AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14064–14085.
DOI:10.3934/math.2023719
Received: 10 December 2022
Revised: 23 February 2023
Accepted: 28 February 2023
Published: 14 April 2023

Research article

Necessary and sufficient conditions for boundedness of commutators of maximal function on the p-adic vector spaces

Qianjun He[1] **and Xiang Li**[2,*]

[1] School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China
[2] School of Science, Shandong Jianzhu University, Jinan 250000, China

* **Correspondence:** Email: lixiang162@mails.ucas.ac.cn.

Abstract: In this paper, we first show that the p-adic version of maximal function $\mathcal{M}^p_{L\log L}$ is equivalent to the maximal function $\mathcal{M}^p(\mathcal{M}^p)$ and that the class of functions for which the maximal commutators and the commutator with the p-adic version of maximal function or the maximal sharp function are bounded on the p-adic vector spaces are characterized and proved to be the same. Moreover, new pointwise estimates for these operators are proved.

Keywords: p-adic vector space; maximal function; commutator; BMO space; Morrey spaces
Mathematics Subject Classification: 11E95, 11K70, 11S80, 42B25, 42B35

1. Introduction and statement of main results

It is well known that the commutators of a great variety of operators appearing in Harmonic Analysis are intimately related to the regularity properties of the solutions of certain partial differential equations, see for example [5,6,9,10,13,32]. A first result in this direction was established by Coifman, Rochberg and Weiss in [11], where the authors studied the commutator $[b, T]$ generated by the classical singular integral operator T and a suitable function b is given by

$$[b, T](f) = bT(f) - T(bf). \tag{1.1}$$

They gave a characterization of $\mathrm{BMO}(\mathbb{R}^d)$ in virtue of the L^q-boundedness of the above commutator. In [22] the author extended the results in [11] to functions belonging to a Lipschitz functional space and gave a characterization in terms of the boundedness of the commutators of singular integral operators with symbols in this class. Milman and Schonbek [27] established a commutator result by real interpolation techniques. As an application, they obtained the L^q-boundedness of the commutators of maximal function $[b, M]$ when $b \in \mathrm{BMO}(\mathbb{R}^d)$ and $b \geq 0$. This operator can be used in studying the product of a function in H^1 and a function in BMO (see [7] for instance). Bastero, Milman and Ruiz [4] studied the necessary and sufficient conditions for the boundedness of the commutators of maximal function $[b, M]$ and sharp maximal function $[b, M^\sharp]$ on $L^q(\mathbb{R}^d)$ spaces when $1 < q < \infty$. Recently, Guliyev et al. [17] gave the characterization of fractional maximal operator and its commutators on Orlicz spaces in the Dunkl setting. For more information about the characterization of the commutator of maximal operator, see also [2, 3, 12, 43–45] and the references therein.

Motivated by [3, 4] and [19], we will study the characterization of BMO functions in the context of p-adic field spaces. For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$. If any non-zero rational number x is represented as $x = p^\gamma \frac{m}{n}$, where m and n are integers which are not divisible by p, and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \qquad |x + y|_p \leq \max\{|x|_p, |y|_p\}$$

It follows from the second property that when $|x|_p \neq |y|_p$, then $|x + y|_p = \max\{|x|_p, |y|_p\}$. From the standard p-adic analysis [40], we see that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.2}$$

where a_j are integers, $0 \leq a_j \leq p - 1$, $a_0 \neq 0$. The series (1.2) converges in the p-adic norm because $|a_j p^j|_p = p^{-\gamma}$.

The space $\mathbb{Q}_p^d$ consists of points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, d$. The p-adic norm on $\mathbb{Q}_p^d$ is $|\mathbf{x}|_p := \max_{1\leq j\leq d} |x_j|_p$ for $\mathbf{x} \in \mathbb{Q}_p^d$. Denote by $B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p \leq p^\gamma\}$, the ball with center at $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ, and by $S_\gamma(\mathbf{a}) := \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\}$ the sphere with center at $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ, $\gamma \in \mathbb{Z}$. It clear that $S_\gamma(\mathbf{a}) = B_\gamma(\mathbf{a})\backslash B_{\gamma-1}(\mathbf{a})$, and $B_\gamma(\mathbf{a}) = \bigcup_{k\leq\gamma} S_k(\mathbf{a})$.

Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Harr measure $d\mathbf{x}$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant. We normalize the measure $d\mathbf{x}$ so that

$$\int_{B_0(\mathbf{0})} dx = |B_0(\mathbf{0})|_H = 1,$$

where $|E|_H$ denotes the Harr measure of a measurable subset E of $\mathbb{Q}_p^d$. From this integral theory, it is easy to obtain that $|B_\gamma(\mathbf{a})|_H = p^{\gamma d}$ and $|S_\gamma(\mathbf{a})|_H = p^{\gamma d}(1 - p^{-d})$ for any $\mathbf{a} \in \mathbb{Q}_p^d$.

In what follows, we say that a (real-valued) measurable function f defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d)$, $1 \leq q \leq \infty$, if it satisfies

$$\begin{aligned} \|f\|_{L^q(\mathbb{Q}_p^d)} &:= \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q d\mathbf{x}\right)^{1/q} < \infty, \quad 1 \leq q < \infty, \\ \|f\|_{L^\infty(\mathbb{Q}_p^d)} &:= \inf\{\alpha : |\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}|_H = 0\} < \infty. \end{aligned} \tag{1.3}$$

Here the integral in (1.3) is defined as

$$\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q d\mathbf{x} = \lim_{\gamma\to\infty} \int_{B_\gamma(\mathbf{0})} |f(\mathbf{x})|^q d\mathbf{x} = \lim_{\gamma\to\infty} \sum_{-\infty<k\leq\gamma} \int_{S_k(\mathbf{0})} |f(\mathbf{x})|^q d\mathbf{x},$$

if the limit exists. We now mention some of the previous works on harmonic analysis on the p-adic field, see [18, 26, 33–35] and the references therein.

For a function $f \in L^1_{\rm loc}(\mathbb{Q}_p^d)$, we defined the Hardy-Littlewood maximal function of f on $\mathbb{Q}_p^d$ by

$$\mathcal{M}^p(f)(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| d\mathbf{y}.$$

In [24, 25], Kim proved L^q boundedness of the version of maximal function $\mathcal{M}^p$ and gave some properties similar to the Euclidean setting.

The maximal commutator of $\mathcal{M}^p$ with a locally integrable function b is defined by

$$\mathcal{M}_b^p(f)(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{x}) - b(\mathbf{y})||f(\mathbf{y})| d\mathbf{y}.$$

The first part of this paper is to study the boundedness of $\mathcal{M}_b^p$ when the symbol belongs to a BMO space (see in Section 2). Some characterizations of the BMO space via such commutator are given. Our first result can be stated as follows.

Theorem 1.1. *Let b be a locally integrable function on $\mathbb{Q}_p^d$. The following statements are equivalent:*
(1) $b \in {\rm BMO}(\mathbb{Q}_p^d)$*;*
(2) $\mathcal{M}_b^p$ *is bounded on* $L^q(\mathbb{Q}_p^d)$ *for all q with* $1 < q \leq \infty$*;*
(3) $\mathcal{M}_b^p$ *is bounded on* $L^q(\mathbb{Q}_p^d)$ *for some q with* $1 < q \leq \infty$*.*

We remark that the boundedenss of the commutators of maximal function unknown until the author made some progress in [19] for a partial case on the p-adic vector space. In an attempt to close the gap in his work, we came up with some new results.

On the other hand, similar to (1.1), we can define the commutator of the p-adic version of maximal function $\mathcal{M}^p$ with a locally integrable function b by

$$[b, \mathcal{M}^p](f)(\mathbf{x}) = b(\mathbf{x})\mathcal{M}^p(f)(\mathbf{x}) - \mathcal{M}^p(bf)(\mathbf{x}).$$

In this paper we show that a slightly extended form of positivity is a necessary and suffcient condition to characterize the boundedness of $[b, \mathcal{M}^p]$. To see what this condition should be we observe that if $\mathcal{M}^p$ were a linear operator, given that everything we do is modulo bounded operators, the correct requirement would appear to be that $b \in {\rm BMO}(\mathbb{Q}_p^d)$ with its negative part b^- bounded. Indeed, the suffciency of the condition $b \in {\rm BMO}(\mathbb{Q}_p^d)$ with b^- bounded formally follows from Theorem 1.1, the fact that $b \in {\rm BMO}(\mathbb{Q}_p^d)$ and the estimate

$$|[b, \mathcal{M}^p](f)(\mathbf{x})| \leq |[|b|, \mathcal{M}^p](f)(\mathbf{x})| + 2b^-(\mathbf{x})\mathcal{M}^p(f)(\mathbf{x}) \leq \mathcal{M}_b^p(f)(\mathbf{x}) + 2b^-(\mathbf{x})\mathcal{M}^p(f)(\mathbf{x}), \tag{1.4}$$

where $b^- = -\min\{0, b\}$. We summarize the previous discussion with the following

Theorem 1.2. *If $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$ and b^- is bounded, then the commutator $[b, \mathcal{M}^p]$ is bounded on $L^q(\mathbb{Q}_p^d)$ for all $q \in (1, \infty]$.*

The purpose of this paper is to prove the converse of Theorem 1.2 and to show that a similar characterization also holds for the sharp maximal operator.

Our main result for $[b, \mathcal{M}^p]$ can be now stated as follows.

Theorem 1.3. *Let b be a locally integrable function on $\mathbb{Q}_p^d$. The following statements are equivalent:*
(1) *$b \in \mathrm{BMO}(\mathbb{Q}_p^d)$ and $b^- \in L^\infty(\mathbb{Q}_p^d)$;*
(2) *$[b, \mathcal{M}^p]$ is bounded on $L^q(\mathbb{Q}_p^d)$ for all q with $1 < q \leq \infty$;*
(3) *$[b, \mathcal{M}^p]$ is bounded on $L^q(\mathbb{Q}_p^d)$ for some q with $1 < q \leq \infty$;*
(4) *There exists $q \in [1, \infty)$ such that*

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y} < \infty, \tag{1.5}$$

where $\mathcal{M}^p_{B_\gamma(\mathbf{x})}$ denote the maximal operator with respect to a p-adic ball which is defined by

$$\mathcal{M}^p_{B_\gamma(\mathbf{x})}(f)(\mathbf{y}) = \sup_{B_\gamma(\mathbf{x})\supseteq B_{\gamma_0}(\mathbf{y})} \frac{1}{|B_{\gamma_0}(\mathbf{y})|_H} \int_{B_{\gamma_0}(\mathbf{y})} |f(\mathbf{z})| d\mathbf{z};$$

Here, the supremum is take over all the p-adic $B_{\gamma_0}(\mathbf{y})$ with $B_{\gamma_0}(\mathbf{y}) \subseteq B_\gamma(\mathbf{x})$ for a fixed p-adic ball $B_\gamma(\mathbf{x})$.
(5) *For all $q \in [1, \infty)$ such that*

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y} < \infty.$$

Recall that the p-adic version of sharp function is given by

$$f_p^\sharp(\mathbf{x}) = \mathcal{M}_p^\sharp(f)(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})}| d\mathbf{y},$$

where $f_{B_\gamma(\mathbf{x})}$ denotes the average of f over $B_\gamma(\mathbf{x})$, i.e., $f_{B_\gamma(\mathbf{x})} = \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} f(\mathbf{y}) d\mathbf{y}$.

Next we consider commutators with the p-adic version of sharp function. The results are similar to those in Theorem 1.3.

Theorem 1.4. *Let b be a locally integrable function on $\mathbb{Q}_p^d$ and $1 \leq \delta < \infty$. The following statements are equivalent:*
(1) *$b \in \mathrm{BMO}(\mathbb{Q}_p^d)$ and $b^- \in L^\infty(\mathbb{Q}_p^d)$;*
(2) *$[b, \mathcal{M}_p^\sharp]$ is bounded on $L^q(\mathbb{Q}_p^d)$ for all q with $1 < q \leq \infty$;*
(3) *$[b, \mathcal{M}_p^\sharp]$ is bounded on $L^q(\mathbb{Q}_p^d)$ for some q with $1 < q \leq \infty$;*
(4) *There exists $q \in [1, \infty)$ such that*

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \left| b(\mathbf{y}) - \frac{p^2}{2(p-1)} (b\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) \right|^q d\mathbf{y} < \infty; \tag{1.6}$$

(5) *For all $q \in [1, \infty)$ such that*

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \left| b(\mathbf{y}) - \frac{p^2}{2(p-1)} (b\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) \right|^q d\mathbf{y} < \infty.$$

It is well-known that the Morrey space introduced by Morrey in [28] in order to study regularity questions which appear in the Calculus of Variations, and the p-adic version of Morrey spaces defined as follows: for $1 \le q \le \infty$ and $0 \le \lambda \le d$,

$$L^{q,\lambda}(\mathbb{Q}_p^d) = \{f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^d) : \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} < \infty\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} := \sup_{\mathbf{x}\in\mathbb{Q}_p^d} \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma(\mathbf{x})|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{x})|^q d\mathbf{x} \right)^{1/q}.$$

Note that $L^{q,0}(\mathbb{Q}_p^d) = L^q(\mathbb{Q}_p^d)$ and $L^{q,d}(\mathbb{Q}_p^d) = L^\infty(\mathbb{Q}_p^d)$.

These spaces These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in PDE, such as applications to the Navier-Stokes equations, the Schrödinger equations, the elliptic equations with discontinuous coefficients and the potential analysis, see [1, 8, 14, 23, 38].

The following theorems we investigate boundedness of maximal commutator and commutator of maximal function on the p-adic version of Morrey spaces.

Theorem 1.5. *Let $1 < q < \infty$ and $0 \le \lambda \le d$. The following statements are equivalent:*
(1) $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$;
(2) $\mathcal{M}_b^p$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^d)$.

Theorem 1.6. *Let $1 < q < \infty$ and $0 \le \lambda \le d$. The following statements are equivalent:*
(1) $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$ and $b^- \in L^\infty(\mathbb{Q}_p^d)$;
(2) $[b, \mathcal{M}^p]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^d)$.

The rest of the present paper is organized as follows: In Section 2, we will give some definitions and lemmas. The proof of Theorems 1.1–1.4 are presented in Section 3. In Secction 4, we will give the proof of Theorems 1.5 and 1.6. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. The positive constants C varies from one occurrence to another. For a real number q, $1 < q < \infty$, q' is the conjugate number of q, that is, $1/q + 1/q' = 1$.

2. Preliminary definitions and lemmas

To prove our main results, we need the following definitions and lemmas.

Definition 2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d)$ be given. If $\|f^\sharp\|_{L^\infty(\mathbb{Q}_p^d)} < \infty$, then we say that f is a function of bounded mean oscillation on $\mathbb{Q}_p^d$. We denote the space of such function by $\mathrm{BMO}(\mathbb{Q}_p^d)$; that is say,*

$$\mathrm{BMO}(\mathbb{Q}_p^d) = \{f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d) : f^\sharp \in L^\infty(\mathbb{Q}_p^d)\}.$$

For $f \in \mathrm{BMO}(\mathbb{Q}_p^d)$, we write

$$\|f\|_{\mathrm{BMO}(\mathbb{Q}_p^d)} = \|f^\sharp\|_{L^\infty(\mathbb{Q}_p^d)} = \sup_{\mathbf{x}\in\mathbb{Q}_p^d} \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})}| d\mathbf{y}.$$

In [24], Kim gave the following property of BMO functions whcih is similar to Euclidean setting.

Lemma 2.1. *If* $1 < q < \infty$ *and* $f \in \mathrm{BMO}(\mathbb{Q}_p^d)$ *is given, then we have the following properties;*
(a) *The norm* $\|f\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}$ *is equivalent to the norm* $\|f\|_{\mathrm{BMO}^q}(\mathbb{Q}_p^d)$*, where the norm* $\|f\|_{\mathrm{BMO}^q}(\mathbb{Q}_p^d)$ *defined by*

$$\|f\|_{\mathrm{BMO}^q}(\mathbb{Q}_p^d) = \sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})}|^q d\mathbf{y}\right)^{1/q}.$$

(b) *For any* λ *with* $0 < \lambda < c_2/\|f\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}$*, where* c_2 *is the constant given by Theorem 5.16 in [24],*

$$\sup_{\mathbf{x}\in\mathbb{Q}_p^d}\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\exp(\lambda|f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})}|)d\mathbf{y} < \infty.$$

Let

$$\mathcal{F}_p = \{B_\gamma(\mathbf{x}) : \gamma \in \mathbb{Z}, \mathbf{x} \in \mathbb{Q}_p^d\}$$

denote the family of all the p-adic balls, which differ from those of the Euclidean case, see [25].

Lemma 2.2. *The family* $\mathcal{F}_p$ *has the following properties:*
(a) *If* $\gamma \le \gamma'$*, then either* $B_\gamma(\mathbf{x}) \cap B_{\gamma'}(\mathbf{y}) = \emptyset$ *or* $B_\gamma(\mathbf{x}) \subset B_{\gamma'}(\mathbf{y})$.
(b) $B_\gamma(\mathbf{x}) = B_\gamma(\mathbf{y})$ *if and only if* $\mathbf{y} \in B_\gamma(\mathbf{x})$.

A continuously increasing function on $[0, \infty]$, say $\Psi : [0, \infty] \to [0, \infty]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $\Psi(\infty) = \infty$, will be referred to as an Orlicz function. If Ψ is a Orlicz function, then

$$\Phi(t) = \sup\{ts - \Psi(s) : s \in [0, \infty]\}$$

is the complementary Orlicz function to Ψ.

The Orlicz space denoted by $L^\Psi(\mathbb{Q}_p^d)$ consists of all measurable function $g : \mathbb{Q}_p^d \to \mathbb{R}$ such that

$$\int_{\mathbb{Q}_p^d}\Psi\left(\frac{|g(\mathbf{x})|}{\alpha}\right)d\mathbf{x} < \infty$$

for some $\alpha > 0$.

Let us define the Ψ-average of g over a p-adic ball $B_\gamma(\mathbf{x})$ of $\mathbb{Q}_p^d$ by

$$\|g\|_{\Psi,B_\gamma(\mathbf{x})} = \inf\left\{\alpha > 0 : \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\Psi\left(\frac{|g(\mathbf{x})|}{\alpha}\right)d\mathbf{x} \le 1\right\}.$$

When Ψ is a Young function, that is, a convex Orlicz function, the quantity

$$\|f\|_\Psi = \inf\left\{\alpha > 0 : \int_{\mathbb{Q}_p^d}\Psi\left(\frac{|g(\mathbf{x})|}{\alpha}\right)d\mathbf{x} \le 1\right\}$$

is well known Luxemburg norm is the space $L^\Psi(\mathbb{Q}_p^d)$ which can be found in [31].

A Young function Ψ is said to satisfy the ∇_2-condition, denoted $\Psi \in \nabla_2$, if for some $K > 1$

$$\Psi(t) \le \frac{1}{2K}\Psi(Kt) \text{ for all } t > 0.$$

It should be noted that $\Psi(t) \equiv t$ fails the ∇_2-condition.

If $f \in L^{\Psi}(\mathbb{Q}_p^d)$, the maximal function of f with respect to Ψ is defined by setting

$$\mathcal{M}_{\Psi}^p(f)(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \|f\|_{\Psi,B_\gamma(\mathbf{x})}.$$

The following generalized Hölder's inequality (see [31])

$$\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})g(\mathbf{y})|d\mathbf{y} \le \|f\|_{\Phi,B_\gamma(\mathbf{x})}\|g\|_{\Psi,B_\gamma(\mathbf{x})}, \tag{2.1}$$

holds for any the complementary Young function Ψ associated to Φ.

The main example that we will consider to use the Young function $\Phi(t) = t(1+\log^+ t)$ with maximal function defined by $\mathcal{M}^p_{L(\log L)}$. The complementary Young function is given by $\Psi(t) \approx e^t$ with the corresponding maximal function denoted by $\mathcal{M}^p_{\exp L}$.

Let $\mathfrak{M}(\mathbb{Q}_p^d)$ denote the set of all measurable functions on $\mathbb{Q}_p^d$. The Zygmund class $L(\log^+ L)(\mathbb{Q}_p^d)$ is the set of all $f \in \mathfrak{M}(\mathbb{Q}_p^d)$ such that

$$\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|(\log^+ |f(\mathbf{x})|)d\mathbf{x} < \infty,$$

where $\log^+ t = \max\{\log t, 0\}$ and $t > 0$. Generally, this is not a linear set. Nevertheless, considering the class

$$L(1+\log^+ L)(\mathbb{Q}_p^d) = \left\{ f \in \mathfrak{M}(\mathbb{Q}_p^d) : \|f\|_{L(1+\log^+ L)(\mathbb{Q}_p^d)} = \int_{\mathbb{Q}_p^d} |f(\mathbf{x})|(1+\log^+ |f(\mathbf{x})|)d\mathbf{x} < \infty \right\},$$

we obtain a linear set, the Zygmund space.

The size of $\mathcal{M}^p(\mathcal{M}^p)$ is given by the following.

Lemma 2.3. *Let $f \in \mathfrak{M}(\mathbb{Q}_p^d)$. Then there exist two constants c and c' such that for any $\mathbf{x} \in \mathbb{Q}_p^d$*

$$c'\mathcal{M}^p_{L\log L}(f)(\mathbf{x}) \le \mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x}) \le c\mathcal{M}^p_{L\log L}(f)(\mathbf{x}). \tag{2.2}$$

This lemma, in the same form but in the context of $\mathbb{R}^d$ and spaces of homogeneous type which can be found in [29, 30]. A similar estimate is also given in both [15, 16, 20, 42]. The idea of deducing $L\log L$ behavior of a function from integrability of its maximal function goes back to E. Stein in [39].

In order to prove Lemma 2.3, we need the following lemma.

Lemma 2.4. *Let $f \in \mathfrak{M}(\mathbb{Q}_p^d)$ and $\alpha > 0$. Then we have the following estimatetes for $\omega_H(\alpha) = |\{\mathbf{x} \in \mathbb{Q}_p^d : \mathcal{M}^p(f)(\mathbf{x}) > \alpha\}|_H$:*

$$c' \int_{\{\mathbf{x}\in\mathbb{Q}_p^d : |f(\mathbf{x})|>\alpha\}} |f(\mathbf{x})|d\mathbf{x} \le \alpha\omega_H(\alpha) \le c \int_{\{\mathbf{x}\in\mathbb{Q}_p^d : |f(\mathbf{x})|>\alpha/2\}} |f(\mathbf{x})|d\mathbf{x} \tag{2.3}$$

with constants c and c' which do not depend on f or α.

Proof. Firstly, we give the proof of the right hand side inequality in (2.3). Write $f = f_1 + f_2$, where

$$f_1(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})| > \alpha/2, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})| \leq \alpha/2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|f_2(\mathbf{x})| \leq \alpha/2$ implies that $\mathcal{M}^p(f_2)(\mathbf{x}) \leq \alpha/2$. Then we have $\mathcal{M}^p(f)(\mathbf{x}) \leq \mathcal{M}^p(f_1)(\mathbf{x}) + \mathcal{M}^p(f_2)(\mathbf{x}) \leq \mathcal{M}^p(f_1)(\mathbf{x}) + \alpha/2$. Thus, by the weak $(1, 1)$ boundedness of maximal function $\mathcal{M}^p$, we have

$$\omega_H(\alpha) \leq |\{\mathbf{x} \in \mathbb{Q}_p^d : \mathcal{M}^p(f_1)(\mathbf{x}) > \alpha/2\}|_H \leq \frac{c}{\alpha/2} \int_{\mathbb{Q}_p^d} |f_1(\mathbf{x})| d\mathbf{x} = \frac{c}{\alpha} \int_{\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha/2\}} |f(\mathbf{x})| d\mathbf{x}$$

which gives the right hand of inequality (2.3).

On the other hand, we may assume that $f \in L^1(\mathbb{Q}_p^d)$ (otherwise we truncate and apply a limiting process). Then we use the p-adic version of Calderón-Zygmund decomposition (see [24, Corollary 3.4]) for f and α. we have non-overlapping p-adic balls $B_j \in \mathcal{F}_p$, such that

$$\alpha |B_j|_H < \int_{B_j} |f(\mathbf{x})| d\mathbf{x} \leq p^d \alpha |B_j|_H$$

for any j, and $|f(\mathbf{x})| \leq \alpha$ for a.e. $\mathbf{x} \notin \bigcup_j B_j$. Now, since $x \in B_j$ implies that $\mathcal{M}^p(f)(\mathbf{x}) > \alpha$, we can write

$$\omega_H(\alpha) \geq \sum_{j=1}^{\infty} |B_j|_H \geq \frac{1}{p^d \alpha} \sum_{j=1}^{\infty} \int_{B_j} |f(\mathbf{x})| d\mathbf{x} \geq \frac{1}{p^d \alpha} \int_{\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}} |f(\mathbf{x})| d\mathbf{x}$$

and (2.3) is proved with $c' = p^{-d}$. □

Proof of Lemma 2.3. Firstly, we give the proof of the left hand side inequality in (2.2). By the definition of the Luxemburg norm, the left hand side of inequality (2.2) will follow from showing that for some constant $c_0 > 1$, c_0 independent of f,

$$\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \frac{|f(\mathbf{y})|}{\lambda_{B_\gamma(\mathbf{x})}} \left(1 + \log^+ \left(\frac{|f(\mathbf{y})|}{\lambda_{B_\gamma(\mathbf{x})}}\right)\right) d\mathbf{y} \leq 1, \tag{2.4}$$

where we denote $\lambda_{B_\gamma(\mathbf{x})} = (c_0/|B_{\gamma+1}(\mathbf{x})|_H) \int_{B_{\gamma+1}(\mathbf{x})} \mathcal{M}^p(f)(\mathbf{y}) d\mathbf{y}$.

Let $h = |f|/\lambda_{B_\gamma(\mathbf{x})}$. Recall that $h_{B_\gamma(\mathbf{x})} = \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} h(\mathbf{y}) d\mathbf{y}$ so that $0 \leq h_{B_{\gamma+1}(\mathbf{x})} \leq 1/c_0$ by the p-adic version of Lebesgue differentiation theorem (see [24, Corollary 2.11]) and the definition of $\lambda_{B_\gamma(\mathbf{x})}$. Using the formula

$$\int_{\mathbb{Q}_p^d} \Phi(h)(\mathbf{y}) d\nu(\mathbf{y}) = \int_0^\infty \Phi'(\lambda) \nu(\{\mathbf{y} \in \mathbb{Q}_p^d : h(\mathbf{y}) > \lambda\}) d\lambda,$$

which holds for any Young function Φ and Harr mearsure ν (see [31, p. 406]), we have

$$\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |h(\mathbf{y})|(1 + \log^+ |h(\mathbf{y})|) d\mathbf{y}$$

$$\begin{aligned}
&\leq \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_0^\infty \min\left(1,\frac{1}{\lambda}\right)h(\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ |h(\mathbf{y})|>\lambda\})d\lambda\\
&=\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_0^{h_{B_{\gamma+1}(\mathbf{x})}} \min\left(1,\frac{1}{\lambda}\right)h(\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ |h(\mathbf{y})|>\lambda\})d\lambda\\
&\quad+\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{h_{B_{\gamma+1}(\mathbf{x})}}^\infty \min\left(1,\frac{1}{\lambda}\right)h(\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ |h(\mathbf{y})|>\lambda\})d\lambda\\
&=: I+II,
\end{aligned}$$

where we use the nonation $h(E) = \int_E h(\mathbf{x})d\mathbf{x}$ for any measurable set E on $\mathbb{Q}_p^d$. Recalling that $h_{B_{\gamma+1}(\mathbf{x})}\leq 1/c_0$, we have

$$I=\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|h(\mathbf{y})|\int_0^{\min(|h(\mathbf{y})|,h_{B_{\gamma+1}(\mathbf{x})})}\min\left(1,\frac{1}{\lambda}\right)d\lambda d\mathbf{y}\leq p h^2_{B_{\gamma+1}(\mathbf{x})}\leq\frac{p^d}{c_0^2}.$$

For the second term II, by using Lemma 2.4, we have

$$\begin{aligned}
II&\leq\frac{1}{c'|B_\gamma(\mathbf{x})|_H}\int_{h_{B_{\gamma+1}(\mathbf{x})}}^\infty \lambda\min\left(1,\frac{1}{\lambda}\right)|\{\mathbf{y}\in B_{\gamma+1}(\mathbf{x}):\ \mathcal{M}^p(h)(\mathbf{y})>\lambda\}|_H d\lambda\\
&\leq\frac{1}{c'|B_\gamma(\mathbf{x})|_H}\int_0^\infty \lambda\min\left(1,\frac{1}{\lambda}\right)|\{\mathbf{y}\in B_{\gamma+1}(\mathbf{x}):\ \mathcal{M}^p(h)(\mathbf{y})>\lambda\}|_H d\lambda\\
&=\frac{1}{c'|B_\gamma(\mathbf{x})|_H}\int_{B_{\gamma+1}(\mathbf{x})}\mathcal{M}^p(h)(\mathbf{y})d\mathbf{y}\\
&=\frac{1}{c'|B_\gamma(\mathbf{x})|_H}\int_{B_{\gamma+1}(\mathbf{x})}\mathcal{M}^p(f)(\mathbf{y})d\mathbf{y}\frac{1}{\lambda_{B_\gamma(\mathbf{x})}}=\frac{p^d}{c'c_0}
\end{aligned}$$

by using the definition of $\lambda_{B_\gamma(\mathbf{x})}$. Therefore, we conclude that

$$I+II\leq\frac{p^d}{c_0^2}+\frac{p^d}{c'c_0}\leq 1$$

if c_0 is large enough.

On the other hand, let $\mathbf{x}\in\mathbb{Q}_p^d$ and fix a p-adic ball $B_\gamma(\mathbf{x})\subset\mathbb{Q}_p^d$. Let $f=f_1+f_2$, where $f_1=f\chi_{B_{\gamma+1}(\mathbf{x})}$. Then

$$\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\mathcal{M}^p(f)(\mathbf{y})d\mathbf{y}\\
&\leq\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\mathcal{M}^p(f_1)(\mathbf{y})d\mathbf{y}+\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\mathcal{M}^p(f_2)(\mathbf{y})d\mathbf{y}\\
&=D_1(\mathbf{x})+D_2(\mathbf{x}).
\end{aligned}$$

Now, $D_2(\mathbf{x})$ is comparable to $\inf_{\mathbf{z}\in B_\gamma(\mathbf{x})}\mathcal{M}^p(f)(\mathbf{z})$ (see [24, p. 1298] for instance) and hence $D_2(\mathbf{x})\leq C\mathcal{M}^p(f)(\mathbf{x})$. To estimate $D_1(\mathbf{x})$ we claim that

$$\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\mathcal{M}^p(f)(\mathbf{y})d\mathbf{y}\leq C\|f\|_{L\log L,B_\gamma(\mathbf{x})} \tag{2.5}$$

for all f such that $\operatorname{supp} f \subset B_\gamma(\mathbf{x})$. By homogeneity we can take f with $\|f\|_{L\log L, B_\gamma(\mathbf{x})} = 1$ which implies

$$\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|(1 + \log^+ |f(\mathbf{y})|) d\mathbf{y} \le 1.$$

Hence, it is enough to prove

$$\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \mathcal{M}^p(f)(\mathbf{y}) d\mathbf{y} \le C\left(1 + \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \log^+ |f(\mathbf{y})| d\mathbf{y}\right) \tag{2.6}$$

for all f with $\operatorname{supp} f \subset B_\gamma(\mathbf{x})$. Indeed, by using Lemma 2.4, we have

$$\begin{aligned}
\int_{B_\gamma(\mathbf{x})} \mathcal{M}^p(f)(\mathbf{y}) d\mathbf{y} &= \int_0^\infty |\{\mathbf{y} \in B_\gamma(\mathbf{x}) : \ \mathcal{M}^p(f)(\mathbf{y}) > \alpha\}|_H d\alpha \\
&= 2 \int_0^\infty |\{\mathbf{y} \in B_\gamma(\mathbf{x}) : \ \mathcal{M}^p(f)(\mathbf{y}) > 2\alpha\}|_H d\alpha \\
&\le 2\left(\int_0^1 |B_\gamma(\mathbf{x})|_H d\alpha + \int_1^\infty \omega_H(2\alpha) d\alpha\right) \\
&\le 2|B_\gamma(\mathbf{x})|_H + 2c \int_1^\infty \frac{1}{\alpha} \int_{\{\mathbf{y}\in B_\gamma(\mathbf{x}) :\, |f(\mathbf{y})|>\alpha\}} |f(\mathbf{y})| d\mathbf{y} d\alpha \\
&= 2|B_\gamma(\mathbf{x})|_H + 2c \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \int_1^{|f(\mathbf{y})|} \frac{d\alpha}{\alpha} d\mathbf{y} \\
&= 2|B_\gamma(\mathbf{x})|_H + 2c \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \log^+ |f(\mathbf{y})| d\mathbf{y}.
\end{aligned}$$

This imlies that (2.6) holds. Hence, by using generalized Hölder's inequality and using (2.5) with $B_\gamma(\mathbf{x})$ replaced by $B_{\gamma+1}(\mathbf{x})$, we have

$$\begin{aligned}
D_1(\mathbf{x}) + D_2(\mathbf{x}) &\le \frac{p^d}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \mathcal{M}^p(f_1)(\mathbf{y}) d\mathbf{y} + C\mathcal{M}^p(f)(\mathbf{x}) \\
&\le C\|f\|_{L\log L, B_\gamma(\mathbf{x})} + C\mathcal{M}^p_{L\log L}(f)(\mathbf{x}) \\
&\le C\mathcal{M}^p_{L\log L}(f)(\mathbf{x}).
\end{aligned}$$

This completes the proof of Lemma 2.3. □

The following p-adic version of Kolmogorov's inequality will be used in the proof Lemma 2.6.

Lemma 2.5. *Let $B_\gamma(\mathbf{x})$ be any p-adic ball and $0 < q_0 < q < \infty$. Then we have*

$$\left(\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|^{q_0} d\mathbf{y}\right)^{1/q_0} \le \|f\|_{L^{q,\infty}(B_\gamma(\mathbf{x}), d\mathbf{y}/|B_\gamma(\mathbf{x})|_H)}.$$

Proof. Let t be some positive real number which will be determined later. Then, by using Lemma 2.4 in [25], we have

$$\int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|^{q_0} d\mathbf{y} = q_0 \int_0^\infty \lambda^{q_0-1} |\{\mathbf{y} \in B_\gamma(\mathbf{x}) : \ |f(\mathbf{y})| > \lambda\}|_H d\lambda$$

$$\leq q_0|B_\gamma(\mathbf{x})|_H \int_0^t \lambda^{q_0-1}d\lambda + q_0\int_t^\infty \lambda^{q_0-1}|\{\mathbf{y}\in B_\gamma(\mathbf{x}):\ |f(\mathbf{y})|>\lambda\}|_H d\lambda$$
$$\leq q_0 t^{q_0}|B_\gamma(\mathbf{x})|_H + q_0 t^{q_0-q}\|f\|^q_{L^{q,\infty}(B_\gamma(\mathbf{x}),d\mathbf{y}/|B_\gamma(\mathbf{x})|_H)}|B_\gamma(\mathbf{x})|_H.$$

Taking $t=\|f\|_{L^{q,\infty}(B_\gamma(\mathbf{x}),d\mathbf{y}/|B_\gamma(\mathbf{x})|_H)}$, we conclude that

$$\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|f(\mathbf{y})|^{q_0}d\mathbf{y}\right)^{1/q_0} \leq \|f\|_{L^{q,\infty}(B_\gamma(\mathbf{x}),d\mathbf{y}/|B_\gamma(\mathbf{x})|_H)}.$$

This completes the proof of the lemma. □

For $\delta>0$ and $f\in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d)$, the p-adic version of maximal function is defined by

$$\mathcal{M}^p_\delta(f)(\mathbf{x}) := \sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|f(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta}.$$

The following lemma is true which play key role in the proof of our results.

Lemma 2.6. *Let* $0<\delta<1$ *and* $b\in \mathrm{BMO}(\mathbb{Q}_p^d)$*. then there exists a constant* $C>0$ *such that*

$$\mathcal{M}^p_\delta(\mathcal{M}^p_b(f))(\mathbf{x}) \leq C\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x})$$

for all $f\in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d)$.

This lemma has been studied in [2,3] for the Euclidean setting which improves the known inequality

$$M^\sharp_\delta(C_b(f))(x) \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^d)}M^2f(x), \tag{2.7}$$

where $M^\sharp_\delta$, C_b and M denote the sharp maximal function, commutator of maximal function and maximal function in Euclidean case, respectively. Inequality (2.7) is key tool to prove the boundedness of commutator of maximal function and it has attracted much more attention, see [21, 36, 37, 41].

Proof. Actually, the method stem from Agcayazi et al. [2], they have investigated the corresponding theorem in Euclidean case. Following their method, it is easy to give this lemma on p-adic vector spaces as well. For completeness, we give the deails.

Let $\mathbf{x}\in\mathbb{Q}_p^d$ and fix p-adic ball $B_\gamma(\mathbf{x})$, it is enough to show that,

$$\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}^p_b(f)(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x}).$$

Now, we split $f=f_1+f_2$, where $f_1=f\chi_{B_{\gamma+1}(\mathbf{x})}$. Since for any $\mathbf{y}\in\mathbb{Q}_p^d$

$$\begin{aligned}\mathcal{M}^p_b(f)(\mathbf{y}) &= \mathcal{M}^p_b((b-b_{B_{\gamma+1}(\mathbf{x})}+b_{B_{\gamma+1}(\mathbf{x})}-b(\mathbf{y}))f)(\mathbf{y})\\ &\leq |b(\mathbf{y})-b_{B_{\gamma+1}(\mathbf{x})}|\mathcal{M}^p(f)(\mathbf{y}) + \mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f_1)(\mathbf{y}) + \mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f_2)(\mathbf{y}),\end{aligned}$$

it follows that

$$\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}^p_b(f)(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta}$$

$$
\begin{aligned}
&\lesssim \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b_{B_{\gamma+1}(\mathbf{x})}|\mathcal{M}^p(f)(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta}\\
&\quad+\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f_1)(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta}\\
&\quad+\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f_2)(\mathbf{y})|^\delta d\mathbf{y}\right)^{1/\delta}\\
&=: A_1(\mathbf{x})+A_2(\mathbf{x})+A_3(\mathbf{x}).
\end{aligned}
$$

For the first term $A_1(\mathbf{x})$, by using Hölder's inequality and Lemma 2.1, we obtain

$$
\begin{aligned}
A_1(\mathbf{x}) &\lesssim \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b_{B_\gamma(\mathbf{x})}|^{\frac{\delta}{1-\delta}}d\mathbf{y}\right)^{\frac{1-\delta}{\delta}}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}^p(f)(\mathbf{y})|d\mathbf{y}\right)\\
&\le \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x}). \qquad (2.8)
\end{aligned}
$$

For the second term $A_2(\mathbf{x})$. Combining Lemma 2.5 and the weak-$(1,1)$ boundedness of $\mathcal{M}^p$ gives that

$$
A_2(\mathbf{x}) \lesssim \|\mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f_1)\|_{L^{1,\infty}(B_{\gamma+1}(\mathbf{x}),d\mathbf{y}/|B_\gamma(\mathbf{x})|_H)} \lesssim \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_{\gamma+1}(\mathbf{x})}|b-b_{B_{\gamma+1}(\mathbf{x})}||f(\mathbf{y})|d\mathbf{y}.
$$

By using generalized Hölder's inequality (2.1), we obtain

$$
A_2(\mathbf{x}) \lesssim \|b-b_{B_{\gamma+1}(\mathbf{x})}\|_{\exp L,B_{\gamma+1}(\mathbf{x})}\|f\|_{L\log L,B_{\gamma+1}(\mathbf{x})}.
$$

Since by (b) of Lemma 2.1, there is a constant $C>0$ such that for any p-adic ball $B_\gamma(\mathbf{x})$,

$$
\|b-b_{B_\gamma(\mathbf{x})}\|_{\exp L,B_\gamma(\mathbf{x})} \le C\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)},
$$

we arrive at

$$
A_2(\mathbf{x}) \lesssim \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p_{L\log L}(f)(\mathbf{x}). \qquad (2.9)
$$

For the third term $A_3(\mathbf{x})$. This case is easy, since $A_3(\mathbf{x})$ is comparable to $\inf_{\mathbf{y}\in B_\gamma(\mathbf{x})}\ \mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f)(\mathbf{y})$ (see [24, p. 1298] for instance), then

$$
A_3(\mathbf{x}) \le \mathcal{M}^p((b-b_{B_{\gamma+1}(\mathbf{x})})f)(\mathbf{x})
$$

Again by using generalized Hölder's inequality (2.1) and (b) of Lemma 2.1, we conclude that

$$
A_3(\mathbf{x}) \lesssim \sup_{\gamma\in\mathbb{Z}}\|b-b_{B_{\gamma+1}(\mathbf{x})}\|_{\exp L,B_{\gamma+1}(\mathbf{x})}\|f\|_{L\log L,B_{\gamma+1}(\mathbf{x})} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p_{L\log L}(f)(\mathbf{x}). \qquad (2.10)
$$

Finally, combining (2.8)–(2.10) together with Lemma 2.3, we conclude that

$$
\mathcal{M}^p_\delta(\mathcal{M}^p_b(f))(\mathbf{x}) \lesssim \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x}).
$$

This finishes the proof of the lemma. □

Lemma 2.7. *Let* $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$. *Then there exists a positive constant C such that*

$$\mathcal{M}_b^p(f)(\mathbf{x}) \le C\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x})$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^d)$.

Proof. By using the p-adic version of Lebesgue differentiation theorem (see [24, Corollary 2.11])

$$\mathcal{M}_b^p(f)(\mathbf{x}) \le \mathcal{M}_\delta^p(\mathcal{M}_b^p(f))(\mathbf{x}),$$

the statement follows from Lemma 2.6. □

Considering the characteristic function $\chi_{B_\gamma(\mathbf{x})}$, we have the following property (see [19]).

Lemma 2.8. *Let* $1 \le q < \infty$ *and* $0 < \lambda < d$, *then there exist a constant* $C > 0$ *such that*

$$\|\chi_{B_\gamma(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} = |B_\gamma(\mathbf{x})|_H^{\frac{d-\lambda}{dq}}.$$

3. Proof of Theorems 1.1–1.4

Proof of Theeorem 1.1. Combining Lemma 2.7 and $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$ together with the stong (q,q)-type boundedness of $\mathcal{M}^p$ $(1 < q \le \infty)$ (see [25, Theorem 1.1]) gives that (2) and (3) hold.

(3) $\Longrightarrow$ (1): Assume that $\mathcal{M}_b^p$ is bounded from $L^q(\mathbb{Q}_p^d)$ to $L^q(\mathbb{Q}_p^d)$ for some $1 < q \le \infty$. For any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$, by using Hölder's inequality implies that

$$\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}|d\mathbf{y}\\
&\le \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y}) - b(\mathbf{z})|d\mathbf{z}\right)d\mathbf{y}\\
&= \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y}) - b(\mathbf{z})|\chi_{B_\gamma(\mathbf{x})}(\mathbf{z})d\mathbf{z}\right)d\mathbf{y}\\
&\le \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\mathcal{M}_b^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})d\mathbf{y}\\
&\le \frac{1}{|B_\gamma(\mathbf{x})|_H^{1+\beta/d}}\left(\int_{B_\gamma(\mathbf{x})}|\mathcal{M}_b^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\left(\int_{B_\gamma(\mathbf{x})}\chi_{B_\gamma(\mathbf{x})}(\mathbf{y})d\mathbf{y}\right)^{1/q'}\\
&\le \frac{C}{|B_\gamma(\mathbf{x})|_H}\|\mathcal{M}_b^p\|_{L^q(\mathbb{Q}_p^d)\to L^q(\mathbb{Q}_p^d)}\|\chi_{B_\gamma(\mathbf{x})}\|_{L^q(\mathbb{Q}_p^d)}\|\chi_{B_\gamma(\mathbf{x})}\|_{L^{q'}(\mathbb{Q}_p^d)}\\
&\le C\|\mathcal{M}_b^p\|_{L^q(\mathbb{Q}_p^d)\to L^q(\mathbb{Q}_p^d)}.
\end{aligned}$$

This together with Lemma 2.1 implies that $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$.

The proof of Theorem 1.1 is completed since (2) $\Longrightarrow$ (1) follows from (3) $\Longrightarrow$ (1). □

Proof of Theorem 1.2. Combining (1.4) and $f \le \mathcal{M}^p(f)$ together with Lemma 2.7 follows sthat

$$|[b,\mathcal{M}^p](f)(\mathbf{x})| \lesssim (\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)} + \|b^-\|_{L^\infty(\mathbb{Q}_p^d)})\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{x}). \tag{3.1}$$

Thus, by using stong (q,q)-type boumdedness of $\mathcal{M}^p$ (see [25, Theorem 1.1]) implies that

$$\|[b,\mathcal{M}^p](f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim (\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)} + \|b^-\|_{L^\infty(\mathbb{Q}_p^d)})\|f\|_{L^q(\mathbb{Q}_p^d)}.$$

We conclude that Theorem 1.2 is proven. □

Proof of Theorem 1.3. Since the implications (2) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (4) follow readily, we only need to prove (1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (5).

(1) $\Longrightarrow$ (2): The conclusion follows from Theorem 1.2.

(3) $\Longrightarrow$ (4): By using Lemma 2.2, it is easy to obtain that

$$\mathcal{M}^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y}) = \chi_{B_\gamma(\mathbf{x})}(\mathbf{y}) \quad \text{and} \quad \mathcal{M}^p(b\chi_{B_\gamma(\mathbf{x})})(\mathbf{y}) = \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})$$

for any fixed p-adic ball $B_\gamma(\mathbf{x}) \subset \mathcal{F}_p$ and all $y \in B_\gamma(\mathbf{x})$. Thus, we have

$$\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y}\\
&= \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y})\mathcal{M}^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y}) - \mathcal{M}^p(b\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\\
&= \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |[b,\mathcal{M}^p](\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\\
&\le \frac{1}{|B_\gamma(\mathbf{x})|_H}\|[b,\mathcal{M}^p](\chi_{B_\gamma(\mathbf{x})})\|^q_{L^q(\mathbb{Q}_p^d)}\\
&\le \frac{C}{|B_\gamma(\mathbf{x})|_H}\|\chi_{B_\gamma(\mathbf{x})}\|^q_{L^q(\mathbb{Q}_p^d)} < \infty,
\end{aligned}$$

which gives that (4) since the p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$ is arbitrary.

(4) $\Longrightarrow$ (1): To prove $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$, it suffices to verify that there is a constant $C > 0$ such that for any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$,

$$\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y} \le C. \tag{3.2}$$

For any fixed p-adic ball $B_\gamma(\mathbf{x})$, let $E = \{\mathbf{y} \in B_\gamma(\mathbf{x}) : b(\mathbf{y}) \le b_{B_\gamma(\mathbf{x})}\}$ and $B_\gamma(\mathbf{x})$, let $F = \{\mathbf{y} \in B_\gamma(\mathbf{x}) : b(\mathbf{y}) > b_{B_\gamma(\mathbf{x})}\}$. The following equality is trivially true:

$$\int_E |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y} = \int_F |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y}.$$

Since for any $\mathbf{y} \in E$ we have $b(\mathbf{y}) \le b_{B_\gamma(\mathbf{x})} \le \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})$, then for any $\mathbf{y} \in E$,

$$|b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| \le |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|.$$

Thus, we can conclude that

$$\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y} = \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{E\bigcup F} |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y}$$

$$
\begin{aligned}
&= \frac{2}{|B_\gamma(\mathbf{x})|_H} \int_E |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}| d\mathbf{y} \\
&\leq \frac{2}{|B_\gamma(\mathbf{x})|_H} \int_E |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})| d\mathbf{y} \\
&\leq \frac{2}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})| d\mathbf{y}. \qquad (3.3)
\end{aligned}
$$

On the other hand, it follows from Hölder's inequality and (1.5) that

$$
\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})| d\mathbf{y} \\
&\leq \frac{1}{|B_\gamma(\mathbf{x})|_H} \left(\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y} \right)^{1/q} |B_\gamma(\mathbf{x})|_H^{1/q'} \\
&\leq \left(\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y} \right)^{1/q} \\
&\leq C.
\end{aligned}
$$

Combining the above ineuality with (3.3) it follows that $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$.

In order to prove $b^- \in L^\infty(\mathbb{Q}_p^d)$, it suffices to show $b^- = 0$, where $b^- = -\min\{b, 0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed p-adic ball $B_\gamma(\mathbf{x})$, observe that

$$
0 \leq b^+(\mathbf{y}) \leq |b(\mathbf{x})| \leq \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})
$$

for $\mathbf{y} \in B_\gamma(\mathbf{x})$ and therefore we have that for $\mathbf{y} \in B_\gamma(\mathbf{x})$,

$$
0 \leq b^-(\mathbf{y}) \leq \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b^+(\mathbf{y}) + b^-(\mathbf{y}) = \mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y}).
$$

Then, it follows from (1.5) that for any p-adic ball $\mathbf{y} \in B_\gamma(\mathbf{x})$,

$$
\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} b^-(\mathbf{y}) d\mathbf{y} \\
&\leq \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |\mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y})| d\mathbf{y} \\
&\leq \left(\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |\mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y}) - b(\mathbf{y})|^q d\mathbf{y} \right)^{1/q} \\
&\leq C.
\end{aligned}
$$

Thus, $b^- \in L^\infty(\mathbb{Q}_p^d)$ follows from the p-adic version of Lebesgue differentiation theorem (see [24, Corollary 2.11]).

(2) $\Longrightarrow$ (5): This proof is similar to (3) $\Longrightarrow$ (4), we omit the details. Hence, the proof of Theorem 1.4 is completed. □

Proof of Theorem 1.4. Similar to prove Theorem 1.3, we only need to give the proof of (1) $\Longrightarrow$ (2), (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Note that for any $\mathbf{x} \in \mathbb{Q}_p^d$, we have

$$|[b, \mathcal{M}_p^\sharp]f(\mathbf{x}) - [|b|, \mathcal{M}_p^\sharp](f)(\mathbf{x})| \leq 2(b^-(\mathbf{x})\mathcal{M}_p^\sharp(f)(\mathbf{x}) + \mathcal{M}_p^\sharp(b^- f)(\mathbf{x})). \tag{3.4}$$

For any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$, by using triangle inequality, we conclude that

$$\begin{aligned}
2\mathcal{M}_{|b|}^p(f)(\mathbf{x}) &\geq \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |(|b(\mathbf{x})| - |b(\mathbf{y})|)f(\mathbf{y}) - |b(\mathbf{x})|f_{B_\gamma(\mathbf{x})} - (f|b|)_{B_\gamma(\mathbf{x})}|d\mathbf{y} \\
&\geq \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} ||b(\mathbf{x})||f(\mathbf{y}) - f_{B_\gamma(\mathbf{x})}| - ||b(\mathbf{y})|f(\mathbf{y}) - (f|b|)_{B_\gamma(\mathbf{x})}||d\mathbf{y} \\
&\geq |[|b|, \mathcal{M}_p^\sharp](f)(\mathbf{x})|. \qquad (3.5)
\end{aligned}$$

Comining (3.4) and (3.5) together with $\mathcal{M}_p^\sharp(f) \leq 2\mathcal{M}^p(f)$ gives that

$$|[b, \mathcal{M}_p^\sharp](f)(\mathbf{x})| \leq 4(b^-(\mathbf{x})\mathcal{M}^p(f)(\mathbf{x}) + \mathcal{M}^p(b^- f)(\mathbf{x})) + 2\mathcal{M}_{|b|}^p(f)(\mathbf{x}).$$

Since $b \in \mathrm{BMO}(\mathbb{Q}_p^d) \Longrightarrow |b| \in \mathrm{BMO}(\mathbb{Q}_p^d)$, then by using Theorem 1.1, $b^- \in L^\infty(\mathbb{Q}_p^d)$ and the stong (q, q)-type boumdedness of $\mathcal{M}^p$ (see [25, Theorem 1.1]), we have

$$\|[b, \mathcal{M}_p^\sharp](f)\|_{L^q(\mathbb{Q}_p^d)} \lesssim (\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)} + \|b^-\|_{L^\infty(\mathbb{Q}_p^d)})\|f\|_{L^q(\mathbb{Q}_p^d)}.$$

(3) $\Longrightarrow$ (4): The proof of this case follows the procedure in [4]. Let $B_\gamma(\mathbf{x})$ be a fixed p-adic ball as before. For another p-adic ball $B_{\gamma'}(\mathbf{y})$, this gives that

$$\frac{1}{|B_{\gamma'}(\mathbf{y})|_H} \int_{B_{\gamma'}(\mathbf{y})} |\chi_{B_\gamma(\mathbf{x})}(\mathbf{z}) - (\chi_{B_\gamma(\mathbf{x})})_{B_{\gamma'}(\mathbf{y})}|d\mathbf{z} = \frac{2|B_{\gamma'}(\mathbf{y})\backslash B_\gamma(\mathbf{x})|_H |B_{\gamma'}(\mathbf{y}) \bigcap B_\gamma(\mathbf{x})|_H}{|B_{\gamma'}(\mathbf{y})|_H^2}.$$

Without loss of generality, we may assume that $\gamma \leq \gamma'$. Then by using Lemma 2.2, we have that $B_\gamma(\mathbf{x}) \cap B_{\gamma'}(\mathbf{y}) = \emptyset$ or $B_\gamma(\mathbf{x}) \subset B_{\gamma'}(\mathbf{y})$. If $B_\gamma(\mathbf{x}) \cap B_{\gamma'}(\mathbf{y}) = \emptyset$, then we obtain

$$\frac{1}{|B_{\gamma'}(\mathbf{y})|_H} \int_{B_{\gamma'}(\mathbf{y})} |\chi_{B_\gamma(\mathbf{x})}(\mathbf{z}) - (\chi_{B_\gamma(\mathbf{x})})_{B_{\gamma'}(\mathbf{y})}|d\mathbf{z} = 0.$$

If $B_\gamma(\mathbf{x}) \subset B_{\gamma'}(\mathbf{y})$, then we have

$$\frac{1}{|B_{\gamma'}(\mathbf{y})|_H} \int_{B_{\gamma'}(\mathbf{y})} |\chi_{B_\gamma(\mathbf{x})}(\mathbf{z}) - (\chi_{B_\gamma(\mathbf{x})})_{B_{\gamma'}(\mathbf{y})}|d\mathbf{z} = \frac{2(p^{\gamma' d} - p^{\gamma d})p^{\gamma d}}{p^{2\gamma' d}} = \frac{2(p^{(\gamma'-\gamma)d} - 1)}{p^{2(\gamma'-\gamma)d}} \leq \frac{2(p-1)}{p^2},$$

where the last inequality is due to $\gamma', \gamma \in \mathbb{Z}$ and $1 \leq \gamma' - \gamma \in \mathbb{Z}$. On the other hand, for $\mathbf{y} \in B_\gamma(\mathbf{x})$, we consider a p-adic ball $B_{\gamma'}(\mathbf{y})$ always containing $B_\gamma(\mathbf{x})$ such that $|B_{\gamma'}(\mathbf{y})|_H = p|B_\gamma(\mathbf{x})|_H$. This implies that $(\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) = 2(p-1)/p^2$ for any $\mathbf{y} \in B_\gamma(\mathbf{x})$. Hence, we have

$$\begin{aligned}
&\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \left| b(\mathbf{y}) - \frac{p^2}{2(p-1)}(b\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) \right|^q d\mathbf{y} \\
&= \left[\frac{p^2}{2(p-1)}\right]^q \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y})(\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - (b\chi_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y})|^q d\mathbf{y}
\end{aligned}$$

$$= \left[\frac{p^2}{2(p-1)}\right]^q \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |[b, \mathcal{M}_p^\sharp](\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}$$
$$\leq \left[\frac{p^2}{2(p-1)}\right]^q \frac{1}{|B_\gamma(\mathbf{x})|_H}\|[b, \mathcal{M}_p^\sharp](\chi_{B_\gamma(\mathbf{x})})\|_{L^q(\mathbb{Q}_p^d)}^q$$
$$\lesssim \frac{1}{|B_\gamma(\mathbf{x})|_H}\|\chi_{B_\gamma(\mathbf{x})}\|_{L^q(\mathbb{Q}_p^d)}^q < \infty.$$

(4) $\Longrightarrow$ (1): We proceed as in the corresponding portion of the proof of Theorem 1.3, but some extra diffculties appear.

First, our claim is to prove that

$$|b_{B_\gamma(\mathbf{x})}| \leq \frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}), \quad \mathbf{y} \in B_\gamma(\mathbf{x}). \tag{3.6}$$

Picking a p-adic ball $B_{\gamma'}(\mathbf{y})$ containing $B_\gamma(\mathbf{x})$ such that $|B_{\gamma'}(\mathbf{y})|_H = p|B_\gamma(\mathbf{x})|_H$. Then, we have

$$\begin{aligned}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) &\geq \frac{1}{|B_{\gamma'}(\mathbf{y})|_H}\int_{B_{\gamma'}(\mathbf{y})} |b(\mathbf{z})\chi_{B_\gamma(\mathbf{x})}(\mathbf{z}) - (b\chi_{B_\gamma(\mathbf{x})})_{B_{\gamma'}(\mathbf{y})}|d\mathbf{z}\\
&= \frac{1}{p|B_\gamma(\mathbf{x})|_H}\left(\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{z}) - \frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z} + \frac{1}{p}|B_{\gamma'}(\mathbf{y})\backslash B_\gamma(\mathbf{x})|_H|b_{B_\gamma(\mathbf{x})}|\right)\\
&= \frac{1}{p|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{z}) - \frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z} + \frac{p-1}{p^2}|b_{B_\gamma(\mathbf{x})}|.\end{aligned} \tag{3.7}$$

On the other hand

$$\begin{aligned}|b_{B_\gamma(\mathbf{x})}| &\leq \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{z}) - \frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z} + \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|\frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z}\\
&= \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{z}) - \frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z} + \frac{1}{p}|b_{B_\gamma(\mathbf{x})}|,\end{aligned}$$

and so

$$\frac{p-1}{p}|b_{B_\gamma(\mathbf{x})}| \leq \frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|b(\mathbf{z}) - \frac{1}{p}b_{B_\gamma(\mathbf{x})}\right| d\mathbf{z}. \tag{3.8}$$

Therefore, (3.7) and (3.8) lead us to (3.6).

We can now achieve that $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$. In fact, let $E = \{y \in B_\gamma(\mathbf{x}) : b(\mathbf{y}) \leq b_{B_\gamma(\mathbf{x})}\}$. Then, by using (3.6) and (1.6) gives that

$$\begin{aligned}\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}|d\mathbf{y} &= \frac{2}{|B_\gamma(\mathbf{x})|_H}\int_E (b_{B_\gamma(\mathbf{x})} - b(\mathbf{y}))d\mathbf{y}\\
&\leq \frac{2}{|B_\gamma(\mathbf{x})|_H}\int_E\left(\frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y})\right)d\mathbf{y}\\
&\leq \frac{2}{|B_\gamma(\mathbf{x})|_H}\int_E\left|\frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y})\right|d\mathbf{y}\\
&\leq \frac{2}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left|\frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y})\right|d\mathbf{y} \leq C.\end{aligned}$$

In order to prove that $b^- \in L^\infty(\mathbb{Q}_p^d)$ we also use (3.6). We start from the following fact

$$\frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y}) \geq |b_{B_\gamma(\mathbf{x})}| - b^+(\mathbf{y}) + b^-(\mathbf{y}), \quad \mathbf{y} \in B_\gamma(\mathbf{x}).$$

Averaging on $B_\gamma(\mathbf{x})$, we have

$$\begin{aligned} C &\geq \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \left| \frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y}) \right| d\mathbf{y} \\ &\geq \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} \left(\frac{p^2}{2(p-1)}(b_{B_\gamma(\mathbf{x})})_p^\sharp(\mathbf{y}) - b(\mathbf{y}) \right) d\mathbf{y} \\ &\geq \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} (|b_{B_\gamma(\mathbf{x})}| - b^+(\mathbf{y}) + b^-(\mathbf{y})) d\mathbf{y} \\ &= |b_{B_\gamma(\mathbf{x})}| - \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} b^+(\mathbf{y}) d\mathbf{y} + \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} b^-(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Letting $\gamma \to -\infty$ with $\mathbf{y} \in B_\gamma(\mathbf{x})$, the p-adic version of Lebesgue differentiation theorem assures that

$$C \geq |b(\mathbf{y})| - b^+(\mathbf{y}) + b^-(\mathbf{y}) = 2b^-(\mathbf{y})$$

and the desired result follows. This finishes the proof of Theorem 1.4. □

4. Proof of Theorems 1.5–1.6

Proof of Theorem 1.5. Applying the similar agrument as in the proof of Theorem 1.1 in [25], we have that for any p-adic ball $B_\gamma(\mathbf{x})$

$$\int_{B_\gamma(\mathbf{x})} |\mathcal{M}^p(f)(\mathbf{y})|^q d\mathbf{y} \lesssim \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|^q d\mathbf{y}. \tag{4.1}$$

Assume that $b \in \mathrm{BMO}(\mathbb{Q}_p^d)$. By using (4.1) and Lemma 2.7, we have

$$\begin{aligned} \frac{1}{|B_\gamma(\mathbf{x})|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |\mathcal{M}_b^p(f)(\mathbf{y})|^q d\mathbf{y} &\lesssim \frac{\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}}{|B_\gamma(\mathbf{x})|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |\mathcal{M}^p(\mathcal{M}^p(f))(\mathbf{y})|^q d\mathbf{y} \\ &\lesssim \frac{\|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)}}{|B_\gamma(\mathbf{x})|_H^{\lambda/d}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|^q d\mathbf{y}. \end{aligned}$$

Thus, we conclude that

$$\|\mathcal{M}_b^p(f)\|_{L^{q,\lambda}(\mathbb{Q}_p^d)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^d)} \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^d)}.$$

Conversely, if $\mathcal{M}_b^p$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{q,\lambda}(\mathbb{Q}_p^d)$, then for any p-adic ball $B_\gamma(\mathbf{x}) \subset \mathbb{Q}_p^d$

$$\left(\frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |b(\mathbf{y}) - b_{B_\gamma(\mathbf{x})}|^q d\mathbf{y} \right)^{1/q}$$

$$
\begin{aligned}
&\leq \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}\left[\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-b(\mathbf{z})|\chi_{B_\gamma(\mathbf{x})}(\mathbf{z})d\mathbf{z}\right]^q d\mathbf{y}\right)^{1/q}\\
&\leq \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}_b^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&= \left(\frac{|B_\gamma(\mathbf{x})|_H^{\lambda/d}}{|B_\gamma(\mathbf{x})|_H}\right)^{1/q}\left(\frac{1}{|B_\gamma(\mathbf{x})|_H^{\lambda/d}}\int_{B_\gamma(\mathbf{x})}|\mathcal{M}_b^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&\leq |B_\gamma(\mathbf{x})|_H^{-1/q+\lambda/(dq)}\|\mathcal{M}_b^p\|_{L^{q,\lambda}(\mathbb{Q}_p^d)\to L^{q,\lambda}(\mathbb{Q}_p^d)}\|\chi_{B_\gamma(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^d)}\\
&\leq C\|\mathcal{M}_b^p\|_{L^{q,\lambda}(\mathbb{Q}_p^d)\to L^{q,\lambda}(\mathbb{Q}_p^d)},
\end{aligned}
$$

where in the last step we have used Lemma 2.8.

It follows from Lemma 2.1 that $b\in \mathrm{BMO}(\mathbb{Q}_p^d)$. This finishes the proof of Theorem 1.5. □

Proof of Theorem 1.6. (1) $\Longrightarrow$ (2): Assume that $b^-\in L^\infty(\mathbb{Q}_p^d)$ and $b\in\mathrm{BMO}(\mathbb{Q}_p^d)$, then by using (1.4) and Theorem 1.5, we show that $[b,\mathcal{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{q,\lambda}(\mathbb{Q}_p^d)$.

(2) $\Longrightarrow$ (1): Assume that $[b,\mathcal{M}^p]$ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^d)$ to $L^{q,\lambda}(\mathbb{Q}_p^d)$. Similar to estimate for (3.3), we have that for any p-adic ball $B_\gamma(\mathbf{x})\subset\mathbb{Q}_p^d$,

$$
\begin{aligned}
&\left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})-\mathcal{M}^p_{B_\gamma(\mathbf{x})}(b)(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&= \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|b(\mathbf{y})\mathcal{M}^p(\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})-\mathcal{M}^p(b\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&= \left(\frac{1}{|B_\gamma(\mathbf{x})|_H}\int_{B_\gamma(\mathbf{x})}|[b,\mathcal{M}^p](\chi_{B_\gamma(\mathbf{x})})(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\\
&\leq \frac{|B_\gamma(\mathbf{x})|_H^{\lambda/(dq)}}{|B_\gamma(\mathbf{x})|_H^{1/q}}\|[b,\mathcal{M}^p](\chi_{B_\gamma(\mathbf{x})})\|_{L^{q,\lambda}(\mathbb{Q}_p^d)}\\
&\leq C\frac{|B_\gamma(\mathbf{x})|_H^{\lambda/(dq)}}{|B_\gamma(\mathbf{x})|_H^{1/q}}\|\chi_{B_\gamma(\mathbf{x})}\|_{L^{q,\lambda}(\mathbb{Q}_p^d)}\leq C,
\end{aligned}
$$

where inn the last step we have used Lemma 2.8. Thus, by using Theorem 1.3, we give that $b\in\mathrm{BMO}(\mathbb{Q}_p^d)$ and $b^-\in L^\infty(\mathbb{Q}_p^d)$. This finish the proof of Theorem 1.6. □

Acknowledgments

This work was in part supported by National Natural Science Foundation of China (Grant No. 12071473) and Shandong Jianzhu University Foundation (Grant No. X20075Z0101).

Conflict of interest

The authors declare that they have no conflict of interest and competing interests. All procedures were in accordance with the ethical standards of the institutional research committee and with the 1964

Helsinki declaration and its later amendments or comparable ethical standards. All authors contributed equally to this work. The manuscript is approved by all authors for publication. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

References

1. D. Adams, A note on Riesz potentials, *Duke Math. J.*, **42** (1975), 765–778. http://dx.doi.org/10.1215/S0012-7094-75-04265-9
2. M. Agcayazi, A. Gogatishvili, K Koca, R. Mustafayev, A note on maximal commutators and commutators of maximal functions, *J. Math. Soc. Japan*, **67** (2015), 581–593. http://dx.doi.org/10.2969/jmsj/06720581
3. M. Agcayazi, A. Gogatishvili, R. Mustafayev, Weak-type estimates in Morrey spaces for maximal commutator and commutator of maximal function, *Tokyo J. Math.*, **41** (2018), 193–218. http://dx.doi.org/10.3836/tjm/1502179258
4. J. Bastero, M. Milman, F. Ruiz, Commutators for the maximal and sharp functions, *Proc. Amer. Math. Soc.*, **128** (2000), 3329–3334.
5. M. Bramanti, M. Cerutti, Commutators of singular integrals and fractional integrals on homogeneous spaces, In: *Harmonic analysis and operator theory*, Provence: American Mathematical Society, 1995, 81–94.
6. M. Bramanti, M. Cerutti, M. Manfredini, L^p estimates for some ultraparabolic operators with discontinuous coefficients, *J. Math. Anal. Appl.*, **200** (1996), 332–354. http://dx.doi.org/10.1006/jmaa.1996.0209
7. A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister, On the product of functions in BMO and H^1, *Ann. Inst. Fourier*, **57** (2007), 1405–1439.
8. L. Caffarelli, Elliptic second order equations, *Seminario Mat. e. Fis. di Milano*, **58** (1988), 253–284. http://dx.doi.org/10.1007/BF02925245
9. F. Chiarenza, M. Frasca, P. Longo, Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients, *Ric. Mat.*, **40** (1991), 149–168.
10. F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients, *Trans. Am. Math. Soc.*, **336** (1993), 841–853.
11. R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, **103** (1976), 611–635. http://dx.doi.org/10.2307/1970954
12. F. Deringoz, K. Dorak and V. Guliyev, Characterization of the boundedness of fractional maximal operator and its commutators in Orlicz and generalized Orlicz-Morrey spacces on spaces of homogeneous type, *Anal. Math. Phys.*, **11** (2021), 63. http://dx.doi.org/10.1007/s13324-021-00497-1
13. G. Difazio, M. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, *J. Funct. Anal.*, **112** (1993), 241–256. http://dx.doi.org/10.1006/jfan.1993.1032

14. D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Berlin: Springer-Verlag, 2001. http://dx.doi.org/10.1007/978-3-642-61798-0

15. L. Grafakos, *Modern Fourier analysis*, New York: Springer-Verlag, 2014. http://dx.doi.org/10.1007/978-1-4939-1230-8

16. I. Greco, T. Iwaniec, New inequalities for the Jacobian, *Ann. Inst. H. Poincare*, **11** (1994), 17–35. http://dx.doi.org/10.1016/S0294-1449(16)30194-9

17. V. Guliyev, Y. Mannadov, F. Muslumova, Characterization of fractional maximal operator and its commutators on Orlicz spaces in the Dunkl setting, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 1699–1717. http://dx.doi.org/10.1007/s11868-020-00364-w

18. Q. He, M. Wei, D. Yan, Characterizations of p-adic central Campanato spaces via commutator of p-adic Hardy type operators, *J. Korean Math. Soc.*, **56** (2019), 767–787. http://dx.doi.org/10.4134/JKMS.j180390

19. Q. He, X. Li, Characterization of Lipschitz spaces via commutators of maximal function on the p-adic vector space, *J. Math.*, **2022** (2022), 7430272. http://dx.doi.org/10.1155/2022/7430272

20. G. Hu, H. Lin, D. Yang, Commutators of the Hardy-Littlewood maximal operator with BMO symbols on spaces of homogeneous type, *Abstr. Appl. Anal.*, **2008** (2008), 237937. http://dx.doi.org/10.1155/2008/237937

21. G. Hu, D. Yang, Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type, *J. Math. Anal. Appl.*, **354** (2009), 249–262. http://dx.doi.org/10.1016/j.jmaa.2008.12.066

22. S. Janson, Mean oscillation and commutators of singular integral operators, *Ark. Mat.*, **16** (1978), 263–270. http://dx.doi.org/10.1007/BF02386000

23. T. Kato, Strong solutions of the Navier-Stokes equation in Morrey spaces, *Bol. Soc. Bras. Mat.*, **22** (1992), 127–155. http://dx.doi.org/10.1007/BF01232939

24. Y. Kim, Carleson measures and the BMO space on the p-adic vector space, *Math. Nachr.*, **282** (2009), 1278–1304. http://dx.doi.org/10.1002/mana.200610806

25. Y. Kim, L^q-estimates of maximal operators on the p-adic vector space, *Commun. Korean Math. Soc.*, **24** (2009), 367–379. http://dx.doi.org/10.4134/CKMS.2009.24.3.367

26. R. Liu, J. Zhou, Weighted multilinear p-adic Hardy operators and commutators, *Open Math.*, **15** (2017), 1623–1634. http://dx.doi.org/10.1515/math-2017-0139

27. M. Milman, T. Schonbek, Second order estimates in interpolation theory and applications, *Proc. Amer. Math. Soc.*, **110** (1990), 961–969.

28. C. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43** (1938), 126–166. http://dx.doi.org/10.2307/1989904

29. C. Pérez, Endpoint estimates for commutators of singular integral operators, *J. Funct. Anal.*, **128** (1995), 163–185. http://dx.doi.org/10.1006/jfan.1995.1027

30. C. Pérez, R. Wheeden, Uncertainty principle estimates for vector fields, *J. Funct. Anal.*, **181** (2001), 146–188. http://dx.doi.org/10.1006/jfan.2000.3711

31. M. Rao, Z. Ren, *Theory of Orlicz spaces*, New York: Marcel Dekker Inc., 1991.

32. C. Rios, The L^p Dirichlet problem and nondivergence harmonic measure, *Trans. Amer. Math. Soc.*, **355** (2003), 665–687. http://dx.doi.org/10.1090/S0002-9947-02-03145-8

33. K. Rim, J. Lee, Estimate of weighted Hard-Littlewood averages on the p-adic vector space, *J. Math. Anal. Appl.*, **324** (2006), 1470–1477. http://dx.doi.org/10.1016/j.jmaa.2006.01.038

34. K. Rogers, A van der Corput lemma for the p-adic numbers, *Proc. Amer. Math. Soc.*, **133** (2005), 3525–3534.

35. K. Rogers, Maximal averages along curves over the p-adic numbers, *Bull. Austral. Math. Soc.*, **70** (2004), 357–375. http://dx.doi.org/10.1017/S0004972700034602

36. C. Segovia, J. Torrea, Weighted inequalities for commutators of fractional and singular integrals, *Publ. Mat.*, **35** (1991), 209–235.

37. C. Segovia, J. Torrea, Higher order commutators for vector-valued Calderón-Zygmund operators, *Trans. Amer. Math. Soc.*, **36** (1993), 537–556.

38. Z. Shen, Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains, *Am. J. Math.*, **125** (2003), 1079–1115.

39. E. Stein, Note on the class $L \log L$, *Stud. Math.*, **32** (1969), 305–310.

40. V. Vladimirov, I. Volovich, E. Zelenov, *p-adic Analysis and Mathematical Physics*, Singapore: World Scientific, 1992. http://dx.doi.org/10.1142/1581

41. D. Wang, J. Zhou, Necessary and sufficient conditions for boundedness of commutators of bilinear Hardy-Littlewood maximal function, arXiv:1708.09549.

42. R. Wheeden, J. Wilson, Weighted norm estimates for gradients of half-space extensions, *Indiana Univ. Math. J.*, **44** (1995), 917–969.

43. P. Zhang, Multiple weighted estimates for commutators of multilinear maximal function, *Acta. Math. Sin.-English Ser.*, **31** (2015), 973–994. http://dx.doi.org/10.1007/s10114-015-4293-6

44. P. Zhang, Characterization of Lipschitz spaces via commutators of Hardy-Littlewood maximal function, *CR Math.*, **355** (2017), 336–344. http://dx.doi.org/10.1016/j.crma.2017.01.022

45. P. Zhang, Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces, *Anal. Math. Phys.*, **9** (2019), 1411–1427. http://dx.doi.org/10.1007/s13324-018-0245-5