AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 14008–14026.
DOI: 10.3934/math.2023716
Received: 27 December 2022
Revised: 30 March 2023
Accepted: 03 April 2023
Published: 14 April 2023

Research article

On Hilbert-Pachpatte type inequalities within ψ-Hilfer fractional generalized derivatives

Yasemin Başçı[1,*] **and Dumitru Baleanu**[2,3,4]

[1] Bolu Abant Izzet Baysal University, Department of Mathematics, Faculty of Arts and Sciences, Golkoy 14280, Bolu, Turkey

[2] Çankaya University, Department of Mathematics, Faculty of Arts and Sciences, Ankara 06530, Turkey

[3] Lebanese American University, Beirut, Lebanon

[4] Institutes of Space Sciences Magurele-Bucharest, Romania

* **Correspondence:** Email: basci_y@ibu.edu.tr.

Abstract: In this manuscript, we discussed various new Hilbert-Pachpatte type inequalities implying the left sided ψ-Hilfer fractional derivatives with the general kernel. Our results are a generalization of the inequalities of Pečarić and Vuković [1]. Furthermore, using the specific cases of the ψ-Hilfer fractional derivative, we proceed with wide class of fractional derivatives by selecting ψ, a_1, b_1 and considering the limit of the parameters α and β.

Keywords: Hilbert-Pachpatte type inequality; ψ-Hilfer fractional derivative; Riemann-Liouville derivative; homogeneous kernel; fractional derivative
Mathematics Subject Classification: 26A24, 26A33, 26B15

1. Introduction

Starting from 1695 [2–4], non-integer calculus has been applied in various fields of science and engineering [5–18]. Introducing new fractional integrals and derivatives leads to an increasing rate of growth and cleanses its numerous applications [12, 19–26]. Various researchers reported integral inequalities involving the different definitions of fractional derivatives [27–34].

We notice that Hilbert [35] presented the following integral inequality:

Theorem 1.1. *[35] (Chapter IX, Theorem 316) If* $g \in L^p(0,\infty)$*,* $h \in L^q(0,\infty)$*,* $g, h \geq 0$*,* $p > 1$ *and*

$q = \frac{p}{(p-1)}$, then

$$\int_0^\infty \int_0^\infty \frac{g(s)h(t)}{s+t} dsdt \leq \frac{\pi}{sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty g^p(s)ds\right)^{1/p} \times \left(\int_0^\infty h^q(t)dt\right)^{1/q}, \tag{1.1}$$

such that $\pi / \sin\left(\frac{\pi}{p_1}\right)$ is the best value.

We recall (1.1) as Hilbert's inequality. It is a fact that Hilbert's inequalities has an important place in analysis. In recent years, various mathematicians investigated Hilbert's inequalities, Hilbert's type inequalities and their several generalizations, see [1, 22, 36–66] and the references therein.

Pachpatte [47] reported the following integral inequality in a similar way as Hilbert's inequality:

Theorem 1.2. *[47] Let $0 \leq r \leq m-1$ and $m \geq 1$ be integers. Besides, let $u_1 \in C^m([0,x])$ and $v_1 \in C^m([0,y])$ such that $x > 0$, $y > 0$ and $u_1^{(i)}(0) = v_1^{(i)}(0) = 0$ for $i \in \{0, 1, ..., m-1\}$. As a results we give*

$$t_0^x \int_0^y \frac{\left|u_1^{(r)}(s)\right| \left|v_1^{(r)}(t)\right|}{s^{2m-2r-1} + t^{2m-2r-1}} dsdt$$
$$\leq M(m,r,x,y) \times \left(\int_0^x (x-s)\left(u_1^{(m)}(s)\right)^2 ds\right)^{1/2} \left(\int_0^y (y-t)\left(v_1^{(m)}(t)\right)^2 dt\right)^{1/2}, \tag{1.2}$$

where

$$M(m,r,x,y) := \frac{\sqrt{xy}}{2\left((m-r-1)!\right)^2 (2m-2r-1)}.$$

Handly et al. [40] derived new type Hilbert-Pachpatte integral inequalities from the results of Pachpatte in [47, 48].

Lü [46] gave some new inequalities related to the Hilbert-Pachpatte inequalities. Same year, Dragomir and Kim [38] extended and proved the results obtained by Handly et al. [40, 41]. So, they obtained new inequalities like Hilbert-Pachpatte-type inequalities.

He and Li [22] established new inequalities related to the Hilbert-Pachpatte inequalities. Also, they presented some new generalizations of Hilbert-Pachpatte inequality.

Anastassiou [36] gave very general weighted Hilbert-Pachpatte type integral inequalities involving Caputo and Riemann-Liouville fractional derivatives as well as fractional partial derivatives of the above mentioned types. Also, in 2021, Anastassiou [37] we presented Hilfer-Polya, ψ-Hilfer Ostrowski as well as ψ-Hilfer-Hilbert-Pachpatte types fractional inequalities implying left and right Hilfer and ψ-Hilfer fractional derivatives.

Zhao et al. [64] obtained some multiple integral Hilbert-Pachpatte type inequalities. One year later, Zhao and Cheung [65] established Hilbert-Pachpatte-type integral inequalities.

Pečarić and Vuković [1] studied of some generalizations of Hilbert-Pachpatte type inequality involving the fractional derivative utilizing the Taylor series of function and refinement of Arithmetic-Geometric Inequality (A.G.I.) from [67]. Their results based on the results of Krnić and Pečarić [45].

A new kind of fractional derivative presented by Sousa and Oliveira [68]. They gave ψ-Hilfer fractional derivative with respect to (w.r.t.) another function, having in mind to combine many fractional derivatives into a single fractional operator, and as a result they open a path for new applications. Later, their authors studied Gronwall inequality and the Cauchy-type problem by using of ψ-Hilfer operator in [69].

Our work is organized as given below. In Section 1 and Section 2, we present the introduction and preliminaries, respectively. Motivated by [1], we developed several new Hilbert-Pachpatte type inequalities for the left sided ψ-Hilfer fractional derivatives with the general kernel. Furthermore, using the particular cases of the ψ-Hilfer fractional derivative, we proceed with the wide class of fractional derivatives by taking into account ψ, a_1, b_1 and taking the limit of the parameters α and β.

2. Basic tools

In the following some basic tools required in this manuscript are given.

Definition 2.1. [68] Let (a_1, b_1) $(-\infty \leq a_1 < x < b_1 \leq \infty)$ be a finite or infinite interval and $\alpha > 0$. Also, let g be an integrable function on $[a_1, b_1]$, and ψ be positive monotone and an increaasing function on $(a_1, b_1]$ such that $\psi'(x)$ is a continuous derivative on (a_1, b_1). The left-sided fractional integral of a function g w.r.t. another function ψ becomes

$$I_{a_1^+}^{\alpha;\psi} g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} g(t) dt. \tag{2.1}$$

Definition 2.2. [68] Let $\psi'(x) \neq 0$ in $-\infty \leq a_1 < x < b_1 \leq \infty$, $m \in \mathbb{N}$ and $\alpha > 0$. The Riemann-Liouville derivatives of a function g w.r.t. ψ of order α correspondent to the Riemann-Liouville is given by

$$\begin{aligned} D_{a_1^+}^{\alpha;\psi} g(x) &= \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^m I_{a_1^+}^{m-\alpha;\psi} g(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^m \int_{a_1}^{x} \psi'(t) (\psi(x) - \psi(t))^{m-\alpha-1} g(t) dt, \end{aligned} \tag{2.2}$$

such that $m = [\alpha] + 1$.

Definition 2.3. [68] Let $m - 1 < \alpha < m$ $(m \in \mathbb{N})$, $I = [a_1, b_1]$ be the interval such that $-\infty \leq a_1 < b_1 \leq \infty$ and $g, \psi \in C^m([a_1, b_1], \mathbb{R})$ two functions such that $\psi'(x) > 0$ and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivative (left-sided) ${}^H D_{a_1^+}^{\alpha,\beta;\psi}(.)$ of function of order α and type $0 \leq \beta \leq 1$, is defined as

$${}^H D_{a_1^+}^{\alpha,\beta;\psi} g(x) = I_{a_1^+}^{\beta(m-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^m I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} g(x). \tag{2.3}$$

Also, (2.3) becomes

$$\begin{aligned} {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(x) &= I_{a_1^+}^{\gamma-\alpha;\psi} D_{a_1^+}^{\gamma;\psi} g(x) \\ &= \frac{1}{\Gamma(\gamma-\alpha)} \int_{a_1}^{x} \psi'(t) (\psi(x) - \psi(t))^{\gamma-\alpha-1} D_{a_1^+}^{\gamma;\psi} g(t) dt \end{aligned} \tag{2.4}$$

with $\gamma = \alpha + \beta(m - \alpha)$. Here, $I_{a_1^+}^{\gamma;\psi}(.)$ and $D_{a_1^+}^{\gamma;\psi}(.)$ are defined in (2.1) and (2.2), respectively.

Theorem 2.1. *[37] Let $\psi, g \in C^m([a_1, b_1])$ such that ψ is increasing and $\psi'(x) \neq 0$ over $[a_1, b_1]$, where $m-1 < \alpha < m$, $0 \le \beta \le 1$ and $\gamma = \alpha + \beta(m-\alpha)$, $x \in [a_1, b_1]$. Then,*

$$g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} \psi'(t)(\psi(x) - \psi(t))^{\alpha-1H} D_{a_1^+}^{\alpha,\beta;\psi} g(t)dt + \sum_{k=1}^{m-1} \frac{(\psi(x) - \psi(a_1))^{\gamma-k}}{\Gamma(\gamma-k+1)} g^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} g\right)(a_1). \tag{2.5}$$

Here notice that

$$I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} g(a_1) = 0.$$

If

$$g^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} g\right)(a_1) = 0, \ (k = 1, 2, ..., m-1),$$

then (2.5) becomes

$$g(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} \psi'(t)(\psi(x) - \psi(t))^{\alpha-1H} D_{a_1^+}^{\alpha,\beta;\psi} g(t)dt. \tag{2.6}$$

Krnić and Pecarić reported the following inequalities in [45]:

$$\int_{\Omega\times\Omega} K(x,y)\, g(x)\, h(y)\, d\mu_1(x)\, d\mu_2(y) \le \left[\int_\Omega \theta^p(x)\, G(x)\, g^p(x)\, d\mu_1(x)\right]^{1/p} \left[\int_\Omega \phi^q(y)\, H(y)\, h^q(y)\, d\mu_2(y)\right]^{1/q} \tag{2.7}$$

and

$$\int_\Omega H^{1-p}(y)\, \phi^{-p}(y) \left[\int_\Omega K(x,y)\, g(x)\, d\mu_1(x)\right]^p d\mu_2(y) \le \int_\Omega \theta^p(x)\, G(x)\, g^p(x)\, d\mu_1(x). \tag{2.8}$$

Here $p > 1$, μ_1 and μ_2 are positive σ-finite measures, $K : \Omega\times\Omega \to \mathbb{R}$, $g, h, \theta, \phi : \Omega \to \mathbb{R}$ are measurable nonnegative functions and

$$G(x) = \int_\Omega \frac{K(x,y)}{\phi^p(y)} d\mu_2(y), \tag{2.9}$$

$$H(y) = \int_\Omega \frac{K(x,y)}{\theta^q(x)} d\mu_1(x). \tag{2.10}$$

On the other hand, we give some definitions of Yang et al. [55]. That is, let $p_1, ..., p_l > 1$ be real parameters such that $\sum_{i=1}^{l} \frac{1}{p_i} = 1$. Also, let $K : \Omega^l \to \mathbb{R}$ and $\omega_{ij} : \Omega \to \mathbb{R}$ $(i, j = 1, ..., l)$ be nonnegative measurable functions. If $\prod_{i,j=1}^{l} \omega_{ij}\left(x_j\right) = 1$, then the following inequality gives for all measurable nonnegative functions $g_1, ..., g_l : \Omega \to \mathbb{R}$:

$$\int_{\Omega^l} K(x_1, ..., x_l) \prod_{i=1}^{l} g_i(x_i)\, dx_1 ... dx_l \le \prod_{i=1}^{l} \left(\int_\Omega G_i(x_i) \left(\omega_{ij} g_i\right)^{p_i}(x_i)\, dx_i\right)^{\frac{1}{p_i}}, \tag{2.11}$$

where

$$G_i(x_i) = \int_{\Omega^{l-1}} K(x_1, ..., x_l) \prod_{j=1, j\neq i}^{l} \omega_{ij}^{p_i}(x_i)\, dx_1 ... dx_{i-1} dx_{i+1} ... dx_{x_l}, \tag{2.12}$$

for $i = 1, ..., l$.

In [67], Krnić et al. showed the following refinements and converses of Young's inequality in quotient and difference form. If $x = (x_1, x_2, ..., x_m)$ and $p = (p_1, p_2, ..., p_m)$, we denote $P_m = \sum_{i=1}^m p_i$,

$$A_m(x) = \frac{\sum_{i=1}^m x_i}{m}, H_m(x) = \left(\prod_{i=1}^m x_i\right)^{1/m},$$

and

$$M_r(x,p) = \begin{cases} \left(\frac{1}{P_m}\sum_{i=1}^m p_i x_i^r\right)^{1/r}, & r \neq 0. \\ \left(\prod_{i=1}^m x_i^{p_i}\right)^{1/P_m}, & r = 0. \end{cases}$$

Lemma 2.1. *[52] Let $x = (x_1, x_2, ..., x_m)$ and $p = (p_1, p_2, ..., p_m)$ be positive m-tuples, such that $\sum_{i=1}^m \frac{1}{p_i} = 1$, and*

$$x^p = (x_1^{p_1}, x_2^{p_2}, ..., x_m^{p_m}), \quad \frac{1}{p} = \left(\frac{1}{p_1}, \frac{1}{p_2}, ..., \frac{1}{p_m}\right).$$

Then, the followings hold:

(i)

$$\left[\frac{A_m(x^p)}{H_m(x^p)}\right]^{m \min_{1\le i\le m}\left\{\frac{1}{p_i}\right\}} \le \frac{M_1\left(x^p, \frac{1}{p}\right)}{M_0\left(x^p, \frac{1}{p}\right)} \le \left[\frac{A_m(x^p)}{H_m(x^p)}\right]^{m \max_{1\le i\le m}\left\{\frac{1}{p_i}\right\}},$$

(ii)

$$\begin{aligned} m \min_{1\le i\le m}\left\{\frac{1}{p_i}\right\}\left[A_m(x^p) - H_m(x^p)\right] &\le M_1\left(x^p, \frac{1}{p}\right) - M_0\left(x^p, \frac{1}{p}\right) \\ &\le m \max_{1\le i\le m}\left\{\frac{1}{p_i}\right\}\left[A_m(x^p) - H_m(x^p)\right]. \end{aligned}$$

Now, define

$$k(\xi) := \int_0^\infty K(1,u) u^{-\xi} ds,$$

such that $K(x,y)$ represents a nonnegative homogeneous function of degree $-s$ ($s > 0$), that is, $K(tx,ty) = t^{-s}K(x,y)$ is satisfied. Suppose that $k(\xi) < \infty$ for $1 - s < \xi < 1$.

Lemma 2.2. *[1] If $\lambda > 0$, $1 - \lambda < \xi < 1$ and $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a nonnegative homogeneous function of degree $-\lambda$, then*

$$\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^\xi dy = x^{1-\lambda} k(\xi),$$

and

$$\int_0^\infty K(x,y)\left(\frac{y}{x}\right)^\xi dx = y^{1-\lambda} k(2 - \lambda - \xi).$$

3. Main results

Below, we will prove various new Hilbert-Pachpatte type inequalities including the left sided ψ-Hilfer fractional derivatives.

Theorem 3.1. *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$, $-\infty\leq a_1<b_1\leq\infty$ and $\gamma\geq\alpha+1$. Assume that $K:[a_1,b_1]\times[a_1,b_1]\to\mathbb{R}$ is nonnegative function, $\theta(x)$, $\phi(y)$ are nonnegative functions on $[a_1,b_1]$, and $g,h,\psi\in C^n([a_1,b_1],\mathbb{R})$ are functions such that ψ is an increasing function with $\psi'(x)\neq 0$, for all $x\in[a_1,b_1]$. Also, let $D_{a_1^+}^{\gamma;\psi}$ for $\alpha=\gamma$ and ${}^HD_{a_1^+}^{\alpha,\beta;\psi}$ be defined by (2.2) and (2.4), respectively. Then, we have*

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left[(x-a_1)^{\frac{1}{q(M_1-m_1)}}+(y-b_1)^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy\\
&\leq\frac{1}{4^{M_1-m_1}}\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(x-a_1)^{\frac{1}{q}}(y-b_1)^{\frac{1}{p}}}dxdy\\
&\leq\frac{1}{4^{M_1-m_1}(\Gamma(\gamma-\alpha))^2}\left(\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\right.\\
&\left.\times\,\theta^p(x)G(x)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}\\
&\times\left(\int_{a_1}^{b_1}\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\phi^q(y)H(y)|D_{a_1^+}^{\gamma;\psi}h(t)|^q dtdy\right)^{1/q}
\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}
&\int_{a_1}^{b_1}H^{1-p}(y)\phi^{-p}(y)\left(\int_{a_1}^{b_1}K(x,y)\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p}dx\right)^p dy\\
&\leq\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\theta^p(x)G(x)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx,
\end{aligned}\tag{3.2}$$

where $m_1=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $M_1=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, and $G(x)$ and $H(y)$ are defined as in (2.9) and (2.10), respectively.

Proof. Applying Hölder's inequality in (2.4) and using an increasing of the function $\psi(x)$, for $x\in[a_1,b_1]$ and $t\in[a_1,x]$ we obtain

$$\begin{aligned}
\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|&=\frac{1}{\Gamma(\gamma-\alpha)}\left|\int_{a_1}^{x}\psi'(t)(\psi(x)-\psi(t))^{\gamma-\alpha-1}D_{a_1^+}^{\gamma;\psi}g(t)dt\right|\\
&\leq\frac{1}{\Gamma(\gamma-\alpha)}\int_{a_1}^{x}(\psi'(t))(\psi(x)-\psi(t))^{\gamma-\alpha-1}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|dt\\
&\leq\frac{1}{\Gamma(\gamma-\alpha)}\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p}\left(\int_{a_1}^{x}1dt\right)^{1/q}.
\end{aligned}$$

From here, we can write

$$\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right| \leq \frac{(x-a_1)^{1/q}}{\Gamma(\gamma-\alpha)}\left(\int_{a_1}^{x}(\psi'(t))^{p}(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^{p}dt\right)^{1/p}. \tag{3.3}$$

Similarly, for $y \in [a_1, b_1]$ and $t \in [a_1, y]$ we can derive

$$|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)| \leq \frac{(y-a_1)^{1/p}}{\Gamma(\gamma-\alpha)}\left(\int_{a_1}^{y}(\psi'(t))^{q}(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\times|D_{a_1^+}^{\gamma;\psi}h(t)|^{q}dt\right)^{1/q}. \tag{3.4}$$

Multiplying (3.3) by (3.4), we get

$$\begin{aligned}\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right| &\leq \frac{(x-a_1)^{1/q}(y-a_1)^{1/p}}{[\Gamma(\gamma-\alpha)]^2}\\ &\times\left(\int_{a_1}^{x}(\psi'(t))^{p}(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^{p}dt\right)^{1/p}\\ &\times\left(\int_{a_1}^{y}(\psi'(t))^{q}(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^{q}dt\right)^{1/q}.\end{aligned} \tag{3.5}$$

Applying Lemma 2.1(i),we obtain

$$4^{M_1-m_1}(x^py^q)^{M_1-m_1} \leq (x^p+y^q)^{2(M_1-m_1)}, \quad x \geq 0, \quad y \geq 0, \tag{3.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$, and $m_1=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $M_1=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$. Replacing x by $x^{\frac{1}{pq(M_1-m_1)}}$ and y by $y^{\frac{1}{pq(M_1-m_1)}}$ in (3.6), taking it into (3.5), ones has

$$\begin{aligned}&\frac{4^{M_1-m_1}\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left[(x-a_1)^{\frac{1}{q(M_1-m_1)}}+(y-a_1)^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}\\ &\leq \frac{\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(x-a_1)^{\frac{1}{q}}(y-a_1)^{\frac{1}{p}}}\\ &\leq \frac{1}{(\Gamma(\gamma-\alpha))^2}\left(\int_{a_1}^{x}(\psi'(t))^{p}(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^{p}dt\right)^{1/p}\\ &\times\left(\int_{a_1}^{y}(\psi'(t))^{q}(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^{q}dt\right)^{1/q}.\end{aligned} \tag{3.7}$$

Multiplying (3.7) by the kernel $K(x,y)$ and integrating x and y on domain $[a_1,b_1]\times[a_1,b_1]$, we obtain

$$\begin{aligned}&4^{M_1-m_1}\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left[(x-a_1)^{\frac{1}{q(M_1-m_1)}}+(y-a_1)^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy\\ &\leq \int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(x-a_1)^{\frac{1}{q}}(y-a_1)^{\frac{1}{p}}}dxdy\end{aligned}$$

$$\leq \frac{1}{(\Gamma(\gamma-\alpha))^2}\int_{a_1}^{b_1}\int_{a_1}^{b_1} K(x,y)\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p}$$
$$\times\left(\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dt\right)^{1/q} dxdy. \tag{3.8}$$

Taking

$$g_1(x)=\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p}$$

and

$$h_1(y)=\left(\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dt\right)^{1/q}.$$

Replacing $g(t)$ by $g_1(t)$ and $h(t)$ by $h_1(t)$ in (2.7), we have

$$\int_{a_1}^{b_1}\int_{a_1}^{b_1} K(x,y)\,g_1(x)h_1(y)dxdy$$
$$\leq\left(\int_{a_1}^{b_1}\theta^p(x)\,G(x)\,g_1^p(x)dx\right)^{1/p}\left(\int_{a_1}^{b_1}\phi^q(y)\,H(y)\,h_1^q(y)dy\right)^{1/q}$$
$$=\left(\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(y)-\psi(t))^{p(\gamma-\alpha-1)}\theta^p(x)\,G(x)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}$$
$$\times\left(\int_{a_1}^{b_1}\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\phi^q(y)\,H(y)\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dtdy\right)^{1/q}. \tag{3.9}$$

Using (3.8) and (3.9), we obtain (3.1). Also, the inequality (3.2) can be proved by applying (2.8). □

Corollary 3.1. *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$ and $\gamma\geq\alpha+1$. Assume that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a nonnegative homogeneous function for degree $-\lambda$, $\lambda>0$, and $g,h,\psi\in C_0^n([0,\infty])$ are functions such that ψ is an increasing function with $\psi'(x)\neq 0$, for all $x\in([0,\infty),\mathbb{R})$. Also, let ${}^{H}D_{a_1^+}^{\alpha,\beta;\psi}$ and $D_{a_1^+}^{\gamma;\psi}$ be defined by (2.2) and (2.4), respectively. Then the inequalities hold:*

$$\int_0^\infty\int_0^\infty \frac{K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left[x^{\frac{1}{q(M_1-m_1)}}+y^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy$$
$$\leq\frac{pq}{4^{M_1-m_1}}\int_0^\infty\int_0^\infty K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|d\left(x^{1/p}\right)d\left(y^{1/q}\right)$$
$$\leq\frac{L}{4^{M_1-m_1}(\Gamma(\gamma-\alpha))^2}\left(\int_0^\infty\int_0^x(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\;x^{p(A_1-A_2)+1-\lambda}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}$$
$$\times\left(\int_0^\infty\int_0^y(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\;y^{q(A_2-A_1)+1-\lambda}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dtdy\right)^{1/q}, \tag{3.10}$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)\left(\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p}dx\right)^p dy$$
$$\leq L^p\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x))^{p(\gamma-\alpha-1)}x^{p(A_1-A_2)+1-\lambda}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx, \tag{3.11}$$

where M_1 and m_1 are defined as Theorem 3.1, and

$$A_1=\left(\frac{1-\lambda}{q},\frac{1}{q}\right),\ \ A_2=\left(\frac{1-\lambda}{p},\frac{1}{p}\right),\ \ L=\left[k(pA_2)\right]^{1/p}\left[k(2-\lambda-qA_1)\right]^{1/q}.$$

Proof. Let $G(x)$ and $H(y)$ be defined by (2.9) and (2.10), respectively. In (3.9), taking $\theta(x)=x^{A_1}$ and $\phi(y)=y^{A_2}$, and from $\psi(x)$ is an increasing function and $\gamma\geq\alpha+1$, we write

$$(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\leq\psi^{p(\gamma-\alpha-1)}(x),for\quad x\in[0,\infty]\quad and\quad t\in[0,x].$$

By using Lemma 2.2, we obtain

$$\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\theta^p(x)G(x)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx$$
$$\leq\int_0^\infty\int_0^x (\psi'(t))^p\psi^{p(\gamma-\alpha-1)}(x)x^{p(A_1-A_2)}\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2}dy\right)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx$$
$$=k(pA_2)\int_0^\infty\int_0^x (\psi'(t))^p\psi^{p(\gamma-\alpha-1)}(x)x^{p(A_1-A_2)+1-\lambda}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx. \tag{3.12}$$

Similarly, we can derive

$$\int_0^\infty\int_0^y (\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\phi^q(y)H(y)\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dtdy$$
$$\leq k(2-\lambda-qA_1)\int_0^\infty\int_0^y(\psi'(t))^q\psi^{q(\gamma-\alpha-1)}(y)y^{q(A_2-A_1)+1-\lambda}|D_{a_1^+}^{\gamma;\psi}h(t)|^q dtdy. \tag{3.13}$$

Therefore, from (3.1), (3.12) and (3.13), we obtain (3.10). Also, the inequality (3.11) can proved applying (3.2). So, we complete the proof of Corollary 3.1. □

Let's choose the special homogeneous function $K(x,y)$. Taking $K(x,y)=\frac{\ln\frac{y}{x}}{y-x}$ in Corollary 3.1, then we get the following Corollary 3.2:

Corollary 3.2. *Let $\frac{1}{p}+\frac{1}{q}=1$ for $p,q>1$ and $\gamma\geq\alpha+1$. Also, let M_1,m_1,g,h as in Corollary 3.1. Then we have*

$$\int_0^\infty\int_0^\infty\frac{\ln\frac{y}{x}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(y-x)\left[x^{\frac{1}{q(M_1-m_1)}}+y^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy$$
$$\leq\frac{pq}{4^{M_1-m_1}}\int_0^\infty\int_0^\infty\frac{\ln\frac{y}{x}}{y-x}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|d\left(x^{1/p}\right)d\left(y^{1/q}\right)$$

$$\leq \frac{L_1}{4^{M_1-m_1}\left(\Gamma(\gamma-\alpha)\right)^2}\left(\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\ x^{p(A_1-A_2)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}$$

$$\times\left(\int_0^\infty\int_0^y (\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\ y^{q(A_2-A_1)}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dtdy\right)^{1/q}, \quad (3.14)$$

and

$$\int_0^\infty y^{p(A_1-A_2)}\left(\int_0^\infty \frac{\ln\frac{y}{x}}{y-x}\left(\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\ \left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p} dx\right)^p dy$$

$$\leq L_1^p\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x))^{p(\gamma-\alpha-1)}\ x^{p(A_1-A_2)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx, \quad (3.15)$$

where $A_1=\left(0,\frac{1}{q}\right)$, $A_2=\left(0,\frac{1}{p}\right)$ *and* $L_1=\pi^2\left(\sin\left(pA_2\pi\right)\right)^{-2/p}\left(\sin\left(qA_1\pi\right)\right)^{-2/q}$.

Furthermore, for the homogeneous function of degree $-\lambda$, $\lambda>0$, taking $K(x,y)=(\max\{x,y\})^{-\lambda}$, $A_1=A_2=\frac{2-\lambda}{pq}$ with the condition $\lambda>2-\min\{p,q\}$ in Corollary 3.1, we can give the following result:

Corollary 3.3. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *for* $p,q>1$ *and* $\gamma\geq\alpha+1$. *Also, let* M_1,m_1,g,h *as in Corollary 3.1. Then inequalities hold:*

$$\int_0^\infty\int_0^\infty \frac{(\max\{x,y\})^{-\lambda}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left[x^{\frac{1}{q(M_1-m_1)}}+y^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy$$

$$\leq \frac{pq}{4^{M_1-m_1}}\int_0^\infty\int_0^\infty \frac{\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(\max\{x,y\})^\lambda}d\left(x^{1/p}\right)d\left(y^{1/q}\right)$$

$$\leq \frac{L_2}{4^{M_1-m_1}\left(\Gamma(\gamma-\alpha)\right)^2}\left(\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\right.$$

$$\left.\times\ x^{\lambda-1}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}\left(\int_0^\infty\int_0^y (\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\ y^{\lambda-1}\left|D_{a_1^+}^{\gamma;\psi}h(t)\right|^q dtdy\right)^{1/q}, \quad (3.16)$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty(\max\{x,y\})^{-\lambda}\left(\int_0^x (\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dt\right)^{1/p} dx\right)^p dy$$

$$\leq L_2^p\int_0^\infty\int_0^x (\psi'(t))^p(\psi(x))^{p(\gamma-\alpha-1)}\ x^{\lambda-1}\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx, \quad (3.17)$$

where $L_2=k\left(\frac{2-\lambda}{q}\right)$ *and* $k(\alpha)=\frac{\lambda}{(1-\alpha)(\lambda+\alpha-1)}$.

If we are applying the second refinement of A.G.I. (see Lemma 2.1 (ii)), then we report the related theorem:

Theorem 3.2. *Assume that the conditions of Theorem 3.1 provided. Then, the following inequalities hold:*

$$\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{\left(\frac{1}{2}\left[(x-a_1)^{\frac{1}{q}}+(y-b_1)^{\frac{1}{p}}\right]+\frac{1}{M_1-m_1}\right)^2}dxdy$$

$$\leq\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}g(x)\right|\left|{}^{H}D_{a_1^+}^{\alpha,\beta;\psi}h(y)\right|}{(x-a_1)^{\frac{1}{q}}(y-b_1)^{\frac{1}{p}}}dxdy$$

$$\leq\frac{1}{(\Gamma(\gamma-\alpha))^2}\left(\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\gamma-\alpha-1)}\ \theta^p(x)G(x)\left|D_{a_1^+}^{\gamma;\psi}g(t)\right|^p dtdx\right)^{1/p}$$

$$\times\left(\int_{a_1}^{b_1}\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\gamma-\alpha-1)}\phi^q(y)H(y)|D_{a_1^+}^{\gamma;\psi}h(t)|^q dtdy\right)^{1/q}. \tag{3.18}$$

Proof. Using Lemma 2.1 (ii), we get

$$x^py^q\leq\left(\frac{x^p+y^q}{2}+\frac{1}{M_1-m_1}\right)^2,\ \ x\geq 0\ \ and\ \ y\geq 0, \tag{3.19}$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and $m_1=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $M_1=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$. If we take (3.19) and proceed as in the proof of Theorem 3.1, then we obtain (3.18). □

Now, we can derive the following theorem, which is the generalization of Theorem 3.1. In the proof of the Theorem 3.3, we used a general Hilbert-type inequality (5) of Yang et al. [51].

Theorem 3.3. *Let $n,l\in\mathbb{R}$, $l\geq 2$, $\sum_{i=1}^{l}\frac{1}{p_i}=1$ with $p_1,...,p_l>1$ and $\gamma\geq\alpha+1$. Also, let α_i $(i=1,...,l)$ be defined by $\alpha_i=\prod_{j=1,j\neq i}^{l}p_j$. Assume that $K:[a_1,b_1]^l\to\mathbb{R}$ is nonnegative function, $\omega_{ij}\left(x_j\right)$ are nonnegative functions on $[a_1,b_1]$ for $i,j=1,...,l$ such that $\prod_{i,j=1}^{l}\omega_{ij}\left(x_j\right)=1$, and $g_i,\psi\in C^n([a_1,b_1])$ are functions such that ψ is an increasing function with $\psi'(x)\neq 0$, for all $x\in[a_1,b_1]$. Then we report the following:*

$$\int_{(a_1,b_1)^l}\frac{K(x_1,...,x_l)\prod_{i=1}^{l}\left|{}^{H}D_{a_1^+}^{\alpha_i,\beta;\psi}g_i(x_i)\right|}{\left(\sum_{i=1}^{l}(x_i-a_1)^{\frac{1}{\alpha_i(M_1-m_1)}}\right)^{l(M_1-m_1)}}dx_1...dx_l$$

$$\leq\frac{1}{l^{(M_1-m_1)l}}\int_{(a_1,b_1)^l}\frac{K(x_1,...,x_l)\prod_{i=1}^{l}\left|{}^{H}D_{a_1^+}^{\alpha_i,\beta;\psi}g_i(x_i)\right|}{\prod_{i=1}^{l}(x_i-a_1)^{\frac{1}{\alpha_i}}}dx_1...dx_l$$

$$\leq\frac{1}{l^{(M_1-m_1)l}\left[\Gamma(\gamma-\alpha)\right]^l}\prod_{i=1}^{l}\left(\int_{a_1}^{b_1}\int_{a}^{x_i}(\psi'(t))^{p_i}(\psi(x_i)-\psi(t))^{p_i(\gamma-\alpha_i-1)}\omega_{ij}^{p_i}(x_i)\right.$$

$$\left.\times G_i(x_i)\left|D_{a_1^+}^{\gamma;\psi}g_i(t)\right|^{p_i}dtdx_i\right)^{1/p_i}, \tag{3.20}$$

where $m_1=\min_{1\leq i\leq l}\left\{\frac{1}{p_i}\right\}$, $M_1=\max_{1\leq i\leq l}\left\{\frac{1}{p_i}\right\}$, and $G_i(x_i)$ is defined by (2.12) for $i=1,...,l$.

Also, using the Taylor series (see Theorem 2.1), we get the following the Hilbert-Pachpatte Type inequalities:

Theorem 3.4. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$, $-\infty\leq a_1<b_1\leq\infty$ *and* $\alpha\geq 1$. *Assume that* $K:[a_1,b_1]\times[a_1,b_1]\to\mathbb{R}$ *is nonnegative function,* $\theta(x)$, $\phi(y)$ *are nonnegative functions on* $[a_1,b_1]$, *and* $g,h,\psi\in C^n([a_1,b_1])$ *are functions such that* ψ *is an increasing with* $\psi'(x)\neq 0$, *for all* $x\in[a_1,b_1]$. *Also, for* $k=1,2,...,m-1$ *suppose that*

$$g^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi}g\right)(a_1)=0$$

and

$$h^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi}h\right)(a_1)=0.$$

Let ${}^HD_{a_1^+}^{\alpha,\beta;\psi}$ *be defined by (2.4). As a results we get*

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\,|g(x)|\,|h(y)|}{\left[(x-a_1)^{\frac{1}{q(M_1-m_1)}}+(y-b_1)^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}}dxdy\\ &\leq\frac{1}{4^{M_1-m_1}}\int_{a_1}^{b_1}\int_{a_1}^{b_1}\frac{K(x,y)\,|g(x)|\,|h(y)|}{(x-a_1)^{\frac{1}{q}}(y-b_1)^{\frac{1}{p}}}dxdy\\ &\leq\frac{1}{4^{M_1-m_1}(\Gamma(\alpha))^2}\left(\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\alpha-1)}\,\theta^p(x)\,G(x)\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)\right|^p dtdx\right)^{1/p}\\ &\times\left(\int_{a_1}^{b_1}\int_{a_1}^{y}(\psi'(t))^q(\psi(y)-\psi(t))^{q(\alpha-1)}\phi^q(y)H(y)|{}^HD_{a_1^+}^{\alpha,\beta;\psi}h(t)|^q dtdy\right)^{1/q}\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}&\int_{a_1}^{b_1}H^{1-p}(y)\,\Phi^{-p}(y)\left(\int_{a_1}^{b_1}K(x,y)\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\alpha-1)}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)\right|^p dt\right)^{1/p}dx\right)^p dy\\ &\leq\int_{a_1}^{b_1}\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\alpha-1)}\,\theta^p(x)\,G(x)\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)\right|^p dtdx,\end{aligned}\tag{3.22}$$

where $m_1=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$, $M_1=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$, *and* $G(x)$ *and* $H(y)$ *are defined as in (2.9) and (2.10) respectively.*

Proof. Applying the Hölder's inequality to (2.6), we obtain

$$\begin{aligned}|g(x)|&=\frac{1}{\Gamma(\alpha)}\left|\int_{a_1}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\,{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)dt\right|\\ &\leq\frac{1}{\Gamma(\alpha)}\int_{a_1}^{x}(\psi'(t))(\psi(x)-\psi(t))^{\alpha-1}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)\right|dt\\ &\leq\frac{1}{\Gamma(\alpha)}\left(\int_{a_1}^{x}(\psi'(t))^p(\psi(x)-\psi(t))^{p(\alpha-1)}\left|{}^HD_{a_1^+}^{\alpha,\beta;\psi}g(t)\right|^p dt\right)^{1/p}\left(\int_{a_1}^{x}1dt\right)^{1/q}\end{aligned}$$

$$= \frac{(x-a_1)^{1/q}}{\Gamma(\alpha)} \left(\int_{a_1}^{x} (\psi'(t))^p (\psi(x)-\psi(t))^{p(\alpha-1)} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(t) \right|^p dt \right)^{1/p}. \tag{3.23}$$

Similarly, we have

$$|h(y)| = \frac{(y-a_1)^{1/p}}{\Gamma(\alpha)} \left(\int_{a_1}^{y} (\psi'(t))^q (\psi(y)-\psi(t))^{q(\alpha-1)} \ |{}^H D_{a_1^+}^{\alpha,\beta;\psi} h(t)|^q dt \right)^{1/q}. \tag{3.24}$$

Repeating the same procedure as in the proof of Theorem 3.1, only multiplying (3.23) by (3.24) and using (3.6), one gets Theorem 3.4. □

Corollary 3.4. *Let $\frac{1}{p}+\frac{1}{q}=1$ with $p,q>1$ and $\alpha \geq 1$. Assume that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ is a nonnegative homogeneous function for degree $-\lambda$, $\lambda>0$, and $g,h,\psi \in C_0^n([0,\infty])$ are functions such that ψ is an increasing with $\psi'(x)\neq 0$, for all $x\in[0,\infty)$. Also, for $k=1,2,...,m-1$ suppose that*

$$g^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} g\right)(a_1)=0$$

and

$$h^{[m-k]}\left(I_{a_1^+}^{(1-\beta)(m-\alpha);\psi} h\right)(a_1)=0.$$

Let ${}^H D_{a_1^+}^{\alpha,\beta;\psi}$ be defined by (2.4). Then the inequalities hold:

$$\int_0^\infty \int_0^\infty \frac{K(x,y)\,|g(x)|\,|h(y)|}{\left[x^{\frac{1}{q(M_1-m_1)}}+y^{\frac{1}{p(M_1-m_1)}}\right]^{2(M_1-m_1)}} dxdy$$

$$\leq \frac{pq}{4^{M_1-m_1}} \int_0^\infty \int_0^\infty K(x,y)\,|g(x)|\,|h(y)|\, d\left(x^{1/p}\right) d\left(y^{1/q}\right)$$

$$\leq \frac{L}{4^{M_1-m_1}\,[\Gamma(\alpha)]^2} \left(\int_0^\infty \int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\alpha-1)} \ x^{p(A_1-A_2)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(t) \right|^p dtdx \right)^{1/p}$$

$$\times \left(\int_0^\infty \int_0^y (\psi'(t))^q (\psi(y)-\psi(t))^{q(\alpha-1)} \ y^{q(A_2-A_1)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} h(t) \right|^q dtdy \right)^{1/q}, \tag{3.25}$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)} \left(\int_0^\infty K(x,y) \left(\int_0^x (\psi'(t))^p (\psi(x)-\psi(t))^{p(\alpha-1)} \ \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(t) \right|^p dt \right)^{1/p} dx \right)^p dy$$

$$\leq L^p \int_0^\infty \int_0^x (\psi'(t))^p (\psi(x))^{p(\alpha-1)} \ x^{p(A_1-A_2)+1-\lambda} \left| {}^H D_{a_1^+}^{\alpha,\beta;\psi} g(t) \right|^p dtdx, \tag{3.26}$$

where M_1 and m_1 are defined as Theorem 3.2. Also,

$$A_1=\left(\frac{1-\lambda}{q},\frac{1}{q}\right),\ \ A_2=\left(\frac{1-\lambda}{p},\frac{1}{p}\right),\ \ L=\left[k(pA_2)\right]^{1/p}\left[k(2-\lambda-qA_1)\right]^{1/q}.$$

Now, we recall that in some particular cases of the ψ-Hilfer fractional derivative operator Eq (2.3), we proceed with wide class of fractional derivatives by taking some special values of $\psi(x)$, a_1, α and β (see [12, 26, 68]).

(**1**) Using the limit $\beta \to 1$ on both sides of the Eq (2.3), we obtain the following ψ-Caputo fractional operator w.r.t. another function:

$$ {}^{H}D_{a_1^+}^{\alpha,1;\psi} g(x) = I_{a_1^+}^{m-\alpha;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^m g(x) = {}^{C} D_{a_1^+}^{\alpha;\psi} g(x). \tag{3.27} $$

(**2**) Using the limit $\beta \to 0$ on both sides of the Eq (2.3), we get the following ψ-Riemann-Liouville fractional operator w.r.t. another function, namely:

$$ {}^{H}D_{a_1^+}^{\alpha,0;\psi} g(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);\psi} g(x) = D_{a_1^+}^{\alpha;\psi} g(x). \tag{3.28} $$

(**3**) For $\psi(x) = x$, considering the limit $\beta \to 1$ on both sides of the Eq (2.3), we end up with the following Caputo fractional operator:

$$ {}^{H}D_{a_1^+}^{\alpha,1;x} g(x) = I_{a_1^+}^{(m-\alpha);x} \left(\frac{d}{dx} \right)^m g(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} (x-t)^{m-\alpha-1} \left(\frac{d}{dt} \right)^m g(t)\, dt = {}^{C} D_{a_1^+}^{\alpha} g(x). \tag{3.29} $$

(**4**) For $\psi(x) = x^\rho$, considering the limit $\beta \to 0$ on both sides of the Eq (2.3), we obtain the below fractional operator (Katugampola):

$$ \begin{aligned} \rho^{\alpha}\ {}^{H}D_{a_1^+}^{\alpha,0;x^\rho} g(x) &= \rho^{\alpha} \left(\frac{1}{\rho x^{\rho-1}} \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);x^\rho} g(x) \\ &= \rho^{\alpha-m+1} \left(\frac{1}{\rho x^{\rho-1}} \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} t^{\rho-1} (x^\rho - t^\rho)^{m-\alpha-1} g(t)\, dt \\ &= {}^{\rho} D_{a_1^+}^{\alpha} g(x). \end{aligned} \tag{3.30} $$

(**5**) When $\psi(x) = x$, considering the limit $\beta \to 0$ on both sides of the Eq (2.3), we obtain the following Riemann-Liouville fractional operator:

$$ {}^{H}D_{a_1^+}^{\alpha,0;x} g(x) = \left(\frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);x} g(x) = \left(\frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} (x-t)^{m-\alpha-1} g(t)\, dt = D_{a_1^+}^{\alpha} g(x). \tag{3.31} $$

(**6**) When $\psi(x) = \ln x$, considering the limit $\beta \to 0$ on both sides of the Eq (2.3), we get the below Hadamard fractional operator:

$$ {}^{H}D_{a_1^+}^{\alpha,0;\ln x} g(x) = \left(x \frac{d}{dx} \right)^m I_{a_1^+}^{(m-\alpha);\ln x} g(x) = \left(x \frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} \left(\ln \left(\frac{x}{t} \right) \right)^{m-\alpha-1} g(t) \frac{dt}{t} = {}^{H} D_{a_1^+}^{\alpha} g(x). \tag{3.32} $$

(**7**) When $\psi(x) = \ln x$, considering the limit $\beta \to 1$ on both sides of the Eq (2.3), we obtain the Caputo-Hadamard fractional operator, namely:

$$ {}^{H}D_{a_1^+}^{\alpha,1;\ln x} g(x) = I_{a_1^+}^{(m-\alpha);\ln x} \left(x \frac{d}{dx} \right)^m g(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} \left(\ln \left(\frac{x}{t} \right) \right)^{m-\alpha-1} \left(t \frac{d}{dt} \right)^m g(t) \frac{dt}{t} = {}^{CH} D_{a_1^+}^{\alpha} g(x). \tag{3.33} $$

(**8**) For $\psi(x) = x^\rho$, considering the limit $\beta \to 1$ on both sides of the Eq (2.3), we get the following fractional operator (Caputo-Katugampola):

$$\rho^\alpha \quad {}^{H}D_{a_1^+}^{\alpha,1;x^\rho} g(x) = \rho^\alpha \quad I_{a_1^+}^{(m-\alpha);x^\rho} \left(\frac{1}{\rho x^{\rho-1}} \frac{d}{dx}\right)^m g(x)$$

$$= \rho^{\alpha-m+1} \frac{1}{\Gamma(m-\alpha)} \int_{a_1}^{x} t^{\rho-1} (x^\rho - t^\rho)^{m-\alpha-1} \left(\frac{1}{t^{\rho-1}} \frac{d}{dt}\right)^m g(t)\, dt = {}^{CK} D_{a_1^+}^{\alpha,\rho} g(x). \tag{3.34}$$

(**9**) For $\psi(x) = x$ and $a_1 = 0$, considering the limit $\beta \to 0$ on both sides of the Eq (2.3), we obtain the Riemann fractional operator:

$${}^{H}D_{0^+}^{\alpha,0;x} g(x) = \left(\frac{d}{dx}\right)^m I_{0^+}^{(m-\alpha);x} g(x) = {}^{R} D_{a_1^+}^{\alpha} g(x). \tag{3.35}$$

(**10**) For $\psi(x) = x$ and $a_1 = c$, considering the limit $\beta \to 0$ on both sides of the Eq (2.3), we report the following Chen fractional operator:

$${}^{H}D_{c^+}^{\alpha,0;x} g(x) = \left(\frac{d}{dx}\right)^m I_{c^+}^{(m-\alpha);x} g(x) = \left(\frac{d}{dx}\right)^m \frac{1}{\Gamma(m-\alpha)} \int_{c}^{x} (x-t)^{m-\alpha-1} g(t)\, dt = D_{c^+}^{\alpha} g(x). \tag{3.36}$$

Remark 3.1. For the above some particular cases of the ψ-Hilfer fractional derivative operator Eq (2.3), in Theorems 3.1–3.4 and Corollaries 3.1–3.4, we obtain new results for them.

4. Conclusions

Studies involving Hilbert's inequalities play an important place in analysis and application several. Recently, such inequalities were generalized and developed by mathematics. In this study, by generalizing of the inequalities in [1], we establish several new Hilbert-Pachpatte type for the left sided ψ-Hilfer fractional derivatives. Furthermore, using the particular cases of the ψ-Hilfer fractional derivative, we proceed with wide class of fractional derivatives by selecting ψ, a_1, b_1 and considering the limit of the parameters α and β.

Conflict of interest

The authors declare no conflict of interests.

References

1. J. Pečarić, P. Vuković, Hilbert-Pachpatte-type inequality due to fractional differential inequalities, *Ann. Univ. Craiova, Math. Comput. Sci. Ser.*, **41** (2014), 280–291.
2. G. W. Leibniz, Letter from Hanover, Germany to G.F.A. L'Hospital, September 30, 1695, In: *Mathematische schriften*, Olms-Verlag, Hildesheim, Germany, 1849, 301–302.
3. G. W. Leibniz, Letter from Hanover, Germany to Johann Bernoulli, December 28, 1695, In: *Mathematische schriften*, Olms-Verlag, Hildesheim, Germany, 1962, 226.

4. G. W. Leibniz, Letter from Hanover, Germany to John Wallis, May 30, 1697, In: *Mathematische schriften*, Olms-Verlag, Hildesheim, Germany, 1962, 25.

5. O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dyn.*, **38** (2004), 323–337. http://dx.doi.org/10.1007/S11071-004-3764-6

6. T. M. Atanackovic, S. Pilipovic, B. Stankovic, D. Zorica, *Fractional calculus with applications in mechanics: vibrations and diffusion processes*, Wiley, London, Hoboken, 2014. http://dx.doi.org/10.1002/9781118577530

7. D. D. Bainov, P. S. Simeonov, *Integral inequalities and applications*, Springer Dordrecht, 1992. https://doi.org/10.1007/978-94-015-8034-2

8. C. Bandle, L. Losonczi, A. Gilányi, Z. Páles, M. Plum, *Inequalities and applications*, Conference on inequalities and applications, Noszvaj (Hungary), September 2007, Birkhäuser Basel, 2009. https://doi.org/10.1007/978-3-7643-8773-0

9. S. Corlay, J. Lebovits, J. L. Véhel, Multifractional stochastic volatility models, *Math. Finance*, **24** (2014), 364–402. http://dx.doi.org/10.1111/mafi.12024

10. G. S. F. Frederico, D. F. M. Torres, Fractional conservation laws in optimal control theory, *Nonlinear Dyn.*, **53** (2008), 215–222.http://dx.doi.org/10.1007/s11071-007-9309-z

11. R. Herrmann, *Fractional calculus: an introduction for physicists*, Singapore: World Scientific Publishing Company, 2011.

12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, 1 Ed., North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.

13. R. L. Magin, C. Ingo, L. Colon-Perez, W. Triplett, T. H. Mareci, Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy, *Micropor. Mesopor. Mat.*, **178** (2013), 39–43. http://doi.org/10.1016/j.micromeso.2013.02.054

14. R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, *Comput. Math. Appl.*, **59** (2010), 1586–1593. http://doi.org/10.1016/j.camwa.2009.08.039

15. A. B. Malinowska, D. F. M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, *Comput. Math. Appl.*, **59** (2010), 3110–3116. https://doi.org/10.1016/j.camwa.2010.02.032

16. F. C. Meral, T. J. oyston, R. Magin, Fractional calculus in viscoelasticity: an experimental study, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 939–945. https://doi.org/10.1016/j.cnsns.2009.05.004

17. F. S. Costa, J. C. S. Soares, A. R. G. Plata, E. C. de Oliveira, On the fractional Harry Dym equation, *Comp. Appl. Math.*, **37** (2018), 2862–2876. https://doi.org/10.1007/s40314-017-0484-3

18. F. S. Costa, E. C. Grigoletto, J. Vaz Jr., E. C. de Oliveira, Slowing-down of neutrons: a fractional model, *Commun. Appl. Ind. Math.*, **6** (2015).

19. A. K. Anatoly, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38** (2001), 1191–1204.

20. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769.

21. A. Atangana, *Derivative with a new parameter: theory, methods and applications*, San Diego: Academic Press, 2015.

22. B. He, Y. Li, On several new inequalities close to Hilbert-Pachpatte's inequality, *J. Inequal. Pure Appl. Math.*, **7** (2006), 154.

23. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.*, **2012** (2012), 142. https://doi.org/10.1186/1687-1847-2012-142

24. U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062

25. U. N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, **6** (2014), 1–15.

26. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives, theory and applications*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

27. G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi, F. Moftakharzahed, Montgomery identities for fractional integrals and related fractional inequalities, *J. Inequal. Pure Appl. Math.*, **10** (2009), 97.

28. Y. Başcı, D. Baleanu, Hardy-type inequalities within fractional derivatives without singular kernel, *J. Inequal. Appl.*, **2018** (2018), 304. https://doi.org/10.1186/s13660-018-1893-6

29. Y. Başcı, D. Baleanu, New aspects of Opial-type integral inequalities, *Adv. Differ. Equ.*, **2018** (2018), 452. https://doi.org/10.1186/s13662-018-1912-4

30. S. Iqbal, K. Krulić, J. Pečarić, Weighted Hardy-type inequalities for monotone convex functions with some applications, *Fract. Differ. Calc.*, **3** (2013), 31–53. http://dx.doi.org/10.7153/fdc-03-03

31. S. Iqbal, K. Krulić, J. Pečarić, On refined-type inequalities with fractional integrals and fractional derivatives, *Math. Slovaca*, **64** (2014), 879–892. https://doi.org/10.2478/s12175-014-0246-2

32. S. Iqbal, K. Krulić, J. Pečarić, On a new class of Hardy-type inequalities with fractional integrals and fractional derivatives, *Rad Hazu. Math. Znan.*, **18** (2014), 91–106.

33. S. Iqbal, J. Pečarić, M. Samraiz, Z. Tomovski, Hardy-type inequalities for generalized fractional integral operators, *Tbilisi Math. J.*, **10** (2017), 75–90. https://doi.org/10.1515/tmj-2017-0005

34. M. Z. Sarıkaya, H. Budak, New inequalities of Opial type for conformable fractional integrals, *Turk. J. Math.*, **41** (2017), 1164–1173. https://doi.org/10.3906/mat-1606-91

35. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge: Cambridge University Press, 1934.

36. G. A. Anastassiou, Hilbert-Pachpatte type fractional integral inequalities, *Math. Comput. Model.*, **49** (2009), 1539–1550. https://doi.org/10.1016/j.mcm.2008.05.059

37. G. A. Anastassiou, Hilfer-Polya, ψ-Hilfer Ostrowski and ψ-Hilfer-Hilbert-Pachpatte fractional inequalities, *Symmetry*, **13** (2021), 463. https://doi.org/10.3390/sym130304

38. S. S. Dragomir, Y. H. Kim, Hilbert-Pachpatte type integral inequalities and their improvement, *J. Inequal. Pure Appl. Math.*, **4** (2003), 16.

39. M. Z. Gao, B. C. Yang, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.*, **126** (1998), 751–759.

40. G. D. Handley, J. J. Koliha, J. E. Pečarić, New Hilbert-Pachpatte type integral inequalities, *J. Math. Anal. Appl.*, **257** (2001), 238–250. https://doi.org/10.1006/jmaa.2000.7350

41. G. D. Handley, J. J. Koliha, J. E. Pečarić, A Hilbert type inequality, *Tamkang J. Math.*, **31** (2000), 311–315. https://doi.org/10.5556/j.tkjm.31.2000.389

42. K. Jichang, Note on new extensions of Hilbert's integral inequality, *J. Math. Anal. Appl.*, **235** (1999), 608–614. https://doi.org/10.1006/jmaa.1999.6373

43. K. Jichang, L. Debnath, On Hilbert type inequalities with non-conjugate parameters, *Appl. Math. Lett.*, **22** (2009), 813–818. https://doi.org/10.1016/j.aml.2008.07.010

44. J. Jin, L. Debnath, On a Hilbert-type linear series operator and its applications, *J. Math. Anal. Appl.*, **371** (2010), 691–704. https://doi.org/10.1016/j.jmaa.2010.06.002

45. M. Krnić, J. Pečarić, General Hilbert's and Hardy's inequalities, *Math. Inequal. Appl.*, **8** (2005), 29–52. https://doi.org/10.7153/mia-08-04

46. Z. Lü, Some new inequalities similar to Hilbert-Pachpatte's type inequalities, *J. Inequal. Pure Appl. Math.*, **4** (2003), 33.

47. B. G. Pachpatte, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.*, **226** (1998), 166–179.

48. B. G. Pachpatte, Inequalities similar to certain extensions of Hilbert's inequality, *J. Math. Anal. Appl.*, **243** (2000), 217–227. https://doi.org/10.1006/jmaa.1999.6646

49. M. Th. Rassias, B. Yang, On a Hilbert-type integral inequality in the whole plane with the equivalent forms, *J. Math. Inequal.*, **13** (2019), 315–334. https://doi.org/10.7153/jmi-2019-13-23

50. M. Th. Rassias, B. Yang, A. Raigorodskii, A Hilbert-type integral inequality in the whole plane related to the arc tangent function, *Symmetry*, **13** (2021), 351. https://doi.org/10.3390/sym13020351

51. B. Yang, I. Brnetić, M. Krnić, J. Pečarić, Generalization of Hilbert and Hardy-Hilbert integral inequalities, *Math. Inequal. Appl.*, **8** (2005), 259–272. https://doi.org/10.7153/mia-08-25

52. B. Yang, On new generalizations of Hilbert's inequality, *J. Math. Anal. Appl.*, **248** (2000), 29–40. https://doi.org/10.1006/jmaa.2000.6860

53. B. Yang, A half-discrete Hilbert-type inequality with a non-homogeneous kernel and two variables, *Mediterr. J. Math.*, **10** (2013), 677–692. https://doi.org/10.1007/s00009-012-0213-5

54. B. Yang, On a relation between Hilbert's inequality and a Hilbert-type inequality, *Appl. Math. Lett.*, **21** (2008), 483–488. https://doi.org/10.1016/j.aml.2007.06.001

55. B. Yang, D. Andrica, O. Bagdasar, M. Th. Rassias, An equivalent property of a Hilbert-type integral inequality and its applications, *Appl. Anal. Discrete Math.*, **16** (2022), 548–563.

56. B. Yang, M. Th. Rassias, *On Hilbert-type and Hardy-type integral inequalities and applications*, Springer Cham, 2019. https://doi.org/10.1007/978-3-030-29268-3

57. B. Yang, M. Th. Rassias, *On extended Hardy-Hilbert integral inequalities and applications*, World Scientific, 2023. https://doi.org/10.1142/13164

58. B. C. Yang, D. Andrica, O. Bagdasar, M. Th. Rassias, On a Hilbert-type integral inequality in the whole plane with the equivalent forms, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, **117** (2023), 57. https://doi.org/10.1007/s13398-023-01388-9

59. W. Yang, Some new Hilbert-Pachpatte's inequalities, *J. Inequal. Pure Appl. Math.*, **10** (2009), 26.

60. C. J. Zhao, Generalizations on two new Hilbert type inequalities, *J. Math.*, **20** (2000), 413–416.

61. C. J. Zhao, L. Debnath, Some new inverse type Hilbert integral inequalities, *J. Math. Anal. Appl.*, **262** (2001), 411–418. https://doi.org/10.1006/jmaa.2001.7595

62. C. J. Zhao, Inequalities similar to Hilbert's inequality, *Abstr. Appl. Anal.*, **2013** (2013), 861948. http://dx.doi.org/10.1155/2013/861948

63. C. J. Zhao, L. Y. Chen, W. S. Cheung, On some new Hilbert-type inequalities, *Math. Slovaca*, **61** (2011), 15–28. https://doi.org/10.2478/s12175-010-0056-0

64. C. J. Zhao, L. Y. Chen, W. S. Cheung, On Hilbert-Pachpatte multiple integral inequalities, *J. Inequal. Appl.*, **2010** (2010), 820857. https://doi.org/10.1155/2010/820857

65. C. J. Zhao, W. J. Cheung, On new Hilbert-Pachpatte type integral inequalities, *Taiwan. J. Math.*, **14** (2010), 1271–1282. https://doi.org/10.11650/twjm/1500405943

66. C. J. Zhao, J. Pečarić, G. S. Leng, Inverses of some new inequalities similar to Hilbert's inequalities, *Taiwan. J. Math.*, **10** (2006), 699–712. https://doi.org/10.11650/twjm/1500403856

67. M. Krnić, N. Lovričević, J. Pečarić, Jensen's functional, its properties and applications, *An. St. Univ. Ovidius Constanta*, **20** (2012), 225–248.

68. J. V. da C. Sousa, E. C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2017), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005

69. J. V. da C. Sousa, E. C. de Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, *Differ. Equ. Appl.*, **11** (2019), 87–106. https://doi.org/10.7153/dea-2019-11-02

AIMS Press