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Research article

Calculation of the value of the critical line using multiple zeta functions

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Abstract: Newton's identities of an infinite polynomial with complex-conjugate roots $n^{-(\sigma+it)}$ and $n^{-(\sigma-it)}$ are multiple zeta functions for $n\in[1,\infty)$, $\sigma\in R$ and $t\in R$. All Newton's identities can be represented by Macdonald determinants. In a special case of the Riemann hypothesis, the multiple zeta function of the first order is equal to zero, $\zeta(\sigma+it)+\zeta(\sigma-it)=0$. The special case includes all non-trivial zeros. The value of the last, infinite multiple zeta function, in the special case, changes the structure of the determinant that can be calculated. The result is the reciprocal of the factorial value $(n!)^{-1}$. The general value of the infinite multiple zeta function is calculated based on Vieta's rules and is equal to $(n!)^{-2\sigma}$. The identity based on the relation of the special case and the general case $(n!)^{-1}=(n!)^{-2\sigma}$ is reduced to the equation $-1=-2\sigma$. The value of the critical line for all non-trivial zeros is singular, $\sigma=½$.

Keywords: non trivial zeros; factorial; imaginary unit; 3.141592…; 2.718281…
Mathematics Subject Classification: 11M26

1. Introduction

In 1737, Leonard Euler (1707–1783) established the relationship of the zeta function $\zeta(s)$ of the argument s and the product $(1-p^{-s})^{-1}$, $\{p\in N|p \text{ is prime}\}$. In his famous 1859 article "On the Number of Primes Less Than a Given Magnitude", Bernhard Riemann (1826–1866) extends Euler's product to a complex variable. According to historical data, Bernhard Riemann calculated the first few non-trivial zeros of the zeta function during his lifetime. The results of his calculations were revealed only after his death. The infinite number of non-trivial zeros, as well as many other important properties of the zeta function, were proved after the death of Bernard Riemann. The functional equation from the famous article from 1859 still represents the starting point of the millennium problem: the Riemann hypothesis.

The non-trivial zeros of the zeta function are organized into complex-conjugate pairs. The product $(1-p^{-s})^{-1}$, for a complex argument s, can be considered as a product of a monomial with one complex argument, i.e., polynomial for the value of real x=1 in the pole of the zeta function and complex roots. The development of this infinite polynomial has no real coefficients. However, if the product is expanded with monomials containing the conjugate of the complex argument, we obtain a polynomial with complex-conjugate roots analogous to the organization of non-trivial zeros of the zeta function.

Synthesis of zeta function and polynomial is more recent. Almost a century ago, George Pólya (1887–1985) proved that the Riemann Hypothesis is equivalent to the hyperbolicity of Jensen polynomials [1]. From the wide opus, we shall single out research on relationships between the zeta function and Apostol-Euler polynomials [2], Bernoulli polynomials [3], Geometric polynomials [4], Legendre polynomials [5], analogies with Chebyshev polynomials [6], etc. In 1990, Kohji Matsumoto introduced the polynomial in the zeta function [7]. Unlike the listed [1–6] and similar unlisted results, the “Matsumoto Zeta function” variety is a generalization of many classical zeta functions (A_p is polynomial). At the same time, “Matsumoto zeta function” directly refers us to the potentially “deep” polynomial structure of the zeta function. The variable of that polynomial can only be x=1, in the pole of the zeta function. The roots of the polynomial are functions of the prime numbers of the argument s. The existence of monomials is obvious (1):

$$\phi(s)=\prod_{prime}\frac{1}{A_p p^{-s}}=\prod_{prime}\frac{1-p^{-s}+p^{-s}}{A_p p^{-s}}=\prod_{prime}\frac{1}{A_p}\underbrace{\left(1-\underbrace{\frac{p^{-s}-1}{p^{-s}}}_{root}\right)}_{monomial}. \tag{1}$$

The development of the multiple zeta function began in 1992. Nobushige Kurokawa proposed the form of multiple zeta functions. Zeros and poles of multiple zeta functions correspond to sums of zeros and poles of the Riemann zeta function [8]. Ken Kamano was the first to develop the harmonic product formula in which the recurrence relation of multiple zeta functions of identical argument was established [9]. However, in the long term, the research of multiple zeta functions remains at the level of double zeta functions with some rare results of triple and quadruple zeta functions [10–13]. Therefore, until now there has been no attempt to calculate the critical axis parameter of the Riemann hypothesis [14,15] using multiple zeta functions of a higher order than two [16,17]. The problem was overcome by introducing the zeta function in the polynomial. Applying the identity of Sir Isaac Newton (1643–1727), Matsumoto et al. [18] confirm Kaman’s harmonic product formula of multiple zeta functions. They define a number of new properties of multiple zeta functions with interasymptotic and intertrivial zeros. From the described chronology, it is obvious that Kurokawa [8] and Kamano [9] developed multiple zeta functions from a deep mathematical intuition and that Matsumoto et al., by applying Newton’s identities [18], subsequently proved the correctness of their approach using polynomials. However, Kurokawa’s and Kaman’s concept is still based on only one complex argument, while Matsumoto et al. limited their concept only to a realistic argument. Synthesis of these two concepts, i.e., the introduction of a complex-conjugate argument into the multiple zeta function can open up new possibilities for the calculation of the critical line of the Riemann hypothesis. Structurally, current solutions of multiple zeta functions are proven to be based on Newton's identities of symmetric polynomials. Therefore, before introducing the complex-conjugate argument to multiple zeta functions, let us consider the polynomial structure of the zeta function.

2. Polynomial structure of the zeta function

The non-trivial zeros of the zeta function are organized identically to polynomials with complex-conjugate roots. For the known property of the zeta function (2):

$$\zeta(s)=\overline{\zeta(\bar{s})}. \tag{2}$$

For all complex roots $s\neq 1$, the zeros of the Riemann zeta function are symmetric around the real axis. Above the critical line of the zeta function of assumed value σ=½, we have a known polynomial. The roots of this polynomial correspond to the non-trivial zeros of only one complex argument of the zeta function. To fully correspond, it is necessary to introduce a conjugated argument. Thanks to the finding of Godfrey Harold Hardy (1877–1947) from 1914, we can project an infinite polynomial with roots-non-trivial zeros of the zeta function (3):

$$P(x)=\prod_{k=1}^{\infty}(x-(\sigma+it_k))(x-(\sigma-it_k)). \tag{3}$$

Values t_1=14.135…, t_2=21.022…, t_3=25.010…, etc. are known from the first finding of Bernard Riemann to the results of Platt and Trudgian [19]. Perhaps this polynomial has no particular theoretical value for the research of the zeta function, but it carries one important illustration–the simultaneous introduction of a complex-conjugate argument into the zeta function is not possible. However, it is possible in a multiple zeta function. The basic form of the zeta function is (4):

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{prime}\frac{1}{1-p^{-s}}=\frac{1}{1-2^{-s}}\cdot\frac{1}{1-3^{-s}}\cdot\frac{1}{1-5^{-s}}\cdots\frac{1}{1-p^{-s}}\cdots. \tag{4}$$

For the application of Newton's identities, the most suitable polynomial is for the real x=1 in the pole of the zeta function and roots $(1-p^s)^{-1}$. The zeta function (4), or the multiple zeta function of the first order also reduces to the polynomial (5):

$$\begin{aligned}\zeta(s)&=\prod_{prime}\frac{1}{1-p^{-s}}=\prod_{prime}\frac{1-p^{-s}+p^{-s}}{1-p^{-s}}=\prod_{prime}\left(\frac{1-p^{-s}}{1-p^{-s}}+\frac{p^{-s}}{1-p^{-s}}\right)\\&=\prod_{prime}\left(1+\frac{p^{-s}}{1-p^{-s}}\right)=\prod_{prime}\left(1+\frac{1}{p^{s}-1}\right)=\prod_{prime}\underbrace{\left(\underbrace{1}_{x=1}-\underbrace{\frac{1}{1-p^{s}}}_{root}\right)}_{monomial}.\end{aligned} \tag{5}$$

Further, we can form the product of zeta functions of complex-conjugate arguments (σ±it) in the value of pole of the zeta function x=1(6):

$$\zeta(s)\zeta(\bar{s})=\prod_{prime}\frac{1}{1-p^{-s}}\frac{1}{1-p^{-\bar{s}}}=\prod_{prime}\left(1-\frac{1}{1-p^{s}}\right)\left(1-\frac{1}{1-p^{\bar{s}}}\right). \tag{6}$$

The association to the polynomial is now complete because the roots in the monomials are complex conjugate. Also, we can easily express the form of the inverse value of the product of the zeta function with complex conjugate arguments, which directly justifies the choice for the value x=1(6):

$$\frac{1}{\zeta(s)\zeta(\bar{s})}=\prod_{prime}\left(1-\frac{1}{p^{s}}\right)\left(1-\frac{1}{p^{\bar{s}}}\right). \tag{7}$$

It must be noted here that the "Matsumoto zeta function" (1) is defined by the product of all complex roots of the polynomial (7), i.e., only for the polynomial $\zeta^{-1}(s)=\Pi(1-p^{-s})$. The product of a polynomial with a conjugate argument is missing. If the polynomial A_p from (1) is projected analogously to the conjugate argument, then (7) can be reduced to (1). The idea of approaching the zeta function with two polynomials has already been established and is not new [20].

Now, based on the results from [18], the product of all the roots points us to the rules of Franciscus Vieta (1540–1603), i.e., to the multiple zeta function of infinite order. This fact will play a key role in calculating the critical line.

For each value of convergence of the value argument (½±it_k) for known values of t_k, $k\in N$: t_1=14.135…, t_2=21.022…, t_3=25.010…, the product of zeta function of the complex-conjugate argument from (6) converges (8). Research on this product [16,17] pointed to the importance of the basic double zeta function and the Hurwitz-Lerch type double zeta function.

$$\zeta(s)\zeta(\bar{s})\to\zeta^{2}(s)\to 0,\qquad \zeta(s)\zeta(\bar{s})\to\zeta^{2}(\bar{s})\to 0\,. \tag{8}$$

At the same time, this approach is a reasonable justification for the long lack of interest in researching multiple zeta functions of higher order than two. The product (8) is an obligatory member of the binomial expansion (9) of the sum of the zeta functions of complex-conjugate arguments:

$$\left(\zeta(s)+\zeta(\bar{s})\right)^{k}=\sum_{j=0}^{k}\binom{k}{j}\zeta^{k-j}(s)\zeta^{j}(\bar{s}). \tag{9}$$

Under the conditions of the Riemann hypothesis, for non-trivial zeros from the polynomial (3), the value of the binomial development expansion (9) is equal to zero. The product searched in (6)–(8), i.e., the double zeta function of the complex-conjugate argument, was created from the sum of the basic zeta functions of the complex-conjugate argument. Therefore, it is necessary to find an adequate polynomial analogous to the results from [18] which, by applying Newton's identities, will establish a relationship between the product and the sum of the zeta functions of the complex-conjugate argument.

3. Coefficients of the polynomial of the multiple zeta function of the complex-conjugate argument

The required polynomial has the form (10). The applied concept is analogous to the development of multiple zeta functions with identical real arguments [18]:

$$P(x,\sigma \pm it) = \prod_{n=1}^{\infty}\left(x - \frac{1}{n^{\sigma+it}}\right)\left(x - \frac{1}{n^{\sigma-it}}\right). \tag{10}$$

When expressing Newton's identities, the standard approach and notation from [18] will be applied, where p_k is equal to the k-th ($k \in R$) power of all the roots of the polynomial (11):

$$p_k = \sum_{n_1=1}^{\infty}\frac{1}{n_1^{k(\sigma+it)}} + \sum_{n_2=1}^{\infty}\frac{1}{n_2^{k(\sigma-it)}} = \zeta(ks) + \zeta(k\bar{s}). \tag{11}$$

Although infinite, we know that the polynomial (8) is of even degree, because of the even number of complex-conjugate roots. The first Newton's identity is equal to the sum of all the roots of the polynomial (10), i.e., sum of individual zeta functions of complex-conjugate arguments (12):

$$\zeta_1(s,\bar{s}) = e_1 = p_1 = \sum_{n_1=1}^{\infty}\frac{1}{n_1^{\sigma+it}} + \sum_{n_2=1}^{\infty}\frac{1}{n_2^{\sigma-it}} = \zeta(s) + \zeta(\bar{s}). \tag{12}$$

Newton's second identity is equal to pairs of products of all roots (13):

$$e_2 = \sum_{1 \le n_1 < n_2}^{\infty}\left(\frac{1}{n_1^{\sigma+it}}\frac{1}{n_2^{\sigma+it}}\right) + \sum_{1 \le n_1 < n_2}^{\infty}\left(\frac{1}{n_1^{\sigma-it}}\frac{1}{n_2^{\sigma-it}}\right) + \sum_{n_1=1}^{\infty}\left(\frac{1}{n_1^{\sigma+it}}\right)\sum_{n_2=1}^{\infty}\left(\frac{1}{n_2^{\sigma-it}}\right). \tag{13}$$

The first and second members are equal to the sum of double zeta functions of identical complex-conjugate arguments, and the third member is equal to the product of zeta functions of complex-conjugate arguments (14):

$$\begin{aligned}
&\sum_{1 \le n_1 < n_2}^{\infty}\left(\frac{1}{n_1^{\sigma+it}}\frac{1}{n_2^{\sigma+it}}\right) = \zeta_2(s) = \frac{\zeta(s)^2 - \zeta(2s)}{2}, \\
&\sum_{1 \le n_1 < n_2}^{\infty}\left(\frac{1}{n_1^{\sigma-it}}\frac{1}{n_2^{\sigma-it}}\right) = \zeta_2(\bar{s}) = \frac{\zeta(\bar{s})^2 - \zeta(2\bar{s})}{2}, \\
&\sum_{n_1=1}^{\infty}\left(\frac{1}{n_1^{\sigma+it}}\right)\sum_{n_2=1}^{\infty}\left(\frac{1}{n_2^{\sigma-it}}\right) = \zeta(s)\zeta(\bar{s}),
\end{aligned} \tag{14}$$

which is in accordance with Newton's second identity e_2 (15):

$$e_2 = \frac{\zeta(s)^2 - \zeta(2s) + \zeta(\bar{s})^2 - \zeta(2\bar{s}) + 2\zeta(s)\zeta(\bar{s})}{2} = \frac{\overbrace{(\zeta(s)+\zeta(\bar{s}))^2}^{p_1^2=e_1^2} - \overbrace{(\zeta(2s)+\zeta(2\bar{s}))}^{p_2}}{2}. \tag{15}$$

The first way of developing Newton's polynomial identities (10), i.e., multiple zeta function of a higher order, is possible by individual calculations as in the case of double zeta functions (13)–(15).

The second way is based on the application of the form for expressing elementary symmetric polynomials in terms of power sums: $e_k=(-1)^k(k!)^{-1}B_k(-p_1,-1!p_2,-2!p_3,\ldots,-(k-1)!p_k)$, where the B_k is the complete exponential polynomial established by Eric Temple Bell (1883−1960).

The third way, in this case the most suitable, is based on the results obtained by Ian Grant Macdonald [21]. Newton's identities, i.e., multiple zeta functions can be calculated based on the product of the reciprocal of the factorial and special determinants. Let us first prepare an expression for the reciprocal value of the factorial (16):

$$\frac{1}{n!}=\frac{1}{\prod_{k=1}^{n}k}=\prod_{k=1}^{n}\frac{1}{k}. \tag{16}$$

By applying (16), according to the results obtained by Macdonald [21], the determinants of Newton's identities are (17)–(22). These identities are simultaneously multiple zeta functions of the complex-conjugate argument:

$$e_1=\zeta_1(s,\bar{s})=\frac{1}{1!}(p_1)=\prod_{n=1}^{1}\frac{1}{n}\left|p_1\right|=\frac{D_1}{1!}, \tag{17}$$

$$e_2=\zeta_2(s,\bar{s})=\frac{1}{2!}(p_1^2-p_2)=\prod_{n=1}^{2}\frac{1}{n}\begin{vmatrix}p_1 & 1\\ p_2 & p_1\end{vmatrix}=\frac{D_2}{2!}, \tag{18}$$

$$e_3=\zeta_3(s,\bar{s})=\frac{1}{3!}(p_1^3-3p_1p_2+2p_3)=\prod_{n=1}^{3}\frac{1}{n}\begin{vmatrix}p_1 & 1 & 0\\ p_2 & p_1 & 2\\ p_3 & p_2 & p_1\end{vmatrix}=\frac{D_3}{3!}, \tag{19}$$

$$e_4=\zeta_4(s,\bar{s})=\frac{1}{4!}(p_1^4-6p_1^2p_2+3p_2^2+8p_1p_3-6p_4)=\prod_{n=1}^{4}\frac{1}{n}\begin{vmatrix}p_1 & 1 & 0 & 0\\ p_2 & p_1 & 2 & 0\\ p_3 & p_2 & p_1 & 3\\ p_4 & p_3 & p_2 & p_1\end{vmatrix}=\frac{D_4}{4!}. \tag{20}$$

Newton's identity of order $k\in\mathrm{N}$, i.e., multiple zeta function of order k is given by (21):

$$e_k=\zeta_k(s,\bar{s})=\prod_{n=1}^{k}\frac{1}{n}\begin{vmatrix}p_1 & 1 & 0 & \vdots & 0 & 0 & 0\\ p_2 & p_1 & 2 & \vdots & 0 & 0 & 0\\ p_3 & p_2 & p_1 & \vdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots\\ p_{k-2} & p_{k-2} & p_{k-2} & \vdots & p_1 & k-2 & 0\\ p_{k-1} & p_{k-2} & p_{k-3} & \vdots & p_2 & p_1 & k-1\\ p_k & p_{k-1} & p_{k-2} & \vdots & p_3 & p_2 & p_1\end{vmatrix}=\frac{D_k}{k!}. \tag{21}$$

Thanks to the findings of Macdonald [21], the last coefficient of the infinite polynomial (10) that converges to the value $e_{n\to\infty}$, i.e., multiple zeta function of order $n\to\infty$, can be presented as a product of the reciprocal value of the factorial from (16) and the infinite determinant D_∞ (22):

$$e_{n\to\infty}=\zeta_n(s,\bar{s})=D_\infty\prod_{n=1}^{\infty}\frac{1}{n}=\prod_{n=1}^{\infty}\frac{1}{n}\begin{vmatrix} p_1 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots \\ p_2 & p_1 & 2 & 0 & \vdots & 0 & 0 & 0 & \vdots \\ p_3 & p_2 & p_1 & 3 & \vdots & 0 & 0 & 0 & \vdots \\ p_4 & p_3 & p_2 & p_1 & \vdots & 0 & 0 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \ddots \\ p_{k-2} & p_{k-3} & p_{k-4} & p_{k-5} & \vdots & p_1 & k-2 & 0 & \vdots \\ p_{k-1} & p_{k-2} & p_{k-3} & p_{k-4} & \vdots & p_2 & p_1 & k-1 & \vdots \\ p_k & p_{k-1} & p_{k-2} & p_{k-3} & \vdots & p_3 & p_2 & p_1 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \ddots \end{vmatrix}. \tag{22}$$

According to the rules of Franciscus Vieta (1540–1603), we know that the last member of a polynomial is equal to the product of all the roots. By applying Vieta's rules, the value of the determinant (22) can be directly expressed by (23). The solution (23) is a general solution of (22) for any value of the real argument σ!

$$e_{n\to\infty}=\zeta_n(s,\bar{s})=\prod_{n=1}^{\infty}\frac{1}{n^{\sigma+it}}\frac{1}{n^{\sigma-it}}=\prod_{n=1}^{\infty}\frac{1}{n^{2\sigma}},\qquad \underbrace{D_\infty\prod_{n=1}^{\infty}\frac{1}{n}}_{special\ form}=\underbrace{\prod_{n=1}^{\infty}\frac{1}{n^{2\sigma}}}_{general\ form}. \tag{23}$$

All the necessary coefficients of the infinite polynomial (10) are thus formed. The general form of the Multiple zeta function $e_{n\to\infty}$ of order $n\to\infty$ is obtained from (23) for the specific value of the first Newton's identity p_1. The Newton's identities p_k of order $k>1$ are functionally dependent on the first Newton's identity.

4. The solution of the infinite determinant under the conditions of the Riemann hypothesis

By using the obtained coefficients (17)–(21), the polynomial (10) in its developed form is equal to (24). Coefficient $e_{n\to\infty}$, i.e., the multiple zeta function of the complex-conjugate argument of order n, has to be positive due to the even number of roots of the polynomial (10). If we expand the product of the monomial (10), a polynomial with coefficients (24) is obtained. The coefficients of this polynomial are multiple zeta functions (24):

$$\begin{aligned} &P(x,\sigma\pm it)=x^n-e_1x^{n-1}+e_2x^{n-2}-e_3x^{n-3}+e_4x^{n-4}\ldots+(-1)^k e_kx^{n-k}+\ldots+e_{n\to\infty}\,,\\ &\qquad e_k=\zeta_k(s,\bar{s}),\quad k\in N. \end{aligned} \tag{24}$$

Under the specific conditions of the Riemann hypothesis (25):

$$\zeta(s)=0\wedge\zeta(\bar{s})=0\ \Rightarrow\ \zeta(s)+\zeta(\bar{s})=p_1=e_1=\zeta_1(s,\bar{s})=0. \tag{25}$$

The first coefficient of the polynomial (22) is equal to zero $p_1=e_1=0$. From (12), hold (26):

$$\begin{aligned} \zeta(s)+\zeta(\bar{s})&=\sum_{n=1}^{\infty}\frac{cos\,t\,ln(n)+i\,sin\,t\,ln(n)}{n^\sigma}+\sum_{n=1}^{\infty}\frac{cos\,t\,ln(n)-i\,sin\,t\,ln(n)}{n^\sigma}\\ &=\sum_{n=1}^{\infty}\frac{2cos\,t\,ln(n)}{n^\sigma}=\underbrace{p_1=e_1=\zeta_1(s,\bar{s})=0}_{from\ (25)}. \end{aligned} \tag{26}$$

By introducing the condition of Riemann hypothesis (26) into the infinite determinant (22), all values on the main diagonal are equal to zero (27):

$$D_\infty = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ p_2 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ p_3 & p_2 & 0 & 3 & 0 & \cdots & 0 & 0 & \cdots \\ p_4 & p_3 & p_2 & 0 & 4 & \cdots & 0 & 0 & \cdots \\ p_5 & p_4 & p_3 & p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ p_{n-1} & p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\ p_n & p_{n-1} & p_{n-2} & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{27}$$

By introducing the conditions of the Riemann hypothesis (26) into the determinant (27), a key question arises: is there a finite value of the infinite determinant D_∞ (27)?

By applying Vieta's rules in (23), it has been proved that the last coefficient of the infinite polynomial (10) shown in (24) as $e_{n\to\infty}$, functionally depends only on the real part σ of the complex argument $s=\sigma\pm it$. Condition (26) includes all pairs of symmetrically distributed non-trivial zeros. The final value of the infinite determinant D_∞ would enable the formation of the special identity emphasized in (23). Now, we add the second column to the first, so we get (28):

$$D_\infty = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ p_2 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ p_3+p_2 & p_2 & 0 & 3 & 0 & \cdots & 0 & 0 & \cdots \\ p_4+p_3 & p_3 & p_2 & 0 & 4 & \cdots & 0 & 0 & \cdots \\ p_5+p_4 & p_4 & p_3 & p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ p_{n-1}+p_{n-2} & p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\ p_n+p_{n-1} & p_{n-1} & p_{n-2} & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{28}$$

Multiple the first row of determinant by $-p_2$ and add it to the second row, then multiple the first row by $-(p_3+p_2)$ and add it to the third row, i.e., multiple the first row by $-(p_n+p_{(n-1)})$ and add to n-th row. We get a determinant that has a value of 1 in the first column only on the main diagonal of the determinant (29):

$$D_{\infty}=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -p_2 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -p_3 & 0 & 3 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -p_4 & p_2 & 0 & 4 & \cdots & 0 & 0 & \cdots \\ 0 & -p_5 & p_3 & p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & -p_{n-1} & p_{n-3} & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\ 0 & -p_n & p_{n-2} & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{29}$$

Then, multiple the third column of determinants by $(p_2+1)/2$ and add to the second column (30):

$$D_{\infty}=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -p_3 & 0 & 3 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & -p_4+p_2\left(\dfrac{p_2+1}{2}\right) & p_2 & 0 & 4 & \cdots & 0 & 0 & \cdots \\ 0 & -p_5+p_3\left(\dfrac{p_2+1}{2}\right) & p_3 & p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & -p_{n-1}+p_{n-3}\left(\dfrac{p_2+1}{2}\right) & p_{n-3} & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\ 0 & -p_n+p_{n-2}\left(\dfrac{p_2+1}{2}\right) & p_{n-2} & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{30}$$

All the elements of the second column from the third row are successively eliminated by multiple the second row with a corresponding number (31):

$$D_{\infty}=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 2p_3 & 3 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & p_2+2p_4-2p_2\left(\dfrac{p_2+1}{2}\right) & 0 & 4 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & p_3+2p_5-2p_3\left(\dfrac{p_2+1}{2}\right) & p_2 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & p_{n-3}+2p_{n-1}-2p_{n-3}\left(\dfrac{p_2+1}{2}\right) & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\ 0 & & p_{n-2}+2p_n-p_{n-2}\left(\dfrac{p_2+1}{2}\right) & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \tag{31}$$

Now, multiple the fourth column of determinants by $(1-2p_3)/3$ and add to the third column (32):

$$D_\infty = \begin{vmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 1 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & 1 & 3 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & p_2 + 2p_4 - 2p_2\left(\dfrac{p_2+1}{2}\right) & 0 & 4 & \cdots & 0 & 0 & \cdots \\
0 & 0 & p_3 + 2p_5 - 2p_3\left(\dfrac{p_2+1}{2}\right) + p_2\left(\dfrac{1-2p_3}{3}\right) & p_2 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
0 & 0 & p_{n-3} + 2p_{n-1} - 2p_{n-3}\left(\dfrac{p_2+1}{2}\right) + p_{n-4}\left(\dfrac{1-2p_3}{3}\right) & p_{n-4} & p_{n-5} & \vdots & 0 & n-1 & \cdots \\
0 & & p_{n-2} + 2p_n - p_{n-2}\left(\dfrac{p_2+1}{2}\right) + p_{n-3}\left(\dfrac{1-2p_3}{3}\right) & p_{n-3} & p_{n-4} & \vdots & p_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix}. \tag{32}$$

All the elements of the third column from the fourth row are successively eliminated by multiple the third row with the corresponding number (33):

$$D_\infty = \begin{vmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 1 & 2 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & 1 & 3 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & 0 & -3p_2 - 6p_4 + 6p_2\left(\dfrac{p_2+1}{2}\right) & 4 & \cdots & 0 & 0 & \cdots \\
0 & 0 & 0 & p_2 - 3p_3 - 6p_5 + 6p_3\left(\dfrac{p_2+1}{2}\right) - 3p_2\left(\dfrac{1-2p_3}{3}\right) & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & p_{n-4} - 3p_{n-3} - 6p_{n-1} + 6p_{n-3}\left(\dfrac{p_2+1}{2}\right) - 3p_{n-4}\left(\dfrac{1-2p_3}{3}\right) & p_{n-5} & \vdots & 0 & n-1 & \cdots \\
0 & 0 & 0 & p_{n-3} - 3p_{n-2} - 6p_n + 6p_{n-2}\left(\dfrac{p_2+1}{2}\right) - 3p_{n-3}\left(\dfrac{1-2p_3}{3}\right) & p_{n-4} & \vdots & p_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix}. \tag{33}$$

For $k \in \mathrm{N}$, the concept is as follows (a are determinant members): to obtain a unit on the main diagonal in the k-th row and k-th column, the (k+1) column is always used where $a_{k,k+1}$=k. We notice that the values $a_{k,1}$, $a_{k,2}$, $a_{k,k-1}$, as well as $a_{k,k+2}$, $a_{k,k+3}$, $a_{k,k+2}$,… are equal to zero. The values $a_{k,k}$ and $a_{k,k+1}$ are only non-zero members in the k-th row. The value $a_{k,k+1}$ is multiple by the corresponding function $a_{k+1,j}=f_{k+1,j}$ (p_2, p_3, …, p_{k+1}), $j \in [k+1,\infty)$, which results in the value on the main diagonal $a_{k,k}$=1. Then, using the values on the main diagonal, we reduce all the values in the k-th column below the main diagonal to zero, whereby in the (k+1) column, new functions $f_{k+2,j}$ (p_2, p_3, …, p_{k+1}, p_{k+2}) are obtained. The values in the other columns do not change because they are multiple by zero.

According to the established concept, values of 1 are obtained on the main diagonal and values of zeros below the main diagonal, i.e., we get the value of the determinant. The value of the determinant converges to 1 (34). We obtain this result due to the initial conditions from the determinant (29) in which all values on the main diagonal are based on (26), i.e., equal to zero. The value that sets 1 on the main diagonal is $a_{1,2}$=1.

$$D_{\infty}=\begin{vmatrix}1&1&0&0&0&\cdots&0&0&\cdots\\0&1&2&0&0&\cdots&0&0&\cdots\\0&0&1&3&0&\cdots&0&0&\cdots\\0&0&0&1&4&\cdots&0&0&\cdots\\0&0&0&0&1&\cdots&0&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\cdots&\cdots&\cdots\\0&0&0&0&0&\vdots&1&n-1&\cdots\\0&0&0&0&0&\vdots&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{vmatrix}\to 1. \tag{34}$$

5. Values of the critical line σ=½ and the factorial function

Expressions (22) and (23) refer to the convergences of an identical multiple zeta function for the same initial conditions (25) and (26) for general and specific conditions. Therefore, with (34) we can form the identity (35):

$$\zeta_n(s,\bar{s})=e_{n\to\infty}=\underbrace{\underbrace{D_{\infty}\prod_{n=1}^{\infty}\frac{1}{n^1}}_{from\ (22)}=\overbrace{1}^{from\,34}\cdot\prod_{n=1}^{\infty}\frac{1}{n^1}}_{special\ \ form}=\underbrace{\underbrace{\prod_{n=1}^{\infty}\frac{1}{n^{2\sigma}}}_{from\ (23)}}_{general\ \ form}\Rightarrow 1=2\sigma\Rightarrow\sigma=\frac{1}{2}. \tag{35}$$

The real value of complex-conjugate arguments of non-trivial zeros of the coefficient $e_{n\to\infty}$ is singular, there is only one solution: σ=½!

The catalytic role of the polynomial (10) for the derivation of multiple zeta functions can also be considered through the polynomial $Q(x,\sigma\pm it)$ with the inverse values of the roots in the monomials (36):

$$Q(x,\sigma\pm it)=\prod_{n=1}^{\infty}(x-n^{\sigma+it})(x-n^{\sigma-it}). \tag{36}$$

In the polynomial (36), qk are the sums of zeta functions of the complex-conjugate argument. q_k is essential for the obtaining Newton's identities and polynomial coefficients (37):

$$q_k=\sum_{n_1=1}^{\infty}n_1^{k(\sigma+it)}+\sum_{n_2=1}^{\infty}n_2^{k(\sigma-it)}=\sum_{n_1=1}^{\infty}\frac{1}{n_1^{-k(\sigma+it)}}+\sum_{n_2=1}^{\infty}\frac{1}{n_2^{-k(\sigma-it)}}=\zeta(-ks)+\zeta(-k\bar{s}). \tag{37}$$

For the polynomial (36), the relations between the Newton's identities represented by the Macdonald determinants (17)–(21) are also valid. It is expressed with the coefficients $e_{(-k)}$, i.e., Newton's identities with a negative index. The concept of Newton's identities with a negative index has already been successfully used in the new "New-nacci" method for calculating the roots of polynomials based on the convergence of Newton's identities [22].

Obtained from the product of monomials, the polynomial (36) has the form (38) with coefficients that are multiple zeta functions, as follows:

$$Q(x,\sigma \pm it) = x^n - e_{(-1)}x^{n-1} + e_{(-2)}x^{n-2} - e_{(-3)}x^{n-3} ... + (-1)^k e_{(-k)}x^{n-k} + ... + e_{(-n)\to\infty}. \tag{38}$$

The significance of the coefficient $e_{n}\to\infty$ for the polynomial, i.e., multiple zeta functions of order $n\to\infty$ (24), is essential, and this coefficient is the lowest common denominator of all polynomial coefficients (10). If we consider that the zeta function (11) is a multiple zeta function of the first order $e_1=\zeta_1(\sigma+it,\ \sigma-it)$ (36), it follows (39):

$$P(x,\sigma \pm it) = e_{n\to\infty}\left(\frac{1}{e_{n\to\infty}}x^n - \frac{e_1}{e_{n\to\infty}}x^{n-1} + \frac{e_2}{e_{n\to\infty}}x^{n-2} ... + (-1)^k \frac{e_k}{e_{n\to\infty}}x^{n-k} + ... + 1\right). \tag{39}$$

For $k\in[1,\ \infty)$ respecting Newton's identities with a negative index, for all multiple zeta functions hold (40):

$$\frac{\zeta_k(s,\bar{s})}{\zeta_{n\to\infty}(s,\bar{s})} = \zeta_{(-n+k)}(s,\bar{s}) = \frac{e_k}{e_{n\to\infty}} = e_{(-n+k)}. \tag{40}$$

The relations between polynomials (10), (24), (36) and (38) in the pole of the zeta function for x=1 are:

$$\begin{aligned} P(1,\sigma \pm it) &= e_{n\to\infty}\left(\frac{1}{e_{n\to\infty}} - \frac{e_1}{e_{n\to\infty}} + \frac{e_2}{e_{n\to\infty}} ... + (-1)^k \frac{e_k}{e_{n\to\infty}} + ... - \frac{e_{n-1}}{e_{n\to\infty}} + 1\right) \\ &= e_{n\to\infty}\underbrace{(e_{(-n)\to\infty} - e_{(-n+1)} + e_{(-n+2)} ... + (-1)^k e_{(-n+k)} + ... - e_{(-1)} + 1)}_{Q(x,\sigma\pm it)}, \end{aligned} \tag{41}$$

which is easily proven from the quotient of the polynomials $P(x,\ \sigma\pm it)$ (10) and $Q(x,\ \sigma\pm it)$ (36) in the pole of the zeta function x=1:

$$\begin{aligned} \frac{P(1,\sigma \pm it)}{Q(1,\sigma \pm it)} &= \frac{\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{\sigma+it}}\right)\left(1-\frac{1}{n^{\sigma-it}}\right)}{\prod_{n=1}^{\infty}(1-n^{\sigma+it})(1-n^{\sigma-it})} = \prod_{n=1}^{\infty}\frac{\left(1-\frac{1}{n^{\sigma+it}}\right)\left(1-\frac{1}{n^{\sigma-it}}\right)}{(1-n^{\sigma+it})(1-n^{\sigma-it})} \\ &= \prod_{n=1}^{\infty}\frac{\frac{(-1)^2(1-n^{\sigma+it})(1-n^{\sigma-it})}{n^{\sigma+it}n^{\sigma-it}}}{(1-n^{\sigma+it})(1-n^{\sigma-it})} = \prod_{n=1}^{\infty}\frac{1}{n^{2\sigma}} = \prod_{n=1}^{\infty}n^{-2\sigma} = e_{n\to\infty} = \zeta_n(s,\bar{s}). \end{aligned} \tag{42}$$

Let us emphasize once again that the lowest common denominator $e_{n\to\infty}$ of the initial polynomial (10) is simultaneously the reciprocal value of the factorial function and the multiple zeta function of the complex-conjugate argument (40). The order of this multiple zeta function is $n\to\infty$. The exponent -2σ for the calculated value $\sigma=½$ (35) is equal to the value $-1=e^{i\pi}$.

6. Conclusions

Since Bernhard Riemann's famous manuscript from 1859, over a period of 163 years, more than 160 failed attempts to prove the Riemann Hypothesis (RH) have been recorded. The group of failed proofs is followed by a large number of extremely important results based on RH. In the opposition, there are stances that present doubts about the sustainability of the RH. The eventual success of the opposing stances would ultimately cause unfathomable consequences in number theory.

For now, the starting point of all the failed evidence of the RH and the results of opposites is common. It is based only on the complex argument, without the conjugate. The introduction of complex-conjugate solutions in RH (25) easily highlighted the singular result of the real part of the non-trivial zeros σ=½ of the complex-conjugate argument (35). However, this solution is based on the convergence of the last member of the special infinite polynomial (10). The existence of this convergence is confirmed, this member is the lowest common denominator of the infinite polynomial (10) or the denominator of all multiple zeta functions. The multiple zeta function of the complex-conjugate argument of the order $n\to\infty$, expressed through Newton's identity with $e_n\to\infty$, converges to the reciprocal value of the infinite factorial, i.e., values of the infinite factorial with the exponent from Leonard Euler's famous formula $-1=e^{i\pi}$ (43):

$$\zeta_n(s,\bar{s}) = e_{n\to\infty} = \prod_{n=1}^{\infty}\frac{1}{n^{2\sigma}} = \prod_{n=1}^{\infty}\frac{1}{n^{2(\frac{1}{2})}} = \prod_{n=1}^{\infty} n^{-1} = \prod_{n=1}^{\infty} n^{e^{i\pi}} . \tag{43}$$

The established convergence of the last member of the infinite polynomial (10, 24) is the main reason for the failure of all authors from supporters to opponents of the RH. In other words, the current conditions of access to the Riemann hypothesis are limiting, at the same time they prevent the proof and negation of RH. Such a state will remain until the Riemann Hypothesis is redefined. The experience of the previous 163 years should be respected. According to the findings in this manuscript, Berhard Riemann's functional equation from 1859 is the result of convergence. The circumstances for solving the Riemann hypothesis are historically analogous to the case of polynomials of the 5th degree and higher. The roots of these polynomials cannot be calculated using mathematical radicals, but can be calculated using other methods based on convergence. Therefore, it is necessary to additionally relax the conditions of proof of the RH with the approved convergence. Although, with this relaxation RH has already been proven with absolute probability [23]. The summary of the alternative calculation of the critical axis of the RH is:

- The basic zeta function is a multiple zeta function of the first order (11).
- Multiple zeta functions of the complex-conjugate argument were applied (17)–(23).
- Initial conditions are based on complex-conjugate non-trivial zeros (26).
- The calculation of the critical axis is based on the convergence of the inverse value of the factorial function (35).

The obtained result goes in favor of RH: Riemann's hypothesis is correct. The critical line of non-trivial zeros is singular. The values of the real part of the complex-conjugate argument of the non-trivial zeros of the zeta function converge to σ=½!

The appearance of the reciprocal value of the factorial at the first degree (2σ=1) in the form of the smallest common denominator of all multiple zeta functions is a confirmation of the monolithic structure of number theory even in "deep" infinity. Any opposite result that deviates from the value σ=½ or the singularity of the critical line (the appearance of two or more critical lines of different values, of which one can be equal to ½) would be in favor of transitory deformations or permanent deviations of the number system. The most critical positions would certainly be in prime numbers.

An alternative approach to the Riemann hypothesis particularly highlights the multiple zeta functions of order $n\to\infty$. The basis of the solution is the factorial of all natural numbers. Its exponent contains the three most important mathematical constants $i=\sqrt{-1}$, π=3.141592… and e=2.718281… from the famous identity $e^{i\pi}+1=0$ by Leonard Euler.

7. Guidelines for further research

Future research should primarily be focused on Eq (26). The first Newton's identity of the polynomial (10), the sum of zeta functions of the complex-conjugate argument, according to current assumptions, has exceptional properties. The value $\zeta(\sigma+it)+\zeta(\sigma-it)$ develops as an oscillatory sum. For the assumed value $\sigma=½$, the oscillation axes of the sum (26) have impulse changes by segments. The segments are approximately determined by the quotient $t/2\pi$, $t/4\pi$, $t/6\pi\ldots$, $t/2k\pi$. It is observed that the segments of the oscillation axes of the sum (26) are functionally connected to the trivial zeros of the zeta function: −2, −4, −6, …. In the last segment for the values $n\in([t/2\pi],\infty)$, the oscillation axis stabilizes. If t corresponds to the values of non-trivial zeros of the zeta function, the axis of oscillation of the last segment will be equal to zero. Oscillations in the last segment seemingly converge. However, these oscillations have divergent periods and amplitudes. Regardless of these divergences, the sums are regularized (Figure 1).

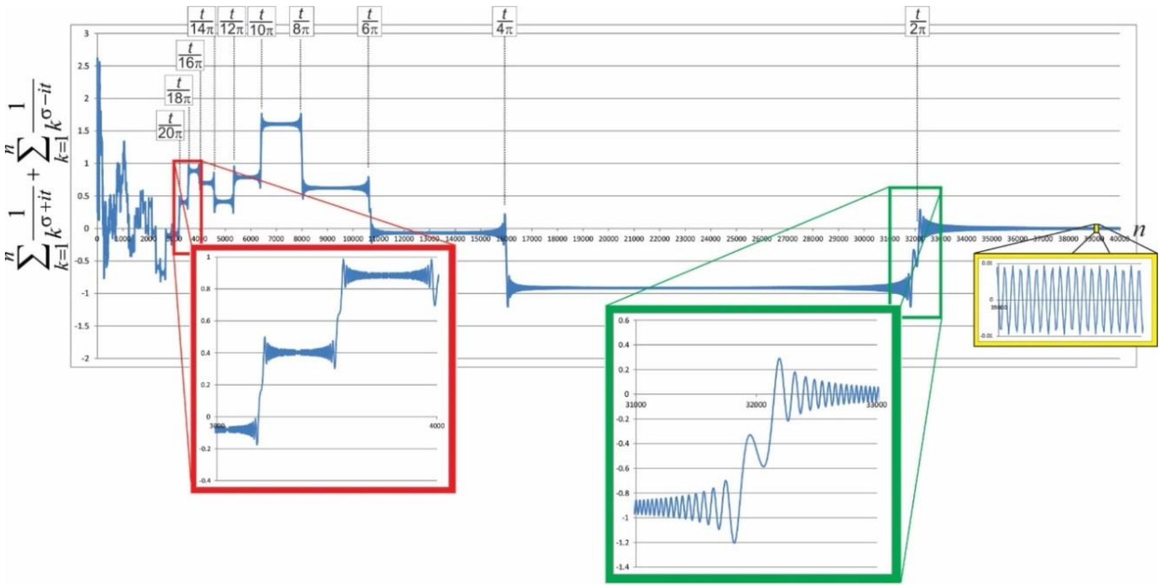

Figure 1. Development of the sum (26) for σ=½ and the 300000th non-trivial zero of the zeta function, imaginary part t=201090.1439061... with enlarged details.

According to previous findings, for the value $\sigma\neq½$, the oscillation axes of the sum (26) have impulse changes by segments, too. However, the stabilized oscillation axis of the last segment $n\in([t/2\pi],\infty)$ cannot be centered in the value zero, because only non-trivial zeros of the zeta function are symmetric, as stated in (2). In general, the zeta function is not symmetric. Therefore, future research is focused on a new, alternative approach to the Riemann hypothesis based on the dual functional Eq (44):

$$\zeta(s)+\zeta(\bar{s})=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)+2^{\bar{s}}\pi^{\bar{s}-1}\sin\left(\frac{\pi\bar{s}}{2}\right)\Gamma(1-\bar{s})\zeta(1-\bar{s})\,. \tag{44}$$

Applying the result (26), the dual functional Eq (44) is equal to (45):

$$(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)+(2\pi)^{\bar{s}-1}\sin\left(\frac{\pi\bar{s}}{2}\right)\Gamma(1-\bar{s})\zeta(1-\bar{s})=\sum_{n=1}^{\infty}\frac{\cos t\ln(n)}{n^{\sigma}}\,. \tag{45}$$

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Griffin, K. Ono, L. Rolen, D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, *Proc. Natl. Acad. Sci. U. S. A.*, **116** (2019), 11103–11110. https://doi.org/10.1073/pnas.1902572116
2. L. Q. Ming, Some formulas for Apostol-Euler polynomials associated with Hurwitz zeta function at rational arguments, *Appl. Anal. Discrete Math.*, **3** (2009), 336–346. https://doi.org/10.2298/AADM0902336L
3. A. P. Veselov, J. P. Ward, On the real zeroes of the Hurwitz zeta-function and Bernoulli polynomials, *J. Math. Anal. Appl.*, **305** (2005), 712–721. https://doi.org/10.1016/j.jmaa.2004.12.046
4. K. N. Boyadzhiev, A. Dil, Geometric polynomials: properties and applications to series with zeta values, *Anal. Math.*, **42** (2016), 203–224. https://doi.org/10.1007/s10476-016-0302-y
5. S. Choi, J. W. Chung, K. S. Kim, Relation between primes and nontrivial zeros in the Riemann hypothesis; Legendre polynomials, modified zeta function and Schrödinger equation, *J. Math. Phys.*, **53** (2012), 122108–122116. https://doi.org/10.1063/1.4770050
6. W. M. Abd-Elhameed, J. A. T. Machado, Y. H. Youssri, Hypergeometric fractional derivatives formula of shifted Chebyshev polynomials: tau algorithm for a type of fractional delay differential equations, *Int. J. Nonlinear Sci. Numer. Simul.*, **23** (2021), 0124. https://doi.org/10.1515/ijnsns-2020-0124
7. K. Matsumoto, *Value-distribution of zeta-functions*, Springer, 1990. https://doi.org/10.1007/BFb0097134
8. N. Kurokawa, Multiple zeta functions: an example, *Adv. Stud. Pure Math.*, **21** (1992), 219–226. https://doi.org/10.2969/aspm/02110219
9. K. Kamano, The multiple Hurwitz zeta function and a generalization of Lerch's formula, *Tokyo J. Math.*, **29** (2006), 61–73. https://doi.org/10.3836/tjm/1166661867
10. Y. Komori, K. Matsumoto, H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, *Publ. Math. Debrecen*, **77** (2010), 15–31. https://doi.org/10.5486/pmd.2010.4532
11. I. Kiuchi, Y. Tanigawa, W. Zhai, Analytic properties of double zeta functions, *Indag. Math.*, **21** (2011), 16–29. https://doi.org/10.1016/j.indag.2010.12.001
12. K. Matsumoto, H. Tsumura, Mean value theorems for the double zeta-function, *J. Math. Soc. Japan*, **67** (2015), 383–406. https://doi.org/10.2969/jmsj/06710383
13. K. Matsumoto, M. Shoji, Numerical computations on the zeros of the Euler double zeta-function II, *Eur. J. Math.*, **6** (2020), 488–507. https://doi.org/10.1007/s40879-019-00339-2

14. E. Bombieri, *Problems of the millennium: the Riemann hypothesis*, Clay Mathematics Institute, 2000.
15. P. Sarnak, *Problems of the millennium: the Riemann hypothesis*, Princeton University Courant Institute of Mathematical Sciences, 2004.
16. K. Matsumoto, On the bounded term in the mean square formula for the approximate functional equation of $\zeta^2(s)$, *Arch. Math.*, **64** (1995), 323–332. https://doi.org/10.1007/BF01198087
17. A. Ivić, K. Matsumoto, On the error term in the mean square formula for the Riemann zeta-function in the critical strip, *Monat. Math.*, **121** (1996), 213–229. https://doi.org/10.1007/BF01298951
18. K. Matsumoto, T. Matsusaka, I. Tanackov, On the Behavior of multiple zeta-functions with identical arguments on the real line, *J. Number Theory*, **239** (2022), 151–182. https://doi.org/10.1013/j.jnt.2021.11.008
19. D. Platt, T. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. London Math. Soc.*, **53** (2021), 792–797. https://doi.org/10.1112/blms.12460
20. K. Matsumoto, L. Weng, *Zeta-functions defined by two polynomials*, Springer, 2002. https://doi.org/10.1007/978-1-4757-3675-5_13
21. I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1995.
22. I. Tanackov, I. Pavkov, Ž. Stević, The new new-nacci method for calculating the roots of a univariate polynomial and solution of quintic equation in radicals, *Mathematics*, **8** (2020), 746. https://doi.org/10.3390/math8050746
23. I. J. Good, R. F. Churchhouse, The Riemann hypothesis and pseudorandom features of the Möbius sequence, *Math. Comput.*, **22** (1968), 857–861. https://doi.org/10.2307/2004584