_b(x, t) dx - \int_{\Omega} \frac{b_1(x) \beta_b(x)}{N_b} S_b(x, t) I_m(x, t) dx + d_2 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) I_b(y, t) dy dx - d_2 \int_{\Omega} I_b(x, t) dx \\ & \quad + \int_{\Omega} \frac{b_1(x) \beta_b(x)}{N_b} S_b(x, t) I_m(x, t) dx - \int_{\Omega} \mu_b(x) I_b(x, t) dx - \int_{\Omega} d_b(x) I_b(x, t) dx \\ &= \int_{\Omega} \Lambda_m(x) dx - \int_{\Omega} \mu_m(x) (S_m + I_m(x, t)) dx + d_1 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) S_b(y, t) dy dx - d_1 \int_{\Omega} S_b(x, t) dx \\ & \quad + \int_{\Omega} \Lambda_b(x) dx - \int_{\Omega} \mu_b(x) (S_b(x, t) + I_b(x, t)) dx + d_2 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) I_b(y, t) dy dx - d_2 \int_{\Omega} I_b(x, t) dx \\ & \quad - \int_{\Omega} d_b(x) I_b(x, t) dx. \end{aligned}$$

Moreover, according to (1.5), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} [S_m(x, t) + I_m(x, t) + S_b(x, t) + I_b(x, t)] dx \\ & \leq \int_{\Omega} \Lambda_m(x) dx - \int_{\Omega} \mu_m(x) (S_m(x, t) + I_m(x, t)) dx + d_1 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) dy S_b(y, t) dx - d_1 \int_{\Omega} S_b(x, t) dx \\ & \quad + \int_{\Omega} \Lambda_b(x) dx - \int_{\Omega} \mu_b(x) (S_b(x, t) + I_b(x, t)) dx + d_2 \int_{\Omega} \int_{\Omega} \mathfrak{Z}(x-y) dy I_b(y, t) dx - d_2 \int_{\Omega} I_b(x, t) dx \\ & \leq (\bar{\Lambda}_m + \bar{\Lambda}_b) |\Omega| - \int_{\Omega} \min\{\underline{\mu}_m, \underline{\mu}_b\} (S_m + I_m(x, t) + S_b(x, t) + I_b(x, t)) dx, \end{aligned}$$

where $|\Omega|$ denotes the volume of Ω ; $\bar{\Lambda}_m$ and $\bar{\Lambda}_b$ represent the upper bound of Λ_m and Λ_b , respectively. $\underline{\mu}_m$ and $\underline{\mu}_b$ denote the lower bound of μ_m and μ_b , respectively. By calculating, we have

$$\limsup_{t \rightarrow \infty} \int_{\Omega} [S_m(x, t) + I_m(x, t) + S_b(x, t) + I_b(x, t)] dx \leq \frac{(\bar{\Lambda}_m + \bar{\Lambda}_b) |\Omega|}{\min\{\underline{\mu}_m, \underline{\mu}_b\}}.$$

Lemma 2.2. If $(S_m(\cdot, t), I_m(\cdot, t), S_b(\cdot, t)I_b(\cdot, t)) \in \mathcal{Y}$ is the solution of system (1.2) with $(S_{m,0}, I_{m,0}, S_{b,0}I_{b,0}) \in \mathcal{Y}_+$, then $(S_m(\cdot, t), I_m(\cdot, t), S_b(\cdot, t)I_b(\cdot, t)) \in \mathcal{Y}_+$ for all $t \in [0, t_0]$.

Proof. By calculation, we have

$$\begin{aligned} S_m(x, t) &= S_{m,0}(x)e^{-\int_0^t (\mu_m(x) + \frac{b_1(x)\beta_m(x)}{N_b} I_b(x, u)) du} + \int_0^t \Lambda_m e^{\int_\tau^t (-\mu_m(x) + \frac{b_1(x)\beta_m(x)}{N_b} I_b(x, u)) du} d\tau, \\ I_m(x, t) &= I_{m,0}(x)e^{-\mu(x)t} + \int_0^t \frac{b_1(x)\beta_m(x)}{N_b} S_m(x, u) I_b(x, u) e^{-\mu(x)(t-\tau)} d\tau, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} S_b(x, t) &= S_{b,0}(x)e^{-\int_0^t (d_1 + \mu_b(x) + \frac{b_1(x)\beta_b(x)}{N_b} I_m(x, u)) du} + \int_0^t (d_1 \int_\Omega \mathfrak{I}(x-y) S_b(y, t) dy + \Lambda_b(x)) \\ &\quad \times e^{-\int_\tau^t (d_1 + \mu_b(x) + \frac{b_1(x)\beta_b(x)}{N_b} I_m(x, u)) du} d\tau, \\ I_b(x, t) &= I_{b,0}(x)e^{-(d_2 + d_b + \mu_b(x))t} + \int_0^t (d_2 \int_\Omega \mathfrak{I}(x-y) I_b(y, t) dy + \frac{b_1(x)\beta_b(x)}{N_b} S_b(x, u) I_m(x, u)) \\ &\quad \times e^{-(d_2 + \mu_b(x))(t-\tau)} d\tau \end{aligned} \quad (2.3)$$

for all $t \in [0, t_0]$ and $x \in \bar{\Omega}$. Due to $(S_{m,0}, I_{m,0}, S_{b,0}I_{b,0}) \in \mathcal{Y}_+$ and $\mathfrak{I}(x) > 0$ on R , it means that $S_m(x, t) \geq 0$, $I_m(x, t) \geq 0$, $S_b(x, t) \geq 0$ and $I_b(x, t) \geq 0$; further, $S_m(\cdot, t) > 0$, $I_m(\cdot, t) > 0$, $S_b(\cdot, t) > 0$ and $I_b(\cdot, t) > 0$ for $t \in [0, t_0]$.

Theorem 2.1. Suppose that $(S_{m,0}, I_{m,0}, S_{b,0}, I_{b,0}) \in \mathcal{Y}$. Then, there exists a $t_0 > 0$ such that system (1.2) has the unique solution $(S_m(\cdot, t), I_m(\cdot, t), S_b(\cdot, t)I_b(\cdot, t))$ for all $t \in [0, t_0]$, and either $t_0 = +\infty$ or $\limsup_{t \rightarrow t_0-0} \|(S_m(\cdot, t), I_m(\cdot, t), S_b(\cdot, t), I_b(\cdot, t))\| = +\infty$.

Proof.

$$F(S_1, S_2, S_3, S_4)(x) = \begin{pmatrix} \Lambda_m(x) - \frac{\beta_m(x)b_1(x)}{N_b} S_1 S_4 \\ \frac{\beta_m(x)b_1(x)}{N_b} S_1 S_4 \\ \Lambda_b(x) - \frac{\beta_b(x)b_1(x)}{N_b} S_2 S_3 \\ \frac{\beta_b(x)b_1(x)}{N_b} S_2 S_3 \end{pmatrix}.$$

Let $F'[\varrho_1, \varrho_2, \varrho_3, \varrho_4]$ be a linear operator on \mathcal{Y} defined as follows:

$$F'[\varrho_1, \varrho_2, \varrho_3, \varrho_4](S_1, S_2, S_3, S_4)(x) = \begin{pmatrix} -\frac{\beta_m(x)b_1(x)}{N_b} \varrho_4 S_1 - \frac{\beta_m(x)b_1(x)}{N_b} \varrho_1 S_4 \\ \frac{\beta_m(x)b_1(x)}{N_b} \varrho_4 S_1 + \frac{\beta_m(x)b_1(x)}{N_b} \varrho_1 S_4 \\ -\frac{\beta_b(x)b_1(x)}{N_b} \varrho_3 S_2 - \frac{\beta_b(x)b_1(x)}{N_b} \varrho_2 S_3 \\ \frac{\beta_b(x)b_1(x)}{N_b} \varrho_3 S_2 + \frac{\beta_b(x)b_1(x)}{N_b} \varrho_2 S_3 \end{pmatrix}.$$

By calculating, we have

$$\begin{aligned} F(S_1, S_2, S_3, S_4)(x) &= F(\varrho_1, \varrho_2, \varrho_3, \varrho_4)(x) + F'[\varrho_1, \varrho_2, \varrho_3, \varrho_4](S_1 - \varrho_1, S_2 - \varrho_2, S_3 - \varrho_3, S_4 - \varrho_4)(x) \\ &\quad + \begin{pmatrix} -\frac{\beta_m(x)b_1(x)}{N_b} (S_1 - \varrho_1)(S_4 - \varrho_4) \\ \frac{\beta_m(x)b_1(x)}{N_b} (S_1 - \varrho_1)(S_4 - \varrho_4) \\ -\frac{\beta_b(x)b_1(x)}{N_b} (S_2 - \varrho_2)(S_3 - \varrho_3) \\ \frac{\beta_b(x)b_1(x)}{N_b} (S_2 - \varrho_2)(S_3 - \varrho_3) \end{pmatrix}; \end{aligned}$$

due to the coefficients being positive and bounded, we have that the last term on the right-hand side of this equation is $o\{(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)^T - (\varrho_1, \varrho_2, \varrho_3, \varrho_4)^T\}$. It means that F is Fréchet-differentiable for $(\varrho_1, \varrho_2, \varrho_3, \varrho_4)^T$ on \mathcal{Y} . Moreover, we have

$$\begin{aligned} & \|F'[\varrho_1, \varrho_2, \varrho_3, \varrho_4] - F'[\tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{\varrho}_3, \tilde{\varrho}_4]\| \\ &= \sup_{\|(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)^T\|_{\mathcal{Y}} \leq 1} \|(F'[\varrho_1, \varrho_2, \varrho_3, \varrho_4] - F'[\tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{\varrho}_3, \tilde{\varrho}_4])(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)\|_{\mathcal{Y}} \\ &= \sup_{\|(\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4)^T\|_{\mathcal{Y}} \leq 1} \left\| \left\{ -\frac{\beta_m(x)b_1(x)}{N_b}(\varrho_4 - \tilde{\varrho}_4)\varsigma_1 - \frac{\beta_m(x)b_1(x)}{N_b}(\varrho_1 - \tilde{\varrho}_1)\varsigma_4, \right. \right. \\ & \quad \left. \frac{\beta_m(x)b_1(x)}{N_b}(\varrho_4 - \tilde{\varrho}_4)\varsigma_1 + \frac{\beta_m(x)b_1(x)}{N_b}(\varrho_1 - \tilde{\varrho}_1)\varsigma_4, -\frac{\beta_b(x)b_1(x)}{N_b}(\varrho_3 - \tilde{\varrho}_3)\varsigma_2 \right. \\ & \quad \left. - \frac{\beta_b(x)b_1(x)}{N_b}(\varrho_2 - \tilde{\varrho}_2)\varsigma_3, \frac{\beta_b(x)b_1(x)}{N_b}(\varrho_3 - \tilde{\varrho}_3)\varsigma_2 + \frac{\beta_b(x)b_1(x)}{N_b}(\varrho_2 - \tilde{\varrho}_2)\varsigma_3 \right\} \| \\ & \leq 2 \frac{\bar{\beta}\bar{b}_1}{N_b} \|(\varrho_1 - \tilde{\varrho}_1, \varrho_2 - \tilde{\varrho}_2, \varrho_3 - \tilde{\varrho}_3, \varrho_4 - \tilde{\varrho}_4)^T\|_{\mathcal{Y}}, \end{aligned}$$

where $(\tilde{\varrho}_1, \tilde{\varrho}_2, \tilde{\varrho}_3, \tilde{\varrho}_4)^T \in \mathcal{Y}$; this implies that F is continuously Fréchet-differentiable [21, Lemma 3.1] on \mathcal{Y} .

Due to $\mathbb{A}_1(t)_{t \geq 0}$, $\mathbb{A}_2(t)_{t \geq 0}$, $\mathbb{A}_3(t)_{t \geq 0}$ and $\mathbb{A}_4(t)_{t \geq 0}$ being generators of uniformly continuous semigroups $\mathbb{A}_1(t)_{t \geq 0}$, $\mathbb{A}_2(t)_{t \geq 0}$, $\mathbb{A}_3(t)_{t \geq 0}$ and $\mathbb{A}_4(t)_{t \geq 0}$, the solution $(S_m(x, t), I_m(x, t), S_b(x, t), I_b(x, t))$ of system (1.2) can be written as follows:

$$w(x, t) = e^{\mathbb{A}(t)}w(\cdot, t)(x) + \int_0^t E^{\mathbb{A}(t-s)}F(w(\cdot, \tau))(x)ds, \quad t \geq 0, \quad x \in \bar{\Omega},$$

where

$$w(x, t) = \begin{pmatrix} S_m(x, t) \\ I_m(x, t) \\ S_b(x, t) \\ I_b(x, t) \end{pmatrix}, \quad \mathbb{A}(t) = \begin{pmatrix} \mathbb{A}_1(t) \\ \mathbb{A}_2(t) \\ \mathbb{A}_3(t) \\ \mathbb{A}_4(t) \end{pmatrix}, \quad F(w(x, t)) = \begin{pmatrix} \Lambda_m(x) - \frac{b_1(x)\beta_m(x)}{N_b}S_m(x, t)I_b(x, t) \\ \frac{b_1(x)\beta_m(x)}{N_b}S_m(x, t)I_b(x, t) \\ \Lambda_b(x) - \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t) \\ \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t) \end{pmatrix}.$$

Due to \mathbb{A} being the infinitesimal generator of $e^{t\mathbb{A}}_{t \geq 0}$ and F being continuously Fréchet-differentiable on \mathcal{Y} , from [22, Proposition 4.16], the result holds.

3. Threshold dynamics of system

3.1. Basic reproduction number

To give the expression of the basic reproduction number in an abstract way using the next-generation matrix method [23], we consider the following linearized equations around the disease-free equilibrium $E^0 = (S_m^0(x), 0, S_b^0(x), 0)$:

$$\begin{cases} \frac{\partial I_m}{\partial t} = \frac{b_1(x)\beta_m(x)}{N_b}S_m^0 I_b(x, t) - \mu_m(x)I_m(x, t), \\ \frac{\partial I_b}{\partial t} = d_2 \int_{\Omega} \mathfrak{I}(x-y)I_b(y, t)dy - d_2 I_b(x, t) + \frac{b_1(x)\beta_b(x)}{N_b}S_b^0 I_m(x, t) - \mu_b(x)I_b(x, t) - d_b I_b(x, t). \end{cases} \quad (3.1)$$

System (3.1) is equivalent to

$$\frac{\partial \Xi}{\partial t} = (M - N)\Xi + Q\Xi, \quad x \in \Omega, t > 0,$$

where

$$\Xi = \begin{pmatrix} I_m \\ I_b \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 \\ 0 & d_2 \int_{\Omega} \mathfrak{I}(x-y)dy \end{pmatrix},$$

and

$$N = \begin{pmatrix} \mu_m & 0 \\ 0 & d_2 + \mu_b + d_b \end{pmatrix} \quad Q = \begin{pmatrix} 0 & \frac{b_1(x)\beta_m(x)}{N_b} S_m^0 \\ \frac{b_1(x)\beta_b(x)}{N_b} S_b^0 & 0 \end{pmatrix}.$$

Let $K(t)$ be the solution semigroup with respect to the following linear reaction-diffusion equation

$$\frac{\partial \Xi}{\partial t} = (M - N)\Xi, \quad x \in \Omega, t > 0. \quad (3.2)$$

Define

$$\mathbb{K}(\mathcal{S})(x) := \int_0^{\infty} Q[K(t)(\mathcal{S})](x)dt.$$

By virtue of the next-infection operator, the spectral radius of \mathbb{K} can be defined as the basic reproduction number

$$R_0 := r(\mathbb{K}).$$

Moreover, by virtue of [24], we can obtain the following lemma:

Lemma 3.1. $\text{sign}(R_0 - 1) = \text{sign}\lambda_0$.

The principal eigenvalue λ_0 is related to eigenfunction $(\Theta_0(x), \Upsilon_0(x))$, which satisfies the following equation:

$$\begin{cases} \lambda\Theta(x) = \frac{b_1(x)\beta_m(x)}{N_b} S_m^0 \Upsilon(x) - \mu_m(x)\Theta(x), \\ \lambda\Upsilon(x) = d_2 \int_{\Omega} \mathfrak{I}(x-y)\Upsilon(y)dy - d_2\Upsilon(x) + \frac{b_1(x)\beta_b(x)}{N_b} S_b^0 \Theta(x) - \mu_b(x)\Upsilon(x) - d_b\Upsilon(x). \end{cases} \quad (3.3)$$

3.2. Global stability of the disease-free equilibrium

Before proving the global asymptotic stability of the disease-free equilibrium point, some lemmas need to be given; we first consider the following eigenvalue problem, which has been studied by García-Melián and Rossi [15].

$$\begin{cases} \int_{R^N} \mathfrak{I}(x-y)(\Psi(y) - \Psi(x))dy = -\lambda_e \Psi(x), & \text{in } \Omega, \\ \Psi(x) = 0, & \text{on } R^N \setminus \Omega. \end{cases} \quad (3.4)$$

Lemma 3.2. For system (3.4), there exists a unique principal eigenvalue λ_1 corresponding to eigenfunction $\Psi(x)$. Furthermore, $0 < \lambda_1 < 1$ and

$$\lambda_1 = \inf_{\psi \in L^2(\Omega), \Psi \neq 0} \frac{\int_{\Omega} \Psi^2(x)dx - \int_{\Omega} \int_{\Omega} \mathfrak{I}(x-y)\Psi(y)\Psi(x)dydx}{\int_{\Omega} \Psi^2(x)}.$$

Now, we have the following global stability result.

Theorem 3.1. *If $R_0 < 1$, the solution $(S_m(x, t), I_m(x, t), S_b(x, t), I_b(x, t))$ of system (1.2) satisfies that*

$$\lim_{t \rightarrow +\infty} S_m(x, t) = S_m^0(x), \lim_{t \rightarrow +\infty} I_m(x, t) = 0, \lim_{t \rightarrow +\infty} S_b(x, t) = S_b^0(x), \lim_{t \rightarrow +\infty} I_b(x, t) = 0.$$

Proof. By virtue of Eqs (2.2) and (2.3), we can obtain that $I_m \rightarrow 0$ and $I_b \rightarrow 0$ as $t \rightarrow +\infty$.

Moreover, we prove that $S_m(x, t) \rightarrow S_m^0(x)$ on x as $t \rightarrow +\infty$; let $p_1(x, t) = S_m(x, t) - S_m^0(x)$; then, we have

$$\frac{\partial p_1(x, t)}{\partial t} = -\mu_m(x)p_1(x, t) - \frac{b_1(x)\beta_m(x)}{N_b}S_m(x, t)I_b(x, t), \quad x \in \Omega. \quad (3.5)$$

By calculating, we have

$$p_1(x, t) = p_{1,0}(x)e^{-\mu_m(x)t} - \frac{1}{\mu_m(x)} \frac{b_1(x)\beta_m(x)}{N_b}S_m(x, t)I_b(x, t),$$

as $t \rightarrow \infty$, we obtain that $p_1(x, t) \rightarrow 0$; hence, $S_m(x, t) \rightarrow S_m^0(x)$.

Next, we prove that $S_b(x, t) \rightarrow S_b^0(x)$ on x as $t \rightarrow +\infty$; let $p_2(x, t) = S_b(x, t) - S_b^0(x)$. Furthermore, we have

$$\frac{\partial p_2(x, t)}{\partial t} = d_1 \int_{\Omega} \mathfrak{I}(x-y)p_2(y, t)dy - d_1 p_2(x, t) - \mu_b(x)p_2(x, t) - \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t), \quad x \in \Omega. \quad (3.6)$$

Let $P(t) = \int_{\Omega} p_2^2(x, t)dx$; we can obtain

$$\begin{aligned} & \frac{dP(t)}{dt} \\ &= 2 \int_{\Omega} p_2(x, t) \frac{\partial p_2(x, t)}{\partial t} dx \\ &= 2 \int_{\Omega} p_2(x, t) \left\{ d_1 \int_{\Omega} \mathfrak{I}(x-y)p_2(y, t)dy - d_1 p_2(x, t) - \mu_b(x)p_2(x, t) - \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t) \right\} dx \\ &= 2 \left\{ d_1 \int_{\Omega} \int_{\Omega} \mathfrak{I}(x-y)p_2(y, t)p_2(x, t)dydx - \int_{\Omega} p_2^2(x, t)dx \right\} \\ & \quad - 2 \int_{\Omega} \left\{ \mu_b(x)p_2(x, t) + \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t) \right\} p_2(x, t)dx \\ &\leq -2d_1\lambda_1 P(t). \end{aligned} \quad (3.7)$$

By calculation we have that

$$P(t) \leq c_0 e^{-2d_1\lambda_1 t}.$$

Hence, there exists a constant c_0 ; we have

$$\|p_2(\cdot, t)\|_{L^2(\Omega)} \leq c_0 e^{-d_1\lambda_1 t}.$$

By virtue of Eq (3.6), we can obtain

$$p_2(x, t) = p_0(x)e^{-(d_1+\mu_b)t} + e^{-(d_1+\mu_b)t} \int_0^t e^{(d_1+\mu_b)s} \left(d_1 \int_{\Omega} \mathfrak{I}(x-y)p_2(y, t)dy - \frac{b_1(x)\beta_b(x)}{N_b}S_b(x, t)I_m(x, t) \right) ds. \quad (3.8)$$

Applying the Hölder inequality to the following equation, there exists some positive constant satisfying that

$$\int_{\Omega} \mathfrak{I}(x-y)p_2(y,t)dy \leq C\|p_2(\cdot, s)\|_{L^2(\Omega)}. \quad (3.9)$$

Combine (3.8) and (3.9); there exists some positive constants $c_i (i = 1, 2)$ we have

$$|p_2(x, t)| \leq c_1 e^{-(d_1 + \mu_b)t} + c_2 e^{-d_1 \lambda_1 t}.$$

Hence, as $t \rightarrow \infty$, $p_2(x, t) \rightarrow 0$ uniformly on $x \in \Omega$. Furthermore, we obtain that $S_b(x, t) \rightarrow S_b^0(x)$.

3.3. Uniform persistence

In this section, we consider the uniform persistence of system (1.2). To achieve these goals, we first consider the following problem.

Theorem 3.2. *If $R_0 > 1$, then there exists a function $\Pi(x)$, such that*

$$\liminf_{t \rightarrow \infty} (S_m(x, t) + I_m(x, t) + S_b(x, t) + I_b(x, t)) \geq \Pi(x);$$

hence, the disease is uniformly persistent.

Proof. Due to $R_0 > 1$, there exists a $\kappa > 0$ such that $\lambda(S_m^* - \kappa, S_b^* - \kappa) > 0$ (where $(S_m^*, I_m^*, S_b^*, I_b^*)$ represents that the endemic equilibrium). It means that there exists a $\tilde{t}_1 > 0$ satisfying that $S_m(x, t) > S_{m,0} - \kappa$ and $S_b(x, t) > S_{b,0} - \kappa$ for $t \geq \tilde{t}_1$ and $x \in \bar{\Omega}$. For $x \in \Omega$, $t > \tilde{t}_1$; according to the comparison principle, we can obtain

$$\begin{cases} \frac{\partial I_m}{\partial t} \geq \frac{b_1(x)\beta_m(x)}{N_b}(S_{m,0} - \kappa)I_b(x, t) - \mu_m(x)I_m(x, t), \\ \frac{\partial I_b}{\partial t} \geq d_2 \int_{\Omega} \mathfrak{I}(x-y)I_b(y, t)dy - d_2 I_b(x, t) + \frac{b_1(x)\beta_b(x)}{N_b}(S_{b,0} - \kappa)I_m(x, t) - \mu_b(x)I_b(x, t) - d_b I_b(x, t). \end{cases}$$

Define $(\tilde{I}_m(x, t), \tilde{I}_b(x, t)) = (Ne^{\tilde{\lambda}t}\tilde{\psi}_1(x), Ne^{\tilde{\lambda}t}\tilde{\psi}_2(x))$, $(\tilde{I}_m(x, t), \tilde{I}_b(x, t))$ to satisfy the following equation

$$\begin{cases} \frac{\partial \tilde{I}_m}{\partial t} = \frac{b_1(x)\beta_m(x)}{N_b}(S_{m,0} - \kappa)I_b(x, t) - \mu_m(x)I_m(x, t), \\ \frac{\partial \tilde{I}_b}{\partial t} = d_2 \int_{\Omega} \mathfrak{I}(x-y)I_b(y, t)dy - d_2 I_b(x, t) + \frac{b_1(x)\beta_b(x)}{N_b}(S_{b,0} - \kappa)I_m(x, t) - \mu_b(x)I_b(x, t) - d_b I_b(x, t), \end{cases}$$

where $(\tilde{\psi}_1(x), \tilde{\psi}_2(x))$ is the eigenfunction with respect to $\tilde{\lambda} < 0$. According to the comparison principle, we know that $I_m(x, t) \geq \tilde{I}_m(x, t)$, $I_b(x, t) \geq \tilde{I}_b(x, t)$ for $x \in \Omega, t > \tilde{t}_1$. Therefore, $I_m(x, t) \geq Ne^{\tilde{\lambda}t}\tilde{\psi}_1(x)$, $I_b(x, t) \geq Ne^{\tilde{\lambda}t}\tilde{\psi}_2(x)$ such that

$$\liminf_{t \rightarrow \infty} I_H(x, t) \geq N\tilde{\psi}_1(x), \quad \liminf_{t \rightarrow \infty} I_V(x, t) \geq N\tilde{\psi}_2(x).$$

On the basis of Lemma (2.1), we know that there exists constants $K > 0$ and \tilde{t}_2 such that

$$I_m(x, t) \leq K, \quad I_b(x, t) \leq K, \quad t \geq \tilde{t}_2, \quad x \in \bar{\Omega}.$$

Then, S_H and S_V satisfy the following equation

$$\begin{cases} \frac{\partial S_m}{\partial t} \geq \Lambda_m(x) - \mu_m(x)S_m - \frac{b_1(x)\beta_m(x)K}{N_b}S_m(x, t), \\ \frac{\partial S_b}{\partial t} \geq d_1 \int_{\Omega} \mathfrak{I}(x-y)S_b(y, t)dy - d_1 S_b(x, t) + \Lambda_b(x) - \mu_b(x)S_b(x, t) - \frac{b_1(x)\beta_b(x)K}{N_b}S_b(x, t), \\ x \in \Omega, t > \tilde{t}_2. \end{cases}$$

By virtue of Eq (3.4), we obtain

$$\begin{cases} \frac{\partial S_m}{\partial t} \geq \Lambda_m(x) - \mu_m(x)S_m - \frac{b_1(x)\beta_m(x)K}{N_b}S_m(x, t), \\ \frac{\partial S_b}{\partial t} \geq -d_1\lambda_e S_b(x, t) + \Lambda_b(x) - \mu_b(x)S_b(x, t) - \frac{b_1(x)\beta_b(x)K}{N_b}S_b(x, t), \\ x \in \Omega, t > \tilde{t}_2. \end{cases}$$

Hence

$$\begin{aligned} \liminf_{t \rightarrow \infty} S_m(x, t) &\geq (\Lambda_m(x))/(\mu_m(x) + \frac{b_1(x)\beta_m(x)K}{N_b}), \\ \liminf_{t \rightarrow \infty} S_b(x, t) &\geq (\Lambda_b(x))/(d_1\lambda_e + \mu_b(x) + \frac{b_1(x)\beta_b(x)K}{N_b}). \end{aligned}$$

Let $\Pi(x) := \min\{(\Lambda_m(x))/(\mu_m(x) + \frac{b_1(x)\beta_m(x)K}{N_b}), (\Lambda_b(x))/(d_1\lambda_e + \mu_b(x) + \frac{b_1(x)\beta_b(x)K}{N_b}), N\tilde{\psi}_1(x), N\tilde{\psi}_2(x)\}$. The disease uniformly persistent is obtained.

4. Numerical simulations

In order to present the theoretical results, some numerical simulations are presented in this section. The parameter values (Table 1) and initial value are selected as follows (all rates are per capita per day (day^{-1})):

Table 1. The partial parameter values.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
b_1	1.6	N_b	150	μ_m	0.15	Λ_m	6.5
μ_b	0.01 [25]	Λ_b	1.5	d_1	0.025	d_2	0.025

Initial value:

$$(S_{m,0}(x), I_{m,0}(x), S_{b,0}(x), I_{b,0}(x)) = (0.03\sin x + 0.05\cos x, 0.02\cos x, 0.01\sin x + 0.03\cos x, 0).$$

Moreover, the nonlocal kernel function [20] is selected as follows:

$$J_x = \begin{cases} A \exp(\frac{1}{x^2 - 1}), & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $A = 2.6423$, $x \in [-1, 1] \subset \mathbb{R}$ and $\int_{\mathbb{R}} J(x)dx = \int_{-1}^1 J(x)dx \approx 1$.

4.1. Dynamics of system (1.2)

In Figure 1, let $\beta_m = 0.12, \beta_b = 0.15$. In Figure 3, set $\beta_m = 0.075(1 - 0.85\cos x), \beta_b = 0.15$. As $t \rightarrow \infty$, the density of the infected mosquitoes and infected bird will converge to 0, that is, the disease will become extinct.

In Figure 2, let $\beta_m = 1.2, \beta_b = 0.88$. In Figure 4, set $\beta_m = 0.75(1 - 0.85\cos x), \beta_b = 0.88$ [18]. It can be seen that the solution of system (1.2) finally converges to a steady state, indicating the persistence of the diseases.

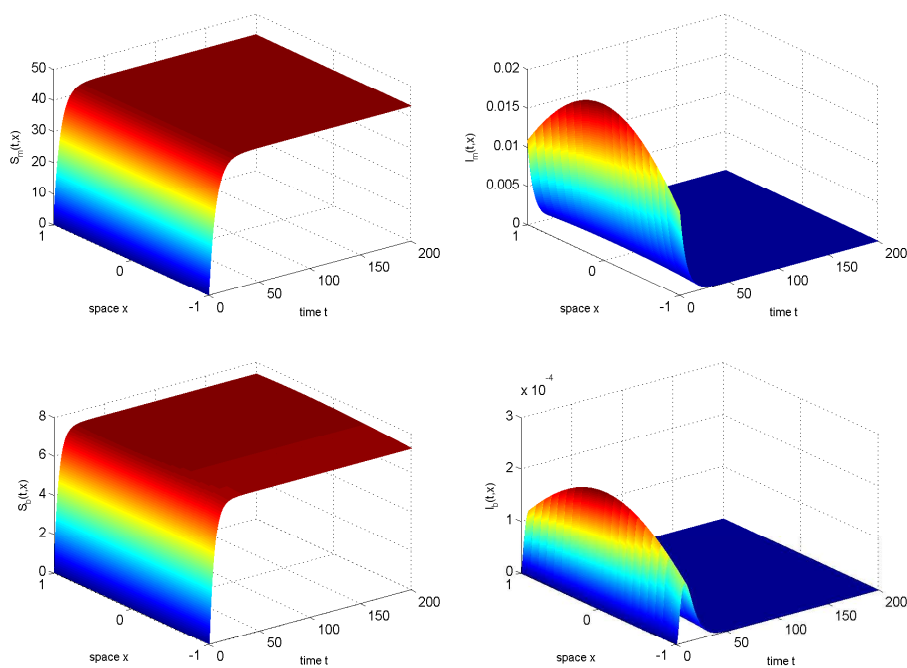


Figure 1. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 0.901080450518523 < 1$.

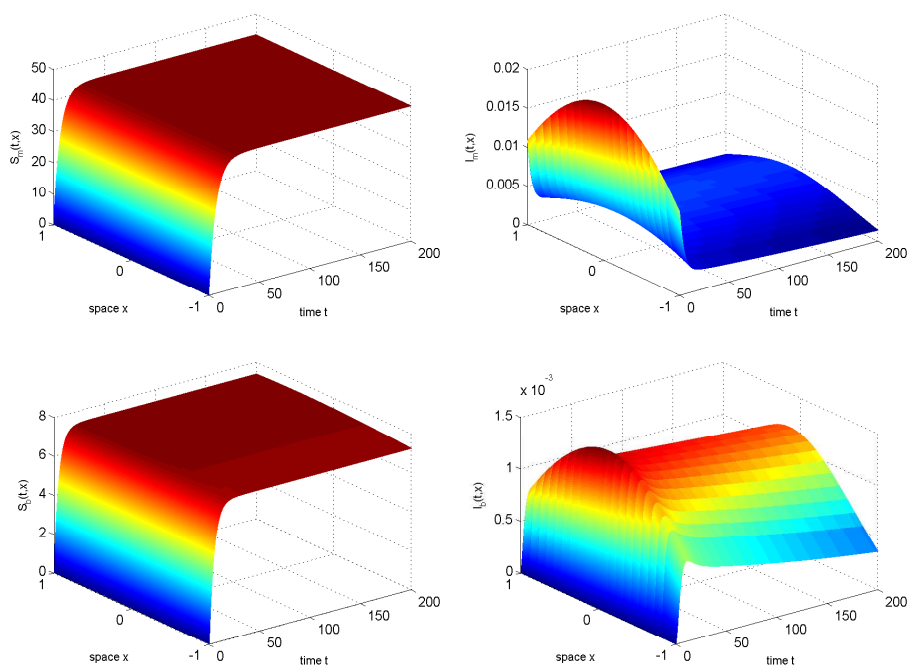


Figure 2. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 6.901750796765820 > 1$.

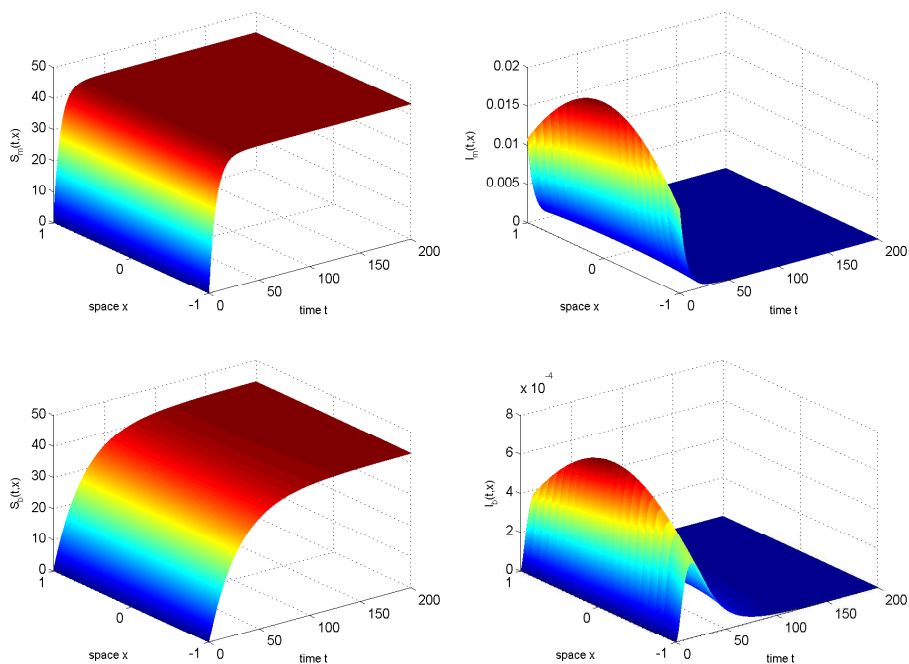


Figure 3. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 0.537588764329462 < 1$.

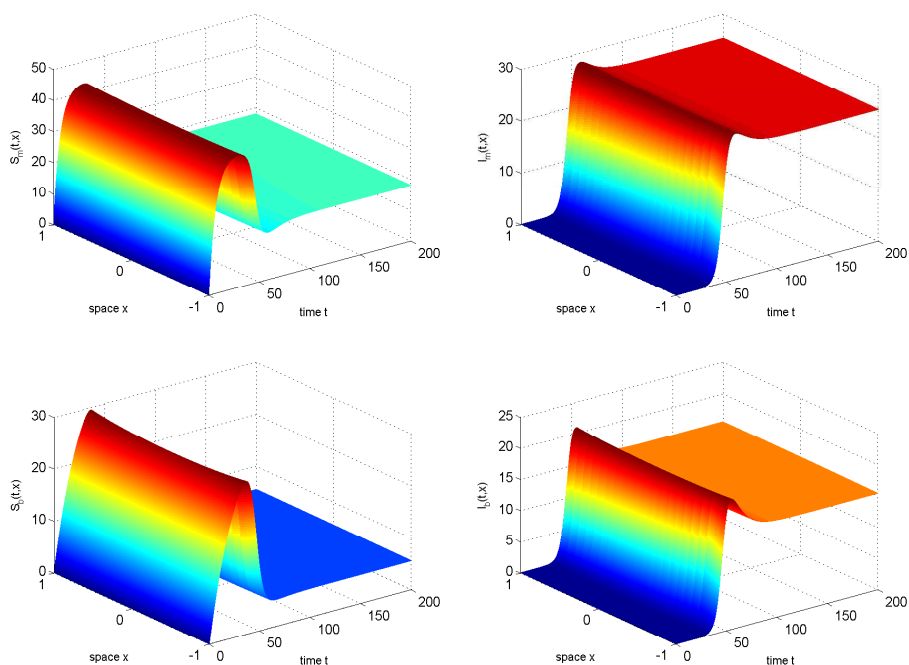


Figure 4. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 4.117616446354086 > 1$.

For $R_0 < 1$, we can see that when the transmission rate is constant, the number of infected drops to 0 relatively quickly; on the contrary, when the transmission rate is spatially dependent, the number of infected tends to 0 for a long time. As $R_0 > 1$, spatial heterogeneity can increase in the number of susceptible mosquitoes and birds, and the number of infected mosquitoes and birds become more evenly distributed in space.

4.2. The impacts of the diffusion rate for mosquitoes and birds

In this section, we aim to examine the impact of the diffusion coefficient on disease persistence considering that the spread of extinct diseases is meaningless. For Figures 4–7, we set $d_1 = d_2 = 0.025$, $d_1 = d_2 = 0.10$, $d_1 = d_2 = 0.20$ and $d_1 = d_2 = 0$, respectively. The results indicate that an increase in the diffusion coefficient leads to an increase in the number of suspected mosquitoes and birds while decreasing the number of infected mosquitoes and birds. However, the differences in mosquito and bird populations across in different spatial distributions become more apparent, resulting in a higher persistence of the disease and making it challenging to control. Therefore, in the event of a disease outbreak, we recommend employing physical or chemical methods to kill mosquitoes in order to better control the spread of the disease.

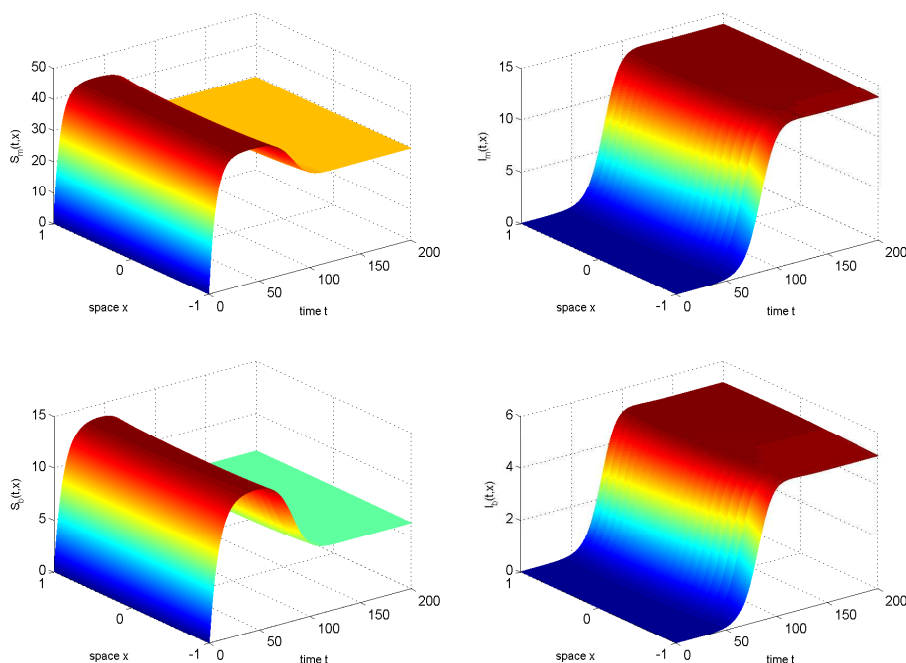


Figure 5. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 2.684224435524666 > 1$.

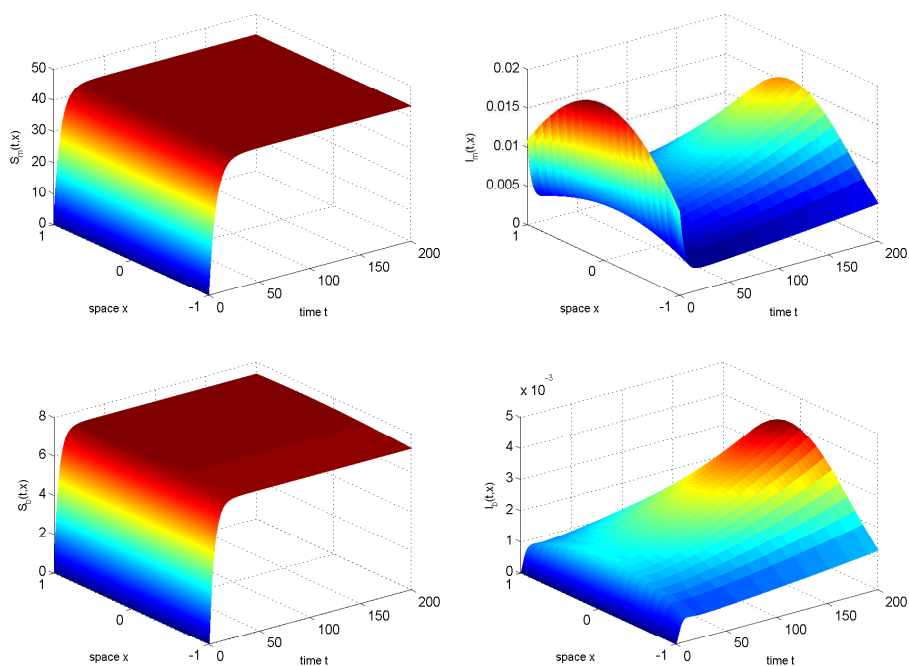


Figure 6. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 2.019457012742855 > 1$.

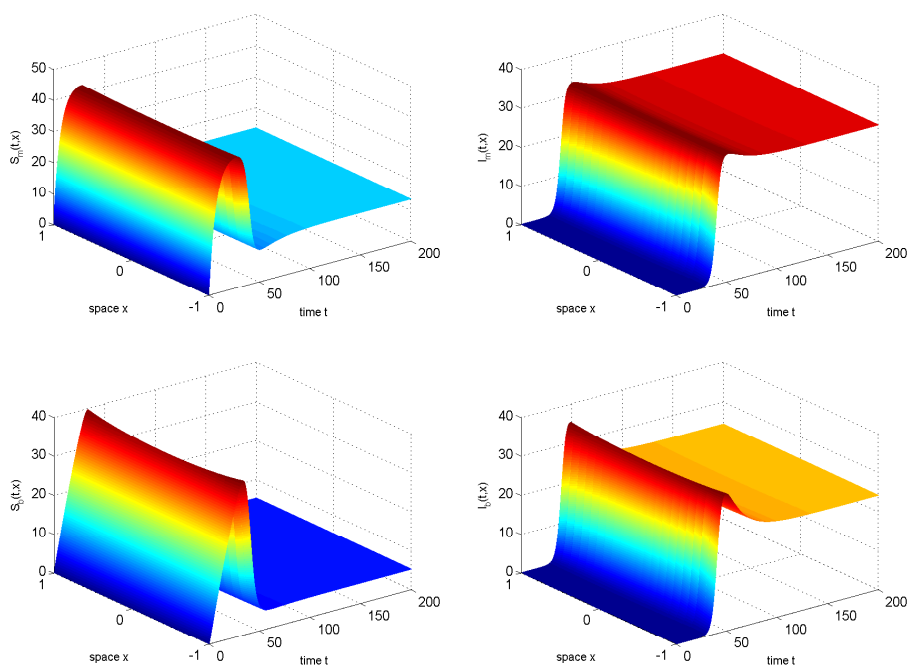


Figure 7. The evolution paths of S_H, I_H, S_V, I_V for system (1.2) with $R_0 = 5.557541598765930 > 1$.

5. Conclusions

We investigated the threshold dynamics of a nonlocal diffusion West Nile virus model with spatial heterogeneity. The well-posedness of the solution is proven using the semigroup theory, and it is continuously Fréchet differentiable. The basic reproduction number is expressed using the next-generation matrix method. Global stability and uniform persistence of the system are proven by constructing a Lyapunov function and applying the comparison principle. Numerical simulations were performed to verify the theorem. The values of the transmission rate were adjusted to examine extinction and persistence of the disease, and the impact of diffusion was considered. Simulation results demonstrated that an increase in the diffusion coefficient leads to an increase in the number of susceptible mosquitoes and birds and a decrease in the number of infected mosquitoes and birds. However, differences in mosquito and bird populations in different spatial distributions become more pronounced, leading to enhanced disease persistence and reduced chances of rapid disease control. Therefore, to better control the spread of the disease, physical or chemical methods for mosquito control are recommended when a disease outbreak occurs.

Conflict of interest

The authors declare that they have no conflict of interest.

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