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Research article

Property $\bar{A}$ of third-order noncanonical functional differential equations with positive and negative terms

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Abstract: In this article, we have derived a new method to study the oscillatory and asymptotic properties for third-order noncanonical functional differential equations with both positive and negative terms of the form

$$(p_2(t)(p_1(t)x'(t))')' + a(t)g(x(\tau(t))) - b(t)h(x(\sigma(t)) = 0$$

Firstly, we have converted the above equation of noncanonical type into the canonical type using the strongly noncanonical operator and obtained some new conditions for Property $\bar{A}$. We furnished illustrative examples to validate our main result.

Keywords: Property $\bar{A}$; third-Order; noncanonical; delay argument
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this paper, we are concerned with the third-order differential equation with a positive and negative term

$$(p_2(t)(p_1(t)x'(t))')' + a(t)g(x(\tau(t))) - b(t)h(x(\sigma(t)) = 0 \tag{E}$$

where $t \geq t_0 > 0$ and we need the following assumptions for our work in the sequel.

$(H1)$ $p_1(t), p_2(t), a(t), b(t) \in \mathbb{C}([t_0, \infty))$ are positive and continuous functions for all $t \geq t_o$;
$(H2)$ $g(u), h(u) \in \mathbb{C}(\mathbb{R}), ug(u) > 0, uh(u) > 0\ for\ u \neq 0$, h is bounded, g is nondecreasing;
$(H3)$ $-g(-uv) \geq g(uv) \geq g(u)g(v)\ for\ uv > 0$;
$(H4)$ $\tau(t), \sigma(t) \in \mathbb{C}([t_0, \infty))$ are continuous functions such that $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\lim_{t\to\infty} \sigma(t) = \infty$;
$(H5)$ the noncanonical case of (E), that is

$$\int_{t_0}^{\infty} \frac{1}{p_2(\zeta)} d\zeta < \infty \text{ and } \int_{t_0}^{\infty} \frac{1}{p_1(\zeta)} d\zeta < \infty.$$

By a solution of (E), we mean a function $x(t)$ with derivatives $p_1(t)x'(t), (p_1(t)x'(t))'$ continuous on $[t_x, \infty), T_x \geq t_0$, which satisfies (E) on $[t_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. A solution x of (E) is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is said to be nonoscillatory otherwise. The equation itself is termed oscillatory if all its solutions oscillate.

In the generalization of the Kiguradze lemma [21], the fixed sign of the highest derivative is used to derive the structure of probable nonoscillatory solutions. Since (E) contains both positive and negative terms, we cannot fix the sign of the third-order quasi-derivative for an eventually positive solution. Therefore, the authors studied the oscillatory properties of (E) when either $a(t) \equiv 0$ or $b(t) \equiv 0$; see, for example [6–16, 18–20, 25, 26, 31–33] and the references cited therein.

Recently, there have been several highly interesting results relating to the oscillatory properties of differential equations [3, 27–33]. Also some applications related to biomathematics can be found in [4, 17, 22–24]. However, third-order differential equations get less attention from researchers compared to second-order differential equations. Now, we recall some studies related to the content of this paper.

In [1] Agarwal et al. obtained some new oscillation criteria for the third-order non-linear differential equation of the form

$$(p_2(t)(p_1(t)x'(t))')' + a(t)x^{\beta}(\tau(t)) = 0$$

under

$$\int_{t_0}^{\infty} \frac{1}{p_2(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{p_1(t)} dt = \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{p_2(t)} dt < \infty, \int_{t_0}^{\infty} \frac{1}{p_1(t)} dt < \infty.$$

In [5] Alzabut et al. produced several results for the asymptotic and oscillatory behaviour of third order differential equations

$$(p_2(t)(y''(t)^{\alpha}))' + a(t)x^{\beta}(\tau(t)) + b(t)x^{\gamma}(\sigma(t)) = 0$$

under $\int_{t_0}^{t} \frac{1}{a^{\frac{1}{\alpha}}(s)} ds = \infty$ where $y(t) = x(t) + P(t)x^{\nu}(\omega(t)) - q(t)x^{\kappa}(\omega(t))$.

In [6], Baculicova and Dzurina considered the equation (E) and they derived new techniques for studying the oscillatory and asymptotic properties of (E) by property A by considering the canonical case

$$\int_{t_0}^{\infty}\frac{1}{p_2(t)}dt=\int_{t_0}^{\infty}\frac{1}{p_1(t)}dt=\infty.$$

In [26], Saranya et al. developed new technique to study the oscillation criteria for the third order quasi-linear delay differential equation

$$(p_2(t)(p_1(t)(x'(t))^{\alpha})')'+a(t)x^{\beta}(\tau(t))=0$$

under

$$\int_{t_0}^{\infty}\frac{1}{p_2(t)}dt<\infty,\int_{t_0}^{\infty}\frac{1}{p_1^{\frac{1}{\alpha}}(t)}dt=\infty.$$

In [27], Saranya et al. attained new criteria for the oscillatory and asymptotic behavior of the solution of the equation

$$(p_2(t)(p_1(t)(x'(t)))')'+a(t)x^{\beta}(\tau(t))=0$$

under the semi-canonical type

$$\int_{t_0}^{\infty}\frac{1}{p_2(t)}dt=\infty\ \int_{t_0}^{\infty}\frac{1}{p_1(t)}dt<\infty.$$

In [30], Santra et al. studied the oscillatory behavior of half-linear neutral conformable differential equations

$$T_{\alpha_3}\Big(p_2(t)(T_{\alpha_2}(p_1(t)T_{\alpha_1}x(t)))^{\beta}\Big)+a(t)x^{\beta}(t)=0,t\geq t_0$$

where $x(t)=y(t)+p(t)x(\delta(t))$ under the canonical case

$$\int_{t_0}^{\infty}\frac{1}{p_1(t)}d_{\alpha_2}t=\int_{t_0}^{\infty}\frac{1}{p_2^{\frac{1}{\beta}}(t)}d_{\alpha_3}t=\infty.$$

However, most of the research work is done under the canonical case

$$\int_{t_0}^{\infty}\frac{1}{p_1(t)}dt=\int_{t_0}^{\infty}\frac{1}{p_2(t)}dt=\infty$$

as the investigation of the oscillatory properties of the canonical equation is much easier than the noncanonical case. Therefore, in this paper, we aim to investigate the oscillatory properties of solutions of (E) in noncanonical form, which is under $(H5)$ using the canonical representation of a strongly noncanonical operator. Thus all the results are new in this paper and complement to those results arrived in [1,2,21,26]. Furthermore, we assume that

$(H6)$ $\int_{t_0}^{\infty}\frac{1}{p_1(t)}\int_{t}^{\infty}\frac{1}{p_2(\zeta)}\int_{\zeta}^{\infty}b(u)du\,d\zeta\,dt<\infty.$

Instead of the above condition, one can easily show that the effect of the negative term is reduced and allows us to study property $\bar{A}$. Property $\bar{A}$ refers to the condition under which every nonoscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty}\frac{x(t)}{\Omega(t)}=0.$

This paper is organized as follows. In Section 2, we present some basic notations and lemmas. We present the main result for the equation (E) in Section 3. In Section 4, we illustrate few examples for our main result.

2. Preliminaries

In this section, initially we provide some lemmas to easily verify the conditions for property $\bar{A}$ of (E).

To simplify our notation, let us denote

$$\Omega_1(t) = \int_t^\infty \frac{1}{p_1(\zeta)}d\zeta,\ \ \Omega_2(t) = \int_t^\infty \frac{1}{p_2(\zeta)}d\zeta,\ \Omega(t) = \int_t^\infty \frac{\Omega_2(\zeta)}{p_1(\zeta)}d\zeta,$$
$$\Omega_*(t) = \int_t^\infty \frac{\Omega_1(\zeta)}{p_2(\zeta)}d\zeta,\ \ t \ge t_0.\ \eta_1(t) = \frac{p_1(t)\Omega^2(t)}{\Omega_*(t)},\ \ \eta_2(t) = \frac{p_2(t)\Omega_*^2(t)}{\Omega(t)},$$
$$Q_1(t) = \int_{t_1}^t \frac{1}{\eta_1(\zeta)}d\zeta = \frac{\Omega_1(t)}{\Omega(t)},\ \ Q_2(t) = \int_{t_1}^t \frac{1}{\eta_2(\zeta)}d\zeta = \frac{\Omega_2(t)}{\Omega_*(t)}$$

and

$$Q(t) = \int_{t_1}^t \frac{Q_2(\zeta)}{\eta_1(\zeta)}d\zeta \text{ for all } t \ge t_1 \ge t_0.$$

Lemma 2.1. *Assume that* $(H1)-(H6)$ *hold, and* $x(t)$ *is an eventually positive solution of* (E) *then, the function*

$$z(t) = x(t) + \int_t^\infty \frac{1}{p_1(\zeta)} \int_\zeta^\infty \frac{1}{p_2(u)} \int_u^\infty b(x)\ h(x(\sigma(x)))dx\ du\ d\zeta \tag{2.1}$$

is a positive solution of the equation

$$\mathbb{L}(z) + a(t)g(x(\tau(t))) = 0 \tag{E_1}$$

where $\mathbb{L}(z) = (p_2(t)(p_1(t)z'(t))')'$.

Proof. It follows from (H6) and the boundedness of $h(u)$ that the definition of function $z(t)$ defined by (2.1) is well defined for all $t \ge T_x \ge t_0$. We write that $z(t) > x(t)$, $z'(t) < x'(t)$ and

$$(p_2(t)(p_1(t)z'(t))')' + a(t)g(x(\tau(t))) = 0$$

which completes the proof. □

Lemma 2.2. *Let* $(H1)-(H6)$ *hold, and the strongly noncanonical operator*

$$\mathbb{L}(z) = (p_2(t)(p_1(t)x'(t))')'$$

has the following unique canonical representation

$$\mathbb{L}(z) = \frac{1}{\Omega_*(t)}\left(\frac{p_2(t)\Omega_*^2(t)}{\Omega(t)}\left(\frac{p_1(t)\Omega^2(t)}{\Omega_*(t)}\left(\frac{z(t)}{\Omega(t)}\right)'\right)'\right)'. \tag{2.2}$$

Proof. The proof can be found in Theorem 3.2 of [8] and is hence omitted.

By Lemma 2.2, (E_1) can be written in the equivalent canonical form as

$$\left(\eta_2(t)\left(\eta_1(t)\left(\frac{z(t)}{\Omega(t)}\right)'\right)'\right)' + \Omega_*(t)a(t)g(x(\tau(t))) = 0.$$

Setting $\mu(t) = \dfrac{z(t)}{\Omega(t)}$, we immediately obtain the follows results (see [8]). □

Lemma 2.3. *Let* $(H1)-(H6)$ *hold, and the strongly noncanonical differential equation* (E_1) *can be written as the equivalent canonical equation*

$$(\eta_2(t)(\eta_1(t)\mu'(t))')' + \Omega_*(t)a(t)g(x(\tau(t))) = 0. \tag{E_2}$$

3. Main Result

Corollary 3.1. *Let* $(H1)-(H6)$ *hold, and the strongly noncanonical equation* (E) *has an eventually positive solution if and only if the canonical equation* (E_2) *has an eventually positive solution. Corollary 3.1 significantly simplifies the examination of* (E) *since for* (E_2) *we deal with only two classes of eventually positive solutions existing of four classes of eventually positive solutions. Thus,* $\mu(t)$ *satisfies either*

$$\mu(t) > 0,\ \eta_1(t)\mu'(t) < 0,\ \eta_2(t)(\eta_1(t)\mu'(t))' > 0,\ (\eta_2(t)(\eta_1(t)\mu'(t))')' < 0$$

and in this case we say $\mu \in S_0$ *or*

$$\mu(t) > 0,\ \eta_1(t)\mu'(t) > 0,\ \eta_2(t)(\eta_1(t)\mu'(t))' > 0,\ (\eta_2(t)(\eta_1(t)\mu'(t))')' < 0$$

and for this property, we denote that $\mu \in S_2$.

Theorem 3.2. *Assume that* $(H1)-(H6)$ *hold, and for all* $t_1 \geq t_0$ *large enough*

$$\int_{t_1}^{\infty} \frac{1}{\eta_1(v)} \int_{v}^{\infty} \frac{1}{\eta_2(\zeta)} \int_{\zeta}^{\infty} \Omega_*(u)\ a(u)\ du\ d\zeta\ dv = +\infty \tag{3.1}$$

and

$$\int_{t_1}^{\infty} \Omega_*(t)\ a(t)\ g(Q(\tau(t)))dt = +\infty. \tag{3.2}$$

If

$$\begin{aligned}
&\limsup_{t\to\infty} \left\{ \frac{1}{Q_2(\tau(t))} \int_{t_1}^{\tau(t)} \Omega_*(\zeta)\ a(\zeta) g(Q(\tau(\zeta)))Q_2(\zeta)\ d\zeta \right. \\
&+ \int_{\tau(t)}^{t} \Omega_*(\zeta)\ a(\zeta)\ g(Q(\tau(\zeta))d\zeta + g(Q_2(\tau(t))) \int_{t}^{\infty} \Omega_*(\zeta)\ a(\zeta)\ g\left(\frac{Q(\tau(\zeta)}{Q_2(\tau(\zeta)} \right) d\zeta \\
&> \limsup_{t\to\infty} \frac{u}{g(u)},
\end{aligned} \tag{3.3}$$

then (E) *holds property* $\bar{A}$.

Proof. Let $x(t)$ be an eventually positive solution of (E) on $[T_x, \infty), T_x \geq t_0$. Then the function $z(t)$ defined on 2.1 satisfies (E_1) with $z(t) > x(t) > 0$ and $z'(t) < x'(t)$ for all $t \geq t_1 \geq T_x$. Setting $\mu(t) = \dfrac{z(t)}{\omega(t)}$, we see that $\mu(t) > 0$ satisfies the equation (E_2) and therefore it satisfies $\mu \in S_0$ or $\mu \in S_2$ for all $t \geq t_1$. First assume that $\mu \in S_2$. Using the fact that $\eta_2(t)(\eta_1(t)\mu'(t))$ is decreasing, we have

$$\eta_1(t)\mu'(t) \geq \int_{t_1}^{t} \eta_2(\zeta)(\eta_1(\zeta)\mu'(\zeta))' \frac{1}{\eta_2(\zeta)} d\zeta \geq \eta_2(t)(\eta_1(t)\mu'(t))' Q_2(t). \tag{3.4}$$

It follows from (3.4) that $\dfrac{\eta_1(t)\mu'(t)}{Q_2(t)}$ is decreasing. Then

$$x(t) \geq \int_{t_1}^{t} x'(\zeta)d\zeta \geq \int_{t_1}^{t} \frac{\eta_1(\zeta)}{Q_2(\zeta)}\Big(\frac{z(\zeta)}{\Omega(\zeta)}\Big)' \frac{Q_2(\zeta)}{\eta_1(\zeta)} d\zeta \geq \frac{\eta_1(t)}{Q_2(t)}\Big(\frac{z(t)}{\Omega(t)}\Big)' Q(t).$$

Setting the last estimate into (E_2), we see that $w(t) = \eta_1(t)\Big(\dfrac{z(t)}{\Omega(t)}\Big)'$ is a positive solution of the differential inequality

$$(\eta_2(t)w'(t))' + \Omega_*(t)a(t)g\Big(\frac{Q(\tau(t)}{Q_2\tau(t)}w(\tau(t))\Big) \leq 0 \tag{3.5}$$

and we have that $\dfrac{w(t)}{Q_2(t)}$ is decreasing and $\eta_2(t)w'(t) > 0$. By integrating (3.5), we get

$$\begin{aligned} w(t) &\geq \int_{t_1}^{t} \frac{1}{\eta_2(u)} \int_{u}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta du \\ &= \int_{t_1}^{t} \frac{1}{\eta_2(u)} \int_{u}^{t} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta du \\ &+ \int_{t_1}^{t} \frac{1}{\eta_2(u)} \int_{t}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta du \\ &= \int_{t_1}^{t} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta \\ &+ Q_2(t) \int_{t}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta. \end{aligned}$$

Hence,

$$\begin{aligned} w(\tau(t)) \geq \int_{t}^{\tau(t)} &\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)Q_2(\zeta)d\zeta \\ &+ Q_2(\tau(t)) \int_{\tau(t)}^{t} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta \\ &+ Q_2(\tau(t)) \int_{t}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta. \end{aligned}$$

In view of (H3) and given that $w(t)$ is non-decreasing and $\dfrac{w(t)}{Q_2(t)}$ is non-increasing, we have

$$\begin{aligned} w(\tau(t)) \geq g\Big(\frac{w(\tau(t)}{Q_2\tau(t)}\Big) &\int_{t}^{\tau(t)} \Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))Q_2(\zeta)d\zeta \\ &+ Q_2(\tau(t))g\Big(\frac{w(\tau(t)}{Q_2\tau(t)}\Big) \int_{\tau(t)}^{t} \Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))d\zeta \\ &+ Q_2(\tau(t))g(w(\tau(t)) \int_{t}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}\Big)d\zeta. \end{aligned} \tag{3.6}$$

Therefore, letting $u = \dfrac{w(\tau(t))}{Q_2(\tau(t))}$, we obtain

$$\frac{u}{g(u)} \geq \frac{1}{Q_2(\tau(t))}\int_{t_1}^{\tau(t)} \Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))Q_2(\zeta)d\zeta \\ + \int_{\tau(t)}^{t} \Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))d\zeta \\ + g(Q_2(\tau(t)))\int_{t}^{\infty} \Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}\Big)d\zeta, \tag{3.7}$$

and condition (3.2) implies that $\dfrac{w(t)}{Q_2(t)} \to 0\ a\zeta\ t \to \infty$. Indeed, if we assume $\dfrac{w(t)}{Q_2(t)} \to L > 0$, then $\dfrac{w(t)}{Q_2(t)} \geq L$ and substituting in (3.5), we have

$$0 \geq (\eta_2(t)w'(t))' + \Omega_*(t)a(t)g(LQ(\tau(t))).$$

Integrating from t_1 to ∞ yields

$$\eta_2(t_1)w'(t_1) \geq g(L)\int_{t_1}^{\infty} \Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))d\zeta$$

which contradicts (3.2). Taking lim sup in (3.7) contradicts (3.3).

Now assume that $\mu(t) \in S_0$. Since $\mu(t)$ is positive and decreasing, there exists $\lim_{t\to\infty}\mu(t) = 2l \geq 0$. It follows from (2.1) that

$$\lim_{t\to\infty}\frac{z(t)}{\Omega(t)} = \lim_{t\to\infty}\frac{x(t)}{\Omega(t)} = 2l.$$

If we assume that $l > 0$, then $\dfrac{x(\tau(t))}{\Omega(\tau(t))} \geq l > 0$, eventually. Intergrating (E_2) yields

$$\eta_2(t)(\eta_1(t)\mu'(t))' \geq \int_t^{\infty} \Omega_*(\zeta)a(\zeta)g(x(\tau(\zeta)))d\zeta \\ \geq g(l)\int_t^{\infty} \Omega_*(\zeta)a(\zeta)g(\Omega(\tau(\zeta)))d\zeta.$$

Integrating from t to ∞ and then from t_1 *to* ∞, we get

$$\mu(t_1) \geq g(l)\int_{t_1}^{\infty}\frac{1}{\eta_1(v)}\int_v^{\infty}\frac{1}{\eta_2(\zeta)}\int_{\zeta}^{\infty}\Omega_*(u)\ a(u)\ du\ d\zeta\ dv$$

which contradicts (3.1). □

Corollary 3.3. *Let* $(H1)-(H6)$ *and* (3.1) *hold, and for all* t_1 *large enough*

$$\int_{t_1}^{\infty}\Omega_*(t)a(t)Q(\tau(t))dt = \infty. \tag{3.8}$$

If

$$\limsup_{t\to\infty}\Big\{\frac{1}{Q_2(\tau(t))}\int_{t_1}^{\tau(t)}\Omega_*(\zeta)a(\zeta)Q(\tau(\zeta))Q_2(\zeta)d\zeta+\int_{\tau(t)}^{t}\Omega_*(\zeta)a(\zeta)Q(\tau(\zeta))d\zeta \qquad (3.9)$$
$$+Q_2(\tau(t))\int_t^\infty\Omega_*(\zeta)a(\zeta)\frac{Q(\tau(\zeta))}{Q_2(\tau(\zeta))}d\zeta\Big\}>1,$$

then (E) *holds property* $\bar{A}$.

Theorem 3.4. *Let* $(H1)-(H6)$ *and* (3.1) *hold, and*

$$\int_{t_1}^\infty\frac{1}{\eta_2(u)}\int_u^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta))}{Q_2(\tau(\zeta))}\Big)d\zeta du=\infty. \qquad (3.10)$$

for all t_1 *large enough. If*

$$\limsup_{t\to\infty}\Big\{g\Big(\frac{1}{Q_2(\tau(t))}\Big)\int_{t_1}^{\tau(t)}\Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta)))Q_2(\zeta)d\zeta \qquad (3.11)$$
$$+Q_2(\tau(t))g\Big(\frac{1}{Q_2(\tau(t))}\Big)\int_{\tau(t)}^{t}\Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta)))d\zeta$$
$$+g(Q_2(\tau(t)))\int_t^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta))}{Q_2(\tau(\zeta))}\Big)d\zeta\Big\}>\limsup_{t\to\infty}\frac{v}{g(v)},$$

then (E) *holds the property* $\bar{A}$.

Proof. Let the positive solution of (E) be $x(t)$ and proceed as in Theorem 3.2; we verified that $\mu(t)$ belongs to either S_0 or S_2. If $\mu(t)\in S_2$, then $w(t)=\eta_1(t)\mu'(t)$ satisfies (3.7). The condition (3.10) imply, $w(t)\to\infty$ as $t\to\infty$. If not, then $w(t)\to M$ as $t\to\infty$. Integrating (3.5), we get

$$\eta_2(t)w'(t)\geq\int_t^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta.$$

Integrating once more, we have

$$M\geq\int_{t_1}^\infty\frac{1}{\eta_2(u)}\int_u^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}w(\tau(\zeta))\Big)d\zeta du$$
$$\geq g(w(\tau(t_1)))\int_{t_1}^\infty\frac{1}{\eta(u)}\int_u^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta)}{Q_2\tau(\zeta)}\Big)d\zeta du$$

which contradicts with (3.10) and we conclude that $w(t)\to\infty$ as $t\to\infty$. Therefore, by setting $v=w(\tau(t))$, we get

$$\frac{v}{g(v)}\geq g\Big(\frac{1}{Q_2(\tau(t))}\Big)\int_{t_1}^{\tau(t)}\Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))Q_2(\zeta)d\zeta$$
$$+Q_2(\tau(t))g\Big(\frac{1}{Q_2(\tau(t))}\Big)\int_{\tau(t)}^{t}\Omega_*(\zeta)a(\zeta)g(Q(\tau(\zeta))d\zeta$$
$$+Q_2(\tau(t))\int_t^\infty\Omega_*(\zeta)a(\zeta)g\Big(\frac{Q(\tau(\zeta))}{Q_2(\tau(\zeta))}\Big)d\zeta.$$

Taking lim sup on both sides contradicts (3.11).

Next, if $\mu(t)\in S_0$, then proceeding by the proof of Theorem 3.2, we verify that $\lim_{t\to\infty}\frac{y(t)}{\Omega(t)}=0$. □

Remark 3.5. *Theorems 3.2 and 3.4 are applicable for*

$$g(s) = |s|^{\alpha} sgn\ s$$

with $0 \leq \alpha \leq 1$ & $\alpha > 1$.

Remark 3.6. *The integral criteria* (3.3) *and* (3.11) *of Theorems 3.2 and 3.4 provide better results than the one term integral criteria that are usually used.*

4. Example

Example 4.1. *Consider the equation*

$$(t^2(t^2x'(t))')' + \frac{a}{t}x\left(\frac{t}{2}\right) - \frac{b}{t^2}\arctan(x(\sigma(t))) = 0, t \geq 1 \tag{4.1}$$

with $a > 0$ *and* $b > 0$.

The condition (H6) holds. Moreover $\Omega_1(t) = \Omega_2(t) = \frac{1}{t}, \Omega(t) = \Omega_*(t) = \frac{1}{2t^2}, \tau(t) = \frac{t}{2}, \eta_1(t) = \eta_2(t) = \frac{1}{2}$, $Q_1(t) = Q_2(t) = 2t, Q(t) = 2t^2, g(u) = u$ and $h(u) = \arctan(u)$. The condition (3.1) becomes

$$2a\int_1^\infty \int_v^\infty \int_s^\infty \frac{1}{u^3}du\ ds\ dv = \infty.$$

Therefore, condition (3.1) holds. The condition (3.8) becomes

$$\frac{a}{4}\int_1^\infty \frac{1}{t}dt = \infty.$$

Therefore, condition (3.8) holds. The condition (3.9) becomes

$$\limsup_{t\to\infty}\left\{\frac{1}{t}\int_1^{\frac{t}{2}} \frac{a}{2}ds + \frac{a}{4}\int_{\frac{t}{2}}^{t} \frac{1}{s}ds + t\int_t^\infty \frac{a}{4s^2}ds\right\} > 1.$$

By integrating the above equation and applying the limits, we get

$$\limsup_{t\to\infty}\left\{\frac{a}{2} - \frac{a}{2t} + \frac{a}{4}\log 2\right\} > 1$$

and

$$a\left(\frac{1}{2} + \frac{1}{4}\log 2\right) > 1.$$

Therefore, condition (3.8) holds if $a\left(\frac{1}{2} + \frac{1}{4}\log 2\right) > 1$. Therefore by Corollary 3.3, any nonoscillatory solution $x(t)$ of (4.1) satisfies

$$\lim_{t\to\infty} t^2x(t) = 0$$

provided that

$$a\left(\frac{1}{2} + \frac{1}{4}\log 2\right) > 1.$$

The results were also applicable when $\tau(t) \equiv t$.

Example 4.2. *Consider the equation*

$$(t^2(t^2x'(t))')' + \frac{a}{t}x(t) - \frac{b}{t^2}\arctan(x(\sigma(t))) = 0, t \geq 1 \tag{4.2}$$

with $a > 0$ *and* $b > 0$.

Clearly the condition (3.1) holds and (3.9) becomes

$$\limsup_{t\to\infty}\left\{\frac{1}{2t}\int_1^t\left(\frac{1}{2s^2}\right)\left(\frac{a}{s}\right)(2s^2)(2s)ds + \int_t^t\left(\frac{1}{2s^2}\right)\left(\frac{a}{s}\right)(2s^2)ds + 2t\int_t^\infty\left(\frac{1}{2s^2}\right)\left(\frac{a}{s}\right)\frac{(2s^2)}{2s}ds\right\} > 1$$

that is,

$$\limsup_{t\to\infty}\left\{\frac{a}{t}\int_1^t ds + at\int_t^\infty \frac{1}{s^2}ds\right\} > 1.$$

Integrating, we get

$$\limsup_{t\to\infty} a\left\{2 - \frac{1}{t}\right\} > 1$$

which implies

$$a > \frac{1}{2}.$$

Therefore, the condition (3.9) holds if $a > \frac{1}{2}$. Hence, any nonoscillatory solution $x(t)$ of (4.2) satisfies

$$\lim_{t\to\infty} t^2x(t) = 0$$

provided that $a > \frac{1}{2}$.

5. Summary

In this paper, we looked at the oscillatory and asymptotic properties of third-order differential equations with positive and negative terms in the noncanonical case. Because the noncanonical case is hard, we turned (E) into a canonical representation by using a strongly noncanonical operator that only works with two classes of eventually positive solutions out of four classes of eventually positive solutions. This paper has better results than the ones that are already out there. The main results are illustrated through the examples.

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Conflict of interest

The authors do not have any conflict of interest.

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