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Research article

On the conjecture of Jeśmanowicz

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Abstract: Let k, l, m_1 and m_2 be positive integers and let both p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ where a is a positive integer with $a \equiv \pm 3 \pmod 8$. In this paper, using only the elementary methods of factorization, congruence methods and the quadratic reciprocity law, we show that Jeśmanowicz' a conjecture holds for the following set of primitive Pythagorean numbers:

$$\frac{q^{2l} - p^{2k}}{2}, p^k q^l, \frac{q^{2l} + p^{2k}}{2}.$$

We also prove that Jeśmanowicz' conjecture holds for non-primitive Pythagorean numbers:

$$n\frac{q^{2l} - p^{2k}}{2}, np^k q^l, n\frac{q^{2l} + p^{2k}}{2},$$

for any positive integer n if for $a = a_1 a_2$ with $a_1 \equiv 1 \pmod 8$ not a square and $\gcd(a_1, a_2) = 1$, then there exists a prime divisor P of a_2 such that $\left(\frac{a_1}{P}\right) = -1$ and $2|m_1, a \equiv 5 \pmod 8$ or $2 \not| m_2, a \equiv 3 \pmod 8$.

Keywords: exponential Diophantine equations; quadratic residue; positive integer solution
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1. Introduction

In 1955/1956, Sierpinśki [1] showed that the equation $3^x + 4^y = 5^z$ has $x = y = z = 2$ as its only solution in positive integers. In the same year, Jeśmanowicz [2] proved that Sierpinśki's result holds also for the following Pythagorean numbers:

$$(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61).$$

Let a, b, c be fixed positive integers. Consider the exponential Diophantine equation

$$a^x + b^y = c^z. \tag{1.1}$$

Jeśmanowicz [2] proposed the following problem:
Conjecture 1.1. Assume that $a^2 + b^2 = c^2$. Then Eq (1.1) has no positive integer solution (x, y, z) other than $x = y = z = 2$.

The pioneering works related to Conjecture 1.1 were obtained by Ke [3]. Ke proved the conjecture for the Pythagorean number $2n + 1, 2n(n + 1), 2n(n + 1) + 1$ if $n \equiv 1, 4, 5, 9, 10 \pmod{12}$ or n is odd and there exist a prime p and a positive integer s such that $2n + 1 = p^s$ or n is the sum of two squares and there exists a prime p that is congruent to 3 modulo 4 such that $2n + 1 \equiv 0 \pmod{p}$.

It is well known that the numbers

$$a = r^2 - s^2, b = 2rs, c = r^2 + s^2 \tag{1.2}$$

form all primitive Pythagorean numbers, where $\gcd(r, s) = 1, r > s$ and r and s have opposite parity.

Józefiak [4] confirmed the conjecture for $(r, s) = (2^m p^n, 1)$, where $m, n \in N, N = \{1, 2, \cdots\}$ denotes the set of positive integers and p is a prime number. Dem'janenko [5] proved the conjecture for $(r, s) = (m, 1)$, where $m \in \mathbb{N}$. Grytczuk and Grelak [6] proved the conjecture for $(r, s) = (2m, 1)$, where $m \in \mathbb{N}$. Takakuwa and Asaeda [7] generalized the result to $(r, s) = (2m, q)$, where $q \equiv 3 \pmod{4}$ is a prime if m is odd and a prime divisor p of a satisfies the conditions $p \equiv 1 \pmod{4}$ and $\left(\frac{q}{p}\right) = -1$. Most of the existing works on Conjecture 1.1 concern the coprimality case, that is, $\gcd(a, b) = 1$. Indeed, all of the above mentioned results treat the coprimality case, and such a case is essential in the study of Eq (1.1). Several authors studied the more general equation

$$(an)^x + (bn)^y = (cn)^z \tag{1.3}$$

under several conditions with $n > 1$ and $a^2 + b^2 = c^2$ with $\gcd(a, b) = 1$.

Deng and Cohen [8] proved that the only solution of (1.3) is $x = y = z = 2$ if a is a prime power and n is a positive integer such that $P(b)|n$ or $P(n) \nmid b$, where $P(n)$ is the product of all distinct prime divisors of n. They proved also that the only solution of (1.3) is $x = y = z = 2$ for each of the Pythagorean triples

$$(a, b, c) = (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)$$

and for any positive integer n. Following Deng and Cohen's work, Le [9] gave the following more general result in 1999: If (x, y, z) is a solution of (1.3) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied:

(1) $\max\{x, y\} > \min\{x, y\} > z, P(n)|c$ and $P(n) < P(c)$;
(2) $x > y > z$ and $P(n)|b$;
(3) $y > z > x$ and $P(k)|a$.

Sixteen years later in 2015, Yang and Fu [10] simplified the conditions given in the above result by removing all conditions on $P(n)$. Meanwhile between 1999 and 2015, many mathematicians considered several specific cases of Eq (1.3). In 2013, Yang and Tang [11] proved the following: Let $n \geq 4$ be a positive integer and $F_n = 2^{2^n} + 1$. Then, for any positive integer N, the Diophantine equation

$$((F_n - 2)N)^x + (2^{2^{n-1}+1}N)^y = (F_nN)^z \tag{1.4}$$

has no solution other than $(x, y, z) = (2, 2, 2)$.

In 2014, Tang and Weng [12] generalized the above result and proved that the unique solution of (1.4), for any positive integers n and N, is $(x, y, z) = (2, 2, 2)$. The same year, Xinwen Zhang and Wenpeng Zhang [13] proved that the only solution of the equation

$$((2^{2m} - 1)N)^x + (2^{m+1}N)^y = ((2^{2m} + 1)N)^z, \tag{1.5}$$

for any positive integers m and N, is $(x, y, z) = (2, 2, 2)$. Finally, another special case for $m = 2$ of Eq (1.4) was recently studied by Yang and Tang [14]. They proved that the only solution of

$$(15N)^x + (8N)^y = (17N)^z$$

is $(x, y, z) = (2, 2, 2)$ for $N \geq 1$. In 2014, Deng [15] considered another special case of Eq (1.4) by putting $m = s + 1$ and accepting some divisibility conditions such as $P(a)|N$ or $P(N) \nmid a, s \geq 0$, and proved that $(x, y, z) = (2, 2, 2)$ is the only solution of the equation

$$((2^{2s+2} - 1)N)^x + (2^{s+2}N)^y = ((2^{2s+2} + 1)N)^z. \tag{1.6}$$

In 2015, Ma and Wu [16] proved that the only solution of the equation

$$((4n^2 - 1)N)^x + (4nN)^y = ((4n^2 + 1)N)^z \tag{1.7}$$

is $x = y = z = 2$ if $P(4n^2 - 1)|N$. Very recently, Miyazaki [17] nicely proved the Jeśmanowicz conjecture when a or b is a power of 2 by extending the result of Tang and Weng [12]. In 2017, Soydan, Demirci, Cangul and Togbe [18] proved that the Diophantine equation

$$\left(\frac{11^2 - 3^{2\cdot 2}}{2}n\right)^x + (3^2 \cdot 11 \cdot n)^y = \left(\frac{11^2 + 3^{2\cdot 2}}{2}n\right)^z$$

has only the solution $x = y = z = 2$ for any positive integer n. In 2022, Feng and Luo [19] proved that the Diophantine equation

$$\left(\frac{q^{2l} - p^{2k}}{2}n\right)^x + (p^k q^l n)^y = \left(\frac{q^{2l} + p^{2k}}{2}n\right)^z \tag{1.8}$$

has only the solution $x = y = z = 2$ for any positive integer n, where p and q are odd primes with $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}, k, l, m_1$ and m_2 are positive integers and $a \equiv 5 \pmod 8$ is a prime.

After these works, Conjecture 1.1 has been proved to be true for various particular cases. For recent results, we only refer to the papers of Deng, Yuan and Luo [20], Hu and Le [21], Miyazaki [17, 22], Miyazaki, Yuan and Wu [23], Terai [24], Yuan and Han [25] and the references given there.

In this paper, we will prove that the result of [19] holds when a is a positive integer with $a \equiv \pm 3 \pmod 8$. We have following:

Theorem 1.1. *Let k, l, m_1, m_2 be positive integers and let p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, where a is a positive integer with $a \equiv \pm 3 \pmod 8$. Then the equation*

$$\left(\frac{q^{2l} - p^{2k}}{2}\right)^x + (p^k q^l)^y = \left(\frac{q^{2l} + p^{2k}}{2}\right)^z \tag{1.9}$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

Theorem 1.2. *Let the assumptions of k, l, m_1, m_2, p, q be as in Theorem 1.1. Suppose that the following conditions hold:*

(i) If $a = a_1a_2$ with $a_1 \equiv 1 \pmod 8$ not a square and $\gcd(a_1, a_2) = 1$, then there exists a prime divisor P of a_2 such that $\left(\frac{a_1}{P}\right) = -1$,
(ii) $2|m_1, a \equiv 5 \pmod 8$ or $2 \not| m_2, a \equiv 3 \pmod 8$.

Then the equation

$$\left(\frac{q^{2l} - p^{2k}}{2}n\right)^x + (p^k q^l n)^y = \left(\frac{q^{2l} + p^{2k}}{2}n\right)^z, \tag{1.10}$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$ for any $n \geq 1$.

Corollary 1.1. *Let the assumptions of k, l, m_1, m_2, p, q be as in Theorem 1.1. If a is a product of a square and a prime that is congruent 3 modulo 8 and m_2 is odd , or if a is a product of a square and a prime that is congruent 5 modulo 8 and m_1 is even, then Eq (1.10) has only the positive integer solution $(x, y, z) = (2, 2, 2)$ for any $n \geq 1$.*

We organize this paper as follows. In Section 2, we present some lemmas which are needed in the proofs of our main results. Consequently, in Sections 3 to 4, we give the proofs of Theorem 1.1 to 1.2 and Corollary 1.1, respectively. In Section 5, we give some applications of Theorem 1.1 and Theorem 1.2.

2. Lemmas

In this section, we present some lemmas that will be used in the proof of results.

Lemma 2.1. *Let k, l, m_1, m_2 be positive integers and let p and q be odd primes such that $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$, where a is a positive integer with $a \equiv \pm 3 \pmod 8$. Then m_1 is odd or m_2 is odd except for $(p, q, k, l, m_1, m_2, a) = (7, 5, 1, 2, 4, 2, 3)$. Moreover if m_1 is odd and m_2 is even, then $3|a$.*

Proof. It is easy to find that is enough to prove that $2 \not| m_1$ and $3|a$ except for $(p, q, k, l, m_1, m_2, a) = (7, 5, 1, 2, 4, 2, 3)$ when m_2 is even.

If m_1 is odd, we claim that $3|a$. On the contrary suppose that $3 \not| a$; then, we get from $q^l = 2^{m_1} + a^{m_2}$ that $3|q^l$, and so $q = 3$ since q is prime. Taking modulo 4 for the equation $3^l = 2^{m_1} + a^{m_2}$ would give $3^l \equiv 1 \pmod 4$. It follows that l is even and

$$1 = \left(\frac{3^l}{a}\right) = \left(\frac{2}{a}\right) = -1,$$

which leads to a contradiction.

If m_1 is also even, then we get from the condition

$$p^k = 2^{m_1} - a^{m_2} = (2^{\frac{m_1}{2}} + a^{\frac{m_2}{2}})(2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}})$$

that

$$2^{\frac{m_1}{2}} + a^{\frac{m_2}{2}} = p^{k_1}, \quad 2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = p^{k_2}, k_1 > k_2 \geq 0.$$

So $2^{\frac{m_1}{2}+1} = p^{k_2}(p^{k_1-k_2} + 1)$, which would thus give $k_2 = 0$ and $2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = 1$. If $m_1 > 4$, then taking the equation $2^{\frac{m_1}{2}} - a^{\frac{m_2}{2}} = 1$ modulo 8 yields $a^{\frac{m_2}{2}} \equiv -1 \pmod 8$, which leads to a contradiction since

$a \equiv \pm 3 \pmod 8$. Hence $m_1 = 4, a = 3, m_2 = 2$, which implies that $p = 7, k = 1, q = 5, l = 2$. This completes the proof. □

Lemma 2.2. *Let $v_2(n)$ denote nonnegative integer t such that $2^t | n$ and $2^{t+1} \nmid n$. Let a be an odd integer with $a \equiv \pm 5 \pmod 8$. Then we have that $v_2(a^m - 1) = v_2(m) + 2$ if $v_2(m) = h \geq 1$.*

Proof. We prove the lemma 2.2 by induction on $h = v_2(m) \geq 1$. For $h = 1$ we get

$$a^m - 1 = (a^{m_1} + 1)(a^{m_1} - 1). \tag{2.1}$$

If $a \equiv 5 \pmod 8$, then we have $a^{m_1} + 1 \equiv 6 \pmod 8$ and $a^{m_1} - 1 \equiv 4 \pmod 8$, where $m = 2m_1, m_1$ is odd. Hence (2.1) implies $v_2(a^m - 1) = 3 = v_2(m) + 2$. If $a \equiv 3 \pmod 8$ then we have $a^{m_1} + 1 \equiv 4 \pmod 8$ and $a^{m_1} - 1 \equiv 2 \pmod 8$. Thus (2.1) also implies $v_2(a^m - 1) = 3 = v_2(m) + 2$. If the result is shown for some positive integer $h = v_2(m)$, then

$$a^{2m} - 1 = (a^m + 1)(a^m - 1).$$

Since $a^m + 1 \equiv 2 \pmod 8$, we get

$$v_2(a^{2m} - 1) = v_2(a^m + 1) + v_2(a^m - 1) = v_2(m) + 3 = v_2(2m) + 2.$$

This completes the proof. □

Lemma 2.3. *If (x, y, z) is a solution of Eq (1.9) with $x \equiv y \equiv z \equiv 0 \pmod 2$, then $(x, y, z) = (2, 2, 2)$.*

Proof. It is easy to find that $m_1 \geq 3$ by the condition $p^k = 2^{m_1} - a^{m_2}$. We may write $x = 2x_1, y = 2y_1, z = 2z_1$ by the assumption $x \equiv y \equiv z \equiv 0 \pmod 2$. It follows from (1.3) that

$$\left((2^{2m_1} + a^{2m_2})^{z_1} + a^{m_2x_1}2^{(m_1+1)x_1}\right)\left((2^{2m_1} + a^{2m_2})^{z_1} - a^{m_2x_1}2^{(m_1+1)x_1}\right) = p^{2ky_1}q^{2ly_1}.$$

As

$$\gcd((2^{2m_1} + a^{2m_2})^{z_1} + a^{m_2x_1}2^{(m_1+1)x_1}, (2^{2m_1} + a^{2m_2})^{z_1} - a^{m_2x_1}2^{(m_1+1)x_1}) = 1,$$

then we have

$$(2^{2m_1} + a^{2m_2})^{z_1} + a^{m_2x_1}2^{(m_1+1)x_1} = q^{2ly_1}$$

and

$$(2^{2m_1} + a^{2m_2})^{z_1} - a^{m_2x_1}2^{(m_1+1)x_1} = p^{2ky_1}.$$

Taking the difference of the above equations gives

$$2^{(m_1+1)x_1+1} \cdot a^{m_2x_1} = ((2^{m_1} + a^{m_2})^{y_1} + (2^{m_1} - a^{m_2})^{y_1})((2^{m_1} + a^{m_2})^{y_1} - (2^{m_1} - a^{m_2})^{y_1}). \tag{2.2}$$

If y_1 is even, then we have that

$$(2^{m_1} + a^{m_2})^{y_1} + (2^{m_1} - a^{m_2})^{y_1} \equiv 2 \pmod 8$$

and

$$(2^{m_1} + a^{m_2})^{y_1} - (2^{m_1} - a^{m_2})^{y_1} \equiv 0 \pmod 8.$$

It follows from Eq (2.2) that

$$(2^{m_1} + a^{m_2})^{y_1} + (2^{m_1} - a^{m_2})^{y_1} = 2 \cdot a_1^{m_2 x_1} \tag{2.3}$$

and

$$(2^{m_1} + a^{m_2})^{y_1} - (2^{m_1} - a^{m_2})^{y_1} = 2^{(m_1+1)x_1} a_2^{m_2 x_1}, \tag{2.4}$$

where $a_1 a_2 = a, a_1 > 1$. Taking modulo a_1 for Eq (2.3) yields to

$$2^{m_1 y_1 + 1} \equiv 0 \pmod{a_1},$$

which leads to a contradiction. Hence y_1 is odd; then, we have that

$$(2^{m_1} + a^{m_2})^{y_1} + (2^{m_1} - a^{m_2})^{y_1} \equiv 0 \pmod 4$$

and

$$(2^{m_1} + a^{m_2})^{y_1} - (2^{m_1} - a^{m_2})^{y_1} \equiv 2 \pmod 4.$$

It follows from Eq (2.2) that

$$(2^{m_1} + a^{m_2})^{y_1} + (2^{m_1} - a^{m_2})^{y_1} = 2^{(m_1+1)x_1} \cdot a_1^{m_2 x_1} \tag{2.5}$$

and

$$(2^{m_1} + a^{m_2})^{y_1} - (2^{m_1} - a^{m_2})^{y_1} = 2a_2^{m_2 x_1}, \tag{2.6}$$

where $a_1 a_2 = a$. Taking modulo a_1 for Eq (2.5) yields $2^{m_1 y_1 + 1} \equiv 0 \pmod{a_1}$. So $a_1 = 1, a_2 = a$. We claim that $y_1 = 1$. On the contrary suppose $y_1 > 1$. Note that y_1 is odd; Eq (2.5) would give that

$$2^{(m_1+1)x_1 - m_1 - 1} = \sum_{r=0}^{(y_1-1)/2} \binom{y_1}{2r} 2^{m_1(y_1-2r-1)} a^{2rm_2}.$$

Thus $y_1 a^{m_2(y_1-1)} \equiv 0 \pmod 2$, which is a contradiction. Therefore $y_1 = 1$ and $2^{(m_1+1)x_1} = 2^{m_1+1}$ yield that $x_1 = 1$. Substituting $x = y = 2$ into Eq (1.9) gives $z = 2$.

This completes the proof. □

In this section, we present two useful results necessary for the proof of our main results.

Lemma 2.4. ([9]) *If (x, y, z) is a solution of (1.3) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied*

(i) $max\{x, y\} > min\{x, y\} > z$; (ii) $x > z > y$; (iii) $y > z > x$.

Lemma 2.5. ([15], [17]) *Assume that $n > 1$. Then (1.3) has no solution (x, y, z) with $max\{x, y\} > min\{x, y\} > z$.*

Lemma 2.6. *The equation*

$$(288n)^x + (175n)^y = (337n)^z \tag{2.7}$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$ for any $n \geq 1$.

Proof. We first consider the case $n = 1$. Assume that (x, y, z) is a positive integer solution. Taking modulo 4 for Eq (2.7) leads to $(-1)^y \equiv 1 \pmod 4$. It follows that y is even. Taking modulo 32 for Eq (2.7) leads to $3^{4|y-z|} \equiv 1 \pmod{32}$. Then we get by Lemma 2.2 that $5 \le v_2(4) + 2 + v_2(|y - z|) = 4 + v_2(|y - z|)$. Thus z is even since y is even. We get from Eq (2.7) that

$$(337^{\frac{z}{2}} + 175^{\frac{y}{2}})(337^{\frac{z}{2}} - 175^{\frac{y}{2}}) = 3^{2x}2^{5x}.$$

If $\frac{y}{2}$ is even, then we have

$$337^{\frac{z}{2}} + 175^{\frac{y}{2}} \equiv 2 \pmod 8$$

and

$$337^{\frac{z}{2}} + 175^{\frac{y}{2}} \equiv 2 \pmod 3.$$

So

$$337^{\frac{z}{2}} + 175^{\frac{y}{2}} = 2, 337^{\frac{z}{2}} - 175^{\frac{y}{2}} = 3^{2x}2^{5x-1},$$

which is impossible. Hence $\frac{y}{2}$ is odd and

$$337^{\frac{z}{2}} - 175^{\frac{y}{2}} = 2 \cdot 3^{2x}, 337^{\frac{z}{2}} + 175^{\frac{y}{2}} = 2^{5x-1}$$

since $337^{\frac{z}{2}} - 175^{\frac{y}{2}} \equiv 2 \pmod 8$ and $337^{\frac{z}{2}} + 175^{\frac{y}{2}} \equiv 2 \pmod 3$. Therefore

$$175^{\frac{y}{2}} = 2^{5x-2} - 3^{2x}.$$

Taking modulo 3 yields $(-1)^{5x-2} \equiv 1 \pmod 3$. It follows that x is even. Thus we get by Lemma 2.3 that $x = y = z = 2$.

We now consider the case $n > 1$. Assume that (x, y, z) is a positive integer solution with $(x, y, z) \ne (2, 2, 2)$. Then we have by Lemmas 2.4 and 2.5 that $x > z > y$ or $y > z > x$. We shall discuss separately two cases.

The case $x > z > y$. Then dividing Eq (2.7) by n^y yields

$$(7 \cdot 5^2)^y = n^{z-y}(337^z - 2^{5x} \cdot 3^{2x} n^{x-z}). \tag{2.8}$$

Since $\gcd((7 \cdot 5^2)^y, 337^z) = 1$, we can observe that the two factors on the right-hand side are co-prime. Hence Eq (2.8) yields $n = 7^u$ for some positive integer u with $y = u(z - y)$ and

$$5^{2y} = 337^z - 2^{5x} \cdot 3^{2x} \cdot 7^{u(x-z)}, \tag{2.9}$$

or $n = 5^v$ for some positive integer v with $2y = v(z - y)$ and

$$7^y = 337^z - 2^{5x} \cdot 3^{2x} \cdot 5^{v(x-z)}, \tag{2.10}$$

or $n = 7^u \cdot 5^v$ for some positive integers u and v and

$$1 = 337^z - 2^{5x} \cdot 3^{2x} \cdot 7^{u(x-z)} \cdot 5^{v(x-z)}. \tag{2.11}$$

If (2.9) holds, then taking modulo 16 leads to $9^y \equiv 1 \pmod{16}$. It follows that y is even. If z is even, then we get from Eq (2.9) that

$$32^x | 337^{\frac{z}{2}} + 5^y \quad or \quad 32^x | 337^{\frac{z}{2}} - 5^y.$$

It follows that $32^x \le 337^{\frac{z}{2}} + 5^y$, which is impossible since

$$32^x = (7+5^2)^{2\cdot x/2} = (674+14\cdot 5^2)^{x/2} > 337^{\frac{z}{2}} + 5^y.$$

Hence z is odd. Taking modulo 5 for Eq (2.9) gives $2^z \equiv -2^x \cdot 2^{u(x-z)} \pmod 5$. So x is also odd since $y = u(z-y)$. By taking Eq (2.9) modulo 7 , we have $2^{2y} \equiv 1 \pmod 7$. It follows that $y \equiv 0 \pmod 3$. Taking modulo 9 for Eq (2.9) yields $2^{2z} \equiv 1 \pmod 9$. It follows that $z \equiv 0 \pmod 3$. Taking modulo 27 for Eq (2.9) leads to $2^y \equiv 13^z \pmod{27}$. Since

$$2^y \equiv \begin{cases} 1 \pmod{27}, y \equiv 0 \pmod{18} \\ 10 \pmod{27}, y \equiv 6 \pmod{18} \\ 11 \pmod{27}, y \equiv 12 \pmod{18} \end{cases},$$

and

$$13^z \equiv \begin{cases} 1 \pmod{27}, z \equiv 9 \pmod{18} \\ 10 \pmod{27}, z \equiv 3 \pmod{18} \\ 11 \pmod{27}, z \equiv 15 \pmod{18} \end{cases},$$

we have $(y,z) \equiv (0,9) \pmod{18}$ or $(y,z) \equiv (6,3) \pmod{18}$ or $(y,z) \equiv (12,15) \pmod{18}$.

(i) $(y,z) \equiv (0,9) \pmod{18}$; then, the congruence modulo 19 of Eq (2.9) leads to

$$2 \equiv -2^{5x} \cdot 3^{2x} \cdot 7^{u(x-z)} \pmod{19}.$$

It follows that

$$-1 = \left(\frac{2}{19}\right) = \left(\frac{-1}{19}\right)\left(\frac{2}{19}\right) = 1,$$

which is a contradiction.

(ii) $(y,z) \equiv (6,3) \pmod{18}$; then, the congruence modulo 19 of Eq (2.9) leads to

$$-1 \equiv -2^{5x} \cdot 3^{2x} \cdot 7^{u(x-z)} \pmod{19}.$$

It follows that

$$-1 = \left(\frac{-1}{19}\right) = \left(\frac{-1}{19}\right)\left(\frac{2}{19}\right) = 1,$$

which is also a contradiction.

(iii) $(y,z) \equiv (12,15) \pmod{18}$; then, the congruence modulo 19 of Eq (2.9) leads to

$$14 \equiv -2^{5x} \cdot 3^{2x} \cdot 7^{u(x-z)} \pmod{19}.$$

It follows that

$$-1 = \left(\frac{2}{19}\right)\left(\frac{7}{19}\right) = \left(\frac{-1}{19}\right)\left(\frac{2}{19}\right) = 1,$$

which leads to a contradiction.

If (2.10) holds, then taking modulo 4 leads to $(-1)^y \equiv 1 \pmod 4$. It follows that y is even. The congruence modulo 5 of Eq (2.10) gives $2^y \equiv 2^z \pmod 5$. So z is also even. Then we get from Eq (2.10) that

$$32^x | 337^{\frac{z}{2}} + 7^{\frac{y}{2}} \quad or \quad 32^x | 337^{\frac{z}{2}} - 7^{\frac{y}{2}}.$$

It follows that $32^x \leq 337^{\frac{z}{2}} + 7^{\frac{y}{2}}$, which is impossible since

$$32^x = (7 + 5^2)^{2 \cdot x/2} = (674 + 14 \cdot 5^2)^{x/2} > 337^{\frac{z}{2}} + 7^{\frac{y}{2}}.$$

If (2.11) holds, then taking modulo 5 for Eq (2.11) would give $1 \equiv 2^z \pmod 5$. It follows that z is also even. Then we get from Eq (2.11) that

$$32^x | 337^{\frac{z}{2}} + 1 \quad or \quad 32^x | 337^{\frac{z}{2}} - 1.$$

It follows that $32^x \leq 337^{\frac{z}{2}} + 1$, which is impossible since

$$32^x = (7 + 5^2)^{2 \cdot x/2} = (674 + 14 \cdot 5^2)^{x/2} > 337^{\frac{z}{2}} + 1.$$

The case $y > z > x$. Then dividing Eq (2.7) by n^x yields

$$2^{5x} 3^{2x} = n^{z-x}(337^z - 175^y n^{y-z}). \tag{2.12}$$

It is easy to see that the two factors on the right-hand side are co-prime. Thus, Eq (2.12) yields $n = 3^s$ for some positive integer s and

$$2^{5x} = 337^z - 175^y 3^{s(y-z)}, \tag{2.13}$$

or $n = 2^r$ for some positive integers r and

$$3^{2x} = 337^z - 175^y 2^{r(y-z)}, \tag{2.14}$$

or $n = 2^r 3^s$ for some positive integers r and s and

$$1 = 337^z - 175^y 2^{r(y-z)} 3^{s(y-z)}. \tag{2.15}$$

If (2.13) holds, then taking modulo 3 for Eq (2.13) would give $(-1)^x \equiv 1 \pmod 3$. It follows that x is even. Taking modulo 5 for Eq (2.13) leads to $2^x \equiv 2^z \pmod 5$. So z is also even. Then we get from Eq (2.13) that

$$25^y | 337^{\frac{z}{2}} + 2^{\frac{5x}{2}} \quad or \quad 25^y | 337^{\frac{z}{2}} - 2^{\frac{5x}{2}}.$$

It follows that $25^y \leq 337^{\frac{z}{2}} + 2^{\frac{5x}{2}}$, which is impossible since

$$25^y = (2^4 + 3^2)^{2 \cdot y/2} = (337 + 2^5 \cdot 3^2)^{y/2} > 337^{\frac{z}{2}} + 2^{\frac{5x}{2}}.$$

If (2.14) holds, then taking modulo 5 for Eq (2.14) leads to $3^{2x} \equiv 2^z \pmod 5$. It follows that $1 = \left(\frac{3^{2x}}{5}\right) = \left(\frac{2}{5}\right)^z = (-1)^z$. Thus z is even. Then we get from Eq (2.14) that

$$25^y | 337^{\frac{z}{2}} + 3^x \quad or \quad 25^y | 337^{\frac{z}{2}} - 3^x.$$

It follows that $25^y \leq 337^{\frac{z}{2}} + 3^x$, which is impossible since

$$25^y = (2^4 + 3^2)^{2 \cdot y/2} = (337 + 2^5 \cdot 3^2)^{y/2} > 337^{\frac{z}{2}} + 3^x.$$

Similarly we can prove that (2.15) is impossible.

This completes the proof. □

3. Proof of Theorem 1.1

Proof. It is easy to find that is enough to prove that $x \equiv y \equiv z \equiv 0 \pmod 2$ by Lemma 2.3. Substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (1.9) gives

$$a^{m_2 x} 2^{(m_1+1)x} + (2^{2m_1} - a^{2m_2})^y = (2^{2m_1} + a^{2m_2})^z. \tag{3.1}$$

Taking modulo 4 for Eq (3.1) gives $(-1)^y \equiv 1 \pmod 4$. It follows that y is even. We now prove that z is also even. Taking modulo 2^{m_1+1} for Eq (3.1) yields

$$a^{2m_2 u} - 1 \equiv 0 \pmod{2^{m_1+1}},$$

where $u = |y - z|$. It follows that $m_1 + 1 \le v_2(2m_2) + v_2(u) + 2 = v_2(m_2) + 3 + v_2(u)$ by Lemma 2.2. Let $v_2(m_2) = h$ and $m_2 = 2^h r, 2 \nmid r$. If $a > 3$ or $r > 1$, then we have

$$2^{m_1} > a^{m_2} = a^{2^h r} > 2^{2^{h+1}}.$$

It follows that $m_1 \ge 2^{h+1} + 1 \ge h + 3$. If $a = 3$ and $r = 1$, one can easily prove from $2^{m_1} = p^k + 3^{2^h}$ that $m_1 \ge h + 3$. Hence $v_2(u) \ge 1$. It follows that z is also even. Finally we prove that x is even. If m_2 is odd, we get from Eq (3.1) that

$$(-1)^{(m_1 m_2 + m_1 + 1)x} = \left(\frac{a}{2^{m_1} - a^{m_2}}\right)^{m_2 x} \cdot \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{(m_1+1)x} = \left(\frac{2^{2m_1} + a^{2m_2}}{2^{m_1} - a^{m_2}}\right)^z = 1,$$

since

$$\left(\frac{2}{2^{m_1} - a^{m_2}}\right) = -1, \left(\frac{a}{2^{m_1} - a^{m_2}}\right) = \left(\frac{2^{m_1}}{a}\right) = (-1)^{m_1}.$$

It follows that x is even since m_2 is odd. If m_2 is even, then we have by Lemmas 2.1 and 2.6 that m_1 is odd and $3|a$. We get from Eq (3.1) that

$$((2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{2m_1} - a^{2m_2})^{\frac{y}{2}})((2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{2m_1} - a^{2m_2})^{\frac{y}{2}}) = a^{m_2 x} 2^{(m_1+1)x}.$$

If $\frac{y}{2}$ is even, then similarly we have

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{2m_1} - a^{2m_2})^{\frac{y}{2}} = 2 \cdot a_1^{m_2 x} \tag{3.2}$$

and

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{2m_1} - a^{2m_2})^{\frac{y}{2}} = 2^{(m_1+1)x-1} \cdot a_2^{m_2 x}, \tag{3.3}$$

where $a_1 a_2 = a$. If $3|a_1$, then taking modulo 3 for Eq (3.2) yields $2 \equiv 0 \pmod 3$, which is a contradiction. Thus $3|a_2$. But this is impossible since

$$2^{(m_1+1)x-1} \cdot a_2^{m_2 x} > 2^{(m_1+1)x} > 2 \cdot a^{m_2 x} > 2 \cdot a_1^{m_2 x}.$$

Therefore $\frac{y}{2}$ is odd; then, we have

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{2m_1} - a^{2m_2})^{\frac{y}{2}} = 2^{(m_1+1)x-1} \cdot a_1^{m_2 x}$$

and

$$(2^{2m_1}+a^{2m_2})^{\frac{z}{2}}-(2^{2m_1}-a^{2m_2})^{\frac{y}{2}}=2\cdot a_2^{m_2x},$$

where $a_1a_2=a$. Thus taking the difference of the above equations yields

$$(2^{m_1}+a^{m_2})^{\frac{y}{2}}(2^{m_1}-a^{m_2})^{\frac{y}{2}}=(2^{\frac{(m_1+1)x}{2}-1}\cdot a_1^{\frac{m_2x}{2}}+a_2^{\frac{m_2x}{2}})(2^{\frac{(m_1+1)x}{2}-1}\cdot a_1^{\frac{m_2x}{2}}-a_2^{\frac{m_2x}{2}}).$$

So

$$(2^{m_1}+a^{m_2})^{\frac{y}{2}}=2^{\frac{(m_1+1)x}{2}-1}\cdot a_1^{\frac{m_2x}{2}}+a_2^{\frac{m_2x}{2}}$$

and

$$(2^{m_1}-a^{m_2})^{y/2}=2^{\frac{(m_1+1)x}{2}-1}\cdot a_1^{\frac{m_2x}{2}}-a_2^{\frac{m_2x}{2}}.$$

Adding the two equations one yields

$$(2^{m_1}+a^{m_2})^{\frac{y}{2}}+(2^{m_1}-a^{m_2})^{\frac{y}{2}}=2^{\frac{(m_1+1)x}{2}}\cdot a_1^{\frac{m_2x}{2}}.$$

We claim that $\frac{y}{2}=y_1=1$. On the contrary suppose $y_1>1$. Note that y_1 is odd; we get that

$$2^{\frac{(m_1+1)(x_1-2)}{2}}a_1^{\frac{m_2x}{2}}=\sum_{r=0}^{(y_1-1)/2}\binom{y_1}{2r}2^{m_1(y_1-2r-1)}a^{2rm_2}.$$

It follows that $y_1\cdot a^{m_2(y_1-1)}\equiv 0 \pmod 2$, which leads to a contradiction. Therefore $y_1=1$ and $2^{\frac{(m_1+1)x}{2}}\cdot a_1^{\frac{m_2x}{2}}=2^{m_1+1}$ yields that $x=2$.

This completes the proof. □

4. Proof of Theorem 1.2 and Corollary 1.1

Proof. Assume that (x,y,z) is a positive integer solution with $(x,y,z)\neq(2,2,2)$. Then we have by Lemmas 2.4, 2.5 and Theorem 1.1 that $n>1$ and either $x>z>y$ or $y>z>x$. By the assumptions and Lemma 2.1, we have that m_2 is odd except for $(p,q,k,l,m_1,m_2,a)=(7,5,1,2,4,2,3)$. We know that the case $(p,q,k,l,m_1,m_2,a)=(7,5,1,2,4,2,3)$ is impossible by Lemma 2.6. We shall discuss separately two cases.

Consider the case $x>z>y$. Then dividing both sides of Eq (1.10) by n^y yields

$$(p^kq^l)^y=n^{z-y}\left(\left(\frac{q^{2l}+p^{2k}}{2}\right)^z-\left(\frac{q^{2l}-p^{2k}}{2}\right)^x n^{x-z}\right). \tag{4.1}$$

If $\gcd(pq,n)=1$, Eq (4.1) and $n>1$ imply that $y=z<x$. We deduce a contradiction to the fact that $y<z$. Therefore, we suppose $\gcd(pq,n)>1$. We write $n=p^uq^v$, where $u+v\geq 1$.

(i) If $u\geq 1, v=0$, then $n=p^u$. Equation (4.1) becomes

$$q^{ly}=\left(\frac{q^{2l}+p^{2k}}{2}\right)^z-\left(\frac{q^{2l}-p^{2k}}{2}\right)^x p^{u(x-z)}. \tag{4.2}$$

Then substituting the conditions $p^k=2^{m_1}-a^{m_2}$ and $q^l=2^{m_1}+a^{m_2}$ into Eq (4.2) would give

$$2^{(m_1+1)x}a^{m_2x}p^{u(x-z)}=(2^{2m_1}+a^{2m_2})^z-(2^{m_1}+a^{m_2})^y. \tag{4.3}$$

Taking modulo 8 for Eq (4.3) gives $a^{m_2 y} \equiv 1 \pmod 8$. So y is even since m_2 is odd. By taking equation $p^k = 2^{m_1} - a^{m_2}$ modulo 8, we have $p \equiv -a \equiv \mp 3 \pmod 8$. Taking modulo p for Eq (4.3) leads to

$$(2 \cdot a^{2m_2})^z \equiv (2^{m_1} + a^{m_2})^y \pmod p.$$

It follows that

$$(-1)^z = \left(\frac{2}{p}\right)^z = \left(\frac{2^{m_1} + a^{m_2}}{p}\right)^y = 1.$$

Therefore z is even. Then we get from Eq (4.3) that

$$2^{(m_1+1)x} a^{m_2 x} p^{u(x-z)} = ((2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{m_1} + a^{m_2})^{\frac{y}{2}})((2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{m_1} + a^{m_2})^{\frac{y}{2}}).$$

If $\frac{y}{2}$ is odd, then we have

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{m_1} + a^{m_2})^{\frac{y}{2}} \equiv 4 \pmod 8$$

and

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{m_1} + a^{m_2})^{\frac{y}{2}} \equiv -2 \pmod 8$$

if $a \equiv 3 \pmod 8$ or

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{m_1} + a^{m_2})^{\frac{y}{2}} \equiv -2 \pmod 8$$

and

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{m_1} + a^{m_2})^{\frac{y}{2}} \equiv 4 \pmod 8$$

if $a \equiv -3 \pmod 8$. It follows that

$$v_2((2^{2m_1} + a^{2m_2})^z - (2^{m_1} + a^{m_2})^y) = 3.$$

But the left hand side of Eq (4.3) is divided by 2^4, which leads to a contradiction. Hence $\frac{y}{2}$ is even and

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{m_1} + a^{m_2})^{\frac{y}{2}} \equiv 2 \pmod 4.$$

Thus it follows that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{\frac{y}{2}};$$

however, this is impossible since

$$2^{(m_1+1)x-1} \geq 2^{(m_1+1)z} = (4 \cdot 2^{2m_1})^{\frac{z}{2}} > (2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - (2^{m_1} + a^{m_2})^{\frac{y}{2}}.$$

(ii) If $u = 0, v \geq 1$, then $n = q^v$. Equation (4.1) becomes

$$p^{ky} = \left(\frac{q^{2l} + p^{2k}}{2}\right)^z - \left(\frac{q^{2l} - p^{2k}}{2}\right)^x q^{v(x-z)}. \tag{4.4}$$

Then substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (4.4) would give

$$2^{(m_1+1)x} a^{m_2 x} q^{v(x-z)} = (2^{2m_1} + a^{2m_2})^z - (2^{m_1} - a^{m_2})^y. \tag{4.5}$$

Taking modulo 8 for Eq (4.5) gives $(-a)^{m_2 y} \equiv 1 \pmod 8$. So y is even since m_2 is odd. By taking the equation $q^k = 2^{m_1} + a^{m_2}$ modulo 8 leads to $q \equiv a \equiv \pm 3 \pmod 8$. Taking modulo q for Eq (4.5) leads to

$$(2 \cdot a^{2m_2})^z \equiv (2^{m_1} - a^{m_2})^y \pmod q.$$

It follows that $(-1)^z = \left(\frac{2}{q}\right)^z = \left(\frac{2^{m_1} - a^{m_2}}{q}\right)^y = 1$. Therefore z is even. Then similarly we get from Eq (4.5) that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - (2^{m_1} + a^{m_2})^{y/2},$$

which is impossible by the above result that has been proved (see discussion of Eq (4.2)).

(iii) If $u \geq 1, v \geq 1$, then $n = p^u q^v$. Equation (4.1) becomes

$$1 = \left(\frac{q^{2l} + p^{2k}}{2}\right)^z - \left(\frac{q^{2l} - p^{2k}}{2}\right)^x p^{u(x-z)} q^{v(x-z)}. \tag{4.6}$$

Then substituting the conditions $p^k = 2^{m_1} - a^{m_2}$ and $q^l = 2^{m_1} + a^{m_2}$ into Eq (4.6) would give

$$2^{(m_1+1)x} a^{m_2 x} p^{u(x-z)} q^{v(x-z)} = (2^{2m_1} + a^{2m_2})^z - 1. \tag{4.7}$$

Taking modulo q for Eq (4.5) leads to

$$(2 \cdot a^{2m_2})^z \equiv 1 \pmod q.$$

It follows that $(-1)^z = \left(\frac{2}{q}\right)^z = \left(\frac{1}{q}\right) = 1$. Therefore z is even. Then similarly we get from Eq (4.7) that

$$2^{(m_1+1)x-1} | (2^{2m_1} + a^{2m_2})^{z/2} - 1,$$

which is impossible by the above result that has been proved (see discussion of Eq (4.2)). This completes the proof of the first case.

Consider the case $y > z > x$. Then dividing both sides of Eq (1.10) by n^x yields

$$a^{m_2 x} 2^{(m_1+1)x} = n^{z-x}((2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y n^{y-z}). \tag{4.8}$$

If $\gcd(2a, n) = 1$, Eq (4.8) and $n > 1$ imply that $x = z < y$. We deduce a contradiction to the fact that $x < z$. Therefore, we suppose $\gcd(2a, n) > 1$. We write $n = 2^r a_1^s$, where $r + s \geq 1, a_1 > 1$ is a divisor of a.

(i) If $r = 0, s \geq 1$, then $n = a_1^s$ and $m_2 x = s(z - x)$. If $a_1 < a$, then Eq (4.8) becomes

$$2^{(m_1+1)x} a_2^{m_2 x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y a_1^{s(y-z)}. \tag{4.9}$$

Since $a \equiv \pm 3 \pmod 8$, we have to consider the eight cases.

Case 1: $(a_1, a_2) \equiv (1, 3) \pmod 8$. Taking modulo $2^{m_1} - a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

It follows that

$$(-1)^{(m_1 + m_1 m_2 + 1)x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} - a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^z = (-1)^z,$$

which leads to $x \equiv z \pmod 2$ since m_2 is odd. So we get from $m_2 x = s(z-x)$ that $x \equiv z \equiv 0 \pmod 2$. Then we get from Eq (4.9) either

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2} a_2^{m_2 x/2}$$

or

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{z/2} - 2^{(m_1+1)x/2} a_2^{m_2 x/2}.$$

Hence

$$(2^{m_1} + a^{m_2})^y \leq (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2} a_2^{m_2 x/2},$$

which is impossible since

$$(2^{m_1} + a^{m_2})^y > (2^{2m_1} + a^{2m_2} + 2^{m_1+1} \cdot a^{m_2})^{z/2} > (2^{2m_1} + a^{2m_2})^{z/2} + 2^{(m_1+1)x/2} a_2^{m_2 x/2}.$$

Case 2: $(a_1, a_2) \equiv (1, 5) \pmod 8$. Taking modulo $2^{m_1} - a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

We know that is impossible by the result proved in Case 1.

Case 3: $(a_1, a_2) \equiv (3, 1) \pmod 8$. Then taking modulo $2^{m_1} - a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

We already prove that is impossible if m_1 is even. If m_1 is odd, then we have

$$1 = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} - a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{z} = (-1)^z,$$

which leads to z is even. Similarly we know that a_2 is not a square. So by the assumption, there is an odd prime divisor P of a_1 such that

$$(-1)^{m_2 x} = \left(\frac{2}{P}\right)^{(m_1+1)x} \left(\frac{a_2}{P}\right)^{m_2 x} = \left(\frac{2}{P}\right)^{z} = 1,$$

which leads to x being even. We know that is impossible by the result proved in Case 1.

Case 4: $(a_1, a_2) \equiv (3, 7) \pmod 8$. Taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} + a^{m_2}}.$$

It follows that

$$(-1)^{(m_1+1)x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} + a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{z} = (-1)^z,$$

which leads to $x \equiv z \pmod 2$. We know that is impossible by the result proved in **Case 1**.

Case 5: $(a_1, a_2) \equiv (5, 1) \pmod 8$. Then taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} + a^{m_2}}.$$

We already proved that that is impossible.

Case 6: $(a_1, a_2) \equiv (5, 7) \pmod 8$. Taking modulo $2^{m_1} - a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}.$$

It follows that

$$(-1)^{(m_1+1)x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} - a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^{z} = (-1)^z, \tag{4.10}$$

which leads to $x \equiv z \pmod 2$ if m_1 is even. We have already proven that that is impossible. If m_1 is odd, then we get from Eq (4.10) that z is even. On the other hand, taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.9) leads to

$$(-1)^{m_2 x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} + a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{z} = 1,$$

which leads to x being even. We know that is impossible by the result proved in Case 1.

Case 7: $(a_1, a_2) \equiv (7, 3) \pmod 8$. Taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} + a^{m_2}}.$$

We already proved that that is impossible.

Case 8: $(a_1, a_2) \equiv (7, 5) \pmod 8$. Taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.9) leads to

$$2^{(m_1+1)x} a_2^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} + a^{m_2}}.$$

It follows that

$$(-1)^{(m_1+m_1 m_2+1)x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{(m_1+1)x} \left(\frac{a_2}{2^{m_1} + a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{z} = (-1)^z.$$

It follows that $x \equiv z \pmod 2$. We already proved that that is impossible.

If $a_1 = a$ then Eq (4.8) becomes

$$2^{(m_1+1)x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y a^{s(y-z)}. \tag{4.11}$$

Taking modulo $2^{m_1} + a^{m_2}$ for Eq (4.11) leads to

$$2^{(m_1+1)x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} + a^{m_2}}.$$

It follows that

$$(-1)^{(m_1+1)x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{(m_1+1)x} = \left(\frac{2}{2^{m_1} + a^{m_2}}\right)^{z} = (-1)^z.$$

It follows that $(m_1 + 1)x \equiv z \pmod 2$. On the other hand, taking modulo a for Eq (4.11) leads to

$$2^{(m_1+1)x} \equiv 2^{2m_1 z} \pmod a.$$

It follows that

$$(-1)^{(m_1+1)x} = \left(\frac{2}{a}\right)^{(m_1+1)x} = \left(\frac{2}{a}\right)^{2m_1 z} = 1,$$

which leads to $(m_1+1)x \equiv z \equiv 0 \pmod 2$. We know that is impossible by the above result.

(ii) If $r \geq 1, s = 0$, then $n = 2^r$ and $(m_1+1)x = r(z-x)$. Equation (4.8) becomes

$$a^{m_2 x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y 2^{r(y-z)}. \tag{4.12}$$

Thus

$$2^{2m_1 z} \equiv 2^{2m_1 y + r(y-z)} \pmod a.$$

It follows that

$$1 = \left(\frac{2}{a}\right)^{2m_1 z} = \left(\frac{2}{a}\right)^{2m_1 y + r(y-z)} = (-1)^{r(y-z)},$$

which yields that $r(y-z)$ is even. Taking modulo $2^{m_1} - a^{m_2}$ for Eq (4.12) leads to $a^{m_2 x} \equiv (2 \cdot a^{2m_2})^z \pmod{2^{m_1} - a^{m_2}}$. It follows that

$$(-1)^{m_1 m_2 x} = \left(\frac{a}{2^{m_1} - a^{m_2}}\right)^{m_2 x} = \left(\frac{2}{a}\right)^{m_1 m_2 x} = \left(\frac{2}{2^{m_1} - a^{m_2}}\right)^z = (-1)^z,$$

which yields $m_1 m_2 x \equiv z \pmod 2$. If $r(y-z) > 2$, we consider Eq (4.12) modulo 8; we have $a^{m_2 x} \equiv 1 \pmod 8$. This means that $m_2 x$ is even. As

$$\gcd((2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + a^{\frac{m_2 x}{2}}, (2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - a^{\frac{m_2 x}{2}}) = 2,$$

we get

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + a^{\frac{m_2 x}{2}} \tag{4.13}$$

or

$$(2^{m_1} + a^{m_2})^y | (2^{2m_1} + a^{2m_2})^{\frac{z}{2}} - a^{\frac{m_2 x}{2}}. \tag{4.14}$$

However, the inequalities

$$(2^{m_1} + a^{m_2})^y > (2^{2m_1} + a^{2m_2} + 2^{m_1+1} a^{m_2})^{\frac{z}{2}} > (2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + a^{\frac{m_2 x}{2}}$$

contradict (4.13) and (4.14). Hence $r(y-z) = 2$; considering Eq (4.12) modulo 8, we obtain $a^{m_2 x} \equiv 5 \pmod 8$. This means that $m_2 x \equiv 0 \pmod 2$ or $a \equiv 5 \pmod 8$. We know that the case $m_2 x \equiv 0 \pmod 2$ is impossible by the proof of the case $r(y-z) > 2$. Hence $a \equiv 5 \pmod 8$. We have that m_1 is even by the assumption. Thus we get from $(m_1+1)x = r(z-x)$ and $r(y-z) = 2$ that $r = 1, y = z+2$. Note that $m_1 m_2 x \equiv z \pmod 2$. We get that both y and z are even. Therefore we get from Eq (4.12) that

$$(2^{2m_1} + a^{2m_2})^{\frac{z}{2}} + (2^{2m_1} - a^{2m_2})^{\frac{y}{2}} | a^{m_2 x},$$

which is impossible.

(iii) If $r \geq 1, s \geq 1$, then $n = 2^r a_1^s$ and $(m_1+1)x = r(z-x), m_2 x = s(z-x)$. Equation (4.8) becomes

$$a_2^{m_2 x} = (2^{2m_1} + a^{2m_2})^z - (2^{2m_1} - a^{2m_2})^y 2^{r(y-z)} a_1^{s(y-z)}. \tag{4.15}$$

Similarly we can prove that Eq (4.15) is impossible. This completes the proof of the second case. This completes the proof. □

By the proof of Theorem 1.2, one can immediately obtain Corollary 1.1.

5. Applications

Corollary 5.1. *Equation (1.1) has only the positive integer solution* $(x, y, z) = (2, 2, 2)$ *if* (a, b, c) *is one of the following primitive Pythagorean numbers*

$$(80, 39, 89), (576, 943, 1105), (320, 999, 1049), (1344, 583, 1465), (1856, 183, 1865),$$

$$(11520, 14359, 18409), (168960, 234919, 289369), (46080, 260119, 264169),$$

$$(1757184, 4010263, 4378345)(33409993656, 936621583, 33423116785),$$

$$(8294400, 93679, 8294929), (25029771264, 8063247823, 26296790545).$$

Proof. By Theorem 1.1 and Table 1, one can immediately obtain the Corollary 5.1 by a simple calculation. □

Table 1. The proof of Corollary 5.1.

$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$	$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$
$3 = 2^3 - 5$	$13 = 2^3 + 5$	$23 = 2^5 - 3^2$	$41 = 2^5 + 3^2$
$3^3 = 2^5 - 5$	$37 = 2^5 + 5$	$11 = 2^5 - 21$	$53 = 2^5 + 21$
$3 = 2^5 - 29$	$61 = 2^5 + 29$	$83 = 2^7 - 45$	$173 = 2^7 + 45$
$347 = 2^9 - 165$	$677 = 2^9 + 165$	$467 = 2^9 - 45$	$557 = 2^9 + 45$
$23 = 2^{11} - 45$	$4073 = 2^{11} + 45$	$1619 = 2^{11} - 429$	$2477 = 2^{11} + 429$
$3623 = 2^{17} - 357^2$	$258521 = 2^{17} + 357^2$	$35591 = 2^{17} - 309$	$226553 = 2^{17} + 309$

Remark 5.1. *There are many prime numbers* p, q *and positive integers* k, l, m_1, m_2, a *satisfying the conditions of Theorem 1.1. One can see Tables 2 and 3.*

Table 2. Some examples of application of Theorem 1.1.

$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$	$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$
$59 = 2^7 - 69$	$197 = 2^7 + 69$	$29 = 2^7 - 99$	$227 = 2^7 + 99$
$107 = 2^7 - 21$	$149 = 2^7 + 21$	$3^3 = 2^7 - 101$	$229 = 2^7 + 101$
$181 = 2^8 - 75$	$331 = 2^8 + 75$	$3 = 2^9 - 509$	$1021 = 2^9 + 509$
$3^3 = 2^9 - 485$	$997 = 2^9 + 485$	$71 = 2^9 - 21^2$	$953 = 2^9 + 21^2$
$251 = 2^9 - 261$	$773 = 2^9 + 261$	$227 = 2^9 - 285$	$797 = 2^9 + 285$
$61 = 2^{10} - 963$	$1987 = 2^{10} + 963$	$1787 = 2^{11} - 261$	$2309 = 2^{11} + 261$
$3709 = 2^{12} - 387$	$4483 = 2^{12} + 387$	$7883 = 2^{13} - 309$	$8501 = 2^{13} + 309$
$7643 = 2^{13} - 549$	$8741 = 2^{13} + 549$	$32507 = 2^{15} - 261$	$33209 = 2^{15} + 261$
$56923 = 2^{16} - 8613$	$74149 = 2^{16} + 8613$	$519527 = 2^{19} - 69^2$	$529049 = 2^{19} + 69^2$
$8060099 = 2^{23} - 69^3$	$8717117 = 2^{23} + 69^3$	$33549671 = 2^{25} - 69^2$	$33559193 = 2^{25} + 69^2$

Table 3. Some examples of application of Theorem 1.1.

$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$
$10887311 = 2^{25} - 69^4$	$56221553 = 2^{25} + 69^4$
$536676431 = 2^{29} - 21^4$	$537065393 = 2^{29} + 21^4$
$536866151 = 2^{29} - 69^2$	$536875673 = 2^{29} + 69^2$
$9007199250656891 = 2^{53} - 21^5$	$9007199258825093 = 2^{53} + 21^5$
$2361183241434822606407 = 2^{71} - 21^2$	$2361183241434822607289 = 2^{71} + 21^2$
$18888953906518175252587 = 2^{74} - 800013^3$	$18888977956438986456981 = 2^{74} + 800013^3$
$9671406556916991286915727 = 2^{83} - 453^4$	$9671406556917070550838830893 = 2^{83} + 453^4$
$2475880078570554658666153799 = 2^{91} - 3^{30}$	$2475880078570966440930343097 = 2^{91} + 3^{30}$
$2535301200455545285745922769853 = 2^{101} - 99^9$	$2535301200457372320240890051651 = 2^{101} + 99^9$
$2^{110} - 800013^3$	$1298074214633707419157584487907221 = 2^{110} + 800013^3$

Corollary 5.2. *Equation (1.1) has only the positive integer solution* $(x, y, z) = (2, 2, 2)$ *if* (a, b, c) *is one of the following Pythagorean numbers*

$$(416n, 87n, 425n), (1728n, 295n, 1753n), (5760n, 2071n, 6121n), (29440n, 3159n, 29609n),$$

$$(31488n, 1255n, 31513n), (47616n, 56887n, 74185n), (59904n, 51847n, 79225n),$$

$$(129536n, 1527n, 129545n), (872448n, 1003207n, 1093945n), (829440n, 884551n, 1212601n),$$

$$(1075200n, 772951n, 1324201n), (1320960n, 632551n, 1464601n), (1382400n, 592951n, 1504201n),$$

$$(2021376n, 74407n, 2022745n), (2058240n, 38551n, 2058601n), (8710976n, 16572007n, 16982425n),$$

$$(4300800n, 16501591n, 17052841n), (5775360n, 16280191n, 17274241n),$$

$$(19562496n, 268079047n, 268791865n), (546963456n, 4277553367n, 4312381225n),$$

$$(1128923136n, 4220783527n, 4369151065n),$$

$$(13435303624704n, 15027021842647n, 171324290231185n),$$

for any $n \in N$.

Proof. By Theorem 1.2 and Table 4, one can immediately obtain Corollary 5.2 by a simple calculation. □

Table 4. The proof of Corollary 5.2.

$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$	$p^k = 2^{m_1} - a^{m_2}$	$q^l = 2^{m_1} + a^{m_2}$
$3 = 2^4 - 13$	$29 = 2^4 + 13$	$5 = 2^5 - 3^3$	$59 = 2^5 + 3^3$
$19 = 2^6 - 45$	$109 = 2^6 + 45$	$13 = 2^7 - 115$	$3^5 = 2^7 + 115$
$5 = 2^7 - 123$	$251 = 2^7 + 123$	$163 = 2^8 - 93$	$349 = 2^8 + 93$
$139 = 2^8 - 117$	$373 = 2^8 + 117$	$3 = 2^8 - 253$	$509 = 2^8 + 253$
$811 = 2^{10} - 213$	$1237 = 2^{10} + 213$	$619 = 2^{10} - 405$	$1429 = 2^{10} + 405$
$499 = 2^{10} - 525$	$1549 = 2^{10} + 525$	$379 = 2^{10} - 645$	$1669 = 2^{10} + 645$
$349 = 2^{10} - 675$	$1699 = 2^{10} + 675$	$37 = 2^{10} - 987$	$2011 = 2^{10} + 987$
$19 = 2^{10} - 1005$	$2029 = 2^{10} + 1005$	$3643 = 2^{12} - 453$	$4549 = 2^{12} + 453$
$3571 = 2^{12} - 525$	$4621 = 2^{12} + 525$	$3391 = 2^{12} - 705$	$4801 = 2^{12} + 705$
$15787 = 2^{14} - 597$	$16981 = 2^{14} + 597$	$61363 = 2^{16} - 4173$	$69709 = 2^{16} + 4173$
$56923 = 2^{16} - 8613$	$74149 = 2^{16} + 8613$	$2592691 = 2^{22} - 117$	$5795917 = 2^{22} + 117$

6. Conclusions

It is easy to see that Jeśmanowicz' a conjecture holds for the following set of primitive Pythagorean numbers:

$$\frac{q^{2l} - p^{2k}}{2}, p^k q^l, \frac{q^{2l} + p^{2k}}{2}.$$

In addition, Jeśmanowicz' conjecture holds for non-primitive Pythagorean numbers:

$$n\frac{q^{2l} - p^{2k}}{2}, np^k q^l, n\frac{q^{2l} + p^{2k}}{2},$$

for any positive integer n if for $a = a_1 a_2$ with $a_1 \equiv 1 \pmod 8$ not a square and $\gcd(a_1, a_2) = 1$, then there exists a prime divisor P of a_2 such that $\left(\frac{a_1}{P}\right) = -1$ and $2|m_1, a \equiv 5 \pmod 8$ or $2 \not| m_2, a \equiv 3$ (mod 8).

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Conflict of interest

All authors declare no conflict of interest regarding the publication of this paper.

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