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Research article

Multivariate Mittag-Leffler function and related fractional integral operators

Gauhar Rahman[1], Muhammad Samraiz[2], Manar A. Alqudah[3] and Thabet Abdeljawad[4,5,6,*]

[1] Department of Mathematics and Statistics, Hazara University, Mansehra, Pakistan
[2] Department of Mathematics, University of Sargodha, P. O. Box, 40100, Sargodha, Pakistan
[3] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P. O. Box, 84428, Riyadh 11671, Saudi Arabia
[4] Department of Mathematics and Sciences, Prince Sultan University, P. O. Box, 66833, Riyadh 11586, Saudi Arabia
[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
[6] Department of Mathematics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea

* **Correspondence:** Email: tabdeljawad@psu.edu.sa.

Abstract: In this paper, we describe a new generalization of the multivariate Mittag-Leffler (M-L) function in terms of generalized Pochhammer symbol and study its properties. We provide a few differential and fractional integral formulas for the generalized multivariate M-L function. Furthermore, by using the generalized multivariate M-L function in the kernel, we present a new generalization of the fractional integral operator. Finally, we describe some fundamental characteristics of generalized fractional integrals.

Keywords: multivariate Mittag-Leffler; generalized pochhammer symbol; fractional integrals
Mathematics Subject Classification: 26A33, 33E12

1. Introduction

The field of mathematical analysis that deals with the study of arbitrary order integrals and derivatives is known as fractional calculus. Because of its numerous applications across a wide range of fields, this field has increased in importance and recognition over the past few years. According to researchers, this field is the most effective at identifying anomalous kinetics and has numerous uses in a variety of fields. Ordinary differential equations with fractional derivatives can be used to simulate a variety of issues, including statistical, mathematical, engineering, chemical, and biological issues. Several distinct forms of fractional integrals and derivative operators (see e.g., [1–4]), including Riemann-Liouville, Caputo, Riesz, Hilfer, Hadamard, Erdélyi-Kober, Saigo, Marichev-Saigo-Maeda and others, have been thoroughly investigated by researchers. From an application perspective, we suggest the readers to see the work related to the fractional differential equations presented by [5–8]. In [9], the authors studied symmetric and antisymmetric solitons in the defocused saturable nonlinearity and the PT-symmetric potential of the fractional nonlinear Schrödinger equation. In [10], the fractional exponential function approach is used to study a time-fractional Ablowitz-Ladik model, and bright and dark discrete soliton solutions, discrete exponential solutions, and discrete peculiar wave solutions are discovered. In [11], the authors presented the rich vector exact solutions for the coupled discrete conformable fractional nonlinear Schrödinger equations by taking into account the conformable fractional derivative.

On the other hand, special functions like Gamma, Beta, Mittag-Leffler, et al. play a vital part in the foundation of fractional calculus. Moreover, the Mittag-Leffler function is regarded as the fundamental function in fractional calculus. The Prabhakar fractional operator containing a three-parameter version of the aforementioned function in the kernel. The M-L function has been extensively studied to construct solutions of fractional PDEs, such as dynamical characteristic of analytical fractional solitons for the space-time fractional Fokas-Lenells equation, soliton dynamics based on exact solutions of conformable fractional discrete complex cubic Ginzburg-Landau equation and Abundant fractional soliton solutions of a space-time fractional perturbed Gerdjikov-Ivanov equation by a fractional mapping method, see [12–14]. Strong generalizations of the univariate and bivariate Mittag-Leffler functions, which are known to be important in fractional calculus, are the multivariate Mittag-Leffler functions.

The well-known one-parameter Mittag-Leffler (M-L) function is defined by [15, 16] as follows

$$\varepsilon_a(z_1) = \sum_{l=0}^{\infty} \frac{z_1^l}{\Gamma(al+1)} \qquad (a \in \mathbb{C}; \Re(a) > 0, z_1 \in \mathbb{C}), \tag{1.1}$$

where $\mathbb{C}$ represents the set of complex numbers and $\Re(a)$ denotes the real part of the complex number.

The generalization of (1.1) with two parameters is defined by [17, 18] as

$$\varepsilon_{a,b}(z_1) = \sum_{l=0}^{\infty} \frac{z_1^l}{\Gamma(al+b)} \qquad (a, b \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0), \tag{1.2}$$

Later on, Agarwal [19], Humbert [20] and Humbert and Agarwal [21] studied the properties and applications of M-L functions. In [22], the generalization of (1.1) and (1.2) is defined by

$$\varepsilon^{c}_{a,b}(z_1) = \sum_{l=0}^{\infty} \frac{(c)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \qquad (a, b, c \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0). \tag{1.3}$$

In [23], the following generalization of the M-L function is defined by

$$\varepsilon^{c,q}_{a,b}(z_1) = \sum_{l=0}^{\infty} \frac{(c)_{lq}}{\Gamma(al+b)} \frac{z_1^l}{l!} \qquad (a, b, c \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, q > 0). \tag{1.4}$$

In [24], Rahman et al. proposed the following generalized of M-L function by

$$\varepsilon_{a,b,p}^{c,q,d}(z_1)=\sum_{l=0}^{\infty}\frac{B_p(c+lq;d-c)(d)_{lq}}{B(c,d-c)\Gamma(al+b)}\frac{z_1^l}{l!},\tag{1.5}$$

where $a,b,c,d\in\mathbb{C};\Re(c)>0,\Re(a)>0,\Re(b)>0,q>0$ and $B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-t-\frac{p}{t}}dt$ is the extension of beta function (see [25]).

Gorenflo et al. [26] and Haubold et al. [27]) studied the various properties of generalized M-L function. In [28], a new generalization of M-L function (1.3) is presented by

$$\varepsilon_{a,b,p}^{c}(z_1)=\sum_{l=0}^{\infty}\frac{(c;p)_l}{\Gamma(al+b)}\frac{z_1^l}{l!}\qquad (p\geq 0,a,b,c\in\mathbb{C};,\Re(a)>0,\Re(b)>0,),\tag{1.6}$$

where $(\lambda;p)_l$ is the Pochhammer symbol defined by Srivastava et al. [29, 30] as

$$(\lambda;p)_\mu=\begin{cases}\frac{\Gamma_p(\lambda+\mu)}{\Gamma(\lambda)}; & (p>0,\lambda,\mu\in\mathbb{C})\\ \\ (\lambda)_\mu; & (p=0,\ \lambda,\mu\in\mathbb{C}\setminus\{0\}.\end{cases}\tag{1.7}$$

The researchers examined the developments of these extension, (1.6) and (1.7) and studied their related features and applications. In [30], Srivastava et al. proposed the following generalized hypergeometric function

$${}_sF_t[(\delta_1;p),\cdots,(\delta_s);(\zeta_1),\cdots,(\zeta_t);z_1]=\sum_{l=0}^{\infty}\frac{(\delta_1;p)_l\cdots(\delta_s)_l}{(\zeta_1)_l\cdots(\zeta_t)_l}\frac{z_1^l}{l!},\tag{1.8}$$

where $\delta_j\in\mathbb{C}$ for j=1, 2,$\cdots$, s, $\zeta_k\in\mathbb{C}$ for $k=1,2,\cdots,t$, and $\zeta_k\neq 0$, -1, -2, $\cdots$.

The integral representation of $(\mu;p)_\eta$ is explained by

$$(\mu;p)_\eta=\frac{1}{\Gamma(\mu)}\int_0^\infty s^{\mu+\eta-1}\ e^{-s-\frac{p}{s}}ds,\tag{1.9}$$

where $\Re(\rho)>0$ and $\Re(\mu+\eta)>0$. In particular, the related confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$${}_2F_1[(\delta_1;p),b;\lambda;z_1]=\sum_{l=0}^{\infty}\frac{(\delta_1;p)_l(b)_l}{(\lambda)_l}\frac{z_1^l}{l!},\tag{1.10}$$

and

$${}_1F_1[(\delta_1;p);\lambda;z_1]=\Phi[(\delta_1;p);\lambda;z_1]=\sum_{l=0}^{\infty}\frac{(\delta_1;p)_l}{(\lambda)_l}\frac{z_1^l}{l!}.\tag{1.11}$$

The expansion of the generalised hypergeometric function ${}_rF_s$, which was studied by [30], has r numerator and s denominator parameters. Researchers recently developed several extensions of special functions, together with their corresponding characteristics and applications. Using extended beta functions as its foundation, Nisar et al. [31], Bohner et al. [32] and Rahman et al. [33] developed an enlargement of fractional derivative operators.

The multivariate M-L function is defined by [34] as follows:

$$\mathcal{E}^{(c_j)}_{(a_j),b}(z_1, z_2, \ldots, z_j) = \mathcal{E}^{(c_1,c_2,\ldots,c_j)}_{(a_1,a_2,\ldots,a_j),b}(z_1, z_2, \ldots z_j)$$
$$= \sum_{m_1,m_2,\ldots,m_j=0}^{\infty} \frac{(c_1)_{m_1}(c_2)_{m_2}\ldots(c_j)_{m_j}(z_1)^{m_1}\ldots(z_j)^{m_j}}{\Gamma(a_1m_1 + a_2m_2 + \ldots a_jm_j + b)m_1!\ldots m_j!}, \tag{1.12}$$

where $z_i, a_i, b, c_i \in \mathbb{C}$; $i = 1, 2, \ldots, j$, $\Re(a_i) > 0$, $\Re(b) > 0$ and $\Re(c_i) > 0$.

In [35–39], the authors have studied various properties and applications of different type of generalized M-L functions. For real (complex) valued functions, the Lebesgue measurable space is defined by

$$\mathsf{L}(r, s) = \left\{\ h : \|h\|_1 = \int_r^s |h(x)|dx < \infty\ \right\}. \tag{1.13}$$

The left and right sides fractional integral operators of the Riemann-Liouville type are defined by [3,4] as follows:

$$(\Im^{\lambda}_{r+}h)(x) = \frac{1}{\Gamma(\lambda)} \int_r^x \frac{h(\varrho)}{(x-\varrho)^{1-\lambda}} d\varrho, \quad (x > r), \tag{1.14}$$

and

$$(\Im^{\lambda}_{s-}h)(x) = \frac{1}{\Gamma(\lambda)} \int_x^s \frac{h(\varrho)}{(\varrho - x)^{1-\lambda}} d\varrho, \quad (x < s),$$

where $h \in \mathsf{L}(r, s)$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

The left and right sides Riemann-Liouville fractional derivatives for the function $h(x) \in \mathsf{L}(r, s)$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$ and $n = [\Re(\lambda)] + 1$ are defined in [3,4] by

$$(\mathfrak{D}^{\lambda}_{r+}h)(x) = (\frac{d}{dx})^n (\Im^{n-\lambda}_{r+}h)(x) \tag{1.15}$$

and

$$(\mathfrak{D}^{\lambda}_{s-}h)(x) = (-\frac{d}{dx})^n (\Im^{n-\lambda}_{s-}h)(x),$$

respectively. The generalized differential operator $\mathfrak{D}^{\lambda,\nu}_{r+}$ of order $0 < \lambda < 1$ and type $0 < \nu < 1$ with respect to x can be found in [2,4] as

$$(\mathfrak{D}^{\lambda,\nu}_{r+}h) = \left(\ \Im^{\nu(1-\lambda)}_{r+} \tfrac{d}{dx}(\Im^{(1-\nu)(1-\lambda)}_{r+}h)\ \right)(x). \tag{1.16}$$

In particular, if $\nu = 0$, then (1.16) will lead to the operator $\mathfrak{D}^{\lambda}_{r+}$ defined in (1.15).

We also take into account the aforementioned well-known results.

Theorem 1.1. *In [40], the following result for the fractional integral is presented by*

$$\Im^{\lambda}_{r+}(\varrho - r)^{\eta-1} = \frac{\Gamma(\eta)}{\Gamma(\lambda + \eta)}(x - r)^{\lambda+\eta-1}, \tag{1.17}$$

where $\lambda, \eta \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\eta) > 0$,

Theorem 1.2. *[41] Suppose that the function* $h(z)$ *has a power series expansion* $h(z) = \sum_{k=0}^{\infty} k_n z^k$ *and it is analytic in the disc* $|z| < R$*, then we have the following result*

$$\mathfrak{D}_z^{\lambda}\{z^{\eta-1}h(z)\} = \frac{\Gamma(\eta)}{\Gamma(\lambda+\eta)}\sum_{n=0}^{\infty}\frac{a_n(\eta)_n}{(\lambda+\eta)_n}z^n.$$

Lemma 1.1. *(Srivastava and Tomovski [42]) Suppose that* $x > r$*,* $\lambda \in (0,1)$*,* $\nu \in [0,1]$*,* $\Re(\eta) > 0$ *and* $\Re(\lambda) > 0$*, then we have*

$$\mathfrak{D}_{r+}^{\lambda}[(\varrho - r)^{\eta-1}](x) = \frac{\Gamma(\eta)}{\Gamma(\eta-\lambda)}(x-r)^{\eta-\lambda-1}. \tag{1.18}$$

The generalized multivariate M-L function (1.12) is then defined in terms of the modified Pochhammer symbol (1.7) and its different features as well as the accompanying integral operators are examined. This is driven by the aforementioned modifications of special functions.

Motivated by the above results and literature, the paper has the following structure: First, we describe and investigate a novel generalization of the multivariate M-L function using a generalized Pochhammer symbol. Secondly, we offer a few differential and fractional integral formulas for the explored multivariate M-L function. By using the new form of the multivariate M-L function, a new generalization of the fractional integral operator is introduced, and some fundamental characteristics of the operator are discussed.

2. The generalized multivariate of M-L function

We are in a position to present the generalized multivariate M-L function by utilizing the extended Pochhammer symbol in (1.7) as follows:

$$\varepsilon_{(a_j),b;p}^{(c_j)}(z_1, z_2, \cdots, z_j) = \sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1;p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1 + a_2l_2 + \cdots + c_jl_j + b)}\frac{z_1^{l_1}z_2^{l_2}\cdots z_j^{l_j}}{l_1!\cdots l_j!}, \tag{2.1}$$

where $a_i, b, c_i \in \mathbb{C}; \Re(a_i) > 0, \Re(b) > 0,, p \geq 0$ for $i = 1, 2, \cdots, j$. The special case for $a_1 = 1$ and $l_2 = \cdots = l_j = 0$ in (2.1) can be reduced to extended confluent hypergeometric function (1.11) as follows:

$$\varepsilon_{1,b;p}^{c_1}(z_1) = \frac{1}{\Gamma(b)}{}_1F_1[(c_1;p);b;z_1] = \frac{1}{\Gamma(b)}\Phi[(c_1;p);b;z_1]. \tag{2.2}$$

In coming results, we demonstrate some fundamental characteristics and integral representations of the generalized multivariate M-L function.

Theorem 2.1. *For the multivariate M-L function defined in* (2.1)*, the following relation holds true:*

$$\varepsilon_{(a_j),b;p}^{(c_j)}(z_1, z_2, \cdots, z_j) = b\varepsilon_{(a_j),b+1;p}^{(c_j)}(z_1, z_2, \cdots, z_j) \tag{2.3}$$
$$+[a_1z_1\frac{d}{dz_1} + \cdots + a_jz_j\frac{d}{dz_j}]\varepsilon_{(a_j),b+1;p}^{(c_j)}(z_1, \cdots, z_j),$$

where $a_i, b, c_i \in \mathbb{C}; \Re(a_i) > 0, \Re(b) > 0,, p \geq 0$ for $i = 1, 2, \cdots, j$.

Proof. From (2.1), we have

$$b\varepsilon^{(c_j)}_{(a_j),b+1,p}(z_1,\cdots,z_j)+[a_1z_1\frac{d}{dz_1}+\cdots+a_jz_j\frac{d}{dz_j}]\varepsilon^{(c_j)}_{(a_j),b+1;p}(z_1,\cdots,z_j)$$

$$=b\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$+[a_1z_1\frac{d}{dz_1}+\cdots+a_jz_j\frac{d}{dz_j}]\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$=b\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$+[a_1z_1\frac{d}{dz_1}+\cdots+a_jz_j\frac{d}{dz_j}]\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_jl_j}{l_1!\cdots l_j!}$$

$$=b\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$+\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}(a_1l_1+\cdots+a_jl_j)$$

$$=\sum_{l_1,\cdots,l_j=0}^{\infty}\frac{(c_1,p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+1)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}(a_1l_1+\cdots+a_jl_j+b)\ \ (\text{using }\Gamma(z_1+1)=z_1\Gamma(z_1))$$

$$=\sum_{l=0}^{\infty}\frac{(c_1,p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$=\varepsilon^{(c_j)}_{(a_j),b,p}(z_1,z_2,\cdots,z_j),$$

which is the desired result (2.3). □

Theorem 2.2. *For the generalized multivariate M-L function defined in* (1.12)*, the following relations hold true:*

$$(\frac{d}{dz_1})^m\cdots(\frac{d}{dz_j})^m\varepsilon^{(c_j)}_{(a_j),b;p}(z_1,z_2,\cdots,z_j)=(c_1)_m\cdots(c_j)_m\varepsilon^{(c_j)+m}_{(a_j),b+(a_j)m;p}(z_1,\cdots,z_j),\tag{2.4}$$

and

$$(\frac{d}{dz_1})^m[z_1^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1z_1^{a_1},\cdots,\varpi_jz_1^{a_j}))]=z_1^{b-m-1}\varepsilon^{(c_j)}_{(a_j),b-m;p}(\varpi_1z_1^{a_1},\cdots,\varpi_jz_1^{a_j}),\tag{2.5}$$

where $a_i,b,c_i\in\mathbb{C};\Re(a_i)>0,\Re(b)>0,,p\geq 0$ for $i=1,2,\cdots,j$, and $\Re(b-m)>0$ with $m\in\mathbb{N}$.

Proof. Differentiating (1.12) m times with respect to $z_1,z_2,\cdots,z_j$ respectively, we get

$$(\frac{d}{dz_1})^m\cdots(\frac{d}{dz_j})^m\varepsilon^{(c_j)}_{(a_j),b;p}(z_1,\cdots,z_j)=(\frac{d}{dz_1})^m\cdots(\frac{d}{dz_j})^m\sum_{l_1=l_2=\cdots=l_j=0}^{\infty}\frac{(c_1;p)_{l_1}(c_2)_{l_2}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}$$

$$= \sum_{l_1=\cdots=l_j=m}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{l_1!\cdots l_j!\, z_1^{l_1-m}\cdots z_j^{l_j-m}}{(l_1-m)!\cdots(l_j-m)!\, l1!\cdots l_j!}$$

$$= \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1;p)_{l_1+m}\cdots(c_j)_{l_j+m}}{\Gamma(a_1(l_1+m)+\cdots a_j(l_j+m)+b)} \frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!} \quad \text{(Replacing } l_i \text{ by } l_i+m)$$

$$= \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1)_m\cdots(c_j)_m\,(c_1+m;p)_{l_1}\cdots(c_j+m)_{l_j}}{\Gamma(a_1l_1+\cdots a_jl_j+b+(a_1+\cdots+a_j)m)} \frac{z_1^{l_1}\cdots z_j^{l_j}}{l_1!\cdots l_j!}.$$

Now using $(\lambda;\sigma)_{\mu+p} = (\lambda)_\mu(\lambda+\mu;\sigma)_p$ and $(\lambda)_{\mu+p} = (\lambda)_\mu(\lambda+\mu)_p$, we get

$$\begin{aligned}
&(\frac{d}{dz_1})^m\cdots(\frac{d}{dz_j})^m \varepsilon^{(c_j)}_{(a_j),b;p}(z_1,\cdots,c_j)\\
=&(c_1)_m\cdots(c_j)_m \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1+m;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots a_jl_j+b+(a_1+\cdots+a_j)m)} \frac{z_1^{l}\cdots z_j^{l_j}}{l_1!\cdots l_j!}\\
=&\ (c_1)_m\cdots(c_j)_m\, \varepsilon^{(c_j)+m}_{(a_j),b+(a_j)m;p}(z_1,z_2,\cdots,z_j),
\end{aligned}$$

which is the desired result (2.4). Similarly, to prove (2.5), we have

$$\begin{aligned}
&(\frac{d}{dz_1})^m[z_1^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1 z_1^{a_1},\cdots\varpi_j z_j^{a_j})]\\
=&(\frac{d}{dz_1})^m z_1^{b-1} \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{(\varpi_1 z_1^{a_1})^{l_1}\cdots(\varpi_j z_1^{a_j})^{l_j}}{l_1!\cdots l_j!}\\
=&(\frac{d}{dz_1})^m \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{z_1^{b-1+a_1l_1+\cdots+a_jl_j}}{l_1!\cdots l_j!}\varpi_1^{l_1}\cdots\varpi_j^{l_j}\\
=&\sum_{l_1=\cdots l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{\varpi_1^{l_1}\cdots\varpi_j^{l_j}}{l_1!\cdots l_j!} \frac{(a_1l_1+\cdots+a_jl_j+b-1)!}{(a_1l_1+\cdots+a_jl_j+b-m-1)!}\, z_1^{a_1l_1+\cdots+a_jl_j+b-m-1}.
\end{aligned}$$

Differentiating m times and using the relation $l(l-1)! = l!$, we get

$$\begin{aligned}
&(\frac{d}{dz_1})^m[z_1^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1 z_1^{a_1},\cdots\varpi_j z_j^{a_j})]\\
=&\sum_{l_1=\cdots l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{\Gamma(a_1l_1+\cdots+a_jl_j+b)}{\Gamma(a_1l_1+\cdots+a_jl_j+b-m)}\varpi_1^{l_1}\cdots\varpi_j^{l_j} \frac{z_1^{a_1l_1+\cdots+a_jl_j+b-1-m}}{l_1!\cdots l_j!}\\
=&z_1^{b-m-1} \sum_{l_1=\cdots l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b-m)} \frac{(\varpi_1 z_1^{a_1})^{l_1}\cdots(\varpi_1 z_1^{a_j})^{l_j}}{l_1!\cdots l_j!}\\
=&z_1^{b-m-1}\varepsilon^{(c_j)}_{(a_j),b-m;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}).
\end{aligned}$$

The proof is completed. □

Corollary 2.1. *The generalized multivariate M-L function has the following integral representations:*

$$\int_0^{z_1} t^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1 t^{a_1},\cdots,\varpi_j t^{a_j})dt = z_1^b\varepsilon^{(c_j)}_{(a_j),b+1;p}(\varpi_1 z_1^{a_1},\cdots,\varpi_j z_1^{a_j}),$$

where $a_i, b, c_i, \varpi_i \in \mathbb{C}$; $\Re(a_i) > 0, \Re(b) > 0, p \geq 0$ for $i = 1, 2, \cdots, j$.

3. Fractional integral and differential formulas of generalized M-L function

In this section, we present some fractional integration and differentiation formulas of generalized M-L function given in (2.1).

Theorem 3.1. *Suppose* $x > r\,(r \in \mathbb{R}_+ = [0,\infty))$, a_i, b, c_i, $\varpi \in \mathbb{C}$, $\Re(a_i) > 0$ *and* $\Re(c_i) > 0$, $\Re(b) > 0$ *and* $\Re(\lambda) > 0$, *then the following relations hold true:*

$$\begin{aligned}&\mathfrak{I}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_j)}_{(a_j),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})\right](x)\\ =&(x-r)^{\lambda+b-1}\varepsilon^{(c_i)}_{(a_i),b+\lambda;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}),\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\mathfrak{D}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})\right](x)\\ =&(x-r)^{b-\lambda-1}\varepsilon^{(c_i)}_{(a_i),b-\lambda;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&\mathfrak{D}^{\lambda,v}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right](x)\\ =&(x-r)^{b-\lambda-1}\varepsilon^{(c_i)}_{(a_i),b-\lambda;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}).\end{aligned}\tag{3.3}$$

Proof. Consider

$$\begin{aligned}&\mathfrak{I}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right](x)\\ =&\frac{1}{\Gamma(\lambda)}\int_r^x\frac{(x-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_1(\varrho-r)^{a_j})}{(x-\varrho)^{1-\lambda}}d\varrho\\ =&\frac{1}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_n}\varpi^{l_1}\cdots\varpi^{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}\int_r^x(\varrho-r)^{b+a_1l_1+\cdots+a_jl_j-1}(x-\varrho)^{\lambda-1}d\varrho\\ =&\sum_{n=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_n}\varpi^{l_1}\cdots\varpi^{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}\left(\mathfrak{I}^{\lambda}_{r+}[(\varrho-r)^{b+a_1l_1+\cdots+a_jl_j-1}]\right).\end{aligned}$$

By the use of (1.17), we have

$$\begin{aligned}&\mathfrak{I}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right](x)\\ =&\sum_{n=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_n}\varpi^{l_1}\cdots\varpi^{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}(x-r)^{b+\lambda+a_1l_1+\cdots+a_jl_j-1}.\frac{\Gamma(a_1l_1+\cdots+a_jl_j+b)}{\Gamma(a_1l_1+\cdots+a_jl_j+b+\lambda)}\end{aligned}$$

$$
\begin{aligned}
=&(x-r)^{b+\lambda-1}\sum_{n=0}^{\infty}\frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b+\lambda)}\frac{[\varpi_1^{l_1}(x-r)^{a_1l_1}\cdots\varpi_j^{l_j}(x-r)^{a_jl_j}]}{l_1!\cdots l_j!}\\
=&(x-r)^{b+\lambda-1}\varepsilon^{(c_i)}_{(a_i),b+\lambda;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}),
\end{aligned}
$$

which gives the proof of (3.1).

Next, we have

$$
\begin{aligned}
&\mathfrak{D}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})\right]\\
=&(\frac{d}{dx})^n\left\{\mathfrak{I}^{n-\lambda}_{r+}[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})]\right\},
\end{aligned}
$$

which on using (3.1) takes the following form:

$$
\begin{aligned}
&\mathfrak{D}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})\right]\\
=&(\frac{d}{dx})^n\left\{(x-r)^{b-\lambda+n-1}\varepsilon^{(c_i)}_{(a_i),b-\lambda+n;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right\}.
\end{aligned}
$$

Applying (2.5), we get

$$
\begin{aligned}
&\mathfrak{D}^{\lambda}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right](x)\\
=&\left\{(x-r)^{\eta-\lambda-1}\varepsilon^{(c_i)}_{(a_i),b-\lambda;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j})\right\},
\end{aligned}
$$

which gives the proof of (3.2).

To obtain (3.3), we have

$$
\begin{aligned}
&\left(\mathfrak{D}^{\lambda,v}_{r+}\left[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})\right]\right)(x)\\
=&\left(\mathfrak{D}^{\lambda,v}_{r+}\left[\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\frac{\varpi^{l_1}\cdots\varpi^{l_j}}{l_1!\cdots l_j!}(\varrho-r)^{a_1l_1+\cdots+a_jl_j+b-1}\right]\right)(x)\\
=&\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\frac{\varpi^{l_1}\cdots\varpi^{l_j}}{l_1!\cdots l_j!}\\
&\times\left(\mathfrak{D}^{\lambda,v}_{r+}[(\varrho-r)^{a_1l_1+\cdots+a_jl_j+b-1}]\right)(x).
\end{aligned}
$$

By applying (1.18), we get

$$
\begin{aligned}
&\left(\mathfrak{D}^{\lambda,v}_{r+}[(\varrho-r)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-r)^{a_1},\cdots,\varpi_j(\varrho-r)^{a_j})]\right)(x)\\
=&\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\frac{\varpi^{l_1}\cdots\varpi^{l_j}}{l_1!\cdots l_j!}\\
&\times\frac{\Gamma(a_1l_1+\cdots+a_jl_j+b)}{\Gamma(a_1l_1+\cdots+a_jl_j+b-\lambda)}(x-r)^{a_1l_1+\cdots+a_jl_j+b-\lambda-1}\\
=&(x-r)^{b-\lambda-1}\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p,v)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b-\lambda)}\frac{\varpi^{l_1}(x-r)^{a_1}\cdots\varpi^{l_j}(x-r)^{a_j}}{l_1!\cdots l_j!}
\end{aligned}
$$

$$=(x-r)^{b-\lambda-1}\varepsilon^{(c_i)}_{(a_i),b-\lambda;p}(\varpi^{l_1}(x-r)^{a_1},\cdots,\varpi^{l_j}(x-r)^{a_j}),$$

which completes the required proof. □

Remark 3.1. *Applying Theorem 3.1 for $p=0$, then we obtain the result presented in [34].*

4. Generalization of fractional integral and its properties

In this section, we define a fractional integral involving newly defined multivariate M-L function and discuss its properties.

Definition 4.1. *Let $b, a_i, c_i, \varpi_i \in \mathbb{C}$, $\Re(c_i) > 0$, $\Re(a_i) > 0$ and $\Re(b) > 0$ and $h \in L(r,s)$. Then the generalized left and right sided fractional integrals are defined by*

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h\right)(x)=\int_r^x (x-\varrho)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})h(\varrho)d\varrho,(x>r) \tag{4.1}$$

and

$$\left(\Re^{(\varpi_i);(c_i)}_{s-;(a_i),b;p}h\right)(x)=\int_x^s (\varrho-x)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-x)^{a_1},\cdots,\varpi_j(\varrho-x)^{a_j})h(\varrho)d\varrho,(x<s), \tag{4.2}$$

respectively.

Remark 4.1. *If we consider $p=0$, then the operators defined in* (4.1) *and* (4.2) *will take the form defined earlier by [34]. Similarly, if we consider $p=0$ and $j=1$, then the operators defined in* (4.1) *and* (4.2) *will take the form defined by [22]. If we take $j=1$, then the work done in this paper will lead to the work presented by [28]. Also, if we consider one of $\varpi_i=0$, for $i=1,2,\cdots,j$, then the operators defined in* (4.1) *and* (4.2) *will take the form of the classical operators.*

Next, we prove the following properties of integral operator defined in (4.1).

Theorem 4.1. *Suppose that $b, a_i, \lambda, c_i, \varpi_i \in \mathbb{C}$, $\Re(a_i) > 0$, $\Re(b) > 0$, $\Re(\lambda) > 0$, $p \geq 0$ and $\Re(c_i) > 0$ for $i=1,2,\cdots,j$, then the following result holds true:*

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}[(\varrho-r)^{\lambda-1}]\right)(x)=(x-r)^{\lambda+b-1}\Gamma(\lambda)\varepsilon^{(c_i);p}_{(a_i),b+\lambda}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}).$$

Proof. By the use of definition (4.1), we have

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h\right)(x)=\int_r^x (x-\varrho)^{b-1}\varepsilon^{(c_i)}_{(a_i),b}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})h(\varrho)d\varrho.$$

Therefore, we get

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}[(\varrho-r)^{\lambda-1}]\right)(x)=\int_r^x (x-\varrho)^{b-1}(\varrho-r)^{\lambda-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})d\varrho$$

$$=\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b))}\frac{\varpi_1^{l_1}\cdots\varpi_j^{l_j}}{l_1!\cdots l_j!}\left(\int_r^x(\varrho-r)^{\lambda-1}(x-\varrho)^{\lambda+a_1l_1+\cdots+a_jl_j-1}d\varrho\right)$$

$$= \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b))} \frac{\varpi_1^{l_1}\cdots\varpi_j^{l_j}}{l_1!\cdots l_j!} \mathfrak{I}_{r+}^{a_1l_1+\cdots+a_jl_j+b}[(\varrho-r)^{\lambda-1}]$$

$$= (x-r)^{b+\lambda-1} \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b))} \frac{[\varpi_1(x-r)^{a_1l_1}\cdots\varpi_j(x-r)^{a_jl_j}]}{l_1!\cdots l_j!}$$

$$\times \frac{\Gamma(\lambda)\Gamma(a_1l_1+\cdots+a_jl_j+b)}{\Gamma(a_1l_1+\cdots+a_jl_j+b+\lambda)}$$

$$=(x-r)^{b+\lambda-1}\Gamma(\lambda)\varepsilon_{(a_i),b+\lambda;p}^{(c_i)}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}),$$

which gives the desired proof. □

Theorem 4.2. *Suppose that $c_i, a_i, b, \varpi_i \in \mathbb{C}$, $\Re(a_i) > 0$, $\Re(b) > 0$, $p \geq 0$ for $i = 1, 2, \cdots, j$, then the following result holds true:*

$$||\Re_{r+;(a_i),b;p}^{(\varpi_i);(c_i)}\Phi||_1 \leq K||\Phi||_1.$$

Where

$$K := (s-r)^{\Re e(b)} \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{|(c_1;p)_{l_1}\cdots(c_j)_{l_j}|}{\Gamma(a_1l_1+\cdots+a_jl_j+b)(\Re(b)+\Re(a_1)l_1+\cdots+\Re(a_j)l_j)}$$

$$\times \frac{|\varpi_1^{l_1}(s-r)^{a_1l_1}\cdots\varpi_j^{l_j}(s-r)^{a_jl_j}|}{l_1!\cdots l_j!}.$$

Proof. By the use of (1.13) and (4.1) and by interchanging integration and summation order, we have

$$||\Re_{r+;(a_i),b;p}^{(\varpi_i);(c_i)}\Phi||_1 = \int_r^s \left|\int_r^x (x-\varrho)^{b-1}\varepsilon_{(a_i),b;p}^{(c_i)}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})\Phi(\varrho)d\varrho\right|dx$$

$$\leq \int_r^s \left[\int_\varrho^x (x-\varrho)^{\Re(b)-1}|\varepsilon_{(a_i),b;p}^{(c_i)}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})|dx\right]|\Phi(\varrho)|d\varrho$$

$$= \int_r^s \left[\int_0^{x-\varrho} u^{\Re(b)-1}|\varepsilon_{(a_i),b;p}^{(c_i)}(\varpi_1u^{a_1},\cdots,\varpi_ju^{a_j})|du\right]|\Phi(\varrho)|d\varrho,$$

by setting $u = x - \varrho$. After simplification, we obtain

$$||\Re_{r+;(a_i),b;p}^{(\varpi_i);(c_i)}\Phi||_1 \leq \int_r^s \Big[\sum_{l_1=\cdots=l_j=0}^{\infty} \frac{|(c_1;p)_{l_1}\cdots(c_j)_{l_j}|}{\Gamma(a_1l_1+\cdots+a_jl_j+b)} \frac{|\varpi_1^{a_1}\cdots\varpi_j^{l_j}|}{l_1!\cdots l_j!}$$

$$\times \left(\frac{(u)^{\Re(b)+\Re(a_1)l_1+\cdots+\Re(a_j)l_j}}{(\Re(b)+\Re(a_1)l_1+\cdots+\Re(a_j)l_j)}\right)_0^{s-r}\Big]|\Phi(\varrho)|d\varrho.$$

It follows that

$$||\Re_{r+;(a_i),b;p}^{(\varpi_i);(c_i)}\Phi||_1 \leq \Big\{(s-r)^{\Re(b)} \sum_{l_1=\cdots=l_j=0}^{\infty} \frac{|(c_1;p)_{l_1}\cdots(c_j)_{l_j}|}{\Gamma(a_1l_1+\cdots+a_jl_j+b)(\Re(b)+\Re(a_1)l_1+\cdots+\Re(a_j)l_j)}$$

$$\times\frac{|\varpi_1^{l_1}(s-r)^{a_1l_1}\cdots\varpi_j^{l_j}(s-r)^{a_jl_j}|}{l_1!\cdots l_j!}\Big\}\int\limits_r^s|\Phi(\varrho)|d\varrho$$
$$=K\|\Phi\|_1,$$

where

$$K=(s-r)^{\Re e(b)}\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{|(c_1;p)_{l_1}\cdots(c_j)_{l_j}|}{\Gamma(a_1l_1+\cdots+a_jl_j+b)(\Re(b)+\Re(a_1)l_1+\cdots+\Re(a_j)l_j)}$$
$$\times\frac{|\varpi_1^{l_1}(s-r)^{a_1l_1}\cdots\varpi_j^{l_j}(s-r)^{a_jl_j}|}{l_1!\cdots l_j!},$$

which gives the desired result. □

Corollary 4.1. *If we take* $a_i, b, c_i, \varpi_i \in \mathbb{C}$, $\Re(a_i) > 0$, $\Re(b) > 0$, $\Re(c_i) > 0$ *with* $i = 1, 2, \cdots, j$, *then the following result holds true:*

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}1\right)(x)=(x-r)^b\varepsilon^{(c_i)}_{(a_i),b+1;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}).$$

Proof. Consider

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}1\right)(x)=\int_r^x(x-\varrho)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-r)^{a_j})d\varrho$$
$$=\int_r^x(x-\varrho)^{b-1}\left(\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}\varpi_1^{l_1}(x-\varrho)^{a_1l_1}\cdots\varpi_j^{l_j}(x-\varrho)^{a_jl_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}\right)d\varrho.$$

It follows that

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}1\right)(x)=\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}\varpi_1^{l_1}\cdots\varpi_j^{l_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}\int_r^x(x-\varrho)^{b+a_1l_1+\cdots+a_jl_j-1}d\varrho$$
$$=(x-r)^b\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{(c_1;p)_{l_1}\cdots(c_j)_{l_j}\varpi_1^{l_1}(x-r)^{a_1l_1}\cdots\varpi_j^{l_j}(x-r)^{a_jl_j}}{\Gamma(a_1l_1+\cdots+a_jl_j+b)(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}$$
$$=(x-r)^b\,\varepsilon^{(c_i)}_{(a_i),b+1;p}(\varpi_1(x-r)^{a_1},\cdots,\varpi_j(x-r)^{a_j}),$$

which gives the desired proof. □

Theorem 4.3. *The generalized fractional operator can be represented in term of Riemann–Liouville fractional integrals for* c_i, a_i, b, $\varpi_i \in \mathbb{C}$ *with* $\Re(a_i) > 0$, $\Re(b) > 0$, $\Re(c_i) > 0$ *for* $i = 1, 2, \cdots, j$, $p \geq 0$ *and* $x > r$ *as follows:*

$$\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}h\right)(x)=\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{\Gamma(c_1+l_1;p)(c_2)_{l_2}\cdots(c_j)_{l_j}\varpi_1^{a_1}\cdots\varpi_j^{a_j}}{\Gamma(c_1)l_1!\cdots l_j!}\Im^{a_1l_1+\cdots+a_jl_j+b}_{r+}h(x).$$

Proof. By utilizing (2.1) in (4.1) and then interchanging the order of summation and integration, we have

$$\begin{aligned}\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}h\right)(x) &= \int_r^x (x-\varrho)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})h(\varrho)d\varrho\\ &= \int_r^x (x-\varrho)^{b-1}\sum_{l_1=\cdots=l_j=0}^{\infty}\frac{\Gamma(c_1+l_1;p)(c_2)_{l_2}\cdots(c_j)_{l_j}\varpi_1^{l_1}(x-\varrho)^{a_1l_1}\cdots\varpi_j^{l_j}(x-\varrho)^{a_jl_j}}{\Gamma(c_1)\Gamma(a_1l_1+\cdots+a_jl_j+b)l_1!\cdots l_j!}h(\varrho)d\varrho\\ &= \sum_{l_1=\cdots=l_j=0}^{\infty}\frac{\Gamma(c_1+l_1;p)(c_2)_{l_2}\cdots(c_j)_{l_j}\varpi_1^{a_1l_1}\cdots\varpi_j^{a_jl_j}}{\Gamma(c_1)l_1!\cdots l_j!}\frac{1}{\Gamma(a_1l_1+\cdots+a_jl_j+b)}\\ &\times\int_r^x (x-\varrho)^{a_1l_1+\cdots+a_jl_j+b-1}h(\varrho)d\varrho\\ &= \sum_{l_1=\cdots=l_j=0}^{\infty}\frac{\Gamma(c_1+l_1;p)(c_2)_{l_2}\cdots(c_j)_{l_j}\varpi_1^{a_1l_1}\cdots\varpi_j^{a_jl_j}}{\Gamma(c_1)l_1!\cdots l_j!}\Im^{a_1l_1+\cdots+a_jl_j+b}_{r+}h(x),\end{aligned}$$

which gives the desired proof. □

Theorem 4.4. *For λ, c_i, a_i, b, $\varpi_i\in\mathbb{C}$ with $\Re(a_i)>0$, $\Re(b)>0$, $\Re(c_i)>0$, $\Re(\lambda)>0$, for $i=1,2,\cdots,j$, $p\geq 0$ and $x>r$, then the following result holds true:*

$$(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h])(x) = (\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b+\lambda}h)(x) = (\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}[\Im^{\lambda}_{r+}h])(x), \tag{4.3}$$

where $h\in L(r,s)$.

Proof. By employing (1.14) and (4.1), we have

$$\begin{aligned}&(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h])(x)\\ =&\frac{1}{\Gamma(\lambda)}\int_r^x\frac{[(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h)(\varrho)]}{(x-\varrho)^{1-\lambda}}d\varrho\\ =&\frac{1}{\Gamma(\lambda)}\int_r^x (x-\varrho)^{\lambda-1}\left[\int_r^{\varrho}(\varrho-u)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-u)^{a_1},\cdots,\varpi_j(\varrho-u)^{a_j})h(u)du\right]d\varrho.\end{aligned}$$

It follows that

$$\begin{aligned}&(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}h])(x)\\ =&\int_r^x\left[\frac{1}{\Gamma(\lambda)}\int_u^x(x-\varrho)^{\lambda-1}(\varrho-u)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(\varrho-u)^{a_1},\cdots,\varpi_j(\varrho-u)^{a_j})d\varrho\right]h(u)du.\end{aligned}$$

By considering $\varrho-u=\theta$, we get

$$\begin{aligned}&(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h])(x)\\ &=\int_r^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-u}(x-u-\theta)^{\lambda-1}\theta^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1\theta^{a_1},\cdots,\varpi_j\theta^{a_j})d\theta\right]h(u)du\\ &=\int_r^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-u}\frac{\theta^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1\theta^{a_1},\cdots,\varpi_j\theta^{a_j})}{(x-u-\theta)^{1-\lambda}}d\theta\right]h(u)du.\end{aligned}$$

Hence, from (1.14) and applying (3.1), we obtain

$$
\begin{aligned}
&(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h])(x)\\
&=\int_r^x \left[\theta^{\lambda+b-1}\varepsilon^{(c_i)}_{(a_i),b+\lambda;p}(\varpi_1\theta^{a_1},\cdots,\varpi_j\theta^{a_j})\right]h(u)du\\
&=\int_r^x (x-u)^{\lambda+b-1}\varepsilon^{(c_i)}_{(a_i),b+\lambda}(\varpi_1(x-u)^{a_1},\cdots,\varpi_j(x-u)^{a_j})h(u)du.
\end{aligned}
$$

Thus, we have

$$
(\Im^{\lambda}_{r+}[\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}h])(x)=\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b+\lambda}h\right)(x). \tag{4.4}
$$

Next, consider the right hand side of (4.3) and employing (4.1) to derive the second part, we have

$$
\begin{aligned}
&(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}[\Im^{\lambda}_{r+}h])(x)\\
&=\int_r^x (x-\varrho)^{b-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})[\Im^{\lambda}_{r+}h](\varrho)d\varrho\\
&=\int_r^x \varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})\left(\frac{1}{\Gamma(\lambda)}\int_r^{\varrho}\frac{h(u)}{(\varrho-u)^{1-\lambda}}du\right)d\varrho.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}[\Im^{\lambda}_{r+}h])(x)\\
&=\int_r^x \frac{1}{\Gamma(\lambda)}\left[\int_u^x (x-\varrho)^{b-1}(\varrho-u)^{\lambda-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1(x-\varrho)^{a_1},\cdots,\varpi_j(x-\varrho)^{a_j})d\varrho\right]h(u)du.
\end{aligned}
$$

By setting $x-\varrho=\theta$, we get

$$
\begin{aligned}
&(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b}[\Im^{\lambda}_{r+}h])(x)\\
&=\int_r^x \frac{1}{\Gamma(\lambda)}\left[\int_{x-u}^0 \theta^{b-1}(x-\theta-u)^{\lambda-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1\theta^{a_1},\cdots,\varpi_j\theta^{a_j})(-d\theta)\right]h(u)du\\
&=\int_r^x \frac{1}{\Gamma(\lambda)}\left[\int_0^{x-u} \theta^{b-1}(x-\theta-u)^{\lambda-1}\varepsilon^{(c_i)}_{(a_i),b;p}(\varpi_1\theta^{a_1},\cdots,\varpi_j\theta^{a_j})d\theta\right]h(u)du.
\end{aligned}
$$

Further, by using (1.14) and applying (3.1), we obtain

$$
(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b;p}[\Im^{\lambda}_{r+}h])(x)=\left(\Re^{(\varpi_i);(c_i)}_{r+;(a_i),b+\lambda}h\right)(x). \tag{4.5}
$$

Thus, (4.4) and (4.5) gives the desired proof. □

5. Conclusions

Nowadays, the theories are developed very rapidly. The scientists are introducing more advanced and generalized forms of the classical ones. In this present study, we introduced a generalized form of the multivariate M-L function (2.1) by employing the generalized Pochhammer symbol and its properties. By using this more extended form of M-L, we introduced a fractional integral operator and studied some of the basic properties of this operator. The special cases of the main results are if we take $p = 0$, then the operators defined in (4.1) and (4.2) will reduce to the work done by [34]. Similarly, if we take $j = 1$ and $p = 0$, then the operators defined in (4.1) and (4.2) will lead to the work done by [22]. If we take $j = 1$, then the work done in this paper will lead to the work presented by [28]. Moreover, if we consider one of $\varpi_i = 0$, for $i = 1, 2, \cdots, j$, then the operators defined in (4.1) and (4.2) will reduce to the classical R-L operators. We believe that our proposed operator will be more applicable in the fields of fractional integral inequalities and operator theory.

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Conflict of interest

The authors declare no conflict of interest.

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