AIMS Mathematics, 8(6): 14180–14187.
DOI:10.3934/math.2023725
Received: 10 February 2023
Revised: 27 March 2023
Accepted: 03 April 2023
Published: 17 April 2023

Mathematics

http://www.aimspress.com/journal/Math

Research article

A local Palais-Smale condition and existence of solitary waves for a class of nonhomogeneous generalized Kadomtsev-Petviashvili equations

Lirong Huang*

Department of Mathematics and Physics, Fujian Jiangxia University, Fuzhou 350108, China

* **Correspondence:** Email: lrhuang515@126.com; Tel: +8613635276326.

Abstract: This paper is concerned with a class of nonhomogeneous generalized Kadomtsev-Petviashvili equations

$$\begin{cases} u_t + (|u|^{p-2}u)_x + u_{xxx} + h_x(x - \tau t, y) + \beta\nabla_y v = 0, \\ v_x = \nabla_y u. \end{cases}$$

By proving a local Palais-Smale condition, we manage to prove the existence of solitary waves with the help of a variational characterization on the smallest positive constant of an anisotropic Sobolev inequality (Huang and Rocha, J. Inequal. Appl., 2018, 163). The novelty is to give an **explicit estimate** on the sufficient condition of h to get the existence of solitary waves.

Keywords: nonhomogeneous generalized Kadomtsev-Petviashvili equation; solitary wave
Mathematics Subject Classification: 35J05, 35J20

1. Introduction

The classical Kadomtsev-Petviashvili [1] equation is a two-dimensional generalization of the Korteweg-de Vries equation appears in mathematical models for the description of long dispersive waves, which travel essentially in one direction but have small transverse effects. A generalized Kadomtsev-Petviashvili (GKP) equation with variable coefficients has been proposed by David et al. [2, 3] to describe water waves that propagate in straits or rivers rather than unbounded surfaces. After that, there are some works studying the existence of solitary waves or soliton solutions of the GKP with variable coefficients; see, for instance [4–6] and the references therein. In $\mathbb{R}^2$, a class of GKP with the form

$$\begin{aligned} &(u(t) + r(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} \\ &+c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0 \end{aligned} \tag{1.1}$$

has been considered by Güngör and Winternitz [4], where $r, q, \sigma, a, b, c, e, f$ and h are functions satisfying some technical conditions.

In the case of GKP without a nonhomogeneous term $h(y,t)$, a lot of mathematicians have studied the existence of solitary waves. A pioneering work was achieved by De Bouard and Saut [7], where the authors have studied the existence of solitary waves for the following:

$$\begin{cases} u_t + f'(u)u_x + u_{xxx} + \beta v_y = 0, \\ v_x = u_y, \end{cases} \quad (x,y) \in \mathbb{R}^N \tag{1.2}$$

with $f(s) = |s|^p s$ and $\beta = -1$, where $1 \le p < 4$ if $N = 2$, and $1 \le p < \frac{4}{3}$ if $N = 3$. Moreover $p = \frac{m}{n}$, where m and n are relatively prime and n is odd. In the three dimensional case, a series of works on soliton solutions, rogue wave solutions, soliton and rogue wave mixed solutions as well as their numerical simulations, have been obtained by Ma et al. [8–12]. Wang and Willem [13] have studied the existence of multiple solitary waves of (1.2) for a more general nonlinear term.

For GKP without a nonhomogeneous term $h(y,t)$ in higher spatial dimensions, Xuan [14] has investigated the existence of solitary waves of

$$w_t + w_{xxx} + (f(w))_x = D_x^{-1}\Delta_y w, \tag{1.3}$$

where $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$, $y = (y_1, \cdots, y_{N-1})$, $N \ge 3$, D_x^{-1} denotes $D_x^{-1}g(x,y) = \int_{-\infty}^{x} g(s,y)ds$ and $\triangle_y = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial y_k^2}$. For various kinds of nonlinearities f, we refer the interested readers to [15–18]. For other kinds of related equations, the existence of solitary waves as well their properties have been investigated by several mathematicians; we refer the interested readers to [19–24].

While for GKP with a nonhomogeneous term, we do not see any results in the literature. The purpose of this paper is to investigate the existence of solitary waves for the following class of GKP equations with a nonhomogeneous term

$$\begin{cases} u_t + (|u|^{p-2}u)_x + u_{xxx} + h_x(x-\tau t, y) + \beta\nabla_y v = 0, \\ v_x = \nabla_y u, \quad (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1}, \end{cases} \tag{1.4}$$

where $2 < p < p_* := 2(2N-1)/(2N-3)$, with $N \ge 2$ and $\nabla_y = \left(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{N-1}}\right)$. The $h(x,y,t)$ is a real valued function satisfying suitable conditions. We recall that a solitary wave of (1.4) is a solution of the form $u(t,x,y) = u(x-\tau t, y)$ with $\tau > 0$. Hence the function u must satisfy the problem

$$\begin{cases} -\tau u_x + (|u|^{p-2}u)_x + u_{xxx} + h_x(x-\tau t, y) + \beta\nabla_y v = 0, \\ v_x = \nabla_y u. \end{cases} \tag{1.5}$$

In the sequel, we will treat the case of $\beta = -1$ and $\tau = 1$. It is easy to see that the above equation can be written as

$$-u_x + (|u|^{p-2}u)_x + u_{xxx} + h_x(x-\tau t, y) - D_x^{-1}\Delta_y u = 0, \tag{1.6}$$

or equivalently

$$(-u_{xx} + D_x^{-2}\Delta_y u + u - (|u|^{p-2}u)_x - h)_x = 0. \tag{1.7}$$

Before stating the main result, we need the following N-dimensional anisotropic Sobolev inequality [25]: for $N \ge 2$ and $2 < p < p_* := 2(2N-1)/(2N-3)$, there is a positive constant α

such that for all $u \in Y$,

$$\int_{\mathbb{R}^N} |u|^p dV \leq \alpha \Big(\int_{\mathbb{R}^N} |u|^2 dV \Big)^{\frac{2(2N-1)+(3-2N)p}{4}} \Big(\int_{\mathbb{R}^N} |u_x|^2 dV \Big)^{\frac{N(p-2)}{4}} \prod_{k=1}^{N-1} \Big(\int_{\mathbb{R}^N} |D_x^{-1} \partial_{y_k} u|^2 dV \Big)^{\frac{p-2}{4}}, \tag{1.8}$$

where $V := (x, y)$ and Y is the closure of $\partial_x(C_0^\infty(\mathbb{R}^N))$ under the norm

$$\|u\|_Y^2 := \int_{\mathbb{R}^N} \left(u_x^2 + |D_x^{-1} \nabla_y u|^2 + |u|^2 \right) dV. \tag{1.9}$$

In [26], we have proven that the smallest positive constant α can be characterized variationally by a minimal action solution of

$$-u + u_{xx} + |u|^{p-2}u = D_x^{-2} \triangle_y u, \quad u \neq 0, \quad u \in Y. \tag{1.10}$$

From [26, Theorem 2.8], we know that the smallest positive constant α satisfies

$$\alpha^{-1} = \left(\frac{T}{2p} \right)^{\frac{T}{4}} \left(\frac{N^N (p-2)^{2N-1}}{(2p)^{2N-3} T^2} \right)^{\frac{p-2}{4}} \left(\int |\phi|^2 dV \right)^{\frac{p-2}{2}}, \tag{1.11}$$

where $T = (3 - 2N)p + 2(2N - 1)$, and ϕ is a minimal action solution of (1.10). Moreover the smallest positive constant α is independent of the choice of ϕ, though we do not have uniqueness of the minimal action solution to (1.10). Besides, from [26, Proposition 2.7], we also have the following inequality:

$$\left(\int |u|^p dV \right)^{\frac{2}{p}} \left(\int |\phi|^p dV \right)^{\frac{p-2}{p}} \leq \|u\|_Y^2, \qquad \text{for all } u \in Y. \tag{1.12}$$

With the help of (1.11) and (1.12), we are ready to propose the following condition:

$$h \in C^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \quad \text{and} \quad \|h\|_{Y^{-1}} < \frac{p-2}{p-1} \left(\frac{1}{p-1} \right)^{\frac{1}{p-2}} \left(\int |\phi|^p dV \right)^{\frac{1}{2}}. \tag{1.13}$$

Theorem 1.1. *Under the condition (1.13), if $N \geq 2$ and $2 < p < 2(2N-1)/(2N-3)$, then (1.4) admits a solitary wave.*

Theorem 1.1 seems to be the first result for the existence of solitary wave GKP with a nonhomogeneous term. The study is based on finding a critical point of the functional I defined on Y; see Section 2. In Section 3, we use the classical minimization argument to prove the existence of solitary waves.

2. Variational framework

Throughout this paper, all integrals are taken over $\mathbb{R}^N$ unless stated otherwise. On Y, we define the following functional

$$I(u) = \frac{1}{2}\|u\|_Y^2 - \frac{1}{p} \int |u|^p dV - \int hu dV. \tag{2.1}$$

Based on the inequality (1.8), we know that I is well defined and C^1 on Y. Moreover there is a one-to-one correspondence between the critical points of I and the solutions to (1.7). Since we study the problem in the whole space $\mathbb{R}^N$, one main difficulty of using a variational method is the lack of the Palais-Smale (PS) condition. In the following proposition, we will prove that I satisfies the $(PS)_c$ condition for a suitable range of c. Define

$$m_0 = \inf_{u\in\Lambda} I(u), \quad \text{where} \quad \Lambda = \{u \in Y\backslash\{0\} \ : \ \langle I'(u), u\rangle = 0\}. \tag{2.2}$$

Proposition 2.1. *Let (u_n) be such that $I(u_n) \to d < m_0 + \frac{p-2}{2p}\int |\phi|^p dV$ and $I'(u_n) \to 0$ in Y^{-1} as $n \to \infty$. Then (u_n) contains a convergent subsequence in Y.*

Proof. For n large enough,

$$\begin{aligned} 1 + d + o(1)\|u_n\|_Y &\geq I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &= \frac{p-2}{2p}\|u_n\|_Y^2 - \frac{p-1}{p}\int hu_n dV \geq \frac{p-2}{4p}\|u_n\|_Y^2 - \frac{(p-1)^2}{p(p-2)}\|h\|_{Y^{-1}}^2, \end{aligned} \tag{2.3}$$

which implies that (u_n) is bounded in Y.

Going if necessary to a subsequence, still denoted by (u_n), we may assume that $u_n \to u_0$ weakly in Y, $u_n \to u_0$ strongly in $L^p_{loc}(\mathbb{R}^N)$ and $u_n \to u_0$ a. e. on $\mathbb{R}^N$. Similar to the proof of [26, Page 6] (see also [27]), we can assume $u_0 \neq 0$. Moreover from $I'(u_n) \to 0$ in Y^{-1} as $n \to \infty$, we also deduce that $I'(u_0) = 0$. Therefore $u_0 \in \Lambda$.

Denote $w_n := u_n - u_0$. Then from the Brezis-Lieb [28] lemma (see also [27]), we obtain from $I'(u_0) = 0$ that for n large enough

$$\begin{aligned} o(1) &= \|u_n\|_Y^2 - \int |u_n|^p dV - \int hu_n dV \\ &= \|u_0\|_Y^2 - \int |u_0|^p dV - \int hu_0 dV + \|w_n\|_Y^2 - \int |w_n|^p dV = \|w_n\|_Y^2 - \int |w_n|^p dV. \end{aligned} \tag{2.4}$$

Suppose that there is a $b \geq 0$ such that $\|w_n\|_Y^2 \to b$ and $\int |w_n|^p dV \to b$ as $n \to \infty$. If $b \neq 0$, then from (1.12), one may deduce that

$$b \geq b^{\frac{2}{p}} \left(\int |\phi|^p dV \right)^{\frac{p-2}{p}}. \tag{2.5}$$

Using $I(u_n) \to d$ as $n \to \infty$, we obtain that for n large enough,

$$\begin{aligned} d + o(1) &= I(u_n) = I(u_0) + \frac{1}{2}\|w_n\|_Y^2 - \frac{1}{p}\int |w_n|^p dV \\ &\geq m_0 + \frac{p-2}{2p} b \geq m_0 + \frac{p-2}{2p}\int |\phi|^p dV, \end{aligned} \tag{2.6}$$

which contradicts the assumption of d. Therefore $b = 0$ and we have proven that $u_n \to u_0$ strongly in Y.

Lemma 2.2. *Under the condition (1.13), then $m_0 + \frac{p-2}{2p}\int |\phi|^p dV > 0$.*

Proof. For any $u \in \Lambda$, we have that

$$I(u) = \frac{1}{2}\|u\|_Y^2 - \frac{1}{p}\int |u|^p dV - \int hudV = \frac{p-2}{2p}\|u\|_Y^2 - \frac{p-1}{p}\int hudV. \tag{2.7}$$

From the condition (1.13), we know that

$$\begin{aligned}\frac{p-1}{p}\int hudV &\leq \frac{p-1}{p}\|h\|_{Y^{-1}}\|u\|_Y \leq \frac{p-2}{p}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\int |\phi|^p dV\right)^{\frac{1}{2}}\|u\|_Y \\ &\leq \frac{p-2}{2p}\|u\|_Y^2 + \frac{p-2}{2p}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\int |\phi|^p dV.\end{aligned} \tag{2.8}$$

Hence

$$I(u) \geq -\frac{p-2}{2p}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\int |\phi|^p dV. \tag{2.9}$$

Therefore we deduce that

$$m_0 + \frac{p-2}{2p}\int |\phi|^p dV \geq \frac{p-2}{2p}\left(1 - \left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\right)\int |\phi|^p dV > 0. \tag{2.10}$$

3. Existence of solitary waves

In this section, we will prove Theorem 1.1. Let $\varphi \in Y$ be such that $\varphi \geq 0$ and $\|\varphi\|_Y = 1$. Then for $s > 0$, we have that

$$I(s\varphi) = \frac{s^2}{2} - \frac{s^p}{p}\int |\varphi|^p dV - s\int h\varphi dV. \tag{3.1}$$

Hence, there is $\rho_0 > 0$ such that for any $s \in (0, \rho_0)$, $I(s\varphi) < 0$. Define the following minimization problem:

$$c_1 = \inf_{\bar{B}_{\rho_0}} I(u). \tag{3.2}$$

Proof of Theorem 1.1. From the above discussion, we have that $-\infty < c_1 < 0$. By the Ekeland variational principle, there is a sequence (v_n) with $v_n \in \bar{B}_{\rho_0}$ such that $I(v_n) \to c_1$ and $I'(v_n) \to 0$ in Y^{-1} as $n \to \infty$.

According to Proposition 2.1 and Lemma 2.2, we know that I satisfies the $(PS)_{c_1}$ condition. Hence there is a convergent subsequence, still denoted by (v_n), and a $v_0 \in Y\backslash\{0\}$ such that $v_n \to v_0$ strongly in Y. Moreover v_0 is a nontrivial solution. This proves Theorem 1.1.

4. Conclusions

The GKP equation is an important model in the description of nonlinear wave propagation in diverse dissipative media. The case of GKP with a nonhomogeneous term has not been investigated in the literature as far as the author's best knowledge. An interesting and important issue is how the nonhomogeneous term affects the existence of solitary waves, and what size of the nonhomogeneous term is enough to get a solitary wave.

In the present paper, with the help of a variational characterization on the smallest positive constant of an anisotropic Sobolev inequality by Huang and Rocha [26], we are able to give an estimate on the size of the nonhomogeneous term, which ensures the existence of solitary waves. We expect that the condition (1.13) proposed here will be helpful to find multiple solitary waves of GKP with a nonhomogeneous term. This can be a work for further study.

Acknowledgments

The author would like to appreciate the unknown referees for their valuable comments and suggestions. This research was supported by the Natural Science Foundation of Fujian Province (No. 2021J011230).

Conflict of interest

The author declares that there are no conflicts of interests.

References

1. B. Kadomtsev, V. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Soviet Physics. Doklady*, **15** (1970), 539–541.
2. D. David, D. Levi, P. Winternitz, Integrable nonlinear equations for water waves in straits of varying depth and width, *Stud. Appl. Math.*, **76** (1987), 133–168. https://doi.org/10.1002/sapm1987762133
3. D. David, D. Levi, P. Winternitz, Solitons in shallow seas of variable depth and in marine straits, *Stud. Appl. Math.*, **80** (1989), 1–23. https://doi.org/10.1002/sapm19898011
4. F. Güngör, P. Winternitz, Generalized Kadomtsev-Petviashvili equation with an infinite-dimensional symmetry algebra, *J. Math. Anal. Appl.*, **276** (2002), 314–328. https://doi.org/10.1016/S0022-247X(02)00445-6
5. B. Tian, Y. T. Gao, Solutions of a variable-coefficient Kadomtsev-Petviashvili equation via computer algebra, *Appl. Math. Comput.*, **84** (1997), 125–130. https://doi.org/10.1016/S0096-3003(96)00115-4
6. Y. Zhang, Y. Xu, K. Ma, New type of a generalized variable coefficient Kadomtsev-Petviashvili equation with self-consistent sources and its Grammian-type solutions, *Commun. Nonlinear Sci.*, **37** (2016), 77–89. https://doi.org/10.1016/j.cnsns.2016.01.008
7. A. De Bouard, J. C. Saut, Solitary waves of generalized Kadomtsev-Petviashvili equations, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **14** (1997), 211–236. https://doi.org/10.1016/S0294-1449(97)80145-X
8. Y. L. Ma, A. M. Wazwaz, B. Q. Li, New extended Kadomtsev-Petviashvili equation: multiple soliton solutions, breather, lump and interaction solutions, *Nonlinear Dyn.*, **104** (2021), 1581–1594. https://doi.org/10.1007/s11071-021-06357-8
9. Y. L. Ma, A. M. Wazwaz, B. Q. Li, A new (3 + 1)-dimensional Kadomtsev-Petviashvili equation and its integrability, multiple-solitons, breathers and lump waves, *Math. Comput. Simulat.*, **187** (2021), 505–519. https://doi.org/10.1016/j.matcom.2021.03.012

10. Y. L. Ma, A. M. Wazwaz, B. Q. Li, Novel bifurcation solitons for an extended Kadomtsev-Petviashvili equation in fluids, *Phys. Lett. A*, **413** (2021), 127585. https://doi.org/10.1016/j.physleta.2021.127585

11. Y. L. Ma, B. Q. Li, Rogue wave solutions, soliton and rogue wave mixed solution for a generalized (3 + 1)-dimensional Kadomtsev-Petviashvili equation in fluids, *Mod. Phys. Lett. B*, **32** (2018), 1850358.

12. Y. L. Ma, B. Q. Li, Mixed lump and soliton solutions for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation, *AIMS Mathematics*, **5** (2020), 1162–1176. https://doi.org/10.3934/math.2020080

13. Z. Q. Wang, M. Willem, A multiplicity result for the generalized Kadomtsev-Petviashvili equation, *Topol. Method. Nonl. An.*, **7** (1996), 261–270. https://doi.org/10.12775/TMNA.1996.012

14. B. Xuan, Nontrivial solitary waves of GKP equation in multi-dimensional spaces, *Revista Colombiana de Matematicas*, **37** (2003), 11–23. https://doi.org/10.15446/recolma

15. C. O. Alves, O. H. Miyagaki, Existence, regularity, and concentration phenomenon of nontrivial solitary waves for a class of generalized variable coefficient Kadomtsev-Petviashvili equation, *J. Math. Phys.*, **58** (2017), 081503. https://doi.org/10.1063/1.4997014

16. X. M. He, W. M. Zou, Nontrivial solitary waves to the generalized Kadomtsev-Petviashvili equations, *Appl. Math. Comput.*, **197** (2008), 858–863. https://doi.org/10.1016/j.amc.2007.08.061

17. Z. P. Liang, J. B. Su, Existence of solitary waves to a generalized Kadomtsev-Petviashvili equation, *Acta Math. Sci.*, **32** (2012), 1149–1156. https://doi.org/10.1016/S0252-9602(12)60087-3

18. W. M. Zou, Solitary waves of the generalized Kadomtsev-Petviashvili equations, *Appl. Math. Lett.*, **15** (2002), 35–39. https://doi.org/10.1016/S0893-9659(01)00089-1

19. M. Marin, A. R. Seadawy, S. Vlase, A. Chirila, On mixed problem in thermoelasticity of type *III* for Cosserat media, *J. Taibah Univ. Sci.*, **16** (2022), 1264–1274. https://doi.org/10.1080/16583655.2022.2160290

20. I. Ali, A. R. Seadawy, S. T. R. Rizvi, M. Younis, K. Ali, Conserved quantities along with Painlevé analysis and optical solitons for the nonlinear dynamics of Heisenberg ferromagnetic spin chains model, *Int. J. Mod. Phys. B*, **34** (2020), 2050283. https://doi.org/10.1142/S0217979220502835

21. A. R. Seadawy, N. Cheemaa, Some new families of spiky solitary waves of one-dimensional higher-order KdV equation with power law nonlinearity in plasma physics, *Indian J. Phys.*, **94** (2020), 117–126. https://doi.org/10.1007/s12648-019-01442-6

22. A. R. Seadawy, Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma, *Comput. Math. Appl.*, **67** (2014), 172–180. https://doi.org/10.1016/j.camwa.2013.11.001

23. S. T. R. Rizvi, A. R. Seadawy, F. Ashraf, M. Younis, H. Iqbal, D. Baleanu, Lump and interaction solutions of a geophysical Korteweg-de Vries equation, *Results Phys.*, **19** (2020), 103661. https://doi.org/10.1016/j.rinp.2020.103661

24. J. Wang, K. Shehzad, A. R. Seadawy, M. Arshad, F. Asmat, Dynamic study of multi-peak solitons and other wave solutions of new coupled KdV and new coupled

Zakharov-Kuznetsov systems with their stability, *J. Taibah Univ. Sci.*, **17** (2023), 2163872. https://doi.org/10.1080/16583655.2022.2163872

25. O. V. Besov, V. P. Il'in, S. M. Nikolski, *Integral representations of functions and imbedding theorems*, New York: Wiley, 1978.
26. L. Huang, E. Rocha, On a class of N-dimensional anisotropic Sobolev inequalities, *J. Inequal. Appl.*, **2018** (2018), 163. https://doi.org/10.1186/s13660-018-1754-3
27. Y. Liu, X. P. Wang, Nonlinear stability of solitary waves of a generalized Kadomtsev-Petviashvili equation, *Commun. Math. Phys.*, **183** (1997), 253–266. https://doi.org/10.1007/BF02506406
28. H. Brezis, E. Lieb, A relation between pointwise convergence of function and convergence of functional, *Proc. Amer. Math. Soc.*, **88** (1983), 486–490. https://doi.org/10.1090/S0002-9939-1983-0699419-3