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Research article

Existence theory for a third-order ordinary differential equation with non-separated multi-point and nonlocal Stieltjes boundary conditions

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Abstract: This study develops the existence of solutions for a nonlinear third-order ordinary differential equation with non-separated multi-point and nonlocal Riemann-Stieltjes boundary conditions. Standard tools of fixed point theorems are applied to prove the existence and uniqueness of results for the problem at hand. Further, we made use of the fixed point theorem due to Bohnenblust-Karlin to discuss the existence of solutions for the multi-valued case. Lastly, we clarify the reported results by means of examples.

Keywords: Stieltjes boundary conditions; nonlocal; multi-point; existence; uniqueness; fixed point; Bohnenblust-Karlin
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1. Introduction

In recent years, solution examinations of boundary value models have increasingly become an interesting area of study. In fact, many real-world applications are modeled through boundary value problems of ordinary differential equations, for instance, problems arising from thermoelasticity, elasticity, fluid dynamics, quantum, and optical physics, chemical engineering, and population dynamics to mention a few, see [1–5] and the references therewith. Moreover, the expensive relevance of these problems entices vast a number of researchers to continuously develop ways to solve the models from the analytical, theoretical, and numerical aspects. In fact, the classical boundary conditions, including the Neumann, Dirichlet, and Robin have relatively become old-fashioned nowadays due to the development of more sophisticated conditions, which perfectly model complicated scenarios. In his regard, we mention some of the most significant boundary conditions comprising nonlocal boundary conditions [6–10], nonlocal multipoint integral boundary conditions [11, 12], discontinuous type integral boundary conditions [13], four-point nonlocal integral boundary conditions [14], multi-strip integral boundary conditions [15], integral boundary conditions [16–18] and the non-classical boundary condition [19] among other. Indeed, these types of conditions take into account all the physical or chemical changes that can occur inside the domain. Also, they can construct boundary data that contain random shapes such as fluid flow problems in the blood vessels. For more developments in the boundary value problems and their related boundary conditions, we refer the reader(s) to [20–24] and the given references therein.

In particular, the boundary conditions featuring integral terms (the nonlocals) are by far more suitable for describing irregularities in parts of curved boundary structures. Indeed, the Riemann-Stieltjes integral boundary condition [6–10] serves as a chief in this class of boundary conditions. Furthermore, they have many applications in physics and statistics (in the area of stochastic processes). We, therefore, refer the reader(s) to see [25–34] that dealt with Riemann-Stieltjes integral boundary conditions. These recently published articles discussed the single-valued case of differential equations equipped with Riemann-Stieltjes integral boundary conditions. The novelty in this study is to study the single and multi-valued case of differential equations with Riemann-Stieltjes integral boundary conditions using modern theories in the arena.

Consequently, this research discusses some existence and uniqueness results for the third-order boundary value problem with nonlocal Stieltjes typed boundary conditions. More explicitly, we make consideration to the following model

$$\begin{cases} y'''(\tau) = f(\tau, y(\tau)), \ a < \tau < T, \ a, T \in \mathbb{R}, \\ \alpha_1 y(a) + \alpha_2 y(T) = \sum_{i=1}^{r} \gamma_i y(\sigma_i) + \int_a^T y(s) d\phi(s) + \lambda_1, \\ \beta_1 y'(a) + \beta_2 y'(T) = \sum_{i=1}^{r} \rho_i y'(\sigma_i) + \int_a^T y'(s) d\phi(s) + \lambda_2, \\ \delta_1 y''(a) + \delta_2 y''(T) = \sum_{i=1}^{r} \nu_i y''(\sigma_i) + \int_a^T y''(s) d\phi(s) + \lambda_3, \end{cases} \tag{1.1}$$

where f is a given continuous function from $[a, T] \times \mathbb{R}$ to $\mathbb{R}$, $a < \sigma_1 < \sigma_2 < \cdots < \sigma_r < T$, ϕ is a bounded variation function. The values $\lambda_j \in \mathbb{R}$ $(j = 1, 2, 3)$, while $\alpha_j, \beta_j, \delta_j \in \mathbb{R}$ $(j = 1, 2)$, and $\gamma_i, \rho_i, \nu_i \in \mathbb{R}^+$ $(i = 1, 2, \ldots, r)$.

More precisely, we aim in this study to extend the traditional third-order boundary value problems to feature non-separated and multi-point Stieltjes boundary data over an interval of choice. Thus, the existence theory for boundary value problems of third-order ordinary differential equations (and inclusions) in the presence of these new boundary conditions will be established in the present study. Besides, Stieltjes' conditions act in the same way with several types of boundary conditions, such as the multi-point and integral boundary conditions. For details on Riemann-Stieltjes integral conditions, we refer the reader(s) to Whyburn [35] and Conti [36]. In addition, the standard tools of functional analysis would be used to prove the existence theory for nonlinear boundary value problems as derived in [37–41].

Lastly, we arrange the paper as follows: In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1.1). The existence and uniqueness of results for the boundary value problem (1.1), together with illustrative examples are proved in Section 3. Section 4 presents the existence of solutions for the parallel multi-valued problem of the problem (1.1), while Section 5 gives some concluding notes.

2. Preliminary result

The solution for the linear variants of the problem (1.1) is defined in the lemma that follows.

Lemma 2.1. *Assume that* $\xi \in C[a,T]$, *and*

$$\Big(\delta_1+\delta_2-\sum_{i=1}^{r}\nu_i-\int_a^T d\phi(s)\Big)\Big(\beta_1+\beta_2-\sum_{i=1}^{r}\rho_i-\int_a^T d\phi(s)\Big)\Big(\alpha_1+\alpha_2-\sum_{i=1}^{r}\gamma_i-\int_a^T d\phi(s)\Big)\neq 0,$$

is satisfied. Then, the following linear problem

$$\begin{cases} y'''(\tau)=\xi(\tau),\ \ a<\tau<T,\\ \alpha_1 y(a)+\alpha_2 y(T)=\sum_{i=1}^{r}\gamma_i y(\sigma_i)+\int_a^T y(s)d\phi(s)+\lambda_1,\\ \beta_1 y'(a)+\beta_2 y'(T)=\sum_{i=1}^{r}\rho_i y'(\sigma_i)+\int_a^T y'(s)d\phi(s)+\lambda_2,\\ \delta_1 y''(a)+\delta_2 y''(T)=\sum_{i=1}^{r}\nu_i y''(\sigma_i)+\int_a^T y''(s)d\phi(s)+\lambda_3,\end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$\begin{aligned} y(\tau) &= \int_a^\tau \frac{(\tau-s)^2}{2}\xi(s)ds\\ &-\frac{1}{\Gamma}\int_a^T\Big[\alpha_2E_1E_2\frac{(T-s)^2}{2}+\beta_2K_1(\tau)(T-s)+\delta_2K_2(\tau)\Big]\xi(s)ds\\ &+\frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_iE_1E_2\frac{(\sigma_i-s)^2}{2}+\rho_iK_1(\tau)(\sigma_i-s)+\nu_iK_2(\tau)\Big]\xi(s)ds\\ &+\frac{1}{\Gamma}\int_a^T\Big[\int_a^s\Big(E_1+-2\frac{(s-t)^2}{2}+K_1(\tau)(s-t)+K_2(\tau)\Big)\xi(t)dt\Big]d\phi(s)\\ &+\frac{1}{E_4}\lambda_1+\frac{1}{\Gamma}K_1(\tau)\lambda_2+\frac{1}{\Gamma}K_2(\tau)\lambda_3, \end{aligned} \tag{2.2}$$

where

$$K_1(\tau)=E_1\Big(E_4(\tau-a)-E_5\Big),\ K_2(\tau)=E_3\Big(E_5-E_4(\tau-a)\Big)-E_2\Big(E_6-E_4\frac{(\tau-a)^2}{2}\Big), \tag{2.3}$$

with

$$\begin{cases} \Gamma = E_1E_2E_4,\ E_1 = \delta_1 + \delta_2 - \sum_{i=1}^{r} \nu_i - \int_a^T d\phi(s) \neq 0, \\ E_2 = \beta_1 + \beta_2 - \sum_{i=1}^{r} \rho_i - \int_a^T d\phi(s) \neq 0, \\ E_4 = \alpha_1 + \alpha_2 - \sum_{i=1}^{r} \gamma_i - \int_a^T d\phi(s) \neq 0, \\ E_3 = \beta_2(T-a) - \sum_{i=1}^{r} \rho_i(\sigma_i - a) - \int_a^T (s-a)d\phi(s), \\ E_5 = \alpha_2(T-a) - \sum_{i=1}^{r} \gamma_i(\sigma_i - a) - \int_a^T (s-a)d\phi(s), \\ E_6 = \alpha_2\frac{(T-a)^2}{2} - \sum_{i=1}^{r} \gamma_i\frac{(\sigma_i - a)^2}{2} - \int_a^T \frac{(s-a)^2}{2}d\phi(s). \end{cases} \tag{2.4}$$

Proof. Integrating $y'''(\tau) = \xi(\tau)$ from a to τ trice, one gets

$$y(\tau) = c_0 + c_1(\tau - a) + c_2\frac{(\tau - a)^2}{2} + \int_a^\tau \frac{(\tau - s)^2}{2}\xi(s)ds, \tag{2.5}$$

where c_0, c_1 and c_2 are chosen real unknown constants. Then, upon combining the third boundary condition of (2.1) with (2.5), we get

$$c_2 = \frac{1}{E_1}\Big[-\delta_2\int_a^T \xi(s)ds + \sum_{i=1}^{r} \nu_i \int_a^{\sigma_i} \xi(s)ds + \int_a^T \Big(\int_a^s \xi(t)dt\Big)d\phi(s) + \lambda_3\Big]. \tag{2.6}$$

Using the second boundary condition of (2.1) together with (2.6) in

$$y'(\tau) = c_1 + c_2(\tau - a) + \int_a^\tau (\tau - s)\xi(s)ds,$$

we obtain

$$\begin{aligned} c_1 \ &= \tfrac{1}{E_1E_2}\Big[E_1\Big(-\beta_2\textstyle\int_a^T (T-s)\xi(s)ds + \sum_{i=1}^{r} \rho_i \int_a^{\sigma_i}(\sigma_i - s)\xi(s)ds \\ &+ \int_a^T \Big(\int_a^s (s-t)\xi(t)dt\Big)d\phi(s) + \lambda_2\Big) \\ &-E_3\Big(-\delta_2\int_a^T \xi(s)ds + \sum_{i=1}^{r} \nu_i \int_a^{\sigma_i} \xi(s)ds + \int_a^T \Big(\int_a^s \xi(t)dt\Big)d\phi(s) + \lambda_3\Big)\Big]. \end{aligned} \tag{2.7}$$

Finally, upon using the first boundary condition of (2.1) in (2.5) together with (2.6) and (2.7), one gets

$$\begin{aligned} c_0 \ &= \tfrac{1}{\Gamma}\Big\{\Big(E_3E_5 - E_2E_6\Big)\Big[-\delta_2\textstyle\int_a^T \xi(s)ds + \sum_{i=1}^{r} \nu_i \int_a^{\sigma_i} \xi(s)ds \\ &+ \int_a^T \Big(\int_a^s \xi(t)dt\Big)d\phi(s) + \lambda_3\Big] - E_1E_5\Big[-\beta_2\int_a^T (T-s)\xi(s)ds \end{aligned}$$

$$+\sum_{i=1}^{r}\rho_i\int_a^{\sigma_i}(\sigma_i-s)\xi(s)ds+\int_a^T\Big(\int_a^s(s-t)\xi(t)dt\Big)d\phi(s)+\lambda_2\Big]$$

$$+E_1E_2\Big[-\alpha_2\int_a^T\frac{(T-s)^2}{2}\xi(s)ds+\sum_{i=1}^{r}\gamma_i\int_a^{\sigma_i}\frac{(\sigma_i-s)^2}{2}\xi(s)ds$$

$$+\int_a^T\Big(\int_a^s\frac{(s-t)^2}{2}\xi(t)dt\Big)d\phi(s)+\lambda_1\Big]\Big\},$$

where Γ and E_j $(j=1,\dots,6)$ are given by (2.4). Therefore, substituting the values of c_0, c_1, and c_2 into (2.5) and using the notations expressed in (2.3), the solution of (2.2) is thus obtained. Moreover, the converse of the lemma follows by direct computation. This completes the proof.

3. Main results

Let $\mathcal{H}=C([a,T],\mathbb{R})$ denote the Banach space, which further contains continuous functions from $[a,T]\to\mathbb{R}$ with the norm introduced by $\|y\|=\sup\{|y(\tau)|,\tau\in[a,T]\}$. Then, on using Lemma 2.1, we transform problem (1.1) into an analogue fixed point problem as

$$y=\mathcal{P}y, \tag{3.1}$$

where $\mathcal{P}:\mathcal{H}\to\mathcal{H}$ is defined by

$$\begin{aligned}(\mathcal{P}y)(\tau) &= \int_a^\tau\frac{(\tau-s)^2}{2}f(s,y(s))ds\\ &-\frac{1}{\Gamma}\int_a^T\Big[\alpha_2E_1E_2\frac{(T-s)^2}{2}+\beta_2K_1(\tau)(T-s)+\delta_2K_2(\tau)\Big]f(s,y(s))ds\\ &+\frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_iE_1E_2\frac{(\sigma_i-s)^2}{2}+\rho_iK_1(\tau)(\sigma_i-s)+\nu_iK_2(\tau)\Big]f(s,y(s))ds\\ &+\frac{1}{\Gamma}\int_a^T\Big[\int_a^s\Big(E_1E_2\frac{(s-t)^2}{2}+K_1(\tau)(s-t)+K_2(\tau)\Big)f(t,y(t))dt\Big]d\phi(s)\\ &+\frac{1}{E_4}\lambda_1+\frac{1}{\Gamma}K_1(\tau)\lambda_2+\frac{1}{\Gamma}K_2(\tau)\lambda_3.\end{aligned} \tag{3.2}$$

If the operator expressed in (3.1) has fixed points, then we can say that problem (1.1) has solutions. Additionally, let us consider the following values, which would be used in the result of this study,

$$\begin{aligned}\Theta &= \frac{(T-a)^3}{3!}+\frac{1}{|E_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+\sum_{i=1}^{r}\gamma_i\frac{(\sigma_i-a)^3}{3!}+\int_a^T\frac{(s-a)^3}{3!}d\phi(s)\Big]\\ &+\frac{k_1}{|\Gamma|}\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^{r}\rho_i\frac{(\sigma_i-a)^2}{2}+\int_a^T\frac{(s-a)^2}{2}d\phi(s)\Big]\\ &+\frac{k_2}{|\Gamma|}\Big[|\delta_2|(T-a)+\sum_{i=1}^{r}\nu_i(\sigma_i-a)+\int_a^T(s-a)d\phi(s)\Big],\end{aligned} \tag{3.3}$$

and

$$\Theta_1 = \Theta + \left|\frac{\lambda_1}{E_4}\right| + \left|\frac{\lambda_2}{\Gamma}\right|k_1 + \left|\frac{\lambda_3}{\Gamma}\right|k_2, \tag{3.4}$$

where $\max_{\tau\in[a,T]}|K_1(\tau)| = k_1$ and $\max_{\tau\in[a,T]}|K_2(\tau)| = k_2$.

3.1. Existence of solutions

In this regard, Krasnoselskii's fixed point theorem [42] would be used to prove the existence of solutions for the problem (1.1) in what follows.

Lemma 3.1. *(Krasnoselskii's fixed point theorem). Let L be a closed bounded, convex and nonempty subset of a Banach space X. Let ϕ_1 and ϕ_2 be operators such that*
(i) $\phi_1 l_1 + \phi_2 l_2 \in L$ whenever $l_1, l_2 \in L$,
(ii) ϕ_1 is compact and continuous,
(iii) ϕ_2 is a contraction mapping. Then, there exists $z \in L$ such that

$$z = \phi_1 z + \phi_2 z.$$

Theorem 3.1. *Let $f : [a,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(C_1) $|f(\tau,y) - f(\tau,x)| \le \ell|y-x|,\ \forall\tau\in[a,T],\ \ell > 0,\ y,x\in\mathbb{R}.$

(C_2) *There exist a map $\kappa \in C([a,T],\mathbb{R}^+)$ with $\|\kappa\| = \sup_{\tau\in[a,T]}|\kappa(\tau)|$ such that $|f(\tau,y)| \le \kappa(\tau)$, $\forall(\tau,y)\in[a,T]\times\mathbb{R}$.*

In addition, it is assumed that

$$\ell\Big(\Theta - \frac{(T-a)^3}{3!}\Big) < 1, \tag{3.5}$$

where Θ is defined by (3.3). *Then, the problem* (1.1) *has at least one solution on* $[a,T]$.

Proof. Consider a closed ball $B_w = \{y\in\mathcal{H} : \|y\| \le w\}$ for fixed $w \ge \Theta_1\|\kappa\|$, where Θ_1 is given by (3.4). Let us define the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on B_w as follows:

$$(\mathcal{P}_1 y)(\tau) = \int_a^\tau \frac{(\tau-s)^2}{2} f(s,y(s))ds,$$

and

$$\begin{aligned}(\mathcal{P}_2 y)(\tau) &= -\frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] f(s,y(s))ds \\ &+\frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i - s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big] f(s,y(s))ds \\ &+\frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-\tau)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) f(t,y(t))dt\Big]d\phi(s) \\ &+\frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3.\end{aligned}$$

Notice that $(\mathcal{P}y)(\tau) = (\mathcal{P}_1 y)(\tau) + (\mathcal{P}_2 y)(\tau)$. For $\tau\in[a,T]$ and $y,x\in B_w$, we find that

$$\|\mathcal{P}_1 y + \mathcal{P}_2 x\| = \sup_{\tau\in[a,T]}\Big\{\Big|\int_a^\tau \frac{(\tau-s)^2}{2} f(s,y(s))ds$$

$$
\begin{aligned}
&-\frac{1}{\Gamma}\int_a^T\Big[\alpha_2E_1E_2\frac{(T-s)^2}{2}+\beta_2K_1(\tau)(T-s)+\delta_2K_2(\tau)\Big]f(s,x(s))ds\\
&+\frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_iE_1E_2\frac{(\sigma_i-s)^2}{2}+\rho_iK_1(\tau)(\sigma_i-s)+\nu_iK_2(\tau)\Big]f(s,x(s))ds\\
&+\frac{1}{\Gamma}\int_a^T\Big[\int_a^s\Big(E_1E_2\frac{(s-t)^2}{2}+K_1(\tau)(s-t)+K_2(\tau)\Big)f(t,x(t))dt\Big]d\phi(s)\\
&+\frac{1}{E_4}\lambda_1+\frac{1}{\Gamma}K_1(\tau)\lambda_2+\frac{1}{\Gamma}K_2(\tau)\lambda_3\Big|\Big\},\\
&\leq\|\kappa\|\sup_{\tau\in[a,T]}\Big\{\frac{(\tau-a)^3}{3!}\\
&+\frac{1}{|E_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+\sum_{i=1}^{r}\gamma_i\frac{(\sigma_i-a)^3}{3!}+\int_a^T\frac{(s-a)^3}{3!}d\phi(s)\Big]\\
&+\frac{1}{|\Gamma|}|K_1(\tau)|\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^{r}\rho_i\frac{(\sigma_i-a)^2}{2}+\int_a^T\frac{(s-a)^2}{2}d\phi(s)\Big]\\
&+\frac{1}{|\Gamma|}|K_2(\tau)|\Big[|\delta_2|(T-a)+\sum_{i=1}^{r}\nu_i(\sigma_i-a)+\int_a^T(s-a)d\phi(s)\Big]\\
&+\Big|\frac{\lambda_1}{E_4}\Big|+\Big|\frac{\lambda_2}{\Gamma}\Big||K_1(\tau)|+\Big|\frac{\lambda_3}{\Gamma}\Big||K_2(\tau)|\Big\}\leq\|\kappa\|\Theta_1\ \leq w,
\end{aligned}
$$

where Θ_1 is given by (3.4). This shows that $\mathcal{P}_1y+\mathcal{P}_2x\in B_w$. Thus, condition (i) in Lemma 3.1 is satisfied. Next, on using the assumption of (C_1) and (3.5), we obtain

$$
\begin{aligned}
\|\mathcal{P}_2y-\mathcal{P}_2x\| &= \sup_{\tau\in[a,T]}\Big\{\frac{1}{|\Gamma|}\int_a^T\Big[|\alpha_2E_1E_2|\frac{(T-s)^2}{2}+|\beta_2K_1(\tau)|(T-s)+|\delta_2K_2(\tau)|\Big]\\
&\quad\times\Big|f(s,y(s))-f(s,x(s))\Big|ds+\frac{1}{|\Gamma|}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_i|E_1E_2|\frac{(\sigma_i-s)^2}{2}\\
&+\ \rho_i|K_1(\tau)|(\sigma_i-s)+\nu_i|K_2(\tau)|\Big]\Big|f(s,y(s))-f(s,x(s))\Big|\\
&+\ \frac{1}{|\Gamma|}\int_a^T\Big[\int_a^s\Big(|E_1E_2|\frac{(s-t)^2}{2}+|K_1(\tau)|(s-t)+|K_2(\tau)|\Big)\\
&\quad\times\Big|f(t,y(t))-f(t,x(t))\Big|dt\Big]d\phi(s)\Big\},\\
&\leq\ \ell\parallel u-v\parallel\Big\{\frac{1}{|E_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+\sum_{i=1}^{r}\gamma_i\frac{(\sigma_i-a)^3}{3!}+\int_a^T\frac{(s-a)^3}{3!}d\phi(s)\Big]\\
&+\ \frac{k_1}{|\Gamma|}\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^{r}\rho_i\frac{(\sigma_i-a)^2}{2}+\int_a^T\frac{(s-a)^2}{2}d\phi(s)\Big]
\end{aligned}
$$

$$+ \quad \frac{k_2}{|\Gamma|}\Big[|\delta_2|(T-a) + \sum_{i=1}^{r} \nu_i(\sigma_i - a) + \int_a^T (s-a)d\phi(s)\Big]\Big\},$$

$$\leq \quad \ell\Big(\Theta - \frac{(T-a)^3}{3!}\Big) \parallel y - x \parallel,$$

which implies that $\mathcal{P}_2$ is a contraction operator. Then, we prove that $\mathcal{P}_1$ is compact and continuous. Notice also that, the continuity of f means that the operator $\mathcal{P}_1$ is continuous. In addition, $\mathcal{P}_1$ is uniformly bounded on B_w as

$$\|\mathcal{P}_1 y\| \leq \|\kappa\| \frac{(T-a)^3}{3!}.$$

Let us fix $\sup_{(\tau,y)\in[a,T]\times B_w} |f(\tau,y)| = \bar{f}$, and take $a < \tau_1 < \tau_2 < T$. Then,

$$\begin{aligned}
|(\mathcal{P}_1 y)(\tau_2) - (\mathcal{P}_1 y)(\tau_1)| &= \Big|\int_a^{\tau_1} \Big[\frac{(\tau_2 - s)^2}{2} - \frac{(\tau_1 - s)^2}{2}\Big] f(s, y(s))ds \\
&+ \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^2}{2} f(s, y(s))ds\Big|, \\
&\leq \bar{f}\Big(\frac{(\tau_2 - \tau_1)^3}{3} + \frac{1}{3!}\Big|(\tau_2 - a)^3 - (\tau_1 - a)^3\Big|\Big), \\
&\to 0 \text{ as } \tau_2 \to \tau_1,
\end{aligned}$$

independently of $y \in B_w$. This implies that $\mathcal{P}_1$ is relatively compact on B_w. Hence, it follows by the Arzelá-Ascoli theorem that the operator $\mathcal{P}_1$ is compact on B_w. Thus, all the hypotheses of Lemma 3.1 have been fulfilled. Consequently, by Krasnoselskii's fixed point theorem, the problem (1.1) has at least one solution on $[a, T]$.

Remark 3.1. *If we switch the role of the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ in the theorem above, then the condition* (3.5) *is replaced with* $\ell\frac{(T-a)^3}{3!} < 1$.

In the following theorem, we prove the uniqueness result of solutions for the problem (1.1) by implementation of the Banach's contraction mapping principle [43].

3.2. Uniqueness of solutions

Theorem 3.2. *Assume that $f : [a, T] \times \mathbb{R} \to \mathbb{R}$ satisfies (C_1) condition. Then, the boundary value problem (1.1) has a unique solution on $[a, T]$ if $\ell < 1/\Theta$, where Θ is given by* (3.3).

Proof. Consider a set $B_\iota = \{y \in \mathcal{H} : \|y\| \leq \iota\}$, where $\iota \geq \dfrac{\Theta_1 q}{1 - \ell\Theta_1}$, $\sup_{\tau\in[a,T]} |f(\tau, 0)| = q$. In the first step, we show that $\mathcal{P}B_\iota \subset B_\iota$, where the operator $\mathcal{P}$ is defined by (3.2). For any $y \in B_\iota, \tau \in [a, T]$, we find that

$$\begin{aligned}
|f(s, y(s))| &= |f(s, y(s)) - f(s, 0) + f(s, 0)|, \\
&\leq |f(s, y(s)) - f(s, 0)| + |f(s, 0)|, \\
&\leq \ell\|y\| + q, \\
&\leq \ell\iota + q.
\end{aligned}$$

Then, for $y \in B_\iota$, we obtain

$$
\begin{aligned}
\|(\mathcal{P}y)\| &= \sup_{\tau\in[a,T]}\Big\{\Big|\int_a^\tau \frac{(t-s)^2}{2} f(s,y(s))ds \\
&- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] f(s,y(s))ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big] f(s,y(s))ds \\
&+ \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1 E_2 \frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) f(t,y(t))dt\Big] d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3\Big|\Big\}, \\
&\le (\ell\iota + q)\Big\{\frac{(T-a)^3}{3!} \\
&+ \frac{1}{|E_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!} + \sum_{i=1}^r \gamma_i \frac{(\sigma_i-a)^3}{3!} + \int_a^T \frac{(s-a)^3}{3!} d\phi(s)\Big] \\
&+ \frac{k_1}{|\Gamma|}\Big[|\beta_2|\frac{(T-a)^2}{2} + \sum_{i=1}^r \rho_i \frac{(\sigma_i-a)^2}{2} + \int_a^T \frac{(s-a)^2}{2} d\phi(s)\Big] \\
&+ \frac{k_2}{|\Gamma|}\Big[|\delta_2|(T-a) + \sum_{i=1}^r \nu_i(\sigma_i - a) + \int_a^T (s-a)d\phi(s)\Big] \\
&+ \Big|\frac{\lambda_1}{E_4}\Big| + \Big|\frac{\lambda_2}{\Gamma}\Big| g_1 + \Big|\frac{\lambda_3}{\Gamma}\Big| g_2\Big\}, \\
&\le (\ell\iota + q)\Theta_1, \\
&\le \iota,
\end{aligned}
$$

where Θ_1 is given by (3.4). This shows that $\mathcal{P}B_\iota \subset B_\iota$.

Now we show that the operator $\mathcal{P}$ is a contraction. Then, for $y, x \in \mathcal{H}$, we have

$$
\begin{aligned}
&\|\mathcal{P}y - \mathcal{P}x\| = \sup_{\tau\in[0,T]}\Big|\mathcal{P}y(\tau) - \mathcal{P}x(\tau)\Big|, \\
&\le \Big\{\int_a^\tau \frac{(\tau-s)^2}{2}\Big|f(s,y(s)) - f(s,x(s))\Big|ds \\
&+\frac{1}{|\Gamma|}\int_a^T \Big[|\alpha_2 E_1 E_2|\frac{(T-s)^2}{2} + |\beta_2 K_1(\tau)|(T-s) + |\delta_2 K_2(\tau)|\Big]\Big|f(s,y(s)) - f(s,x(s))\Big|ds \\
&+\frac{1}{|\Gamma|}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i |E_1 E_2|\frac{(\sigma_i-s)^2}{2} + \rho_i |K_1(\tau)|(\sigma_i - s) + \nu_i |K_2(\tau)|\Big]\Big|f(s,y(s)) - f(s,x(s))\Big|
\end{aligned}
$$

$$+\frac{1}{|\Gamma|}\int_a^T \Big[\int_a^s \Big(|E_1E_2|\frac{(s-t)^2}{2}+|K_1(\tau)|(s-t)+|K_2(\tau)|\Big)$$
$$\times\Big|f(t,y(t))-f(t,x(t))\Big|dt\Big]d\phi(s)\Big\},$$
$$\leq \ell \parallel y-x \parallel \Big\{\frac{(T-a)^3}{3!}+\frac{1}{|E_4|}\Big[|\alpha_2|\frac{(T-a)^3}{3!}+\sum_{i=1}^r \gamma_i\frac{(\sigma_i-a)^3}{3!}+\int_a^T \frac{(s-a)^3}{3!}d\phi(s)\Big]$$
$$+\frac{1}{|\Gamma|}k_1\Big[|\beta_2|\frac{(T-a)^2}{2}+\sum_{i=1}^r \rho_i\frac{(\sigma_i-a)^2}{2}+\int_a^T \frac{(s-a)^2}{2}d\phi(s)\Big]$$
$$+\frac{1}{|\Gamma|}k_2\Big[|\delta_2|(T-a)+\sum_{i=1}^r \nu_i(\sigma_i-a)+\int_a^T (s-a)d\phi(s)\Big]\Big\},$$
$$\leq \ell\Theta \parallel y-x \parallel,$$

where we have used (3.3). By the given assumption: $\ell < 1/\Theta$, it follows that the operator $\mathcal{P}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{P}$ has a fixed point, which corresponds to a unique solution of the problem (1.1) on $[a, T]$.

Example 3.1. *Consider the following non-separated multi-point and Stieltjes boundary value problem:*

$$\begin{cases} y'''(\tau)=\dfrac{3}{\sqrt{\tau^3+441}}\sin y(\tau)+\dfrac{e^{-\tau}}{21(1+e^{\tau})}\dfrac{y}{(1+y)}+\cos\tau,\ \ \tau\in[0,2], \\ \alpha_1 y(a)+\alpha_2 y(T)=\displaystyle\sum_{i=1}^4 \gamma_i y(\sigma_i)+\int_a^T y(s)d\phi(s)+\frac{3}{2}, \\ \beta_1 y'(a)+\beta_2 y'(T)=\displaystyle\sum_{i=1}^4 \rho_i y'(\sigma_i)+\int_a^T y'(s)d\phi(s)+3, \\ \delta_1 y''(a)+\delta_2 y''(T)=\displaystyle\sum_{i=1}^4 \nu_i y''(\sigma_i)+\int_a^T y''(s)d\phi(s)+\frac{1}{2}, \end{cases} \tag{3.6}$$

where $a=0$, $T=2$, $r=4$, $\alpha_1=2/9$, $\alpha_2=4/9$, $\beta_1=1/7$, $\beta_2=3/7$, $\delta_1=1/8$, $\delta_2=1/4$, $\gamma_1=1/9$, $\gamma_2=1/3$, $\gamma_3=5/9$, $\gamma_4=2/3$, $\rho_1=2/7$, $\rho_2=3/7$, $\rho_3=4/7$, $\rho_4=5/7$, $\nu_1=3/8$, $\nu_2=1/2$, $\nu_3=5/8$, $\nu_4=3/4$, $\sigma_1=1/3, \sigma_2=2/3$, $\sigma_3=1$, $\sigma_4=4/3$, $\lambda_1=3/2$, $\lambda_2=3$, $\lambda_3=\frac{1}{2}$, $\phi(s)=\frac{(s-a)^3}{3}$. *Clearly,* $|f(\tau,y)|\leq \frac{3}{\sqrt{\tau^3+441}}+\Big|\frac{e^{-\tau}}{21(1+e^{\tau})}\Big|+|\cos\tau|$, $|f(\tau,y)-f(\tau,x)|\leq \ell|y-x|$, *with* $\ell=1/6$. *Using the given data, we find that* $|E_1|=4.541667\neq 0$, $|E_2|=4.095238\neq 0$, $|E_4|=3.666667\neq 0$, $|E_3|=1.047619$, $|E_5|=4.814815$, $|E_6|=3.261729$ *and* $|\Gamma|=68.197099$ *(*Γ *and* E_i $(i=1,\ldots,6)$ *are given by* (2.4)*)*, $\Theta=4.386978$, $\Theta-\frac{(T-a)^3}{3!}=3.053645$ *(*Θ *is defined by* 3.3*)*.

Furthermore, we note that all the conditions of Theorem 3.1 *are satisfied with* $\ell\Big(\Theta-\frac{(T-a)^3}{3!}\Big)\approx 0.508941<1$. *Hence the conclusion of Theorem* 3.1 *applies to the problem (3.6).*

We also observe that all the conditions of Theorem 3.2 *hold true with* $\ell\Theta\approx 0.731163<1$. *Hence we deduce by the conclusion of Theorem 3.2 that there exists a unique solution for the problem (3.6) on* $[0,2]$.

4. The multi-valued case

Here we discuss the existence of solutions for the multi-valued analogue (inclusions) case of the problem (1.1) given by

$$\begin{cases} y'''(\tau) \in F(\tau, y(\tau)), \; -\infty < a < \tau < T < \infty, \\ \alpha_1 y(a) + \alpha_2 y(T) = \sum_{i=1}^{r} \gamma_i y(\sigma_i) + \int_a^T y(s) d\phi(s) + \lambda_1, \\ \beta_1 y'(a) + \beta_2 y'(T) = \sum_{i=1}^{r} \rho_i y'(\sigma_i) + \int_a^T y'(s) d\phi(s) + \lambda_2, \\ \delta_1 y''(a) + \delta_2 y''(T) = \sum_{i=1}^{r} \nu_i y''(\sigma_i) + \int_a^T y''(s) d\phi(s) + \lambda_3, \end{cases} \tag{4.1}$$

where $F : [a, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and the other quantities are the same as defined in the problem (1.1). We then apply Bohnenblust-Karlin fixed point theorem to prove the existence of solutions for the problem (4.1).

Furthermore, for the convenience of the reader, we outline some basic concepts about multi-valued analysis [44–47] as follows:

I) A multi-valued map $\mathcal{S} : U \to \mathcal{P}(U)$ is
(i) convex (closed) valued if $\mathcal{S}(u)$ is convex (closed) for all $u \in U$, where $(U, \|.\|)$ is a Banach space,
(ii) bounded on a bounded set if $\mathcal{S}(\mathbb{Z}) = \cup_{u \in \mathbb{Z}} \mathcal{S}(u)$ is bounded in U for all $Z \in \mathcal{P}_b(U)$ (that is, $\sup_{u \in \mathbb{Z}} \{\sup\{|v| : v \in \mathcal{S}(u)\}\} < \infty$),
(iii) upper semi-continuous (u.s.c.) on U if for each $u_0 \in U$, the set $\mathcal{S}(u_0)$ is a nonempty closed subset of U, and if for each open set A of U containing $\mathcal{S}(u_0)$, there exists an open neighborhood $\mathcal{A}_0$ of u_0 such that $\mathcal{S}(\mathcal{A}_0) \subseteq A$,
(iv) completely continuous if $\mathcal{S}(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(U)$.

II) If the multi-valued map $\mathcal{S}$ is completely continuous with nonempty compact values, then $\mathcal{S}$ is u.s.c. if and only if $\mathcal{S}$ has a closed graph; that is, $u_n \to u_*$, $v_n \to v_*$, $v_n \in \mathcal{S}(u_n)$ implies $v_* \in \mathcal{S}(u_*)$.

III) A multi-valued map $\mathcal{S} : U \to \mathcal{P}(U)$ has a fixed point if there is $u \in U$ such that $u \in \mathcal{S}(u)$.

IV) In the sequel, we denote the set of all nonempty bounded, closed and convex subset of U by $BCC(U)$, and $L^1([a, T], \mathbb{R})$ denotes the Banach space of functions $u : [a, T] \to \mathbb{R}$, which are Lebesgue integrable, and normed by $\|u\|_{L^1} = \int_0^1 |u(\tau)| d\tau$.

V) Consider the following assumptions, which are needed in the forthcoming analysis:

(A$_1$) Let $F : [a, T] \times \mathbb{R} \to BCC(\mathbb{R}); (\tau, y) \to F(\tau, y)$ be measurable with respect to τ for each $y \in \mathbb{R}$, u.s.c. with respect to y for a.e. $\tau \in [a, T]$, and for each fixed $y \in \mathbb{R}$, the set $S_{F,y} := \{g \in L^1([a, T], \mathbb{R}) : g(\tau) \in F(\tau, y)$ for all $\tau \in [a, T]\}$ is nonempty.

(A$_2$) For each $r > 0$, there exists a function $\psi_r \in L^1([a, T], \mathbb{R}^+)$ such that $\|F(\tau, y)\| = \sup\{|g| : g(\tau) \in F(\tau, y)\} \le \psi_r(\tau)$ for each $(\tau, y) \in [a, T] \times \mathbb{R}$ with $|y| \le r$ and

$$\liminf_{r \to +\infty} \Big(\frac{\int_a^T \psi_r(\tau) d\tau}{r} \Big) = \varepsilon < \infty. \tag{4.2}$$

VI) Lastly, in relation to (4.1), we define

$$\Lambda = \left|\frac{\lambda_1}{E_4}\right| + \left|\frac{\lambda_2}{\Gamma}\right| g_1 + \left|\frac{\lambda_3}{\Gamma}\right| g_2. \tag{4.3}$$

Now, we state the following lemmas, which are needed to prove the main result:

Lemma 4.1. *(Bohnenblust-Karlin [48]) Let $D \subset U$ be nonempty bounded, closed, and convex. Let $\mathcal{S} : D \to \mathcal{B}(U)$ be u.s.c. with closed, convex values such that $\mathcal{S}(D) \subset D$ and $\overline{\mathcal{S}(D)}$ is compact. Then, $\mathcal{S}$ has a fixed point.*

Lemma 4.2. *[49] Let F be a multi-valued map satisfying the condition (A_1) and ϕ is linear continuous from $L^1([a,T],\mathbb{R}) \to C([a,T],\mathbb{R})$. Then, the operator $\phi \circ S_F : C([a,T],\mathbb{R}) \to BCC(C([a,T],\mathbb{R})), y \mapsto (\phi \circ S_F)(y) = \phi(S_{F,y})$ is a closed graph operator in $C([a,T],\mathbb{R}) \times C([a,T],\mathbb{R})$.*

Theorem 4.1. *Assume that (A_1) and (A_2) hold and that*

$$\varepsilon\Theta < 1, \tag{4.4}$$

where Θ and ε are given by (3.3) *and* (4.2) *respectively. Then, the problem* (4.1) *has at least one solution on* $[a,T]$.

Proof. Transforming the problem (4.1) into a fixed point problem, define an operator $\mathcal{G} : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ by

$$\mathcal{G}(y) = \left\{ \begin{array}{l} h \in \mathcal{H} : \\ h(\tau) = \left\{ \begin{array}{l} \displaystyle\int_a^\tau \frac{(\tau-s)^2}{2} g(s)ds \\ \displaystyle -\frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] g(s)ds \\ \displaystyle +\frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big] g(s)ds \\ \displaystyle +\frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) g(t)dt\Big] d\phi(s) \\ \displaystyle +\frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3, \end{array} \right. \end{array} \right\},$$

for $g \in S_{F,y}$. It is obvious that the fixed points of $\mathcal{G}$ are solutions of the boundary value problem (4.1).

We will show that $\mathcal{G}$ satisfies the assumptions of Lemma 4.1 and hence it will have a fixed point which guarantees the existence of a solution for the problem (4.1).

In the first step, we show that $\mathcal{G}(y)$ is convex for each $y \in \mathcal{H}$. For $h_1, h_2 \in \mathcal{G}$, there exist $g_1, g_2 \in S_{F,y}$ such that for all $t \in [a,T]$, we have

$$\begin{aligned} h_i(\tau) &= \int_a^\tau \frac{(\tau-s)^2}{2} g_i(s)ds \\ &- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] g_i(s)ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_i E_1 E_2 \frac{(\sigma_i - s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big] g_i(s)ds \\
&+ \frac{1}{\Gamma}\int_a^{T}\Big[\int_a^{s}\Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big)g_i(t)dt\Big]d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3, \quad i = 1, 2.
\end{aligned}
$$

For $0 \leq \varsigma \leq 1$ and all $\tau \in [a, T]$, we obtain

$$
\begin{aligned}
[\varsigma h_1 + (1-\varsigma)h_2](\tau) &= \int_a^{\tau}\frac{(\tau - s)^2}{2}[\varsigma g_1(s) + (1-\varsigma)g_2(s)]ds \\
&- \frac{1}{\Gamma}\int_a^{T}\Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] \\
&\times[\varsigma g_1(s) + (1-\varsigma)g_2(s)]ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_i E_1 E_2 \frac{(\sigma_i - s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big] \\
&\times[\varsigma g_1(s) + (1-\varsigma)g_2(s)]ds \\
&+ \frac{1}{\Gamma}\int_a^{T}\Big[\int_a^{s}\Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) \\
&\times[\varsigma g_1(t) + (1-\varsigma)g_2(t)]dt\Big]d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3.
\end{aligned}
$$

As $S_{F,y}$ is convex (F has convex values), so one can deduce that $\Big(\varsigma h_1 + (1-\varsigma)h_2\Big) \in \mathcal{G}(y)$.

Now we show that there exists a positive number r such that $\mathcal{G}(B_r) \subseteq B_r$, where $B_r = \{y \in \mathcal{H} : \|y\| \leq r\}$, and B_r is a bounded closed convex set in $\mathcal{H}$. If it is not true, then we can find a function $y_r \in B_r, h_r \in \mathcal{G}(y_r)$ with $\|\mathcal{G}(y_r)\| > r$ for each positive number r such that

$$
\begin{aligned}
h_r(\tau) &= \int_a^{\tau}\frac{(\tau - s)^2}{2}g_r(s)ds \\
&- \frac{1}{\Gamma}\int_a^{T}\Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big]g_r(s)ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i}\Big[\gamma_i E_1 E_2 \frac{(\sigma_i - s)^2}{2} + \rho_i K_1(\tau)(\sigma_i - s) + \nu_i K_2(\tau)\Big]g_r(s)ds \\
&+ \frac{1}{\Gamma}\int_a^{T}\Big[\int_a^{s}\Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big)g_r(t)dt\Big]d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3,
\end{aligned}
$$

for some $g_r \in S_{F,y_r}$. On the other hand, using (A_2), we get

$$
\begin{aligned}
r &< \|\mathcal{G}(y_r)\|,\\
&\le \int_a^\tau \frac{(\tau-s)^2}{2}\psi_r(s)ds\\
&- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big]\psi_r(s)ds\\
&+ \frac{1}{\Gamma}\sum_{i=1}^{r}\int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i-s) + \nu_i K_2(\tau)\Big]\psi_r(s)ds\\
&+ \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1 E_2 \frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big)\psi_r(t)dt\Big]d\phi(s)\\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3,\\
&\le \Theta\int_a^T \psi_r(s)ds + \Lambda,
\end{aligned}
$$

where Θ and Λ are given by (3.3) and (4.3) respectively. Dividing both sides of the above inequality by r yields

$$1 \le \Theta\Big(\frac{\int_a^T \psi_r(s)ds}{r}\Big) + \frac{\Lambda}{r}.$$

Now, taking the limit inf as $r \to \infty$ together with the notation (4.2), we obtain

$$1 \le \varepsilon\Theta,$$

which contradicts (4.4). Consequently, there exists a positive number r_1 such that $\mathcal{G}(B_{r_1}) \subseteq B_{r_1}$.

Next, we show that $\mathcal{G}(B_{r_1})$ is equicontinuous set of $\mathcal{H}$. Let $\tau_1, \tau_2 \in [a, T]$ with $\tau_1 < \tau_2$, $y \in B_{r_1}$ and $h \in \mathcal{G}(y)$, there exists $g \in S_{F,y}$ such that for each $\tau \in [a, T]$, we have

$$
\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\le \psi_r(\tau)\Big\{\frac{(\tau_2-\tau_1)^3}{3} + \frac{1}{3!}\Big|(\tau_2-a)^3 - (\tau_1-a)^3\Big|\\
&+ \frac{1}{|E_2|}(\tau_2-\tau_1)\Big[|\beta_2|\frac{(T-a)^2}{2} + \sum_{i=1}^{r}\rho_i\frac{(\sigma_i-a)^2}{2} + \int_a^T \frac{(s-a)^2}{2}d\phi(s)\Big]\\
&+ \frac{1}{|\Gamma|}\Big(|E_3E_4|(\tau_2-\tau_1) + \frac{|E_2E_4|}{2}((\tau_2-a)^2 - (\tau_1-a)^2)\Big)\Big[|\delta_2|(T-a)\\
&+ \sum_{i=1}^{r}\nu_i(\sigma_i-a) + \int_a^T (s-a)d\phi(s)\Big]\Big\}\\
&+ \frac{(\tau_2-\tau_1)}{|E_2|}\lambda_2 + \frac{1}{|\Gamma|}\Big(|E_3E_4|(\tau_2-\tau_1) + \frac{|E_2E_4|}{2}((\tau_2-a)^2 - (\tau_1-a)^2)\Big)\lambda_3,
\end{aligned}
$$

$$\to \ 0 \text{ as } (\tau_2-\tau_1)\to 0,$$

independently of $y \in B_{r_1}$. Subsequently, the Ascoli-Arzelá theorem applies since the above three conditions are satisfied. Thus the operator $\mathcal{G} : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ is compact multi-valued map. In order to prove that $\mathcal{G}$ is u.s.c. we have to show that $\mathcal{G}$ has a closed graph as follows, where $\mathcal{G}$ is completely continuous (see Proposition 1.2 [44]). Let $y_n \to y_*$, $h_n \in \mathcal{G}(y_n)$ and $h_n \to h_*$. Then, we prove that $h_* \in \mathcal{G}(y_*)$. Associated with $h_n \in \mathcal{G}(y_n)$, there exists $g_n \in S_{F,y_n}$ for all $\tau \in [a, T]$, such that

$$\begin{aligned}
h_n(\tau) &= \int_a^\tau \frac{(\tau-s)^2}{2} g_n(s)ds \\
&- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] g_n(s)ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i-s) + \nu_i K_2(\tau)\Big] g_n(s)ds \\
&+ \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) g_n(t)dt\Big] d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3.
\end{aligned}$$

Therefore, it is enough to prove that there exists $g_* \in S_{F,y_*}$ such that for all $\tau \in [a, T]$, we have

$$\begin{aligned}
h_*(\tau) &= \int_a^\tau \frac{(\tau-s)^2}{2} g_*(s)ds \\
&- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] g_*(s)ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i-s) + \nu_i K_2(\tau)\Big] g_*(s)ds \\
&+ \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big) g_*(t)dt\Big] d\phi(s) \\
&+ \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3.
\end{aligned}$$

Set $Q : L^1([a, T], \mathbb{R}) \to \mathcal{H}$ as a continuous linear operator given by

$$\begin{aligned}
g \mapsto Q(g)(\tau) &= \int_a^\tau \frac{(\tau-s)^2}{2} g(s)ds \\
&- \frac{1}{\Gamma}\int_a^T \Big[\alpha_2 E_1 E_2 \frac{(T-s)^2}{2} + \beta_2 K_1(\tau)(T-s) + \delta_2 K_2(\tau)\Big] g(s)ds \\
&+ \frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i} \Big[\gamma_i E_1 E_2 \frac{(\sigma_i-s)^2}{2} + \rho_i K_1(\tau)(\sigma_i-s) + \nu_i K_2(\tau)\Big] g(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big)g(t)dt\Big]d\phi(s)\\
&+ \quad \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|h_n - h_*\| \quad \leq \quad & \int_a^\tau \frac{(\tau-s)^2}{2}\Big|g_n(s) - g_*(s)\Big|ds\\
- \quad & \frac{1}{|\Gamma|}\int_a^T \Big[|\alpha_2E_1E_2|\frac{(T-s)^2}{2} + |\beta_2K_1(\tau)|(T-s) + |\delta_2K_2(\tau)|\Big]\\
& \times\Big|g_n(s) - g_*(s)\Big|ds + \frac{1}{|\Gamma|}\sum_{i=1}^r \int_a^{\sigma_i}\Big[\gamma_i|E_1E_2|\frac{(\sigma_i-s)^2}{2}\\
+ \quad & \rho_i|K_1(\tau)|(\sigma_i - s) + \nu_i|M_2\tau)|\Big]\Big|g_n(s) - g_*(s)\Big|ds\\
+ \quad & \frac{1}{|\Gamma|}\int_a^T \Big[\int_a^s \Big(|E_1E_2|\frac{(s-t)^2}{2} + |K_1(\tau)|(s-t) + |K_2(\tau)|\Big)\\
& \times\Big|g_n(t) - g_*(t)\Big|dt\Big]d\phi(s) \to 0 \ \text{ as } \ n\to\infty.
\end{aligned}
$$

Thus, by Lemma 4.2, $Q \circ S_F$ is a closed graph operator and $h_n(\tau) \in Q(S_{F,y_n})$ as $y_n \to y_*$. Consequently, we have

$$
\begin{aligned}
h_*(\tau) \quad = \quad & \int_a^\tau \frac{(\tau-s)^2}{2}g_*(s)ds\\
- \quad & \frac{1}{\Gamma}\int_a^T \Big[\alpha_2E_1E_2\frac{(T-s)^2}{2} + \beta_2K_1(\tau)(T-s) + \delta_2K_2(\tau)\Big]g_*(s)ds\\
+ \quad & \frac{1}{\Gamma}\sum_{i=1}^r \int_a^{\sigma_i}\Big[\gamma_iE_1E_2\frac{(\sigma_i-s)^2}{2} + \rho_iK_1(\tau)(\sigma_i-s) + \nu_iK_2(\tau)\Big]g_*(s)ds\\
+ \quad & \frac{1}{\Gamma}\int_a^T \Big[\int_a^s \Big(E_1E_2\frac{(s-t)^2}{2} + K_1(\tau)(s-t) + K_2(\tau)\Big)g_*(t)dt\Big]d\phi(s)\\
+ \quad & \frac{1}{E_4}\lambda_1 + \frac{1}{\Gamma}K_1(\tau)\lambda_2 + \frac{1}{\Gamma}K_2(\tau)\lambda_3,
\end{aligned}
$$

for some $g_* \in S_{F,y_*}$. The hypothesis of Lemma 4.1 holds true and we can conclude that $\mathcal{G}$ is a compact multi-valued map, u.s.c. with convex closed values. Thus, the operator $\mathcal{G}$ has a fixed point y which is indeed a solution of problem (4.1). This completes the proof.

Example 4.1. *Consider the following boundary value problem*

$$\begin{cases} y'''(\tau) \in F(\tau, y(\tau)), \ \tau \in [0, 2], \\ \alpha_1 y(0) + \alpha_2 y(2) = \displaystyle\sum_{i=1}^{4} \gamma_i y(\sigma_i) + \int_0^2 y(s) d\phi(s) + \frac{3}{2}, \\ \beta_1 y'(0) + \beta_2 y'(2) = \displaystyle\sum_{i=1}^{4} \rho_i y'(\sigma_i) + \int_0^2 y'(s) d\phi(s) + 3, \\ \delta_1 y''(0) + \delta_2 y''(2) = \displaystyle\sum_{i=1}^{4} \nu_i y''(\sigma_i) + \int_0^2 y''(s) d\phi(s) + \frac{1}{2}, \end{cases} \tag{4.5}$$

where all the constants take the same values as in Example 3.1.

So, $\|F(\tau, y)\| \leq \frac{3}{(2+\tau)^4}|y| + \cos\tau$ in this case, with condition (4.4) satisfied, since $\varepsilon\Theta \approx 0.482568 < 1$. Hence, it is deduced from the conclusion of Theorem 4.1 that there exists at least one solution for the problem (4.5) on $[0, 2]$.

5. Conclusions

In conclusion, the present study made use of the fixed point theorems to develop and further prove the existence and uniqueness results for the generalized nonlinear third-order ordinary differential equation with non-separated multi-point and nonlocal Riemann-Stieltjes (integral) boundary conditions. Additionally, the existence of solutions for the multi-valued case has equally been established through the application of Bohnenblust-Karlin's version of the fixed point theorem. Lastly, some supportive examples are supplied to clarify the applicability of the proven results. Besides, the present study is relevant to a variety of physical models in science and engineering applications.

Conflict of interest

The author declares that she has no conflict of interest.

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