AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(6): 13525–13542.
DOI: 10.3934/math.2023686
Received: 15 November 2022
Revised: 22 March 2023
Accepted: 27 March 2023
Published: 07 April 2023

*Research article*

# A $q$-Type $k$-Lidstone series for entire functions

**Zeinab S. I. Mansour**[1] **and Maryam Al-Towailb**[2,*]

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[2] Department of Computer Science and Engineering, College of Applied Studies and Community Service, King Saud University, Riyadh, KSA

* **Correspondence:** Email: mtowaileb@ksu.edu.sa.

**Abstract:** In this paper, we consider the $q$-type $k$-Lidstone series. The series follows from expanding certain classes of entire functions in terms of Jackson $q^{-1}$- derivatives at integers congruent to r modulo k, where $k$ is a positive integer. We study the main properties of the fundamental polynomials that appear in the series expansion. We include a detailed study for the case $k = 3$ with some examples.

**Keywords:** $q$-calculus; Lidstone series; convergence series; $q$-type Lidstone polynomials; $q$-difference operator
**Mathematics Subject Classification:** 05A30, 40A05, 41A58, 41A60

## 1. Introduction and preliminaries

A Lidstone series provides a generalization of Taylor's theorem that approximates an entire function $f$ of exponential type less than $\pi$ in a neighborhood of two points instead of one:

$$f(z) = \sum_{n=0}^{\infty} \Big[ f^{(2n)}(1) A_n(z) + f^{(2n)}(0) A_n(1-z) \Big], \tag{1.1}$$

where the polynomials $(A_n(z))_n$ are called Lidstone polynomials (see [14]).

Several authors including Boas [5, 6], Poritsky [19], Schoenberg [22], Whittaker [23], and Widder [24] gave necessary and sufficient conditions for representation of functions by Lidstone series (1.1).

In [13], Leeming and Sharma introduced an extension of Lidstone series. They proved that for a given integer $k \geq 2$, the following representation holds for a certain class of entire functions:

$$f(z) = \sum_{n=0}^{\infty} \Big[ f^{(kn)}(1)\, C_{kn}(z) + \sum_{v=0}^{k-2} f^{(kn+v)}(1)\, A_{kn+v}(z) \Big], \tag{1.2}$$

where $(C_{kn}(z))_n$ and $(A_{kn+v}(z))_n$ are certain polynomials which they called the fundamental polynomials of the series defined on the right-hand side of (1.2).

Recently, Ismail and Mansour [11] introduced a $q$-analog of the Lidstone expansion theorem for a certain class of entire functions as in the following formula:

$$f(z) = \sum_{n=0}^{\infty} \Big[ (D_{q^{-1}}^{2n} f)(1) A_n(z) - (D_{q^{-1}}^{2n} f)(0) B_n(z) \Big], \tag{1.3}$$

where $(A_n)_n$ and $(B_n)_n$ are the $q$-Lidstone polynomials defined by the generating functions

$$\frac{E_q(zw) - E_q(-zw)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} A_n(z) w^{2n}, \tag{1.4}$$

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} B_n(z) \frac{w^n}{[n]_q!}, \tag{1.5}$$

respectively, and $E_q(\cdot)$ is one of Jackson's $q$-exponential function defined by

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(z(1-q))^n}{(q;q)_n} \quad (z \in \mathbb{C}). \tag{1.6}$$

On the other hand, Al-Towailb [3] has constructed another $q$-type Lidstone theorem by expanding a class of entire functions in terms of $q$-derivatives of even orders at 0 and $q$-derivatives of odd orders at 1. Also, in [3], we proved that

$$f(z) = \sum_{n=0}^{\infty} \Big[ (D_{q^{-1}}^{r_n} f)(1)\, \pi_n(z;q) + (D_{q^{-1}}^{s_n} f)(0)\, \zeta_n(z;q), \Big]$$

where $f$ is an entire function satisfying some prescribed conditions, the sequences $(r_n)_n$ and $(s_n)_n$ are two sequences of non-negative integers, and $\{\pi_n(z;q), \zeta_n(z;q)\}_n$ are the set of polynomials (called a $q^{-1}$-standard set) that satisfies the following conditions:

$$(D_{q^{-1}}^{r_k} \pi_n)(1) = \delta_{n,k} \quad \text{and} \quad (D_{q^{-1}}^{s_k} \pi_n)(0) = 0;$$

$$(D_{q^{-1}}^{s_k} \zeta_n)(0) = \delta_{n,k} \quad \text{and} \quad (D_{q^{-1}}^{r_k} \zeta_n)(1) = 0,$$

where $\delta_{n,k}$ is the Kronecker delta ($k \in \mathbb{N}$). In particular, the set of polynomials $\{A_n(z), B_n(z)\}_n$ which defined in (1.4) and (1.5) form a $q^{-1}$-standard set of polynomials in relation to the pair of sequences $(r_n; s_n) = (2n; 2n)_{n \in \mathbb{N}_0}$.

For details and more results to the $q$-Lidstone's theorem, we also refer the reader to [2, 4, 15, 16].

Our aim here is to introduce another extension of $q$-Lidstone series, which will be called $q$-type $k$-Lidstone series, and determine the class of functions for which this series is valid, to obtain a $q$-analog of Leeming and Sharma's result. Furthermore, we consider the problem of expanding an entire function in the $q$-type 3-Lidstone series. These results will be derived by using Cauchy's integral formula and complex contour integration.

Throughout this paper, we assume that $q$ is a positive number less than one and $\mathbb{N}$ is the set of positive integers. We follow Gasper and Rahman [9] for the definitions, notations and properties of the $q$-shifted factorials $(a;q)_n$, $q$-gamma function $\Gamma_q(n)$, $q$-numbers $[n]_q$ and $q$-factorial $[n]_q!$.

The Jackson's $q$-derivative of a function $f$ is defined by

$$D_q f(z) := \frac{f(z) - f(qz)}{z - qz} \text{ for } z \neq 0,$$

and $D_q f(0)$ is usually defined as $f'(0)$ if $f$ is differentiable at zero (see [9]).

We start by stating some definitions in Section 2 and introduce a $q$ analog of the generalized circular functions of order $k$ ($k \in \mathbb{N}$), which we need in our investigation. In Section 3, we state and prove the principle theorem, and define the fundamental polynomials of a $q$-type $k$-Lidstone series. Then, we present some properties of these polynomials. Section 4 studies the problem of expanding an entire function in the $q$-type 3-Lidstone series. Also, we give six tables that deal with the generating functions of the fundamental polynomials associated with the six kinds of $q$-type 3-Lidstone series.

## 2. $q$-analogs of circular functions of high orders

$q$-analogs of the trigonometric functions $\sin z$ and $\cos z$ are defined by

$$\begin{aligned} \mathrm{Sin}_q z &:= \frac{E_q(iz) - E_q(-iz)}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} (z(1-q))^{2n+1}, \\ \mathrm{Cos}_q z &:= \frac{E_q(iz) + E_q(-iz)}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(2n-1)}}{(q;q)_{2n}} (z(1-q))^{2n}, \end{aligned} \tag{2.1}$$

respectively, where $E_q(z)$ is defined as in (1.6). $q$-analogs of the hyperbolic functions $\mathrm{Sinh}_q z$ and $\mathrm{Cosh}_q z$ are defined by

$$\mathrm{Sinh}_q(z) := -i\,\mathrm{Sin}_q(iz), \quad \mathrm{Cosh}_q(z) := \mathrm{Cos}_q(iz). \tag{2.2}$$

In 1948, Mikusinski [17] introduced the generalized circular functions of order $k$ ($k \in \mathbb{N}$) by

$$M_{k,j}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{kn+j}}{(kn+j)!}; \tag{2.3}$$

$$N_{k,j}(z) = \sum_{n=0}^{\infty} \frac{z^{kn+j}}{(kn+j)!}. \tag{2.4}$$

Note that there exists a relationship between these functions and the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta + \alpha n)}, \quad \alpha, \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,$$

(see [8, Section 18.1]), that is

$$M_{k,j}(z) = z^j E_{k,j+1}(-z^k) \quad \text{and} \quad N_{k,j}(z) = z^j E_{k,j+1}(z^k).$$

We consider the following $q$-special functions $M_{k,j}(z;q)$ and $N_{k,j}(z;q)$ ($k \in \mathbb{N}$), which are $q$-analogs of the functions (2.3) and (2.4), respectively.

$$M_{k,j}(z;q) = \sum_{m=0}^{\infty} (-1)^m q^{\frac{(km+j)(km+j-1)}{2}} \frac{z^{km+j}}{\Gamma_q(km+j+1)}; \tag{2.5}$$

$$N_{k,j}(z;q) = \sum_{m=0}^{\infty} q^{\frac{(km+j)(km+j-1)}{2}} \frac{z^{km+j}}{\Gamma_q(km+j+1)}. \tag{2.6}$$

Observe that $M_{1,0}(z;q) = E_q(-z)$, $M_{2,0}(z;q) = \mathrm{Cos}_q z$, and $M_{2,1}(z;q) = \mathrm{Sin}_q z$. Also, it is easy to conclude that

$$D^k_{q^{-1}} N_{k,j}(z;q) = N_{k,j}(z;q). \tag{2.7}$$

*Remark* 2.1. The function $N_{k,j}(z;q)$ is a special case of the big $q$-Mittag-Leffler function which is introduced in [21], and defined by

$$E_{q;\alpha,\beta}(z;c) = \sum_{n=0}^{\infty} \frac{q^{(\alpha n+\beta-1)(\alpha n+\beta-2)/2}}{(-c;q)_{\alpha n+\beta-1}} \frac{z^{\alpha n+\beta-1}\,(c/z;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}},$$

where $q, z, c, \alpha, \beta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $|q| < 1$. More precisely,

$$N_{k,j}(z;q) = (1-q) E_{q;k,j+1}(\frac{z}{1-q}; 0).$$

**Proposition 2.2.** *Let $k \in \mathbb{N}$, $j = 0, 1, \ldots, k-1$, and $\omega = \exp(2\pi i/k)$. Then, the following results hold:*

$$\omega^{-(j/2)} \sum_{m=0}^{k-1} \omega^{-mj} E_q(\omega^{m+1/2} z) = k M_{k,j}(z;q); \tag{2.8}$$

$$\omega^{j/2} M_{k,j}(z\omega^{-(1/2)}; q) = N_{k,j}(z;q); \tag{2.9}$$

$$\sum_{m=0}^{k-1} \omega^{-mj} E_q(\omega^m z) = k N_{k,j}(z;q). \tag{2.10}$$

*Proof.* To prove Eq (2.8), we use (1.6) to obtain

$$\sum_{m=0}^{k-1} \omega^{-mj} E_q(\omega^{m+1/2} z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n\, \omega^{n/2}}{\Gamma_q(n+1)} \sum_{m=0}^{k-1} \omega^{m(n-j)}. \tag{2.11}$$

Since $\omega = \exp(2\pi i/k)$, then $\omega^k = 1$ and $1 + \omega + \omega^2 + \ldots + \omega^{k-1} = 0$. Therefore,

$$\sum_{n=0}^{k-1} \omega^{(i-j)n} = \begin{cases} k, & i = j \pmod{k}; \\ 0, & i \neq j \pmod{k}. \end{cases} \tag{2.12}$$

We obtain the required result by substituting from (2.12) into (2.11) and then multiplying (2.11) with $\omega^{-(j/2)}$.

Formula (2.9) follows immediately from definitions (2.5) and (2.6). Finally, we get (2.10) and complete the proof from the results (2.8) and (2.9). □

Now, we consider the following boundary value problems:

$$\begin{gathered} D_{q^{-1}}^{k}y(x)+\lambda^{k}y(x)=0,\\ y(0)=D_{q^{-1}}y(0)=D_{q^{-1}}^{2}y(0)=\ldots=D_{q^{-1}}^{k-2}y(0)=y(1)=0, \end{gathered} \tag{2.13}$$

and the adjoint problem:

$$\begin{gathered} (-1)^{k}q^{k}D_{q}^{k}z(x)+\lambda^{k}z(x)=0,\\ z(1)=D_{q}z(1)=D_{q}^{2}z(1)=\ldots=D_{q}^{k-2}z(1)=z(0)=0. \end{gathered} \tag{2.14}$$

Then, the real eigenvalues $(\lambda_m)_{m=1}^{\infty}$ are zeros of the $q$-circular function $M_{k,k-1}(x;q)$ (defined in Eq (2.5)). The eigenfunctions of Problem (2.13) are

$$\{M_{k,k-1}(\lambda_m x;q)\}_{m=1}^{\infty},$$

and the eigenfunctions of Problem (2.14) are $\{\widetilde{M}_{k,k-1}(x,\lambda_k;q)\}_{m=1}^{\infty}$, where in general

$$\widetilde{M}_{k,j}(x,\lambda;q):=\sum_{n=0}^{\infty}(-1)^{n}q^{\frac{(nk+j)(nk+j-1)}{2}}\frac{(-\lambda x)^{nk+j}(1/x;q)_{nk+j}}{\Gamma_q(nk+j+1)}. \tag{2.15}$$

Notice, Ismail in [10] defined a $q$-translation operator by

$$\varepsilon_q^{y}x^{n}=x^{n}(-y/x;q)_{n},$$

and acts on polynomials as a linear operator. Therefore, one can verify that

$$\widetilde{M}_{k,j}(x,\lambda;q)=\varepsilon_q^{-1}M_{k,j}(\lambda x;q).$$

We use $\widetilde{M}_{k,j}(x;q)$ to denote $\widetilde{M}_{k,j}(x,1;q)$. One can also verify that

$$D_{q^{-1}}^{r}M_{k,j}(x)=\begin{cases} M_{k,j-r}(x), & r\le j,\\ -M_{k,j-r+k}(x), & j<r<k. \end{cases}$$

In the following, we construct the addition formula of the $q$-circular functions. For this, we define the function

$$K_{k,j}(x,\lambda,y;q):=$$
$$\sum_{m=0}^{\infty}(-1)^{m}\frac{\lambda^{km+j}}{\Gamma_q(km+j+1)}\sum_{r=0}^{km+j}\begin{bmatrix}km+j\\ r\end{bmatrix}_q q^{\binom{r}{2}}x^{r}(-y)^{km+j-r}(1/y;q)_{km+j-r}.$$

One can verify that $K_{k,j}(x,\lambda,1;q)=M_{k,j}(\lambda x)$, $K_{k,j}(0,\lambda,y;q)=\widetilde{M}_{k,j}(y,\lambda;q)$. Moreover,

$$D_{q^{-1}}^{r}K_{k,j}(x,\lambda,y)=\begin{cases} K_{k,j-r}(x,\lambda,y), & 0\le r\le j\le k-1;\\ -K_{k,k-r+j}(x,\lambda,y), & k>r>j. \end{cases}$$

**Theorem 2.3.** *The following result hold for* $j=0,1,\ldots,k-1$.

$$K_{k,j}(x,\lambda,y;q)=$$
$$\sum_{l=0}^{j}\widetilde{M}_{k,j-l}(y,\lambda;q)M_{k,l}(\lambda x;q)-\sum_{l=j+1}^{k-1}\widetilde{M}_{k,k-j+l}(y,\lambda;q)M_{k,l}(\lambda x;q).$$

*Proof.* For fixed $y$, the functions $\{y_j = K_{k,j}(x,\lambda,y)\}_{j=0}^{k-1}$ form a fundamental set of solutions of the initial value problem

$$D^k_{q^{-1}}y_j(x) + \lambda^k y_j(x) = 0,\ D^r_{q^{-1}}y_j(0) = \delta_{j,r},\ \{j,\ r\} \subset \{0,1,\dots,k-1\}.$$

Therefore, there exist some constants $c_n$ $(0 \le n \le k-1)$ such that

$$K_{k,j}(x,\lambda,y) = c_0 M_{k,0}(\lambda x; q) + c_1 M_{k,1}(\lambda x; q) + \ldots + c_{k-1} M_{k,k-1}(\lambda x; q).$$

Hence, for $r \in \{0,1,\dots,k-1\}$

$$\begin{aligned} &D^r_{q^{-1}} K_{k,j}(x,\lambda,y;q) = \\ &c_0 D^r_{q^{-1},x} M_{k,0}(\lambda x; q) + c_1 D^r_{q^{-1},x} M_{k,1}(\lambda x; q) + \ldots + c_{k-1} D^r_{q^{-1},x} M_{k,k-1}(\lambda x; q) \\ &= -\sum_{l=0}^{r-1} c_l M_{k,k-r+l}(\lambda x; q) + \sum_{l=r}^{k-1} c_l M_{k,l-r}(\lambda x; q). \end{aligned}$$

If we set $x = 0$ on the previous identity, we get

$$c_r = D^r_{q^{-1}} K_{k,j}(x,\lambda,y;q)|_{x=0} = \begin{cases} -\widetilde{M}_{k,k-r+j}(y,\lambda;q), & k > r > j, \\ \widetilde{M}_{k,j-r}(y,\lambda;q), & 0 \le r \le j. \end{cases}$$

□

**Theorem 2.4.** *The following biorthogonal property holds:*

$$\int_0^1 M_{k,k-1}(x\lambda_m; q)\widetilde{M}_{k,k-1}(x,\lambda_j;q)\, d_q x = -\frac{M_{k,k-2}(\lambda_m)}{k}\delta_{j,m},$$

*where $\delta_{j,m}$ is the Kronecker's delta, and $(\lambda_m)_{m=1}^{\infty}$ are the set of the real zeros of the function $M_{k,k-1}(x;q)$.*

*Proof.* We set $y(x) = M_{k,k-1}(\lambda x; q)$, $z(x) = \widetilde{M}_{k,k-1}(y,\lambda;q)$. Then, we have

$$\begin{aligned} (-1)^k q^k D^k_q z(x) &= -\lambda^k z(x), \\ D^k_{q^{-1}} y(x) &= -\lambda^k y(x). \end{aligned} \tag{2.16}$$

Consequently,

$$\int_0^1 \left[ z(x) D^k_{q^{-1}} y(x) + (-1)^k q^k y(x) D^k_q z(x) \right] d_q x = -2\lambda^k \int_0^1 y(x) z(x)\, d_q x. \tag{2.17}$$

Applying the $q$-integration by parts on Eq (2.17) $j$ times, we obtain

$$\begin{aligned} &\sum_{j=1}^{k-1} \int_0^1 \left[ (-1)^j q^j D^j_q z(x) D^{k-j}_{q^{-1}} y(z) + (-1)^{k-j} q^k - j D^j_{q^{-1}} y(x) D^{k-j}_q z(x) \right] d_q x \\ &= -2(k-1)\lambda^k \int_0^1 y(x) z(x)\, d_q x. \end{aligned}$$

That is,

$$2\int_0^1 \sum_{j=1}^{k-1}(-1)^j q^j D_q^j z(x) D_{q^{-1}}^{k-j} y(x)\, d_q x = -2(k-1)\lambda^k \int_0^1 y(x)z(x)\, d_q x.$$

But

$$\begin{aligned}\sum_{j=1}^{k-1}(-1)^j q^j D_q^j z(x) D_{q^{-1}}^{k-j} y(x) &= \lambda^k \sum_{j=1}^{k-1} M_{k,k-1-j}(\lambda x; q)\widetilde{M}_{k,j-1}(y,\lambda; q)\\ &= \sum_{j=0}^{k-2} M_{k,k-2-j}(\lambda x; q)\widetilde{M}_{k,j}(y,\lambda; q) = K_{k,k-2}(x,\lambda,x;q) + y(x)z(x).\end{aligned}$$

Set $(x * y)^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k (-y)^{n-k}(1/y;q)_{n-k}$. Then

$$\begin{aligned}(x * x)^n &= x^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-1)^{n-k}(1/x;q)_{n-k}\\ &= x^n q^{n(n-1)/2} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(1/x;q)_k q^k = q^{n(n-1)/2}.\end{aligned} \tag{2.18}$$

Hence, $K_{k,k-2}(x,\lambda,x;q) = M_{k,k-2}(\lambda;q)$, and then we obtain

$$\int_0^1 y(x)z(x)\, d_q x = \frac{M_{k,k-2}(\lambda;q)}{-k}.$$

□

## 3. $q$-type $k$-Lidstone expansion theorem

Recall that $\Psi$ is a comparison function if $\Psi(t) = \sum_{n=0}^{\infty} \Psi_n t^n$ and $\Psi_n > 0$ ($n \in \mathbb{N}_0$) such that $\left(\Psi_{n+1}/\Psi_n\right)$ is a decreasing sequence that converges to zero (see [6]). We denote by $\mathcal{R}_\Psi$, the class of all entire functions $f$ such that, for some numbers $\tau$,

$$|f(re^{i\theta})| \leq M\Psi(\tau r), \tag{3.1}$$

as $r \to \infty$. The infimum of the numbers $\tau$ for which (3.1) holds is the $\Psi$-type of the function $f$. This type can be computed by applying Nachbin's theorem [18] which states that a function $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is of $\Psi$-type $\tau$ if and only if

$$\tau = \limsup_{n\to\infty} \left|\frac{f_n}{\Psi_n}\right|^{\frac{1}{n}}.$$

We will use the following result from [6, Theorem 2.9].

**Theorem 3.1.** *Let $\Psi$ be a comparison function and $f$ is a function in the class $\mathcal{R}_\Psi$. Suppose that*

$$\psi(z) = \sum_{n=0}^{\infty} \psi_n z^n \ \ \textit{and} \ \ f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

*If $D(f)$ is a closed set consisting of the union of all singular points of $F$ and all points exterior to the domain of $F$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(zw)F(w)\, dw,$$

*where $\Gamma$ encloses $D(f)$ and*

$$F(w) = \sum_{n=0}^{\infty} \frac{f_n}{\Psi_n w^{n+1}}.$$

Ramis [20] defined an entire function $f$ to have a $q$-exponential growth ($q > 1$) of order $\gamma$ and a finite type if there exist positive numbers $\alpha$ and $K > 0$ such that

$$|f(z)| < K|z|^{\alpha} \exp\Big(\frac{\gamma \ln^2 |z|}{2 \ln^2 q}\Big). \tag{3.2}$$

Also, from [20, Lemma 2.2], if the series $f(z) := \sum_{n=0}^{\infty} a_n z^n$ satisfies (3.2), then

$$|a_n| \leq K q^{\frac{(n-\alpha)^2}{2\gamma}} \quad (n \in \mathbb{N}). \tag{3.3}$$

**Proposition 3.2.** *Let $\mu_1$ be the zero with the smallest positive absolute magnitude of $N_{k,k-1}(w;q)$, defined in* (2.5)*, and let $w$ be a complex number such that $|w| < |\mu_1|$. Assume that*

$$E_q(wz) = \sum_{j=0}^{k-2} w^j \psi_j(z, w^k) + E_q(w)\,\varphi(z, w^k) \quad (k \geq 2), \tag{3.4}$$

*where $E_q$ is the $q$-exponential function defined in* (1.6)*. Then, for $j \in \{0, 1, \ldots, k-2\}$:*

$$\begin{aligned} w^j \psi_j(z, w^k) &= N_{k,j}(wz;q) - N_{k,j}(w;q)\Big[\frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)}\Big]; \\ \varphi(z, w^k) &= \frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)}, \end{aligned} \tag{3.5}$$

*where $N_{k,j}(z;q)$ are the functions defined in* (2.6)*.*

*Proof.* Replace $w$ in Eq (3.4) by $\omega w, \omega^2 w, \ldots, \omega^{k-1} w$, with $\omega^k = 1$ ($\omega \neq 1$), and note that the functions $\varphi$ and $\psi_j$ remain unchanged. Then, we obtain the following system of $k$ equations:

$$E_q(\omega^n wz) = E_q(\omega^n w)\,\varphi(z, w^k) + \sum_{j=0}^{k-2} \omega^{nj} w^j \psi_j(z, w^k), \tag{3.6}$$

where $n \in \{0, 1, \ldots, k-1\}$. If we multiply Eq (3.6) by $\omega^{-nj}$ and then adding these equations for $n \in \{0, 1, \ldots, k-1\}$, we obtain

$$\sum_{n=0}^{k-1} \omega^{-nj} E_q(\omega^n wz) = \sum_{n=0}^{k-1} \omega^{-nj} E_q(\omega^n w)\,\varphi(z, w^k) + \sum_{i=0}^{k-2} w^i \psi_i(z, w^k) \sum_{n=0}^{k-1} \omega^{(i-j)n}.$$

Therefore, from (2.10), we get

$$\sum_{n=0}^{k-1} \omega^{-nj} E_q(\omega^n wz) = kN_{k,j}(z;q)\,\varphi(z,w^k) + \sum_{i=0}^{k-2} w^i\,\psi_i(z,w^k) \sum_{n=0}^{k-1} \omega^{(i-j)n}.$$

Thus, the result follows at once from (2.12).

Obviously, $\varphi(z,w^k)$ and $w^j\psi_j(z,w^k)$ are analytic functions for $|w| < |\mu_1|$. □

According to the above results, we can prove the following main theorem.

**Theorem 3.3.** *Let $\mu_1$ be the zero with the smallest positive absolute magnitude of $N_{k,k-1}(w;q)$, defined in* (2.5)*. If $f$ is an entire function of $q^{-1}$-exponential growth of order less than* 1*, or of order* 1 *and a finite type $\alpha$ such that*

$$\alpha < \Big(\frac{1}{2} - \frac{\log|\mu_1(1-q)|}{\log q}\Big), \tag{3.7}$$

*then, for all $z \in \mathbb{C}$, the following representation holds*

$$f(z) = \sum_{n=0}^{\infty} \Big[(D_{q^{-1}}^{kn} f)(1)\, A_{kn}(z) + \sum_{j=0}^{k-2} (D_{q^{-1}}^{kn+j} f)(0)\, B_{kn+j}(z)\Big], \tag{3.8}$$

*where $A_{kn}(z)$ and $B_{kn+j}(z)$ are the polynomials defined by the following generating functions:*

$$\begin{aligned} \sum_{n=0}^{\infty} w^{kn} A_{kn}(z) &= \frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)} \equiv \varphi(z,w^k), \\ \sum_{n=0}^{\infty} w^{kn+j} B_{kn+j}(z) &= N_{k,j}(wz;q) - N_{k,j}(w;q)\Big[\frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)}\Big] \\ &\equiv w^j\psi_j(z,w^k) \quad (k \geq 2,\ j = 0,1,\ldots,k-2), \end{aligned} \tag{3.9}$$

*and $w$ is a complex number such that $|w| < |\mu_1|$. Furthermore, the series on right-hand side of* (3.8) *converges to $f(z)$ uniformly on all compact subsets of the plane.*

*Proof.* We apply Theorem 3.1. Set $\Psi(z) = E_q(z)$ and $f(z) := \sum_{n=0}^{\infty} a_n z^n$. Then

$$\Psi_n = \frac{q^{\frac{n(n-1)}{2}}}{\Gamma_q(n+1)}, \quad \frac{\Psi_{n+1}}{\Psi_n} = \frac{q^n(1-q)}{1-q^{n+1}} = \frac{q^n}{[n+1]_q}$$

is decreasing and vanishes at $\infty$. Since $\Psi(z)$ has a $q^{-1}$-exponential growth of order 1, then Eq (3.1) holds if $f$ is a function of $q^{-1}$-exponential growth $\gamma$, $\gamma \leq 1$. Then from (3.3), the type $\tau$ of the function $f$ , $\tau \geq 0$ of the function $f$ has the upper bound

$$\begin{aligned} \tau &:= \limsup_{n\to\infty} \Big|\frac{a_n}{\Psi_n}\Big|^{\frac{1}{n}} \\ &\leq \frac{q^{\frac{1}{2}-\alpha}}{(1-q)} \limsup_{n\to\infty} \Big(K\,(q;q)_n q^{\alpha^2/2}\Big)^{\frac{1}{n}} q^{\frac{n}{2}(\frac{1}{\gamma}-1)}. \end{aligned}$$

Consequently, if $\gamma < 1$, then $\tau = 0$, if $\gamma = 1$, then $\tau \leq \frac{q^{\frac{1}{2}-\alpha}}{1-q}$. So, $D(f)$ lies in the closed disk $|w| \leq \tau \leq \dfrac{q^{\frac{1}{2}-\alpha}}{1-q} < \mu_1$ and we choose $\Gamma$ to be the circle $|w| = \tau + \epsilon < \mu_1$, $\epsilon > 0$ which encloses $D(f)$. Note that the inequality $\frac{q^{\frac{1}{2}-\alpha}}{1-q} < |\mu_1|$ satisfies the condition (3.7) on the type of $f(z)$. Then, we obtain

$$f(z) = \frac{1}{2\pi i}\int_\Gamma E_q(zw)F(w)\,dw.$$

Therefore,

$$\begin{aligned} D^{kn}_{q^{-1}} f(1) &= \frac{1}{2\pi i}\int_\Gamma w^{kn}\, E_q(w)F(w)\,dw, \\ D^{kn+j}_{q^{-1}} f(0) &= \frac{1}{2\pi i}\int_\Gamma w^{kn+j}\, F(w)\,dw \quad j = 0, 1, \dots, k-2. \end{aligned}$$

By setting

$$\sum_{n=0}^{\infty} w^{kn} A_{kn}(z) \equiv \varphi(z, w^k), \qquad \sum_{n=0}^{\infty} w^{kn+j} B_{kn+j}(z) \equiv w^j \psi_j(z, w^k),$$

and using Proposition 3.2, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\Big[(D^{kn}_{q^{-1}} f)(1)\, A_{kn}(z) + \sum_{j=0}^{k-2}(D^{kn+j}_{q^{-1}} f)(0)\, B_{kn+j}(z)\Big] \\ = &\ \frac{1}{2\pi\, i}\int_\Gamma \left\{E_q(w)\sum_{n=0}^{\infty} w^{kn} A_{kn}(z) + \sum_{n=0}^{\infty}\sum_{j=0}^{k-2} w^{kn+j}\, B_{kn+j}(z)\right\} F(w)\,dw \\ = &\ \frac{1}{2\pi\, i}\int_\Gamma \left\{E_q(w)\,\varphi(z, w^k) + \sum_{j=0}^{k-2} w^j\, \psi_j(z, w^k)\right\} F(w)\,dw \\ = &\ \frac{1}{2\pi\, i}\int_\Gamma E_q(wz)F(w)\,dw = f(z). \end{aligned}$$

Finally, from the definitions of $\phi$ and $\psi_j$, we have that the right-hand side of (3.4) is analytic in the disk $|w| < |\mu_1|$. Therefore, the series defined by $\phi$ and $\psi_j$ converges uniformly in every compact subset of the disk $|w| < |\mu_1|$. □

We will say that the formula (3.8) is $q$-type of $k$-Lidstone series of the function $f$, and the functions $A_{kn}(z)$ and $B_{kn+j}(z)$ ($j = 0, 1, \dots, k-2$) are the fundamental polynomials of this series.

*Remark* 3.4. If we set $k = 2$ in Eq (3.8), we have the $q$-Lidstone series expansion (1.3).

The following result gives some properties of the fundamental polynomials of the $q$-type $k$-Lidstone series.

**Proposition 3.5.** *For $n \in \mathbb{N}$, $k \geq 2$, and $j \in \{0, 1, 2, \dots, k-2\}$, the fundamental polynomials of the $q$-type of $k$-Lidstone series, $A_{kn}(x)$ and $B_{kn+j}(x)$, satisfy the following properties:*

*(i)* $A_0(x) = x^{k-1}$ *and* $B_0(x) = 1 - x^{k-1}$;

*(ii)* $D^k_{q^{-1}} A_{kn}(x) = A_{k(n-1)}(x)$ *and* $D^k_{q^{-1}} B_{kn+j}(x) = B_{k(n-1)+j}(x)$;

*(iii)* $A_{kn}(1) = 0$ *and* $B_{kn+j}(1) = 0$;

*(iv)* $D^r_{q^{-1}} A_{kn}(0) = 0$ *and* $D^r_{q^{-1}} B_{kn+j}(0) = 0$ *for* $r \in \{0, 1, \ldots, k-2\}$.

*Proof.* The proof of (i) follows from the substitution with $w = 0$ in the generating functions in (3.9). To prove (ii), we act on the two sides of (3.9) by the operator $D^k_{q^{-1},x}$ and use (2.7). To prove the first identity in (iii), we substitute with $z = 1$ in the first equation in (3.9) and using that $A_0(1) = 1$, consequently, $\sum_{n=1}^{\infty} w^{kn} A_{kn}(1) = 0$, This yields $A_{kn}(1) = 0$. The substitution with $z = 1$ in the second identity in (3.9) yields $\sum_{n=0}^{\infty} w^{kn+j} B_{kn+j}(1) = 0$. Hence, $B_{kn+j}(1) = 0$ for all $n \in \mathbb{N}_0$. Finally, the proof of (iv) follows at once from (ii) and (iii). □

## 4. Special case: A $q$-type 3-Lidstone series

In this section, we give several problems that illustrate the $q$-type of 3-Lidstone series. One of them can be derived immediately from Theorem 3.3 (see Table 1).

In the following, we discuss another problem of expanding an entire function in a $q$-type 3-Lidstone series.

**Theorem 4.1.** *Let* $\mu_{1,3}$ *be the zero with the smallest positive absolute magnitude of* $M_{3,0}(z; q)$ *which defined in* (2.5)*. Then, for every entire function* $f(z)$ *of* $q^{-1}$*-exponential growth of order less than* 1*, or of order* 1 *and a finite type* $\alpha$ *such that* $\alpha < \left(\frac{1}{2} - \frac{\log|\mu_{1,3}(1-q)|}{\log q}\right)$ *the following representation holds:*

$$f(z) = \sum_{n=0}^{\infty} \Big[ D^{3n}_{q^{-1}} f(1)\, \tilde{A}_{3n}(z) + D^{3n+1}_{q^{-1}} f(0)\, \tilde{B}_{3n+1}(z) + D^{3n+2}_{q^{-1}} f(0)\, \tilde{B}_{3n+2}(z) \Big], \tag{4.1}$$

*where* $\tilde{A}_{3n}(z)$, $\tilde{B}_{3n+1}(z)$ *and* $\tilde{B}_{3n+2}(z)$ *are polynomials defined by the following generating functions:*

$$\begin{aligned} \sum_{n=0}^{\infty} w^{3n} \tilde{A}_{3n}(z) &= \frac{N_{3,0}(wz; q)}{N_{3,0}(w; q)}, \\ \sum_{n=0}^{\infty} w^{3n+1} \tilde{B}_{3n+1}(z) &= N_{3,1}(wz; q) - N_{3,1}(w; q)\Big[\frac{N_{3,0}(wz; q)}{N_{3,0}(w; q)}\Big], \\ \sum_{n=0}^{\infty} w^{3n+2} \tilde{B}_{3n+2}(z) &= N_{3,2}(wz; q) - N_{3,2}(w; q)\Big[\frac{N_{3,0}(wz; q)}{N_{3,0}(w; q)}\Big]. \end{aligned} \tag{4.2}$$

*The series on* (4.1) *converges to* $f(z)$ *for all* $z$ *and the convergence is uniform on all compact subsets of the plane.*

*Proof.* First, we consider the functions $\sum_{n=0}^{\infty} w^{3n}\tilde{A}_{3n}(z) = \tilde{\varphi}(z, w^3) \equiv \tilde{\varphi}$,

$$\sum_{n=0}^{\infty} w^{3n+1}\tilde{B}_{3n+1}(z) = w\,\tilde{\psi}_1(z, w^3) \equiv w\,\tilde{\psi}_1,$$
$$\sum_{n=0}^{\infty} w^{3n+2}\tilde{B}_{3n+2}(z) = w^2\,\tilde{\psi}_2(z, w^3) \equiv w^2\,\tilde{\psi}_2,$$

and assume that the function $E_q(wz)$ has the following representation:

$$E_q(wz) = E_q(w)\varphi + w\psi_1 + w^2\psi_2. \tag{4.3}$$

Let $\omega^3 = 1$ with $\omega \neq 1$ and replacing $w$ by $\omega w$ and $\omega^2 w$ in Eq (4.3), we obtain

$$E_q(w\omega z) = E_q(w\omega)\tilde{\varphi} + w\omega\tilde{\psi}_1 + w^2\omega^2\tilde{\psi}_2, \tag{4.4}$$
$$E_q(w\omega^2 z) = E_q(w\omega^2)\tilde{\varphi} + w\omega^2\tilde{\psi}_1 + w^2\omega\tilde{\psi}_2. \tag{4.5}$$

If we add the Eqs (4.3)–(4.5), we obtain $\tilde{\varphi}$. In order to get the function $w\tilde{\psi}_1$, we multiply the Eqs (4.3)–(4.5) by $1, \omega^{-1}$ and $\omega^{-2}$, respectively and add. $w^2\tilde{\psi}_2$ obtained by multiplying the same equations by $1, \omega^{-2}$ and $\omega^{-1}$, respectively. The proof is then completed similar to the proof of Theorem 3.3. □

**Corollary 4.2.** *For $n \in \mathbb{N}$, the polynomials $\tilde{A}_{3n}$ and $\tilde{B}_{3n+j}$ $(j = 1, 2)$ satisfy the q-difference equations*

$$D^3_{q^{-1}}\tilde{A}_{3n}(z) = \tilde{A}_{3(n-1)}(z), \quad D^3_{q^{-1}}\tilde{B}_{3n+j}(z) = \tilde{B}_{3(n-1)+j}(z),$$

*with the boundary conditions*

$$\tilde{A}_{3n}(1) = 0 = \tilde{B}_{3n+j}(1), \quad D_{q^{-1}}\tilde{A}_{3n}(0) = 0 = D_{q^{-1}}\tilde{B}_{3n+j}(0),$$
$$D^2_{q^{-1}}\tilde{A}_{3n}(0) = 0 = D^2_{q^{-1}}\tilde{B}_{3n+j}(0).$$

*Proof.* The proof follows by using the generating functions in (4.2). □

*Remark* 4.3. According to formula (4.1), if $P$ any polynomial of degree less than or equals to 6, then we have

$$\begin{aligned} P(z) = P(1)\tilde{A}_0(z) + D^3_{q^{-1}}P(1)\,\tilde{A}_3(z) + D^6_{q^{-1}}P(1)\,\tilde{A}_6(z) + \\ D_{q^{-1}}P(0)\,\tilde{B}_1(z) + D^2_{q^{-1}}P(0)\,\tilde{B}_2(z) + D^4_{q^{-1}}P(0)\,\tilde{B}_4(z) + D^5_{q^{-1}}P(0)\,\tilde{B}_5(z). \end{aligned} \tag{4.6}$$

So, by setting $P(z) = 1, z, \ldots, z^6$, successively, in (4.6) we get

$$\tilde{A}_0(z) = 1,\ \tilde{B}_1(z) = z - 1,\ \tilde{B}_2(z) = \frac{q}{[2]_q!}(z^2 - 1),\ \tilde{A}_3(z) = \frac{q^3}{[3]_q!}(z^3 - 1),$$
$$\tilde{B}_4(z) = \frac{q^6}{[4]_q!}(z^4 - 1) - \frac{q^3}{[3]_q!}(z^3 - 1),\ \tilde{B}_5(z) = \frac{q^{10}}{[5]_q!}(z^5 - 1) - \frac{q^4}{[3]_q!\,[2]_q!}(z^3 - 1),$$
$$\tilde{A}_6(z) = \frac{q^{15}}{[6]_q!}(z^6 - 1) - \frac{q^6}{[3]_q!\,[3]_q!}(z^2 - 1).$$

*Example* 4.4. We apply Theorem 4.1 on the function $f(z) = (z;q)_{3n}$ ($n \in \mathbb{N}$) and using that for any $m \in \mathbb{N}_0$,

$$D^k_{q^{-1}}(z;q)_m = \begin{cases} (-1)^k \dfrac{[m]_q!}{[m-k]_q!}(z;q)_{m-k}, & k \le m, \\ 0, & k > m. \end{cases}$$

This gives

$$\frac{(z;q)_{3n}}{[3n]_q!} = (-1)^n A_{3n}(z;q) + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{\widetilde{B}_{3k+1}(z)}{[3n-3k-1]_q!} + \sum_{k=0}^{n-1}(-1)^k \frac{\widetilde{B}_{3k+2}(z)}{[3n-3k-2]_q!}.$$

*Example* 4.5. Consider the Al-Salam-Carlitz II polynomials (see [12]) defined by

$$V_n^{(a)}(x;q) = (-a)^n q^{\binom{-n}{2}} \sum_{k=0}^{n} \frac{(q^{-n}, x;q)_k}{(q;q)_k} \left(\frac{q^n}{a}\right)^k.$$

Since, $D^k_{q^{-1}} V_m^{(a)}(x;q) = q^{-\binom{m}{2}+\binom{m-k}{2}} \frac{[m]_q!}{[m-k]_q!} V^{(a)}_{m-k}(x;q)$ if $0 \le k \le m$, we obtain

$$D^k_{q^{-1}} V_m^{(a)}(1;q) = q^{-\binom{m}{2}} \frac{[m]_q!}{[m-k]_q!}(-a)^{m-k},$$

$$D^k_{q^{-1}} V_m^{(a)}(0;q) = q^{-\binom{m}{2}} \frac{[m]_q!}{[m-k]_q!}(-a)^{m-k}(1/a;q)_{m-k},$$

for $k \le m$. Consequently, applying Theorem 4.1 yields

$$(-a)^{-3n} q^{\binom{3n}{2}} \frac{V_{3n}^{(a)}(x;q)}{[3n]_q!} = \sum_{k=0}^{n} \frac{(-a)^{-3k}}{[3n-3k]_q!}\widetilde{A}_{3k}(x) + \sum_{k=0}^{n-1} \frac{(-a)^{-3k-1}}{[3n-3k-1]_q!}\widetilde{B}_{3k+1}(x) + \sum_{k=0}^{n-1} \frac{(-a)^{-3k-2}}{[3n-3k-2]_q!}\widetilde{B}_{3k+2}(x).$$

*Remark* 4.6. For the $q$-type of 3-Lidstone series, we can consider six problems:

$$\sum_{n=0}^{\infty} \Big[ D^{3n}_{q^{-1}} f(1)\, A^{(1)}_{3n}(z) + D^{3n}_{q^{-1}} f(0)\, B^{(1)}_{3n}(z) + D^{3n+1}_{q^{-1}} f(0)\, B^{(1)}_{3n+1}(z) \Big],$$

$$\sum_{n=0}^{\infty} \Big[ D^{3n}_{q^{-1}} f(1) A^{(2)}_{3n}(z) + D^{3n+1}_{q^{-1}} f(0) B^{(2)}_{3n+1}(z) + D^{3n+2}_{q^{-1}} f(0)\, B^{(2)}_{3n+2}(z) \Big],$$

$$\sum_{n=0}^{\infty} \Big[ D^{3n+1}_{q^{-1}} f(1) A^{(3)}_{3n+1}(z) + D^{3n}_{q^{-1}} f(0) B^{(3)}_{3n}(z) + D^{3n+2}_{q^{-1}} f(0)\, B^{(3)}_{3n+2}(z) \Big],$$

$$\sum_{n=0}^{\infty} \Big[ D^{3n+2}_{q^{-1}} f(1) A^{(4)}_{3n+2}(z) + D^{3n}_{q^{-1}} f(0) B^{(4)}_{3n}(z) + D^{3n+1}_{q^{-1}} f(0)\, B^{(4)}_{3n+1}(z) \Big],$$

$$\sum_{n=0}^{\infty} \Big[ D^{3n}_{q^{-1}} f(1)\, A^{(5)}_{3n}(z) + D^{3n}_{q^{-1}} f(0)\, B^{(5)}_{3n}(z) + D^{3n+2}_{q^{-1}} f(0)\, B^{(5)}_{3n+2}(z) \Big],$$

$$\sum_{n=0}^{\infty} \Big[ D^{3n+1}_{q^{-1}} f(1) A^{(6)}_{3n+1}(z) + D^{3n}_{q^{-1}} f(0) B^{(6)}_{3n}(z) + D^{3n+1}_{q^{-1}} f(0)\, B^{(6)}_{3n+1}(z) \Big],$$

where $f$ is assumed to be entire function and satisfies the conditions on Theorem 4.1.

The tables below give the generating functions of the $q$-type fundamental polynomials and the values at $n = 0$ in these problems, respectively.

**Table 1.** $q$-type 3-Lidstone series (1).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n} A_{3n}^{(1)}(z) = \frac{N_{3,2}(wz;q)}{N_{3,2}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n} B_{3n}^{(1)}(z) = N_{3,0}(wz;q) - N_{3,0}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,2}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n+1} B_{3n+1}^{(1)}(z) = N_{3,1}(wz;q) - N_{3,1}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,2}(w;q)}$ |
| $q$-Polynomials | $A_0^{(1)}(z) = z^2, \quad B_0^{(1)}(z) = 1 - z^2$ and $B_1^{(1)}(z) = z(1-z)$ |

**Table 2.** $q$-type 3-Lidstone series (2).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n} A_{3n}^{(2)}(z) = \frac{N_{3,0}(wz;q)}{N_{3,0}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n+1} B_{3n+1}^{(2)}(z) = N_{3,1}(wz;q) - N_{3,1}(w;q)\frac{N_{3,0}(wz;q)}{N_{3,0}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n+2} B_{3n+2}^{(2)}(z) = N_{3,2}(wz;q) - N_{3,2}(w;q)\frac{N_{3,0}(wz;q)}{N_{3,0}(w;q)}$ |
| $q$-Polynomials | $A_0^{(2)}(z) = 1, \quad B_1^{(2)}(z) = z - 1$ and $B_2^{(2)}(z) = \frac{q}{[2]_q!}(z^2 - 1)$ |

**Table 3.** $q$-type 3-Lidstone series (3).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n+1} A_{3n+1}^{(3)}(z) = \frac{N_{3,1}(wz;q)}{N_{3,0}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n} B_{3n}^{(3)}(z) = N_{3,0}(wz;q) - N_{3,2}(w;q)\frac{N_{3,1}(wz;q)}{N_{3,0}(w;q)}$ <br> $\sum_{n=0}^{\infty} w^{3n+2} B_{3n+2}^{(3)}(z) = N_{3,2}(wz;q) - N_{3,1}(w;q)\frac{N_{3,1}(wz;q)}{N_{3,0}(w;q)}$ |
| $q$-Polynomials | $A_1^{(3)}(z) = z, \quad B_0^{(3)}(z) = 1$ and $B_2^{(3)}(z) = \frac{q}{[2]_q}z^2 - z$ |

**Table 4.** $q$-type 3-Lidstone series (4).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n+2} A_{3n+2}^{(4)}(z) = \frac{N_{3,2}(wz;q)}{N_{3,0}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n} B_{3n}^{(4)}(z) = N_{3,0}(wz;q) - N_{3,1}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,0}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n+1} B_{3n+1}^{(4)}(z) = N_{3,1}(wz;q) - N_{3,2}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,0}(w;q)}$ |
| $q$-Polynomials | $A_2^{(4)}(z) = \frac{qz^2}{[2]_q}$, $B_0^{(4)}(z) = 1$ and $B_1^{(4)}(z) = z$ |

**Table 5.** $q$-type 3-Lidstone series (5).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n} A_{3n}^{(5)}(z) = \frac{N_{3,1}(wz;q)}{N_{3,1}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n} B_{3n}^{(5)}(z) = N_{3,0}(wz;q) - N_{3,0}(w;q)\frac{N_{3,1}(wz;q)}{N_{3,1}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n+2} B_{3n+2}^{(5)}(z) = N_{3,2}(wz;q) - N_{3,2}(w;q)\frac{N_{3,1}(wz;q)}{N_{3,1}(w;q)}$ |
| $q$-Polynomials | $A_0^{(5)}(z) = z$, $B_0^{(5)}(z) = 1 - z$ and $B_2^{(5)}(z) = \frac{qz(z-1)}{[2]_q}$ |

**Table 6.** $q$-type 3-Lidstone series (6).

| The functions | The results |
|---|---|
| Generating | $\sum_{n=0}^{\infty} w^{3n+1} A_{3n+1}^{(6)}(z) = \frac{N_{3,2}(wz;q)}{N_{3,1}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n} B_{3n}^{(6)}(z) = N_{3,0}(wz;q) - N_{3,2}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,1}(w;q)}$<br>$\sum_{n=0}^{\infty} w^{3n+1} B_{3n+1}^{(6)}(z) = N_{3,1}(wz;q) - N_{3,0}(w;q)\frac{N_{3,2}(wz;q)}{N_{3,1}(w;q)}$ |
| $q$-Polynomials | $A_1^{(6)}(z) = \frac{qz^2}{[2]_q}$ $B_0^{(6)}(z) = 1$ and $B_1^{(6)}(z) = \frac{z([2]_q - qz)}{[2]_q}$ |

## 5. Conclusions

In this paper, we introduced an extension of $q$-Lidstone series which was called a $q$-type $k$-Lidstone series:

$$f(z) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{kn} f(1)\, A_{kn}(z) + \sum_{j=0}^{k-2} D_{q^{-1}}^{kn+j} f(0)\, B_{kn+j}(z)\Big], \tag{5.1}$$

where $(A_{kn}(z))_n$ and $(B_{kn+j}(z))_n$ are polynomials defined by the following generating functions:

$$\sum_{n=0}^{\infty} w^{kn} A_{kn}(z) = \frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)} \quad (k \geq 2),$$

$$\sum_{n=0}^{\infty} w^{kn+j} B_{kn+j}(z) = N_{k,j}(wz;q) - N_{k,j}(w;q)\Big[\frac{N_{k,k-1}(wz;q)}{N_{k,k-1}(w;q)}\Big],$$

and determined the class of functions for which (5.1) is valid.

Notice, by following the same manner as a proof of (5.1), we can conclude that the function $f$ can be given also by the convergent another $q$-type $k$-Lidstone series expansion with different $q$-polynomials. More precisely, we can obtain the following result.

**Theorem 5.1.** *Let $\mu_1$ be the zero with the smallest positive absolute magnitude of $N_{k,0}(w;q)$ which defined in* (2.5)*. If the function $f(z)$ is an entire function of $q^{-1}$-exponential growth of order less than* 1*, or of order* 1 *and a finite type $\alpha$ such that*

$$\alpha < \Big(\frac{1}{2} - \frac{\log|\mu_1(1-q)|}{\log q}\Big),$$

*then, for all $z \in \mathbb{C}$ the following representation holds*

$$f(z) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{kn} f(0)\, C_{kn}(z) + \sum_{j=1}^{k-1} D_{q^{-1}}^{kn+j} f(1)\, P_{kn+j}(z)\Big],$$

*where $(C_{kn}(z))_n$ and $(P_{kn+j}(z))_n$ are polynomials defined by the following generating functions:*

$$\sum_{n=0}^{\infty} w^{kn} C_{kn}(z) = \frac{1}{k} N_{k,0}(wz;q),$$

$$\sum_{n=0}^{\infty} w^{kn+j} P_{kn+j}(z) = \Big[\frac{N_{k,j}(wz;q)}{N_{k,0}(w;q)}\Big] \quad (k \geq 1,\ j = 1,2,\ldots,k-1).$$

As a special case, we considered six problems of expanding an entire function in the $q$-type 3-Lidstone series. The Lidstone polynomials are used in many interpolation and boundary value problems. See for example, [1, 7]. We studied boundary value problems includes the 2 type $q$-Lidstone polynomials in [15], and we aim to study boundary value problems associated with the $k$ type $q$-Lidstone polynomials, $k > 2$.

## Acknowledgments

The authors would like to express appreciation to the editor and the referees for their helpful comments and suggestions that improved this article.


## Conflict of interest

The authors declare no conflicts of interest.

## References


1. R. Agarwal, P. Wong, Lidstone polynomials and boundary value problems, *Comput. Math. Appl.*, **17** (1989), 1397–1421.
2. M. Al-Towailb, A $q$-difference equation and Fourier series expansions of $q$-Lidstone polynomials, *Symmetry*, **14** (2022), 782. https://doi.org/10.3390/sym14040782
3. M. AL-Towailb, A generalization of the $q$-Lidstone series, *AIMS Mathematics*, **7** (2022), 9339–9352. https://doi.org/10.3934/math.2022518
4. M. AL-Towailb, Z. S. I. Mansour, The $q$-Lidstone series involving $q$-Bernoulli and $q$-Euler polynomials generated by the third Jackson $q$-Bessel function, *Kjm*, 2022, In press.
5. R. P. Boas, Representation of functions by Lidstone series, *Duke Math. J.*, **10** (1943), 239–245.
6. R. P. Boas, R. C. Buck, *Polynomial expansions of analytic functions*, 2 Eds., Berlin: Springer-Verlag, 1964.
7. F. A. Costabile, M. I. Gualtieri, A. Napoli, Lidstone–Euler second-type boundary value problems: Theoretical and computational tools, *Mediterr. J. Math.*, **18** (2021), 180. https://doi.org/10.1007/s00009-021-01822-5
8. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, New York: McGraw-Hill, 1955.
9. G. Gasper, M. Rahman, *Basic hypergeometric series*, 2Eds., Cambridge: Cambridge University Press, 2004.
10. M. E. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge: Cambridge University Press, 2009.
11. M. Ismail, Z. S. I. Mansour, $q$-analogs of Lidstone expansion theorem, two point Taylor expansion theorem, and Bernoulli polynomials, *Anal. Appl.*, **17** (2019), 853–895. https://doi.org/10.1142/S0219530518500264
12. R. Koekoek, R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue, 1998. Available from: `https://arxiv.org/pdf/math/9602214`.
13. D. Leeming, A. Sharma, A generalization of the class of completely convex functions, *Symp. Inequalities*, **3** (1972), 177–199.
14. G. Lidstone, *Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types*, *Proc. Edinburgh Math. Soc.*, **2** (1929), 16–19.

15. Z. S. I. Mansour, M. AL-Towailb, $q$-Lidstone polynomials and existence results for $q$-boundary value problems, *Bound. Value Probl.*, **2017** (2017), 178. https://doi.org/10.1186/s13661-017-0908-4
16. Z. S. I. Mansour, M. AL-Towailb, The complementary $q$-Lidstone interpolating polynomials and applications, *Math. Comput. Appl.*, **25** (2020), 34. https://doi.org/10.3390/mca25020034
17. J. G. Mikusinski, Sur les fonctions $k_n(x) = \sum_{\nu=0}^{\infty}(-l)^{\nu}x^{n+k\nu}/(n+k\nu)!, (k = 1, 2, ...; n = 0, 1, ..., k-1)$, *Ann. Soc. Pol. Math.*, **21** (1948), 46–51.
18. L. Nachbin, An extension of the notion of integral functions of finite exponential type, *Anais Acad. Brasil. Ciencias*, **16** (1944), 143–147.
19. H. Portisky, On certain polynomial and other approximations to analytic functions, *PNAS*, **16** (1930), 83–85. https://doi.org/10.1073/pnas.16.1.83
20. J. P. Ramis, About the growth of entire functions solutions of linear algebraic $q$-difference equations, *Ann. Fac. Sci. Toulouse: Math.*, **1** (1992), 53–94. https://doi.org/10.5802/afst.739
21. P. M. Rajković, S. D. Marinković, M. S. Stanković, On $q$-analogues of Caputo derivative and Mittag-Leffler function, *Fract. Calc. Appl. Anal.* **10** (2007), 359–374.
22. I. Schoenberg, On certain two-point expansions of integral functions of exponential type, *Bull. Amer. Math. Soc.*, **42** (1936), 284–288. https://doi.org/10.1090/S0002-9904-1936-06293-2
23. J.M. Whittaker, On Lidstone' series and two-point expansions of analytic functions, *Proc. London Math. Soc.*, **2** (1934), 451–469.
24. D. Widder, Completely convex functions and Lidstone series, *Trans. Am. Math. Soc.*, **51** (1942), 387–398.