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Research article

Nonlocal integro-multistrip-multipoint boundary value problems for $\overline{\psi}_*$-Hilfer proportional fractional differential equations and inclusions

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Abstract: In the present paper, we establish the existence criteria for solutions of single valued and multivalued boundary value problems involving a $\overline{\psi}_*$-Hilfer fractional proportional derivative operator, subject to nonlocal integro-multistrip-multipoint boundary conditions. We apply the fixed-point approach to obtain the desired results for the given problems. The obtained results are well-illustrated by numerical examples. It is important to mention that several new results appear as special cases of the results derived in this paper (for details, see the last section).

Keywords: fractional differential equations; fractional differential inclusions; $\overline{\psi}_*$-Hilfer fractional derivative; proportional fractional derivative; nonlocal boundary conditions; existence; fixed point
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1. Introduction

Fractional differential equations have been of great interest during the last few years as such equations provide appropriate mathematical models for real world problems arising in physics, engineering, economics, robotics, control theory, etc. A systematic development of fractional calculus and fractional differential equations can be found in the monographs [1–5]. Unlike the classical derivative operator, one can find a variety of its fractional counterparts, such as the Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Hilfer, Caputo-Hadamard, etc. In [6–8], the authors discussed the concept of a proportional fractional derivative and its generalizations. The authors in [9] proposed a Hilfer type generalized proportional fractional derivative. In [10], the authors discussed a generalization of the $\overline{\psi}_*$-Hilfer fractional derivative.

Initial and boundary value problems for differential equations and inclusions involving different fractional derivative operators have also been investigated by many researchers, such as [11–16]. Recently, in [17], the authors studied a nonlocal initial value problem of order in $(0,1)$ for $\overline{\psi}_*$-Hilfer generalized proportional fractional derivative of a function with respect to another function. Very recently, in [18], the authors studied a nonlocal mixed boundary value problem for $\overline{\psi}_*$-Hilfer fractional proportional differential equations and inclusions of order in $(1,2]$ given by

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\pi(z) = \Pi(z,\pi(z)) \quad or \quad \in \mathbb{H}(z,\pi(z)), \quad z \in [a_1,b_1],\ b_1 > a_1 \geq 0, \\ \pi(a_1) = 0, \\ \pi(b_1) = \sum_{j=1}^{m} \epsilon_j \pi(\zeta_j) + \sum_{i=1}^{n} \zeta_i\, \mathbb{I}_{a_1+}^{\phi_i,\vartheta_*,\overline{\psi}_*}\pi(\theta_i) + \sum_{k=1}^{r} \lambda_k\, \mathbb{D}_{a_1+}^{\delta_k,\varphi,\vartheta_*,\overline{\psi}_*}\pi(\mu_k). \end{cases}$$

Here, $\mathbb{D}_{a_1+}^{\omega,\varphi,\vartheta_*,\overline{\psi}_*}$ is the $\overline{\psi}_*$-Hilfer fractional proportional derivative operator of order $\omega \in \{\rho,\delta_k\}$, $\rho,\delta_k \in (1,2]$, and type $\varphi \in [0,1]$, $\vartheta_* \in (0,1]$, and $\epsilon_j,\zeta_i,\lambda_k \in \mathbb{R}$. Π: $[a_1,b_1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function (or $\mathbb{H}$: $[a_1,b_1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map), $\mathbb{I}_{a_1+}^{\phi_i;\vartheta_*,\overline{\psi}_*}$ is the fractional integral operator of order $\phi_i > 0$, and $\zeta_j,\theta_i,\mu_k \in (a_1,b_1)$, $j = 1,2,\ldots,m$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,r$.

The objective of the present work is to enrich the literature on boundary value problems involving $\overline{\psi}_*$-Hilfer fractional proportional derivative operators. In precise terms, we consider and investigate a new problem consisting of a $\overline{\psi}_*$-Hilfer fractional proportional differential equation and nonlocal integro-multistrip-multipoint boundary conditions given by

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\sigma(z) = \Psi(z,\sigma(z)), \quad z \in [a_1,b_1], \\ \sigma(a_1) = 0, \\ \int_{a_1}^{b_1} \overline{\psi}'_*(s)\sigma(s)ds = \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s)\sigma(s)ds + \sum_{j=1}^{m} \theta_j \sigma(\zeta_j), \end{cases} \tag{1.1}$$

where $\mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}$ denotes the $\overline{\psi}_*$-Hilfer fractional proportional derivative operator of order $\rho \in (1,2]$ and type $\varphi \in [0,1]$, $\vartheta_* \in (0,1]$, $a_1 < \zeta_j < \xi_i < \eta_i < b_1$, $\varphi_i,\theta_j \in \mathbb{R}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $\overline{\psi}_*$: $[a_1,b_1] \to \mathbb{R}$ is an increasing function with $\overline{\psi}'_*(z) \neq 0$ for all $z \in [a_1,b_1]$, and $\overline{\psi}_*$: $[a_1,b_1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. As a second problem, we investigate the multivalued analogue (4.1) of the problem (1.1) in Section 4.

Here, we emphasize that the problems (1.1) and (4.1) investigated in this paper are new in the sense of integro-multistrip-multipoint boundary conditions. In order to establish the existence and uniqueness results for the problem (1.1), our strategy is to convert it into a fixed point problem and then apply the fixed point theorems due to Banach, Krasnosel'skiĭ, Schaefer and Leray-Schauder alternative. In the case of the multi-valued problem (4.1), we prove two existence results via nonlinear alternative for Kakutani maps and Covitz-Nadler fixed point theorem for convex and non-convex valued multivalued maps in (4.1), respectively. It is imperative to mention that the tools of the fixed point theory provide an excellent platform for analyzing the nonlinear problems. All the results accomplished in the present study are novel and give rise to some new results as special cases (for details, see Section 5). We also demonstrated the application of the main results by constructing numerical examples.

The rest of the paper is constructed as follows. In Section 2, some basic definitions and preliminary results related to our work are recalled. Section 3 contains the existence and uniqueness results for the single valued problem (1.1), while the existence results for the multi-valued analogue of the problem (1.1) are proved in Section 4. The paper concludes with some interesting observations.

2. Preliminaries

Let us begin this section with basic definitions.

Definition 2.1. *([7, 8]) For $\vartheta_* \in (0,1]$ and $\rho \in \mathbb{R}^+$, the fractional proportional integral of $\hat{h} \in L^1([a_1,b_1],\mathbb{R})$ with respect to $\overline{\psi}_*$ of order ρ is given by*

$$(\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h})(z) = \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\int_{a_1+}^{t} e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(s))}(\overline{\psi}_*(z)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}_*'(s)\hat{h}(s)ds,\ z>a_1. \tag{2.1}$$

Definition 2.2. *([7, 8]) Let $\overline{\psi}_* \in C([a_1,b_1],\mathbb{R})$ with $\overline{\psi}_*'(z) > 0$, $\vartheta_* \in (0,1]$, and $\rho \in \mathbb{R}^+$. The fractional proportional derivative for $\hat{h} \in C([a_1,b_1],\mathbb{R})$ with respect to $\overline{\psi}_*$, of order ρ, is given by*

$$(\mathbb{D}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h})(z) = \frac{\mathbb{D}^{n,\vartheta_*,\overline{\psi}_*}}{\vartheta_*^{n-\rho}\Gamma(n-\rho)}\int_{a_1+}^{t} e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(s))}(\overline{\psi}_*(z)-\overline{\psi}_*(s))^{n-\rho-1}\overline{\psi}_*'(s)\hat{h}(s)ds,\ z>a_1, \tag{2.2}$$

where $n=[\rho]+1$, and $[\rho]$ denotes the integer part of the real number ρ.

Definition 2.3. *([17]) Let $\overline{\psi}_*$ be positive and strictly increasing with $\overline{\psi}_*'(z) \neq 0$, for all $z \in [a_1,b_1]$ and $\hat{h}, \overline{\psi}_* \in C^m([a_1,b_1],\mathbb{R})$. The $\overline{\psi}_*$-Hilfer fractional proportional derivative for $\hat{h}$ with respect to another function $\overline{\psi}_*$, of order ρ and type φ, is defined by*

$$(\mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\hat{h})(z) = (\mathbb{I}_{a_1+}^{\varphi(n-\rho),\vartheta_*,\overline{\psi}_*}(\mathbb{D}^{n,\vartheta_*,\overline{\psi}_*})\mathbb{I}_{a_1+}^{(1-\varphi)(n-\rho),\vartheta_*,\overline{\psi}_*}\hat{h})(z), \tag{2.3}$$

where $n-1<\rho<n$, $0 \le \varphi \le 1$, $n \in \mathbb{N}$, and $\vartheta_* \in (0,1]$. Also,

$$\mathbb{D}^{\vartheta_*,\overline{\psi}_*}\hat{h}(z) = (1-\vartheta_*)\hat{h}(z) + \vartheta_*\frac{\hat{h}'(z)}{\overline{\psi}_*'(z)},$$

$\mathbb{I}_{a_1+}^{(.)}$ is the fractional proportional integral operator defined in (2.1).

Now we recall some known results.

Lemma 2.1. *([17]) The $\overline{\psi}_*$-Hilfer fractional proportional derivative can be expressed as*

$$(\mathbb{D}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\hat{h})(z) = (\mathbb{I}_{a_1+}^{\varphi(n-\rho),\vartheta_*,\overline{\psi}_*}(\mathbb{D}^{n,\vartheta_*,\overline{\psi}_*})(\mathbb{I}_{a_1+}^{(1-\varphi)(n-\rho),\vartheta_*,\overline{\psi}_*}\hat{h})(z) = (\mathbb{I}^{\varphi(n-\rho),\vartheta_*,\overline{\psi}_*}\mathbb{D}_{a_1+}^{\gamma_1,\vartheta_*,\overline{\psi}_*}\hat{h})(z),$$

where $\gamma_1 = \rho+\varphi(n-\rho)$.

Remark 2.1. *([17]) The following relations hold:*

$$\gamma_1 = \rho + \varphi(n-\rho), \;\; n-1 < \rho, \;\; \gamma_1 \le n, \;\; 0 \le \varphi \le 1,$$

and

$$\gamma_1 \ge \rho, \;\; \gamma_1 > \varphi, \;\; n - \gamma_1 < n - \varphi(n-\rho).$$

Lemma 2.2. *([17]) Let*

$$n-1 < \rho < n, \; n \in \mathbb{N}, \; \vartheta_* \in (0,1], \; 0 \le \varphi \le 1$$

and $\gamma_1 = \rho + \varphi(n-\rho)$ *be such that* $n-1 < \gamma_1 < n$. *If* $\hat{h} \in C([a_1,b_1],\mathbb{R})$ *and*

$$\mathbb{I}_{a_1+}^{n-\gamma_1,\vartheta_*,\overline{\psi}_*} f \in C^n([a_1,b_1],\mathbb{R}),$$

then

$$\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\hat{h}(z) = \hat{h}(z) - \sum_{k=1}^{n} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-k}}{\vartheta_*{}^{\gamma_1-k}\Gamma(\gamma_1-k+1)}(\mathbb{I}_{a_1+}^{k-\gamma_1,\vartheta_*,\overline{\psi}_*}\hat{h})(a).$$

3. Single-valued case

In this section we will establish existence and uniqueness results for the boundary value problem (1.1).

Lemma 3.1. *Let* $\hat{h}_0 \in C([a_1,b_1],\mathbb{R})$ *and*

$$\begin{aligned}
\mathbb{L} \;\; := & \int_{a_1}^{b_1} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))^{\gamma_1-1}\overline{\psi}_*'(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}ds \\
- & \sum_{i=1}^{n}\varphi_i \int_{\xi_i}^{\eta_i} \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))^{\gamma_1-1}\overline{\psi}_*'(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}ds \\
- & \sum_{j=1}^{m}\theta_j \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \neq 0.
\end{aligned} \tag{3.1}$$

Then, σ *is a solution of the linear nonlocal integro-multistrip-multipoint* $\overline{\psi}_*$*-Hilfer generalized proportional fractional boundary value problem*

$$\begin{cases}
\mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\sigma(z) = \hat{h}_0(z), \quad z \in [a_1,b_1], \\
\sigma(a_1) = 0, \\
\displaystyle\int_{a_1}^{b_1} \overline{\psi}_*'(s)\sigma(s)ds = \sum_{i=1}^{n}\varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}_*'(s)\sigma(s)ds + \sum_{j=1}^{m}\theta_j\sigma(\zeta_j),
\end{cases} \tag{3.2}$$

if and only if

$$\begin{aligned}
\sigma(z) \;\; = \;\; & \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds \\
+ & \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds\Bigg\}, \quad z \in [a_1,b_1].
\end{aligned} \tag{3.3}$$

Proof. From Lemma 2.2 with $n = 2$, we have

$$\begin{aligned}\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\sigma(z) &= \sigma(z) - \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}(\mathbb{I}_{a_1+}^{1-\gamma_1,\vartheta_*,\overline{\psi}_*}\sigma)(a_1)\\ &- \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-2}}{\vartheta_*^{\gamma_1-2}\Gamma(\gamma_1-1)}(\mathbb{I}_{a_1+}^{2-\gamma_1,\vartheta_*,\overline{\psi}_*}\sigma)(a_1),\end{aligned}$$

which implies that

$$\begin{aligned}\sigma(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(z) + c_0\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\\ &+ c_1\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-2}}{\vartheta_*^{\gamma_1-2}\Gamma(\gamma_1-1)},\end{aligned} \tag{3.4}$$

where

$$c_0 = (\mathbb{I}_{a_1+}^{1-\gamma_1,\vartheta_*,\overline{\psi}_*}\sigma)(a_1)$$

and

$$c_1 = (\mathbb{I}_{a_1+}^{2-\gamma_1,\vartheta_*,\overline{\psi}_*}\sigma)(a_1).$$

Using (3.4) in the condition $\sigma(a_1) = 0$, we get $c_1 = 0$ since $\gamma_1 \in [\rho, 2]$. Hence, (3.4) takes the form:

$$\sigma(z) = \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(z) + c_0\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}. \tag{3.5}$$

Inserting (3.5) in the condition:

$$\int_{a_1}^{b_1}\overline{\psi}'_*(s)\sigma(s)ds = \sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\sigma(s)ds + \sum_{j=1}^{m}\theta_j\sigma(\zeta_j),$$

we get

$$\begin{aligned}&\int_{a_1}^{b_1}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds + c_0\int_{a_1}^{b_1}\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))^{\gamma_1-1}\overline{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}ds\\ =\ &\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds + c_0\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(s)-\overline{\psi}_*(a_1))^{\gamma_1-1}\overline{\psi}'_*(s)}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}ds\\ +\ &\sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(\zeta_j) + c_0\sum_{j=1}^{m}\theta_j\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)},\end{aligned}$$

which, together with notation (3.1), yields

$$c_0 = \frac{1}{\mathbb{L}}\bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds + \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\hat{h}_0(s)ds\bigg\}.$$

Substituting the above value of c_0 in (3.5) leads to the solution (3.3). The converse of the lemma can be established by direct computation. □

Denote by

$$\mathbb{X} = C([a_1, b_1], \mathbb{R})$$

the Banach space of all continuous functions from $[a_1, b_1]$ to $\mathbb{R}$ endowed with the norm

$$\|\sigma\| := \max_{z\in[a_1,b_1]} |\sigma(z)|.$$

In view of Lemma 3.1, we define an operator $\mathbb{S}$: $\mathbb{X} \to \mathbb{X}$ as

$$\begin{aligned}
\mathbb{S}(\sigma)(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(z,\sigma(z)) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(s,\sigma(s))ds + \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(\zeta_j,\sigma(\zeta_j)) \\
&- \int_{a_1}^{b_1} \overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(s,\sigma(s))ds \Bigg\},\ z \in [a_1, b_1]. \qquad (3.6)
\end{aligned}$$

For convenience, in the sequel, the following notation is used:

$$\begin{aligned}
\Omega &= \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho + 1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{ \sum_{i=1}^{n} |\varphi_i| \frac{\left[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho + 2)} \\
&+ \sum_{j=1}^{m} |\theta_j| \frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho + 1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho + 2)} \Bigg\}. \qquad (3.7)
\end{aligned}$$

3.1. Uniqueness result

Here, we establish the existence of a unique solution for the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer generalized proportional fractional boundary value problem (1.1) by using Banach's fixed point theorem [19].

Theorem 3.1. *Assume that:*

(H_1) $|\Psi(z,\sigma_1) - \Psi(z,\sigma_2)| \leq \Lambda_0|\sigma_1 - \sigma_2|$ *for some constant* $\Lambda_0 > 0$ *for all* $z \in [a_1, b_1]$ *and* $\sigma_i \in \mathbb{R},\ i = 1, 2$.

If $\Lambda_0\Omega < 1$, *where* Ω *is given by (3.7), then the nonlinear nonlocal integro-multistrip-multipoint* $\overline{\psi}_*$*-Hilfer fractional proportional boundary value problem (1.1) has a unique solution on* $[a_1, b_1]$.

Proof. Let

$$\Psi_0 = \max_{z\in[a_1,b_1]} |\Psi(z, 0)| < \infty.$$

Then, by (H_1), we have

$$|\Psi(z,\sigma(z))| \leq \Lambda_0|\sigma(z)| + |\Psi(z, 0)| \leq \Lambda_0\|\sigma\| + \Psi_0. \qquad (3.8)$$

We give the proof in two steps.

Step I: Consider

$$\mathbb{B}_{\hat{r}} = \{\sigma \in \mathbb{X} : \|\sigma\| < \hat{r}\}$$

with $\hat{r} \geq \Psi_0\Omega/(1-\Lambda_0\Omega)$. Then, we show that $\mathbb{S}(\mathbb{B}_{\hat{r}}) \subset \mathbb{B}_{\hat{r}}$. For $\sigma \in \mathbb{B}_{\hat{r}}$ and using

$$0 < e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(\cdot)-\overline{\psi}_*(\cdot))} \leq 1,$$

we get

$$\begin{aligned}
|\mathbb{S}(\sigma)(z)| &\leq \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(z,\sigma(z))| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds \\
&+ \sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\xi_j,\sigma(\zeta_j))| + \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds\Bigg\} \\
&\leq \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}(\Lambda_0\|\sigma\|+\Psi_0) + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\
&+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}(\Lambda_0\|\sigma\|+\Psi_0) \\
&\leq \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\
&+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\Bigg](\Lambda_0\hat{r}+\Psi_0) \\
&= \Omega(\Lambda_0\hat{r}+\Psi_0) \leq \hat{r}.
\end{aligned}$$

Consequently,

$$\|\mathbb{S}(\sigma)\| = \max_{z\in[a_1,b_1]}|\mathbb{S}(\sigma)(z)| \leq \hat{r},$$

which implies that $\mathbb{S}(\mathbb{B}_{\hat{r}}) \subset \mathbb{B}_{\hat{r}}$.

Step II: We show that the operator $\mathbb{S}$ is a contraction. For $\sigma_1, \sigma_2 \in \mathbb{X}$ and for any $z \in [a_1, b_1]$, we have

$$\begin{aligned}
|\mathbb{S}(\sigma_2)(z)-\mathbb{S}(\sigma_1)(z)| &\leq \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(z,\sigma_2(z))-\Psi(z,\sigma_1(z))| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma_2(s))-\Psi(s,\sigma_1(s))|ds \\
&+ \sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,\sigma_2(\zeta_j))-\Psi(\zeta_j,\sigma_1(\zeta_j))|
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{a_1}^{b_1} \overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma_2(s)) - \Psi(s,(s,\sigma_1(s))|ds\Bigg\} \\
&\leq \left[\frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\right. \\
&\times \left\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)}\right. \\
&\left.\left. + \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\right\}\right]\Lambda_0\|\sigma_2 - \sigma_1\| \\
&= \Lambda_0\Omega\|\sigma_2 - \sigma_1\|.
\end{aligned}
$$

Consequently,

$$
\|\mathbb{S}(\sigma_2) - \mathbb{S}(\sigma_1)\| = \max_{z\in[a_1,b_1]}|\mathbb{S}(\sigma_2)(z) - \mathbb{S}(\sigma_1)(z)| \leq \Lambda_0\Omega\|\sigma_2 - \sigma_1\|,
$$

which, by the assumption $\Lambda_0\Omega < 1$, shows that the operator $\mathbb{S}$ is a contraction. Hence, by Banach's contraction mapping principle, the operator $\mathbb{S}$ has a unique fixed point. Therefore, the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) has a unique solution on $[a_1, b_1]$. □

3.2. Existence results

Here we present three existence results which are proved with the aid of Krasnosel'skiĭ's fixed point theorem [20], Schaefer's fixed point theorem [21] and Leray-Schauder nonlinear alternative [22].

Theorem 3.2. *Suppose that the continuous function $\overline{\psi}_*$: $[a_1, b_1] \times \mathbb{R} \to \mathbb{R}$ satisfies (H_1). In addition, we suppose that*

(H_2) $|\Psi(z,\sigma)| \leq \phi_*(z)$, *for each* $(z,\sigma) \in [a_1, b_1] \times \mathbb{R}$, *and* $\phi_* \in C([a_1, b_1], \mathbb{R}^+)$.

Then, the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) has at least one solution on $[a_1, b_1]$, provided that

$$
\begin{aligned}
\Omega_1\Lambda_0 &= \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\left\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)}\right. \\
&\left. + \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\right\}\Lambda_0 < 1. \qquad (3.9)
\end{aligned}
$$

Proof. For $r_0 \geq \Omega\|\phi_*\|$ with

$$
\|\phi_*\| = \sup_{z\in[a_1,b_1]}|\phi_*(z)|,
$$

we consider a ball

$$
\mathbb{B}_{r_0} = \{\sigma \in C([a_1, b_1], \mathbb{R}) : \|\sigma\| \leq r_0\}.
$$

The operator $\mathbb{S}$ given by (3.6) can be decomposed as $\mathbb{S} = \mathbb{S}_1 + \mathbb{S}_2$, where

$$
\begin{aligned}
(\mathbb{S}_1\sigma)(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(z,\sigma(z)), \quad z \in [a_1, b_1], \\
(\mathbb{S}_2\sigma)(z) &= \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(s,\sigma(s))ds \\
&+ \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(\zeta_j,\sigma(\zeta_j)) - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}\Psi(s,\sigma(s))ds\Bigg\} \quad z \in [a_1, b_1].
\end{aligned}
$$

For any $\sigma, y \in \mathbb{B}_{r_0}$, we have

$$
\begin{aligned}
|(\mathbb{S}_1\sigma)(z) &+ (\mathbb{S}_2 y)(z)| \\
&\leq \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(b,\sigma(b))| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,y(s))|ds \\
&+ \sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,y(\zeta_j))| + \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,y(s))|ds\Bigg\} \\
&\leq \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\Big[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\Big]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\
&+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\Bigg]\|\phi_*\| \\
&= \Omega\|\phi_*\| \leq r_0,
\end{aligned}
$$

where Ω is given by (3.7). Therefore, $\|\mathbb{S}_1\sigma + \mathbb{S}_2 y\| \leq r_0$, which shows that $\mathbb{S}_1\sigma + \mathbb{S}_2 y \in \mathbb{B}_{r_0}$. As in the proof of Theorem 3.1, it can be shown by using the condition (3.9) that the operator $\mathbb{S}_2$ is a contraction mapping. Since $\overline{\psi}_*$ is continuous, the operator $\mathbb{S}_1$ is continuous. Moreover, $\mathbb{S}_1$ is uniformly bounded on $\mathbb{B}_{r_0}$, since

$$
\|\mathbb{S}_1\sigma\| \leq \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}\|\phi_*\|.
$$

In the final step we will prove that the operator $\mathbb{S}_1$ is completely continuous. For $z_1, z_2 \in [a_1, b_1]$, with $z_1 < z_2$, we obtain

$$
\begin{aligned}
|\mathbb{S}_1\sigma(z_2) - \mathbb{S}_1\sigma(z_1)| &\leq \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\Bigg|\int_{a_1+}^{t_2} e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))}(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds \\
&- \int_{a_1+}^{t_1} e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z_1)-\overline{\psi}_*(s))}(\overline{\psi}_*(z_1)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds\Bigg| \\
&\leq \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\Bigg|\int_{a_1+}^{t_1}\Big[(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(s))^{\rho-1}\Big]\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\left|\int_{t_1}^{t_2}(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds\right| \\
&\leq \quad \frac{\|\phi_*\|}{\vartheta_*^{\rho}\Gamma(\rho+1)}\Big[2(\overline{\psi}_*(z_2)-\overline{\psi}_*(z_1))^{\rho}+|(\overline{\psi}_*(z_2)-\overline{\psi}_*(a_1))^{\rho}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(a_1))^{\rho}|\Big],
\end{aligned}
$$

which tends to zero independently of $\sigma \in \mathbb{B}_{r_0}$ when $z_1 \to z_2$. Consequently, we deduce that $\mathbb{S}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, it is compact on $\mathbb{B}_{r_0}$. Thus, the hypotheses of Krasnosel'skiĭ's fixed point theorem [20] are verified, and hence, its conclusion implies that there exists at least one solution for the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) on $[a_1, b_1]$. □

Theorem 3.3. *Assume that the continuous function $\overline{\psi}_*$: $[a_1, b_1]\times\mathbb{R} \to \mathbb{R}$ is bounded, i.e., $|\Psi(z,u)| \leq \mathfrak{M}$ for all $z \in [a_1, b_1], u \in \mathbb{R},\ \mathfrak{M} > 0$.*

Then, there exists at least one solution for the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) on $[a_1, b_1]$.

Proof. We will give the proof in two steps. In the first step, we establish the complete continuity of the operator $\mathbb{S}$: $\mathbb{X} \to \mathbb{X}$ given by (3.6). For the continuity of $\mathbb{S}$, let $\{\sigma_n\}$ be a sequence such that $\sigma_n \to \sigma$ in $\mathbb{X}$. Then, for each $z \in [a_1, b_1]$, we get

$$
\begin{aligned}
|\mathbb{S}(\sigma_n)(z)-\mathbb{S}(\sigma)(z)| \leq\ & \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(z,\sigma_n(z))-\Psi(z,\sigma(z))| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\
\times\ & \left\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma_n(s))-\Psi(s,\sigma(s))|ds\right. \\
+\ & \sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,\sigma_n(\zeta_j))-\Psi(\zeta_j,\sigma(\zeta_j))| \\
+\ & \left.\int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma_n(s))-\Psi(s,\sigma(s))|ds\right\}.
\end{aligned}
$$

Since $|\Psi(s,\sigma_n(s))-\Psi(s,\sigma(s))| \to 0\ as\ \sigma_n \to \sigma$, as $\overline{\psi}_*$ is continuous, we have

$$
\|\mathbb{S}(\sigma_n)-\mathbb{S}(\sigma)\| \to 0\ as\ \sigma_n \to \sigma,
$$

which proves that $\mathbb{S}$ is continuous.

Now, we show that $\mathbb{S}$ transforms bounded sets into bounded sets in $\mathbb{X}$. For $r_0 > 0$, let

$$
\mathbb{B}_{r_0} = \{\sigma \in \mathbb{X} : \|\sigma\| \leq r_0\}.
$$

Then, for $z \in [a_1, b_1]$, we have

$$
\begin{aligned}
|\mathbb{S}(\sigma)(z)| \leq\ & \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(z,\sigma(z))| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\left\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds\right. \\
+\ & \left.\sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,\sigma(\zeta_j))| + \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}M+\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)}\\
&+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}+\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\mathfrak{M},
\end{aligned}
$$

which leads to $\|\mathbb{S}(\sigma)\| \leq \Omega\mathfrak{M}$.

Finally, we show that $\mathbb{S}$ transforms bounded sets into equicontinuous sets. For $z_1, z_2 \in [a_1, b_1]$, $z_1 < z_2$ and $\sigma \in \mathbb{B}_{r_0}$, we obtain

$$
\begin{aligned}
|\mathbb{S}(\sigma)(z_2) &- \mathbb{S}(\sigma)(z_1)|\\
&\leq \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\left|\int_{a_1+}^{t_1}\left[(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(s))^{\rho-1}\right]\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds\right|\\
&+ \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\left|\int_{t_1}^{t_2}(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}_*'(s)\Psi(s,\sigma(s))ds\right|\\
&+ \frac{(\overline{\psi}_*(z_2)-\overline{\psi}_*(a_1))^{\gamma_1-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}|\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds\\
&+ \sum_{j=1}^{m}|\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,\sigma(\zeta_j))|+\int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds\Bigg\}\\
&\leq \frac{\mathfrak{M}}{\vartheta_*^{\rho}\Gamma(\rho+1)}\left[2(\overline{\psi}_*(z_2-\overline{\psi}_*(z_1))^{\rho}+|(\overline{\psi}_*(z_2)-\overline{\psi}_*(a_1))^{\rho}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(a_1))^{\rho}|\right]\\
&+ \frac{(\overline{\psi}_*(z_2)-\overline{\psi}_*(a_1))^{\gamma_1-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\\
&\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)}\\
&+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}+\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\mathfrak{M},
\end{aligned}
$$

which tends to zero, independently of $\sigma \in \mathbb{B}_{r_0}$, as $z_1 \to z_2$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathbb{S}\colon \mathbb{X} \to \mathbb{X}$ is completely continuous.

In the second step, it will be established that the set

$$
\mathcal{E} = \{\sigma \in \mathbb{X} \mid \sigma = \nu\mathbb{S}(\sigma), 0 \leq \nu \leq 1\}
$$

is bounded. Let $\sigma \in \mathcal{E}$, and then $\sigma = \nu\mathbb{S}(\sigma)$. For any $z \in [a_1, b_1]$, we have $\sigma(z) = \nu\mathbb{S}(\sigma)(z)$. As in first step, one can find that

$$
|\sigma(z)| \leq \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)}+\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}
$$

$$\times \left\{ \sum_{i=1}^{n} |\varphi_i| \frac{\left[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1} \right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} + \sum_{j=1}^{m} |\theta_j| \frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)} \right\} \Bigg] \mathfrak{M}.$$

Therefore,

$$\|\sigma\| \leq \Omega\mathfrak{M},$$

and consequently the set $\mathcal{E}$ is bounded. Hence, by Schaefer's fixed point theorem [21], the operator $\mathbb{S}$ has at least one fixed point which is a solution for the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) on $[a_1, b_1]$. This completes the proof. □

Theorem 3.4. *Suppose that the following conditions hold:*

(H_3) $|\Psi(z,\sigma)| \leq \mathfrak{p}(z)Y(\|\sigma\|)$ *for each* $(z,\sigma) \in [a_1,b_1] \times \mathbb{R}$, *where* $\mathfrak{p} \in C([a_1,b_1],\mathbb{R}^+)$ *and* $Y\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous nondecreasing function;*

(H_4) $\dfrac{\mathfrak{K}}{\Omega\|\mathfrak{p}\|Y(\mathfrak{K})} > 1$, *for a constant* $\mathfrak{K} > 0$, *where* Ω *is defined by (3.7).*

Then, the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$*-Hilfer fractional proportional boundary value problem (1.1) has at least one solution on* $[a_1, b_1]$.

Proof. In Theorem 3.3, it was shown that the operator $\mathbb{S}$ is completely continuous. So, we only need to prove that there exists an open set $U \subseteq C([a_1,b_1],\mathbb{R})$ with $\sigma \neq \mu\mathbb{S}(\sigma)$ for $\mu \in (0,1)$ and $\sigma \in \partial U$.

Let $\sigma \in C([a_1,b_1],\mathbb{R})$ be such that $\sigma = \mu\mathbb{S}(\sigma)$ for some $0 < \mu < 1$. Then, for each $z \in [a_1,b_1]$, we have

$$\begin{aligned}
|\sigma(z)| \leq\ & \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(z,\sigma(z))| + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \left\{ \sum_{i=1}^{n} |\varphi_i| \int_{\xi_i}^{\eta_i} \overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds \right. \\
& \left. + \sum_{j=1}^{m} |\theta_j| \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(\zeta_j,\sigma(\zeta_j))| + \int_{a_1}^{b_1} \overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|\Psi(s,\sigma(s))|ds \right\} \\
\leq\ & \left[\frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \right. \\
& \times \left\{ \sum_{i=1}^{n} |\varphi_i| \frac{\left[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1} \right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \right. \\
& \left. \left. + \sum_{j=1}^{m} |\theta_j| \frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)} \right\} \right] \|\mathfrak{p}\|Y(\|\sigma\|).
\end{aligned}$$

Therefore, we have

$$\frac{\|\sigma\|}{\Omega\|\mathfrak{p}\|Y(\|\sigma\|)} \leq 1,$$

where Ω is given in (3.7). By (H_4), we have that $\|\sigma\| \neq \mathfrak{K}$. Consider the set

$$U = \{\sigma \in C([a_1,b_1],\mathbb{R}) : \|\sigma\| < \mathfrak{K}\}.$$

The operator $\mathbb{S}\colon \overline{U} \to C([a_1, b_1], \mathbb{R})$ is continuous and completely continuous. By the definition of U, we cannot find any $\sigma \in \partial U$ satisfying $\sigma = \mu\mathbb{S}(\sigma)$ for some $\mu \in (0, 1)$. In consequence, by the application of the Leray-Schauder nonlinear alternative [22], we deduce that there exists a fixed point $\sigma \in \overline{U}$ for the operator $\mathbb{S}$, which is a solution of the problem (1.1). This finishes the proof. □

3.3. Illustrative examples: the single-valued case

Consider the following $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem:

$$\begin{cases} \mathbb{D}_{\frac{1}{8}}^{\frac{3}{2},\frac{3}{4},\frac{1}{4},\frac{z+1}{z+2}}\sigma(z) = \Psi(z, \sigma(z)), \quad z \in \left[\frac{1}{8}, \frac{5}{4}\right], \\ \sigma\left(\frac{1}{8}\right) = 0, \\ \displaystyle\int_{\frac{1}{8}}^{\frac{5}{4}} \frac{\sigma(s)}{(s+2)^2} ds = \frac{1}{66}\int_{\frac{3}{4}}^{1} \frac{\sigma(s)}{(s+2)^2} ds + \frac{2}{77}\int_{\frac{7}{8}}^{\frac{9}{8}} \frac{\sigma(s)}{(s+2)^2} ds \\ \qquad\qquad + \frac{3}{55}\sigma\left(\frac{1}{4}\right) + \frac{5}{77}\sigma\left(\frac{3}{8}\right) + \frac{7}{99}\sigma\left(\frac{5}{8}\right), \end{cases} \tag{3.10}$$

where

$$\rho = 3/2, \varphi = 3/4,\ \vartheta_* = 1/4,\ \overline{\psi}_*(z) = (z+1)/(z+2)$$

with

$$\overline{\psi}_*'(z) = 1/(z+2)^2,\ a_1 = 1/8,\ b_1 = 5/4,\ n = 2,\ m = 3,\ \varphi_1 = 1/66,\ \varphi_2 = 2/77,\ \eta_1 = 1,$$

$$\eta_2 = 9/8,\ \xi_1 = 3/4,\ \xi_2 = 7/8,\ \theta_1 = 3/55,\ \theta_2 = 5/77,\ \theta_3 = 7/99,\ \zeta_1 = 1/4,\ \zeta_2 = 3/8,\ \zeta_3 = 5/8.$$

Using the given data, it is found that

$$\gamma_1 = 15/8,\ \mathbb{L} \approx 0.00072023486,\ \Omega \approx 176.2956671 \text{ and } \Omega_1 \approx 164.531941.$$

(i) Let the nonlinear function $\overline{\psi}_*\colon [1/8, 5/4] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\Psi(z, \sigma) = \frac{1}{8z+1}\left(\frac{|\sigma|}{k + |\sigma|}\right) + g(z), \tag{3.11}$$

where $k > 0$ and $g\colon [1/8, 5/4] \to \mathbb{R}$. Then, we have

$$|\Psi(z, \sigma_1) - \Psi(z, \sigma_2)| \le \frac{1}{2k}|\sigma_1 - \sigma_2|,$$

that is, the Lipschitz condition (H_1) in Theorem 3.1 is satisfied with $\Lambda_0 = 1/2k$. Therefore, by Theorem 3.1, if $k > 88.14783355$, then the boundary value problem (3.10) with the function $\overline{\psi}_*$ given by (3.11) has a unique solution on $[1/8, 5/4]$.

(ii) Let us consider

$$\Psi(z, \sigma) = \frac{1}{8z+1}\left(\frac{|\sigma|}{90 + |\sigma|}\right) + z^3 + 3z + \frac{1}{2}. \tag{3.12}$$

Clearly, the function $\overline{\psi}_*$ in (3.12) satisfies the condition (H_2) in Theorem 3.2 as

$$|\Psi(z, \sigma)| \le \frac{1}{8z+1} + |g(z)| := \phi_*(z).$$

Moreover, $\Omega_1\Lambda_0 \approx 0.91406634 < 1$. Hence, by Theorem 3.2, the boundary value problem (3.10) with the function $\overline{\psi}_*$ given by (3.12) has at least one solution on $[1/8, 5/4]$.

(*iii*) Consider the nonlinear function

$$\Psi(z,\sigma) = \frac{1}{15(8z+2)}\left(\frac{1}{15}\frac{\sigma^{2022}+1}{\sigma^{2022}+3} + \frac{1}{12}\right), \tag{3.13}$$

and note that $|\Psi(z,\sigma)| \leq 1/300$. Clearly, the conclusion of Theorem 3.3 leads to the existence of at least one solution for the boundary value problem (3.10) with the function $\overline{\psi}_*$ given by (3.13) on an interval $[1/8, 5/4]$.

(*iv*) Let the function $\overline{\psi}_*$ be expressed as

$$\Psi(z,\sigma) = \frac{1}{15(8z+2)}\left(\frac{1}{15}\frac{\sigma^{2024}+1}{\sigma^{2022}+3} + \frac{1}{12}\right). \tag{3.14}$$

Observe that the function $\overline{\psi}_*$ does not satisfy the Lipschitz condition, but we can find the quadratic bound as

$$|\Psi(z,\sigma)| \leq \frac{1}{15(8z+2)}\left(\frac{1}{15}\sigma^2 + \frac{1}{12}\right).$$

Choosing

$$\mathfrak{p}(z) = 1/(15(8z+2))$$

and

$$Y(u) = (1/15)u^2 + (1/12),$$

we have $\|\mathfrak{p}\| = 1/45$. Then, there exists a constant

$$\mathfrak{K} \in (0.360396904, 3.468398274),$$

which satisfies the condition (H_4) in Theorem 3.4. Thus, the boundary value problem (3.10) with the function $\overline{\psi}_*$ given by (3.14) has at least one solution on $[1/8, 5/4]$.

4. Multi-valued case

In this section, we study the multi-valued variant of the nonlinear nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional boundary value problem (1.1) given by

$$\begin{cases} \mathbb{D}_{a_1+}^{\rho,\varphi,\vartheta_*,\overline{\psi}_*}\sigma(z) \in \mathbb{H}(z,\sigma(z)), \quad z \in [a_1,b_1], \\ \sigma(a_1) = 0, \\ \displaystyle\int_{a_1}^{b_1} \overline{\psi}_*'(s)\sigma(s)ds = \sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\sigma(s)ds + \sum_{j=1}^{m}\theta_j\sigma(\zeta_j), \end{cases} \tag{4.1}$$

where $\mathbb{H}$: $[a_1,b_1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ denotes the family of all nonempty subsets of $\mathbb{R}$, and the other symbols are the same as defined in the problem (1.1).

Let $(\mathbb{X}, \|\cdot\|)$ be a normed space. We denote, respectively, the classes of all closed, bounded, compact, and compact and convex sets in $\mathbb{X}$ by $\mathcal{P}_{cl}$, $\mathcal{P}_b$, $\mathcal{P}_{cp}$ and $\mathcal{P}_{cp,c}$.

The set of selections of $\mathbb{H}$, for each $\sigma \in C([a_1,b_1],\mathbb{R})$, is defined as

$$S_{\mathbb{H},\sigma} := \{\hat{z} \in L^1([a_1,b_1],\mathbb{R}) : \hat{z}(z) \in \mathbb{H}(z,\sigma(z)) \text{ for a.e. } z \in [a_1,b_1]\}.$$

For details on multi-valued analysis, see [23–25].

4.1. Existence results for the multi-valued problem (4.1)

Definition 4.1. *A function $\sigma \in C([a_1, b_1], \mathbb{R})$ is called a solution of the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1) if there exists a function $v \in L^1([a_1, b_1], \mathbb{R})$ with $v(z) \in \mathbb{H}(z, \sigma)$ almost everywhere (a.e.) on $[a_1, b_1]$ such that*

$$\begin{aligned} \sigma(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Bigg\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \\ &+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \Bigg\}. \end{aligned}$$

4.1.1. Case 1: the upper semicontinuous case

Applying the nonlinear alternative for Kakutani maps [22] together with a closed graph operator theorem [26], we establish an existence result for the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1).

Theorem 4.1. *Let the following assumptions be satisfied:*

(A_1) *The multifunction $\mathbb{H}$: $[a_1, b_1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(A_2) $\|\mathbb{H}(z, \sigma)\|_{\mathcal{P}} := \sup\{|z| : z \in \mathbb{H}(z, \sigma)\} \leq \mathfrak{q}(z)\mathfrak{g}(\|\sigma\|)$ *for each* $(z, \sigma) \in [a_1, b_1] \times \mathbb{R}$, *where* $\mathfrak{g} \in C([a_1, b_1], \mathbb{R}^+)$ *is a nondecreasing function, and* $\mathfrak{q} : [a_1, b_1] \to \mathbb{R}^+$ *is a continuous function;*

(A_3) $\dfrac{M}{\mathfrak{g}(M)\|\mathfrak{q}\|\Omega} > 1$, *for a positive number M, where Ω is given by (3.7).*

Then, the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1) has at least one solution on $[a_1, b_1]$.

Proof. Let us introduce a multi-valued operator $\mathbb{W}$: $C([a_1, b_1], \mathbb{R}) \to \mathcal{P}(C([a_1, b_1], \mathbb{R}))$ as

$$\mathbb{W}(\sigma) = \left\{ \begin{array}{l} \varpi \in C([a_1, b_1], \mathbb{R}) : \\ \varpi(z) = \left\{ \begin{array}{l} \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(z) + \dfrac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\ \times \Bigg\{ \displaystyle\sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \\ + \displaystyle\sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \Bigg\}, \quad v \in S_{\mathbb{H},\sigma}. \end{array} \right. \end{array} \right\}$$

It will be verified through several steps that the operator $\mathbb{W}$ satisfies the hypotheses of the Leray-Schauder nonlinear alternative for Kakutani maps [22].

Step 1. $\mathbb{W}$ *transforms bounded sets into bounded sets of* $C([a_1, b_1], \mathbb{R})$.

For $\varepsilon > 0$, we consider a bounded set

$$\mathfrak{B}_\varepsilon = \{\sigma \in C([a_1, b_1], \mathbb{R}) : \|\sigma\| \leq \varepsilon\}$$

in $C([a_1,b_1],\mathbb{R})$. For each $\varpi \in \mathbb{W}(\sigma)$ and $\sigma \in \mathfrak{B}_\varepsilon$, there exists $v \in S_{\mathbb{H},\sigma}$ such that

$$\begin{aligned}\varpi(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^n \varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds \\ &+ \sum_{j=1}^m \theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds\Bigg\}.\end{aligned}$$

For $z \in [a_1,b_1]$, using the assumption (A_2), we obtain

$$\begin{aligned}|\varpi(z)| &\le \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v(z)| + \frac{(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^n |\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v(s)|ds \\ &+ \sum_{j=1}^m |\theta_j|\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v(\zeta_j)| + \int_{a_1}^{b_1}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v(s)|ds\Bigg\} \\ &\le \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\ &\times \Bigg\{\sum_{i=1}^n |\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\ &+ \sum_{j=1}^m |\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\Bigg]\|\mathfrak{q}\|\mathfrak{g}(\|\sigma\|),\end{aligned}$$

which leads to

$$\|\varpi\| \le \|\mathfrak{q}\|\mathfrak{g}(\varepsilon)\Omega.$$

Step 2. $\mathbb{W}$ *maps bounded sets into equicontinuous sets of* $C([a_1,b_1],\mathbb{R})$.

Let $\sigma \in \mathfrak{B}_\varepsilon$ and $\varpi \in \mathbb{W}(\sigma)$. Then, there exists $v \in S_{\mathbb{H},\sigma}$ such that

$$\begin{aligned}\varpi(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^n \varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds \\ &+ \sum_{j=1}^m \theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds\Bigg\}.\end{aligned}$$

Let $z_1,\ z_2 \in [a_1,b_1],\ z_1 < z_2$. Then,

$$\begin{aligned}|\varpi(z_2)-\varpi(z_1)| &\le \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\left|\int_{a_1+}^{t_1}\left[(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(s))^{\rho-1}\right]\overline{\psi}'_*(s)v(s)ds\right| \\ &+ \frac{1}{\vartheta_*^{\rho}\Gamma(\rho)}\left|\int_{t_1}^{t_2}(\overline{\psi}_*(z_2)-\overline{\psi}_*(s))^{\rho-1}\overline{\psi}'_*(s)v(s)ds\right| \\ &+ \frac{(\overline{\psi}_*(z_2)-\overline{\psi}_*(a_1))^{\gamma_1-1}-(\overline{\psi}_*(z_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^n |\varphi_i|\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v(s)|ds\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{m} |\theta_j| \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} |v(\zeta_j)| + \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} |v(s)| ds \Big\} \\
&\leq \frac{\|\mathfrak{q}\| \mathfrak{g}(\varepsilon)}{\vartheta_*^{\rho}\Gamma(\rho+1)} \Big[2(\overline{\psi}_*(z_2 - \overline{\psi}_*(z_1))^{\rho} + |(\overline{\psi}_*(z_2) - \overline{\psi}_*(a_1))^{\rho} - (\overline{\psi}_*(z_1) - \overline{\psi}_*(a_1))^{\rho}| \Big] \\
&+ \frac{(\overline{\psi}_*(z_2) - \overline{\psi}_*(a_1))^{\gamma_1 - 1} - (\overline{\psi}_*(z_1) - \overline{\psi}_*(a_1))^{\gamma_1 - 1}}{|\mathbb{L}| \vartheta_*^{\gamma_1 - 1}\Gamma(\gamma_1)} \\
&\times \Bigg\{ \sum_{i=1}^{n} |\varphi_i| \frac{\Big[(\overline{\psi}_*(\eta_i) - \overline{\psi}_*(a_1))^{\rho+1} - (\overline{\psi}_*(\xi_i) - \overline{\psi}_*(a_1))^{\rho+1} \Big]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\
&+ \sum_{j=1}^{m} |\theta_j| \frac{(\overline{\psi}_*(\zeta_j) - \overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1) - \overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)} \Bigg\} \|\mathfrak{q}\| \mathfrak{g}(\varepsilon) \to 0,
\end{aligned}
$$

as $z_1 \to z_2$ independently of $\sigma \in \mathfrak{B}_{\varepsilon}$. Hence, $\mathbb{W}$: $C([a_1, b_1], \mathbb{R}) \to \mathcal{P}(C([a_1, b_1], \mathbb{R}))$ is completely continuous by virtue of the Arzelá-Ascoli theorem.

Step 3. *For each* $\sigma \in C([a_1, b_1], \mathbb{R})$, $\mathbb{W}(\sigma)$ *is convex.*

Since F has convex values, $S_{\mathbb{H},\sigma}$ is convex.

Step 4. $\mathbb{W}$ *has a closed graph.*

Let $\sigma_n \to \sigma_*$, $\varpi_n \in \mathbb{W}(\sigma_n)$ and $\varpi_n \to \varpi_*$. Then, we show that $\varpi_* \in \mathbb{W}(\sigma_*)$. Observe that $\varpi_n \in \mathbb{W}(\sigma_n)$ implies that there exists $v_n \in S_{\mathbb{H},\sigma_n}$ such that, for each $z \in [a_1, b_1]$, we have

$$
\begin{aligned}
\varpi_n(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Big\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(s) ds \\
&+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(s) ds \Big\}.
\end{aligned}
$$

For each $z \in [a_1, b_1]$, we must have $v_* \in S_{\mathbb{H},\sigma_*}$ such that

$$
\begin{aligned}
\varpi_*(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_*(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Big\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_*(s) ds \\
&+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_*(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_*(s) ds \Big\}.
\end{aligned}
$$

Consider a continuous linear operator Φ: $L^1([a_1, b_1], \mathbb{R}) \to C([a_1, b_1], \mathbb{R})$ as

$$
\begin{aligned}
v \to \Phi(v)(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Big\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \\
&+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \Big\}.
\end{aligned}
$$

Clearly, $\|\varpi_n - \varpi_*\| \to 0$ as $n \to \infty$, and consequently, by the closed graph operator theorem [26], $\Phi \circ S_{\mathbb{H},\sigma}$ is a closed graph operator. Also, we have $\varpi_n \in \Phi(S_{\mathbb{H},\sigma_n})$ and

$$\begin{aligned}\varpi_*(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v_*(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v_*(s)ds \\ &+ \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v_*(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v_*(s)ds\Bigg\},\end{aligned}$$

for some $v_* \in S_{\mathbb{H},\sigma_*}$. Thus, $\mathbb{W}$ has a closed graph. By [23, Proposition 1.2], that is, if a completely continuous operator has a closed graph, then it is upper semicontinuous, we deduce that the operator $\mathbb{W}$ is upper semicontinuous.

Step 5. *There exists an open set $U \subseteq C([a_1,b_1],\mathbb{R})$, such that, for any $\kappa \in (0,1)$ and all $\sigma \in \partial U$, $\sigma \notin \kappa\mathbb{W}(\sigma)$.*

Let $\sigma \in \kappa\mathbb{W}(\sigma)$, $\kappa \in (0,1)$. Then, there exists $v \in L^1([a_1,b_1],\mathbb{R})$ with $v \in S_{\mathbb{H},\sigma}$ such that, for $z \in [a_1,b_1]$, we have

$$\begin{aligned}\sigma(z) &= \kappa\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(z) + \kappa\frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds \\ &+ \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}v(s)ds\Bigg\}.\end{aligned}$$

As in Step 1, for each $z \in [a_1,b_1]$, we have

$$\begin{aligned}|\sigma(z)| &\le \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\ &\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\left[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\right]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\ &+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\Bigg]\|\mathfrak{q}\|\mathfrak{g}(\|\sigma\|) \\ &= \|\mathfrak{q}\|\mathfrak{g}(\|\sigma\|)\Omega.\end{aligned}$$

Thus, we have

$$\frac{\|\sigma\|}{\mathfrak{g}(\|\sigma\|)\|\mathfrak{q}\|\Omega} \le 1.$$

By (A_3), $\|\sigma\| \neq M$ for some M. We define a set

$$\Theta = \{\sigma \in C([a_1,b_1],\mathbb{R}) : \|\sigma\| < M\}.$$

Obviously $\mathbb{W}\colon \overline{\Theta} \to \mathcal{P}(C([a_1,b_1],\mathbb{R}))$ is a compact, convex valued and upper semicontinuous multi-valued map. By the definition of Θ, we cannot find any $\sigma \in \partial\Theta$ for some $\kappa \in (0,1)$ satisfying $\sigma \in \kappa\mathbb{W}(\sigma)$. Therefore, by the Leray-Schauder nonlinear alternative for Kakutani maps [22], the operator $\mathbb{W}$ has a fixed point $\sigma \in \overline{\Theta}$. So, the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1) has at least one solution on $[a_1,b_1]$. □

4.1.2. Case 2: the Lipschitz case

Here, we discuss the existence of solutions for the integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1) with a possible non-convex valued multi-valued map. The main tool of our study in this case is a fixed point theorem for contractive multivalued maps due to Covitz and Nadler [27].

Definition 4.2. *([28]) Let* $(\mathbb{X}, d)$ *be a metric space induced from the normed space* $(\mathbb{X}, \|\cdot\|)$ *and*

$$H_{\bar{d}} : \mathcal{P}(\mathbb{X}) \times \mathcal{P}(\mathbb{X}) \to \mathbb{R} \cup \{\infty\}$$

be defined by

$$H_{\bar{d}}(A, B) = \max\{\sup_{c\in A} \bar{d}(c, d), \sup_{d\in B} \bar{d}(c, d)\},$$

where $\bar{d}(c, d) = \inf_{c\in A} \bar{d}(c, d)$ *and* $\bar{d}(c, d) = \inf_{d\in B} \bar{d}(c, d)$*.*

Theorem 4.2. *Assume that*

(B_1) $\mathbb{H}$*:* $[a_1, b_1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $\mathbb{H}(\cdot, \sigma)$*:* $[a_1, b_1] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $\sigma \in \mathbb{R}$*, where* $\mathcal{P}_{cp}(\mathbb{R})$ *denotes a class of compact sets in* $\mathbb{R}$*;*

(B_2) $H_{\bar{d}}(\mathbb{H}(z, \sigma), \mathbb{H}(z, \bar{\sigma})) \leq \varrho(z)|\sigma - \bar{\sigma}|$ *for almost all* $z \in [a_1, b_1]$ *and* $\sigma, \bar{\sigma} \in \mathbb{R}$ *with* $\varrho \in C([a_1, b_1], \mathbb{R}^+)$ *and* $\bar{d}(0, \mathbb{H}(z, 0)) \leq \varrho(z)$ *for almost all* $z \in [a_1, b_1]$*.*

Then, the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$*-Hilfer fractional proportional inclusion boundary value problem (4.1) has at least one solution on* $[a_1, b_1]$ *if*

$$\Omega\|\varrho\| < 1,$$

where Ω *is given by (3.7).*

Proof. Consider the operator

$$\mathbb{W} : C([a_1, b_1], \mathbb{R}) \to \mathcal{P}(C([a_1, b_1], \mathbb{R})),$$

defined by (4.2). We complete the proof in two steps.

Step I. $\mathbb{W}$ is nonempty and closed for every $v \in S_{\mathbb{H},\sigma}$.

By the measurable selection theorem ([29, Theorem III. 6]), the set-valued map $\mathbb{H}(\cdot, \sigma(\cdot))$ is measurable and admits a measurable selection v: $[a_1, b_1] \to \mathbb{R}$. By the assumption ($B_2$), we get

$$|v(z)| \leq \varrho(z)(1 + |\sigma(z)|),$$

that is, $v \in L^1([a_1, b_1], \mathbb{R})$, and hence, $\mathbb{H}$ is integrably bounded. Therefore, $S_{\mathbb{H},\sigma} \neq \emptyset$.

For each $\sigma \in C([a_1, b_1], \mathbb{R})$, we verify that

$$\mathbb{W}(\sigma) \in \mathcal{P}_{cl}(C([a_1, b_1],\ \mathbb{R})).$$

Let

$$\{u_n\}_{n\geq 0} \in \mathbb{W}(\sigma) \text{ with } u_n \to u \ (n \to \infty) \text{ in } \ C([a_1, b_1], \mathbb{R}).$$

Then, $u \in C([a_1, b_1], \mathbb{R})$, and we can find $v_n \in S_{\mathbb{H},\sigma_n}$ such that, for each $z \in [a_1, b_1]$,

$$\begin{aligned} u_n(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Bigg\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(s) ds \\ &+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_n(s) ds \Bigg\}. \end{aligned}$$

Since $\mathbb{H}$ has compact values, we can obtain a subsequence (if necessary) v_n converging to v in $L^1([a_1, b_1], \mathbb{R})$. Thus, $v \in S_{\mathbb{H},\sigma}$, and for each $z \in [a_1, b_1]$, we have

$$\begin{aligned} \varpi_n(z) \to v(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Bigg\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \\ &+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v(s) ds \Bigg\}. \end{aligned}$$

Thus, $u \in \mathbb{W}(\sigma)$.

Step II. Here, we establish that there exists $0 < \overline{m_0} < 1$ ($\overline{m_0} = \Omega\|\varrho\|$) such that

$$H_{\bar{d}}(\mathbb{W}(\sigma), \mathbb{W}(\bar{\sigma})) \leq \overline{m_0}\|\sigma - \bar{\sigma}\| \ \text{ for each } \ \sigma, \bar{\sigma} \in C([a_1, b_1], \mathbb{R}).$$

Let $\sigma, \bar{\sigma} \in C([a_1, b_1], \mathbb{R})$ and $\varpi_1 \in \mathbb{W}(\sigma)$. Then, there exists $v_1(z) \in \mathbb{H}(z, \sigma(z))$ such that, for each $z \in [a_1, b_1]$,

$$\begin{aligned} \varpi_1(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_1(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z) - \overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \Bigg\{ \sum_{i=1}^{n} \varphi_i \int_{\xi_i}^{\eta_i} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_1(s) ds \\ &+ \sum_{j=1}^{m} \theta_j \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_1(\zeta_j) - \int_{a_1}^{b_1} \overline{\psi}'_*(s) \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_1(s) ds \Bigg\}. \end{aligned}$$

By (B_2), we have

$$H_{\bar{d}}(\mathbb{H}(z, \sigma), \mathbb{H}(z, \bar{\sigma})) \leq \varrho(z)|\sigma(z) - \bar{\sigma}(z)|.$$

So, there exists $z \in \mathbb{H}(z, \bar{w}(z))$ such that

$$|v_1(z) - z| \leq \varrho(z)|\sigma(z) - \bar{\sigma}(z)|, \quad z \in [a_1, b_1].$$

Let us define $\mathcal{V}$: $[a_1, b_1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(z) = \{z \in \mathbb{R} : |v_1(z) - z| \leq \varrho(z)|\sigma(z) - \bar{\sigma}(z)|\}.$$

Since the multivalued operator $\mathcal{V}(z) \cap \mathbb{H}(z, \bar{\sigma}(z))$ is measurable by Proposition III.4 in [29], there exists a function $v_2(z)$ which is a measurable selection of $\mathcal{V}$. Thus, $v_2(z) \in \mathbb{H}(z, \bar{\sigma}(z))$, and for each $z \in [a_1, b_1]$, we have

$$|v_1(z) - v_2(z)| \leq \varrho(z)|\sigma(z) - \bar{\sigma}(z)|.$$

Therefore, for each $z \in [a_1, b_1]$, we get

$$\begin{aligned}\varpi_2(z) &= \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_2(z) + \frac{e^{\frac{\vartheta_*-1}{\vartheta_*}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))}(\overline{\psi}_*(z)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{\mathbb{L}\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_2(s)ds \\ &\quad + \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_2(\zeta_j) - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*} v_2(s)ds\Bigg\}.\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}|\varpi_1(z) &- \varpi_2(z)| \\ &\le \mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v_2(z)-v_1(z)| + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg\{\sum_{i=1}^{n}\varphi_i\int_{\xi_i}^{\eta_i}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v_2(s)-v_1(s)|ds \\ &+ \sum_{j=1}^{m}\theta_j\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v_2(\zeta_j)-v_1(\zeta_j)| - \int_{a_1}^{b_1}\overline{\psi}_*'(s)\mathbb{I}_{a_1+}^{\rho,\vartheta_*,\overline{\psi}_*}|v_2(s)-v_1(s)|ds\Bigg\} \\ &\le \Bigg[\frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\gamma_1-1}}{|\mathbb{L}|\vartheta_*^{\gamma_1-1}\Gamma(\gamma_1)} \\ &\times \Bigg\{\sum_{i=1}^{n}|\varphi_i|\frac{\Big[(\overline{\psi}_*(\eta_i)-\overline{\psi}_*(a_1))^{\rho+1}-(\overline{\psi}_*(\xi_i)-\overline{\psi}_*(a_1))^{\rho+1}\Big]}{\vartheta_*^{\rho}\Gamma(\rho+2)} \\ &+ \sum_{j=1}^{m}|\theta_j|\frac{(\overline{\psi}_*(\zeta_j)-\overline{\psi}_*(a_1))^{\rho}}{\vartheta_*^{\rho}\Gamma(\rho+1)} + \frac{(\overline{\psi}_*(b_1)-\overline{\psi}_*(a_1))^{\rho+1}}{\vartheta_*^{\rho}\Gamma(\rho+2)}\Bigg\}\Bigg]\|\varrho\|\|\sigma-\bar{\sigma}\|,\end{aligned}$$

which yields

$$\|\varpi_1-\varpi_2\| \le \Omega\|\varrho\|\|\sigma-\bar{\sigma}\|.$$

On switching the roles of σ and $\bar{\sigma}$, we have

$$H_{\bar{d}}(\mathbb{W}(\sigma),\mathbb{W}(\bar{\sigma})) \le \Omega\|\varrho\|\|\sigma-\bar{\sigma}\|,$$

which verifies that $\mathbb{W}$ is a contraction. Hence, it follows by Covitz-Nadler's fixed point theorem [27] that the operator $\mathbb{W}$ has a fixed point σ, which is indeed a solution of the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.1). □

4.2. Illustrative examples: the multi-valued case

Let us consider the $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem:

$$\begin{cases} \mathbb{D}_{\frac{1}{8}}^{\frac{3}{2},\frac{3}{4},\frac{1}{4},\frac{z+1}{z+2}}\sigma(z) \in \mathbb{H}(z,\sigma(z)), \quad z \in \left[\frac{1}{8},\frac{5}{4}\right], \\ \sigma\left(\frac{1}{8}\right) = 0, \\ \displaystyle\int_{\frac{1}{8}}^{\frac{5}{4}}\frac{\sigma(s)}{(s+2)^2}ds = \frac{1}{66}\int_{\frac{3}{4}}^{1}\frac{\sigma(s)}{(s+2)^2}ds + \frac{2}{77}\int_{\frac{7}{8}}^{\frac{9}{8}}\frac{\sigma(s)}{(s+2)^2}ds \\ \qquad + \frac{3}{55}\sigma\left(\frac{1}{4}\right) + \frac{5}{77}\sigma\left(\frac{3}{8}\right) + \frac{7}{99}\sigma\left(\frac{5}{8}\right). \end{cases} \tag{4.2}$$

(i) Let $\mathbb{H}$: $[1/8, 5/4] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be defined by

$$\mathbb{H}(z, \sigma) = \left[0, \frac{1}{2(8z+4)^2}\left\{\frac{1}{16}\left(\frac{\sigma^{174}+1}{\sigma^{172}+2}\right) + \frac{1}{14}\left(\frac{\sigma^{152}+3}{|\sigma|^{151}+4}\right) + \frac{1}{15}\right\}\right]. \tag{4.3}$$

Obviously, the multifunction $\mathbb{H}$ is L^1-Carathéodory. Moreover, we have

$$\|\mathbb{H}(z, \sigma)\|_{\mathcal{P}} \leq \frac{1}{2(8z+4)^2}\left(\frac{1}{16}\sigma^2 + \frac{1}{14}|\sigma| + \frac{1}{15}\right).$$

Choosing

$$\mathfrak{q}(z) = 1/(2(8z+4)^2)$$

and

$$\mathfrak{g}(\sigma) = (1/16)\sigma^2 + (1/14)|\sigma| + (1/15),$$

there exists a constant

$$M \in (0.350343537, 3.044630641)$$

which satisfies the condition (A_3) of Theorem 4.1. Thus, we deduce that the nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional proportional inclusion boundary value problem (4.2) with multifunction $\mathbb{H}$ defined by (4.3) has at least one solution on $[1/8, 5/4]$.

(ii) Let the multifunction $\mathbb{H}(z, \sigma)$ be given by

$$\mathbb{H}(z, \sigma) = \left[0, \frac{1}{5(8z+5)^2}\left(\frac{\sigma^2 + 2|\sigma|}{2(1+|\sigma|)} + \frac{3}{4}\right)\right]. \tag{4.4}$$

It is obvious that $\mathbb{H}$ is measurable. Also, we have

$$H_{\bar{d}}(\mathbb{H}(z, \sigma), \mathbb{H}(z, \bar{\sigma})) \leq \frac{1}{5(8z+5)^2}|\sigma - \bar{\sigma}|,$$

for almost all $z \in [1/8, 5/4]$ and $\sigma, \bar{\sigma} \in \mathbb{R}$. Letting

$$\varrho(z) = 1/(5(8z+5)^2),$$

we have

$$\|\varrho\| = 1/180$$

and

$$\bar{d}(0, \mathbb{H}(z, 0)) = (3/4)\varrho(z) \leq \varrho(z)$$

for almost all $z \in [1/8, 5/4]$. Hence, we have that

$$\Omega\|\varrho\| \approx 0.9794203728 < 1.$$

Therefore, by Theorem 4.2, the nonlocal inclusion boundary value problem (4.2) with multifunction $\mathbb{H}$ defined by (4.4) has at least one solution on $[1/8, 5/4]$.

5. Conclusions

In this paper, we have presented the criteria ensuring the existence and uniqueness of solutions for a $\overline{\psi}_*$-Hilfer fractional proportional differential equation complemented with nonlocal integro-multistrip-multipoint boundary conditions. The desired results for the given problem are derived by applying the fixed point theorems due to Banach, Krasnosel'skiĭ, Schaefer and Leray-Schauder alternative. Also, two existence results for the $\overline{\psi}_*$-Hilfer fractional proportional differential inclusion problem with nonlocal nonlocal integro-multistrip-multipoint boundary conditions are proved when the multivalued map takes convex as well as non-convex values. Examples are constructed for illustrating all the abstract results presented in this paper. We emphasize that our results are new and contribute to the literature on the nonlocal integro-multistrip-multipoint boundary value problems involving $\overline{\psi}_*$-Hilfer fractional proportional differential equations and inclusions.

Fixing the parameters involved in the given problems, some new results follow as special cases. For example, our results correspond to the ones for

(i) integral multi-strip nonlocal $\overline{\psi}_*$-Hilfer fractional proportional boundary value problems of order in $(1, 2]$ if $\theta_j = 0, j = 1, 2, \ldots, m$;

(ii) integral multi-point nonlocal $\overline{\psi}_*$-Hilfer fractional proportional boundary value problems of order in $(1, 2]$ if $\varphi_i = 0, i = 1, 2, \ldots, n$;

(iii) integral multi-strip nonlocal Hilfer fractional proportional boundary value problems of order in $(1, 2]$ if $\overline{\psi}_*(z) = z$;

(iv) nonlocal integro-multistrip-multipoint $\overline{\psi}_*$-Hilfer fractional boundary value problems of order in $(1, 2]$ if $\vartheta_* = 1$.

In the future, we plan to investigate the systems of $\overline{\psi}_*$-Hilfer fractional proportional differential equations and inclusions equipped with nonlocal integro-multistrip-multipoint boundary conditions.

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Conflict of interest

The authors declare no conflicts of interest.

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