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Research article

A novel approach of multi-valued contraction results on cone metric spaces with an application

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Abstract: In this paper, we present some generalized multi-valued contraction results on cone metric spaces. We use some maximum and sum types of contractions for a pair of multi-valued mappings to prove some common fixed point theorems on cone metric spaces without the condition of normality. We present an illustrative example for multi-valued contraction mappings to support our work. Moreover, we present a supportive application of nonlinear integral equations to validate our work. This new theory, can be modified in different directions for multi-valued mappings to prove fixed point, common fixed point and coincidence point results in the context of different types of metric spaces with the application of different types of integral equations.

Keywords: common fixed point; contraction conditions; integral equations; cone metric space
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1. Introduction

It was recognized that in 1922, Banach proved a "contraction mapping principle for fixed points (FPs)" in his Ph.D. dissertation; see also [1]. It is one of the most significant results in functional analysis and its applications in other branches of mathematics. Specifically, this principle is considered as the basic source of metric FP theory. The study of FP and common fixed point (CFP) results satisfying a certain metric contraction condition has received the attention of many authors; see, for instance [2–10].

Huang and Zhang [11] in 2007, introduced the notion of a cone metric space (CM-space) which generalized the notion of a metric space (M-space). They presented some basic properties and proved a cone Banach contraction theorem for FPs in terms of the interior points of the underlying cone. After the publication of this article, many researchers contributed their work to the problems on CM-spaces. Abbas and Jungck [12], Ilić and Rakocević [13] and Vetro [14] generalized the concept of Huang and Zhang [11] and proved some FP, CFP and coincidence point results on CM-spaces by using different types of contraction conditions. Abbas et al. [15], Abdeljawad et al. [16, 17], Altun et al. [18], Janković et al. [19], Karapinar [20–22], Khamsi [23], Kumar and Rathee [24], and Rezapour and Hamlbarani [25] proved different contractive-type FP and CFP results on CM-spaces.

In 1969, Nadler [26] initially introduced the concept of multi-valued contraction mappings in the theory of FP by using the Hausdorff metric. He proved some multi-valued FP results on complete M-spaces. In other papers [27–31], the authors contributed their ideas to the theory of FP and established multi-valued contraction results in the context of M-spaces. In [32], Rezapour and Haghi proved FP results for multi-functions on CM-spaces. Later on, Klim and Wardowski [33] established some FP results for set-valued nonlinear contraction mappings on CM-spaces. After that, Latif and Shaddad [34] proved some FP results for multi-valued maps on CM-spaces. Cho and Bae [35] presented modified FP theorems for multi-valued mappings on CM-spaces. Meanwhile, Wardowski [36] proved some Nadler type contraction results for set-valued mappings on CM-spaces. Mehmood et al. [37, 38], proved some multi-valued contraction results for FPs on CM-space and order CM-spaces with an application. In 2015, Fierro [39] established some FP theorems on topological vector spaces valued CM-spaces for set-valued mappings. Recently, Rehman et al. [40] proved some multi-valued contraction theorems for FPs and CFPs on $\mathcal{H}-$CM-spaces.

In this paper, we study some new types of generalized multi-valued contraction results on complete CM-spaces. We prove some CFP theorems for a pair of multi-valued contraction mappings on CM-spaces with the condition of normality of the cone. We present an illustrative example to support our work. Further, we present an application of nonlinear integral equations to validate our work. This concept can be extended for different types of multi-valued contraction mappings in the context of M-spaces with the application of different types of integral equations and differential equations. This paper is organized as follows: in Section 2, we introduce the preliminary concepts related to our main work. In Section 3, we establish some CFP theorems for a pair of multi-valued contraction mappings on CM-spaces with an illustrative example. In Section 4, we present a supportive application of nonlinear integral equations to unify our main work. Finally, in Section 5, we present the conclusion of our work.

2. Preliminaries

Definition 2.1. [11] Let $\mathbb{E}$ be a real Banach space. A subset $\mathbb{P} \subseteq \mathbb{E}$ is called a cone if the following are satisfied:

(i) $\mathbb{P}$ is closed, nonempty and $\mathbb{P} \neq \{\theta\}$, where θ is the zero element of $\mathbb{E}$;

(ii) If $0 \leq b_1, b_2 < \infty$ and $u_1, u_2 \in \mathbb{P}$, then $b_1u_1 + b_2u_2 \in \mathbb{P}$;

(iii) $\mathbb{P} \cap -\mathbb{P} = \{\theta\}$.

Given a cone $\mathbb{P} \subseteq \mathbb{E}$, define a partial ordering $\leq$ on $\mathbb{E}$ with respect to $\mathbb{P}$ by $u_1 \leq u_2$ if and only if $u_2 - u_1 \in \mathbb{P}$. We shall write $u_1 < u_2$ if $u_1 \leq u_2$ and $u_1 \neq u_2$ while $u_1 \ll u_2$, and if and only if $u_2 - u_1 \in int(\mathbb{P})$, where $int(\mathbb{P})$ denotes the interior of $\mathbb{P}$. A nonempty cone $\mathbb{P}$ is called normal if there is $K > 1$ such that $\forall\, u_1, u_2 \in \mathbb{E}$, $\|u_1\| \leq K\|u_2\|$, whenever $\theta \leq u_1 \leq u_2$.

A cone $\mathbb{P}$ is known as regular if every non-decreasing sequence which is bounded from above is convergent, i.e., if $\{u_n\}$ is a sequence such that for some $v \in \mathbb{E}$, we have $u_1 \leq u_2 \leq \cdots \leq v$. Then there exists $u^* \in \mathbb{E}$ such that

$$\lim_{n\to+\infty} \|u_n - u^*\| = 0.$$

Equivalently, a cone $\mathbb{P}$ is regular if and only if every non-increasing sequence which is bounded from below is convergent.

Throughout this paper, we assume that $\mathbb{E}$ is a real Banach space, $\mathbb{P}$ is a cone in $\mathbb{E}$ with $int(\mathbb{P}) \neq \emptyset$ and $\leq$ is the partial ordering on $\mathbb{E}$ with respect to $\mathbb{P}$.

Definition 2.2. [11] Let U be a nonempty set. Let δ: $U \times U \to \mathbb{E}$ be called a cone metric if the following hold

(i) $\delta(u_1, u_2) \geq \theta$ and $\delta(u_1, u_2) = \theta \Leftrightarrow u_1 = u_2$;

(ii) $\delta(u_1, u_2) = \delta(u_2, u_1)$;

(iii) $\delta(u_1, u_2) \leq \delta(u_1, u_3) + \delta(u_3, u_2)$;

for all $u_1, u_2, u_3 \in U$. The a pair (U, δ) is called a CM-space.

Definition 2.3. [11] Let (U, δ) be a CM-space. Let $\upsilon \in U$ and $\{u_n\}$ be a sequence in U. Then the following are true:

(i) $\{u_n\}$ is said to be convergent to υ if for every $\zeta \in \mathbb{E}$ with $\zeta \gg \theta$, there is a positive integer N such that $\delta(u_n, \upsilon) \ll \zeta$ for $n \geq$ N. We denote this by $\lim_{n\to+\infty} u_n = \upsilon$ or $u_n \to \upsilon$ as $n \to +\infty$.

(ii) $\{u_n\}$ is said to be a Cauchy sequence if for every $\zeta \in \mathbb{E}$ with $\zeta \gg \theta$, there is a positive integer N such that $\delta(u_n, u_m) \ll \zeta$ for $m, n \geq$ N.

(iii) (U, δ) is called complete if every Cauchy sequence is convergent in U.

Lemma 2.4. *[11] Let (U, δ) be a CM-space and $\mathbb{P}$ be a normal cone. Let $\{u_n\}$ be a sequence in U and $u, v \in U$. Then the following are true:*

(i) $\lim_{n\to+\infty} u_n = u \Leftrightarrow \lim_{n\to+\infty} \delta(u_n, u) = \theta$.

(ii) $\{u_n\}$ *is a Cauchy sequence iff* $\lim_{m,n\to+\infty} \delta(u_n, u_m) = \theta$.

(iii) If $\lim_{n\to+\infty} u_n = u$ *and* $\lim_{n\to+\infty} u_n = v$*, then* $u = v$*.*

In what follows, $\mathfrak{B}$ denotes (resp. $B(U)$, $CB(U)$) the set of nonempty (resp. bounded, sequentially closed and bounded) subsets of (U, δ).

Let (U, δ) be a CM-space and we denote

$$s(u_1) = \{u_2 \in \mathbb{E} : \; u_1 \leq u_2\}$$

for $u_1 \in \mathbb{E}$, and

$$s(x, B) = \cup_{y \in B} \, s\left(\delta(x, y)\right)$$

for $x \in U$ and $B \in \mathfrak{B}$. For $A, B \in B(U)$, we represent

$$s(A, B) = \left(\cap_{x \in A} \, s(x, B)\right) \bigcap \left(\cap_{y \in B} \, s(y, A)\right).$$

Lemma 2.5. *[35] Let* (U, δ) *be a CM-space and* $\mathbb{P}$ *be a cone in Banach space* $\mathbb{E}$*. Then the following are true:*

(i) For all $u_1, u_2 \in \mathbb{E}$*, if* $u_1 \leq u_2$*, then* $s(u_2) \subseteq s(u_1)$*.*

(ii) For all $u \in U$ *and* $A \in \mathfrak{B}$*, if* $\theta \in s(u, A)$*, then* $u \in A$*.*

(iii) For all $u_1 \in \mathbb{P}$ *and* $A, B \in B(U)$ *and* $x \in A$*, if* $u_1 \in s(A, B)$*, then* $u_1 \in s(x, B)$*.*

(iv) If $u_n \in \mathbb{E}$ *with* $u_n \to \theta$*, then for each* $\zeta \in int(\mathbb{P})$ *there exists* N *such that* $u_n \ll \zeta$ *for all* $n >$ N*.*

Remark 2.6. [35] Let (U, δ) be a CM-space.

(i) If $\mathbb{E} = \mathbb{R}$ and $\mathbb{P} = [0, +\infty)$, then (U, δ) is an M-space. Moreover, for $A, B \in CB(U)$, $H_\delta(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by δ.

(ii) $s\left(\{x\}, \{y\}\right) = s(\delta(x, y))$ for $x, y \in U$.

Definition 2.7. Let $T\colon U \to CB(U)$ be a multi-valued map. An element $u_0 \in U$ is called an FP of T if $u_0 \in Tu_0$.

Theorem 2.8. *[26] Let* (U, δ) *be a complete M-space. Let* $T\colon U \to CB(U)$ *satisfy*

$$H_\delta(T\mu, T\nu) \leq \eta\delta(\mu, \nu), \; \forall \, \mu, \nu \in U, \tag{2.1}$$

where $\eta \in [0, 1)$*. Then* T *has an FP.*

Definition 2.9. [28] An element $u_0 \in U$ is a CFP of the mappings $S, T\colon U \to CB(U)$ if $u_0 \in Tu_0 \cap Su_0$.

3. Main results

First we define that $\delta(u, A) := \inf_{v \in A} \delta(u, v)$. Now, we present our first main result.

Theorem 3.1. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings satisfying*

$$\begin{pmatrix} b_1\delta(\mu, \nu) \\ +b_2[\delta(\mu, S\mu) + \delta(\nu, T\nu)] \\ +b_3[\delta(\nu, S\mu) + \delta(\mu, T\nu)] \end{pmatrix} \in s(S\mu, T\nu) \tag{3.1}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2, b_3 \geq 0$ with $b_1 + 2b_2 + 2b_3 < 1$. Then S and T have a CFP in U.

Proof. Fix $\mu_0 \in U$ and let there exists $\mu_1 \in U$ such that $\mu_1 \in S\mu_0$. Then, from (3.1), we have

$$\begin{pmatrix} b_1\delta(\mu_0, \mu_1) \\ +b_2[\delta(\mu_0, S\mu_0) + \delta(\mu_1, T\mu_1)] \\ +b_3[\delta(\mu_1, S\mu_0) + \delta(\mu_0, T\mu_1)] \end{pmatrix} \in s(S\mu_0, T\mu_1).$$

Since $\mu_1 \in S\mu_0$ and by Lemma 2.5(iii), we have

$$\begin{pmatrix} b_1\delta(\mu_0, \mu_1) \\ +b_2[\delta(\mu_0, \mu_1) + \delta(\mu_1, T\mu_1)] \\ +b_3[\delta(\mu_1, \mu_1) + \delta(\mu_0, T\mu_1)] \end{pmatrix} \in s(\mu_1, T\mu_1).$$

Then there exists $\mu_2 \in T\mu_1$ such that

$$\begin{pmatrix} b_1\delta(\mu_0, \mu_1) \\ +b_2[\delta(\mu_0, S\mu_0) + \delta(\mu_1, \mu_2)] \\ +b_3[\delta(\mu_1, S\mu_0) + \delta(\mu_0, \mu_2)] \end{pmatrix} \in s(\delta(\mu_1, \mu_2)).$$

This implies that

$$\begin{aligned} \delta(\mu_1, \mu_2) &\leq b_1\delta(\mu_0, \mu_1) + b_2[\delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2)] + b_3\delta(\mu_0, \mu_2) \\ &\leq b_1\delta(\mu_0, \mu_1) + b_2[\delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2)] + b_3[\delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2)]. \end{aligned}$$

After simplification, we obtain

$$\delta(\mu_1, \mu_2) \leq \beta\delta(\mu_0, \mu_1), \quad \text{where } \beta = \frac{b_1 + b_2 + b_3}{1 - (b_2 + b_3)} < 1. \tag{3.2}$$

Again from (3.1), we have

$$\begin{pmatrix} b_1\delta(\mu_2, \mu_1) \\ +b_2[\delta(\mu_2, S\mu_2) + \delta(\mu_1, T\mu_1)] \\ +b_3[\delta(\mu_1, S\mu_2) + \delta(\mu_2, T\mu_1)] \end{pmatrix} \in s(S\mu_2, T\mu_1).$$

Since $\mu_2 \in T\mu_1$, and by Lemma 2.5(iii), we have

$$\begin{pmatrix} b_1\delta(\mu_2, \mu_1) \\ +b_2[\delta(\mu_2, S\mu_2) + \delta(\mu_1, \mu_2)] \\ +b_3[\delta(\mu_1, S\mu_2) + \delta(\mu_2, \mu_2)] \end{pmatrix} \in s(\mu_2, S\mu_2).$$

Then there exists $\mu_3 \in S\mu_2$ such that

$$\begin{pmatrix} b_1\delta(\mu_2,\mu_1) \\ +b_2[\delta(\mu_2,\mu_3)+\delta(\mu_1,\mu_2)] \\ +b_3[\delta(\mu_1,\mu_3)+\delta(\mu_2,\mu_2)] \end{pmatrix} \in s(\delta(\mu_2,\mu_3)).$$

This implies that

$$\begin{aligned}\delta(\mu_2,\mu_3) &\leq b_1\delta(\mu_2,\mu_1)+b_2[\delta(\mu_2,\mu_3)+\delta(\mu_1,\mu_2)]+b_3\delta(\mu_1,\mu_3)\\ &\leq b_1\delta(\mu_2,\mu_1)+b_2[\delta(\mu_2,\mu_3)+\delta(\mu_1,\mu_2)]+b_3[\delta(\mu_1,\mu_2)+\delta(\mu_2,\mu_3)].\end{aligned}$$

After simplification, we obtain

$$\delta(\mu_2,\mu_3) \leq \beta\delta(\mu_1,\mu_2), \tag{3.3}$$

where

$$\beta = \frac{b_1+b_2+b_3}{1-(b_2+b_3)} < 1.$$

From (3.2) and (3.3), we have

$$\delta(\mu_2,\mu_3) \leq \beta\delta(\mu_2,\mu_1) \leq \beta^2\delta(\mu_0,\mu_1).$$

By repeatedly applying the above arguments we construct a sequence $\{\mu_n\}$ in U such that

$$\mu_{2n+1} \in S\mu_{2n}, \text{ and } \mu_{2n+2} \in T\mu_{2n+1}, \ \forall\, n \in \mathrm{N}.$$

And

$$\delta(\mu_n,\mu_{n+1}) \leq \beta\delta(\mu_{n-1},\mu_n), \tag{3.4}$$

where β is as in (3.3). Thus, by induction, we obtain

$$\delta(\mu_n,\mu_{n+1}) \leq \beta^n\delta(\mu_0,\mu_1). \tag{3.5}$$

We claim that $\{\mu_n\}$ is a Cauchy sequence. Let $m > n$; then, by the triangular inequality and from (3.5), we have

$$\begin{aligned}\delta(\mu_n,\mu_m) &\leq \delta(\mu_n,\mu_{n+1})+\delta(\mu_{n+1},\mu_{n+2})+\cdots+\delta(\mu_{m-1},\mu_m)\\ &\leq \beta^n\delta(\mu_0,\mu_1)+\beta^{n+1}\delta(\mu_0,\mu_1)+\cdots+\beta^{m-1}\delta(\mu_0,\mu_1)\\ &\leq \beta^n(1+\beta+\beta^2+\cdots+\beta^{m-n-1}+\cdots)\delta(\mu_0,\mu_1)\\ &\leq \frac{\beta^n}{1-\beta}\delta(\mu_0,\mu_1) \to \theta \quad \text{as } n \to +\infty.\end{aligned}$$

By Lemma 2.4(ii), $\{\mu_n\}$ is a Cauchy sequence in (U,δ). Since (U,δ) is complete, there exists $\omega_1 \in U$ such that $\mu_n \to \omega_1$ as $n \to +\infty$. Therefore,

$$\lim_{n\to+\infty}\mu_{2n+1} = \lim_{n\to+\infty}\mu_{2n+2} = \omega_1. \tag{3.6}$$

Now, we have to prove that $\omega_1 \in S\omega_1$. From (3.1), we have

$$\begin{pmatrix} b_1\delta(\omega_1,\mu_{2n+1}) \\ +b_2[\delta(\omega_1,S\omega_1)+\delta(\mu_{2n+1},T\mu_{2n+1})] \\ +b_3[\delta(\omega_1,T\mu_{2n+1})+\delta(\mu_{2n+1},S\omega_1)] \end{pmatrix} \in s(T\mu_{2n+1},S\omega_1).$$

Since $\mu_{2n+2} \in T\mu_{2n+1}$ and by Lemma 2.5(iii), we have

$$\begin{pmatrix} b_1\delta(\omega_1, \mu_{2n+1}) \\ +b_2[\delta(\omega_1, S\omega_1) + \delta(\mu_{2n+1}, \mu_{2n+2})] \\ +b_3[\delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+1}, S\omega_1)] \end{pmatrix} \in s(\mu_{2n+2}, S\omega_1).$$

Then there exists $v_n \in Sw_1$ such that

$$\begin{pmatrix} b_1\delta(\omega_1, \mu_{2n+1}) \\ +b_2[\delta(\omega_1, v_n) + \delta(\mu_{2n+1}, \mu_{2n+2})] \\ +b_3[\delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+1}, v_n)] \end{pmatrix} \in s(\delta(\mu_{2n+2}, v_n)).$$

This implies that

$$\begin{aligned} \delta(\mu_{2n+2}, v_n) &\leq b_1\delta(\omega_1, \mu_{2n+1}) + b_2[\delta(\omega_1, v_n) + \delta(\mu_{2n+1}, \mu_{2n+2})] + b_3[\delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+1}, v_n)] \\ &\leq b_1\delta(\omega_1, \mu_{2n+1}) + b_2[\delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+2}, v_n) + \delta(\mu_{2n+1}, \omega_1) + \delta(\omega_1, \mu_{2n+2})] \\ &+ b_3[\delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+1}, \omega_1) + \delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+2}, v_n)] \\ &= 2(b_2 + b_3)\delta(\omega_1, \mu_{2n+2}) + (b_1 + b_2 + b_3)\delta(\omega_1, \mu_{2n+1}) + (b_2 + b_3)\delta(\mu_{2n+2}, v_n). \end{aligned}$$

After simplification, we get that

$$\delta(\mu_{2n+2}, v_n) \leq \frac{2(b_2 + b_3)}{1 - b_2 - b_3}\delta(\omega_1, \mu_{2n+2}) + \frac{b_1 + b_2 + b_3}{1 - b_2 - b_3}\delta(\omega_1, \mu_{2n+1}).$$

Now, by taking the limit as $n \to +\infty$, we get that

$$\lim_{n\to+\infty} \delta(\mu_{2n+2}, v_n) = \theta.$$

Therefore, since

$$\delta(\omega_1, v_n) \leq \delta(\omega_1, \mu_{2n+2}) + \delta(\mu_{2n+2}, v_n)$$

by Lemma 2.4, we deduce that $\lim_{n\to+\infty} v_n = \omega_1$. Since $S\omega_1$ is closed, sequentially, we obtain $\omega_1 \in S\omega_1$.

Similarly, we can prove that $\omega_1 \in T\omega_1$. Hence, it is proved that the mappings S and T have a CFP in U, that is, $\omega_1 \in S\omega_1 \cap T\omega_1$.

By putting the constants $b_3 = 0$ and $b_2 = 0$ in Theorem 3.1, we get the following two corollaries, respectively.

Corollary 3.2. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings satisfying*

$$b_1\delta(\mu, \nu) + b_2[\delta(\mu, S\mu) + \delta(\nu, T\nu)] \in s(S\mu, T\nu) \tag{3.7}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2 \geq 0$ with $(b_1 + 2b_2) < 1$. Then S and T have a CFP in U.

Corollary 3.3. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings satisfying*

$$b_1\delta(\mu, \nu) + b_3[\delta(\nu, S\mu) + \delta(\mu, T\nu)] \in s(S\mu, T\nu) \tag{3.8}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_3 \geq 0$ with $(b_1 + 2b_3) < 1$. Then S and T have a CFP in U.

If we put $S = T$ in Theorem 3.1, we get the following corollary:

Corollary 3.4. *Let (U,δ) be a complete CM-space. Let $S\colon U \to CB(U)$ be a multi-valued mapping such that*

$$\begin{pmatrix} b_1\delta(\mu,\nu) \\ +b_2[\delta(\mu,S\mu)+\delta(\nu,S\nu)] \\ +b_3[\delta(\nu,S\mu)+\delta(\mu,S\nu)] \end{pmatrix} \in s(S\mu,S\nu) \tag{3.9}$$

for all $\mu,\nu \in U$, $b_1 \in (0,1)$ and $b_2,b_3 \geq 0$ with $(b_1+2b_2+2b_3) < 1$. Then S has an FP in U.

Remark 3.5. In the context of complete M-spaces instead of complete CM-spaces, if we put $b_2 = b_3 = 0$ and $S = T$ in Theorem 3.1, then we obtain Nadler's result [26].

In the sense of Nadler's multi-valued concept [26], Theorem 3.1 can be stated as follows:

Corollary 3.6. *Let (U,δ) be a complete CM-space. Let $S,T\colon U \to CB(U)$ be a pair of multi-valued mappings such that:*

$$H_\delta(S\mu,T\nu) \leq b_1\delta(\mu,\nu)+b_2[\delta(\mu,S\mu)+\delta(\nu,T\nu)]+b_3[\delta(\nu,S\mu)+\delta(\mu,T\nu)] \tag{3.10}$$

for all $\mu,\nu \in U$, $b_1 \in (0,1)$, and $b_2,b_3 \geq 0$ with $(b_1+2b_2+2b_3) < 1$. Then S and T have a CFP in U.

Now, we present our second main result.

Theorem 3.7. *Let (U,δ) be a complete CM-space. Let $S,T\colon U \to CB(U)$ be a pair of multi-valued mappings verifying*

$$\left(b_1\delta(\mu,\nu)+b_2\max\left\{ \begin{array}{c} \delta(\mu,S\mu),\delta(\nu,T\nu), \\ \delta(\nu,S\mu),\delta(\mu,T\nu) \end{array} \right\} \right) \in s(S\mu,T\nu) \tag{3.11}$$

for all $\mu,\nu \in U$, $b_1 \in [0,1)$ and $b_2 \geq 0$ with $(b_1+2b_2) < 1$. Then S and T have a CFP in U.

Proof. Fix $\mu_0 \in U$ and $\mu_1 \in S\mu_0$. Then, from (3.11), we have

$$\left(b_1\delta(\mu_0,\mu_1)+b_2\max\left\{ \begin{array}{c} \delta(\mu_0,S\mu_0),\delta(g\mu_1,T\mu_1), \\ \delta(\mu_1,S\mu_0),\delta(\mu_0,T\mu_1) \end{array} \right\} \right) \in s(S\mu_0,T\mu_1).$$

Thus by Lemma 2.5(iii), we have

$$\left(b_1\delta(\mu_0,\mu_1)+b_2\max\left\{ \begin{array}{c} \delta(\mu_0,\mu_1),\delta(g\mu_1,T\mu_1), \\ \delta(\mu_1,\mu_1),\delta(\mu_0,T\mu_1) \end{array} \right\} \right) \in s(\mu_1,T\mu_1).$$

Then there exists $\mu_2 \in T\mu_1$ such that

$$\left(b_1\delta(\mu_0,\mu_1)+b_2\max\{\delta(\mu_0,\mu_1),\delta(\mu_1,\mu_2),\delta(\mu_0,\mu_2)\} \right) \in s(\delta(\mu_1,\mu_2)).$$

This implies that

$$\delta(\mu_1,\mu_2) \leq b_1\delta(\mu_0,\mu_1)+b_2\max\{\delta(\mu_0,\mu_1),\delta(\mu_1,\mu_2),\delta(\mu_0,\mu_2)\}. \tag{3.12}$$

We may have the following three cases:

(a) If $\delta(\mu_0, \mu_1)$ is the maximum term of $\{\delta(\mu_0, \mu_1), \delta(\mu_1, \mu_2), \delta(\mu_0, \mu_2)\}$, then, from (3.12), we get that

$$\delta(\mu_1, \mu_2) \leq (b_1 + b_2)\delta(\mu_0, \mu_1). \tag{3.13}$$

(b) If $\delta(\mu_1, \mu_2)$ is the maximum term of $\{\delta(\mu_0, \mu_1), \delta(\mu_1, \mu_2), \delta(\mu_0, \mu_2)\}$, then, from (3.12), we get that

$$\delta(\mu_1, \mu_2) \leq \frac{b_1}{1 - b_2}\delta(\mu_0, \mu_1). \tag{3.14}$$

(c) If $\delta(\mu_0, \mu_2)$ is the maximum term of $\{\delta(\mu_0, \mu_1), \delta(\mu_1, \mu_2), \delta(\mu_0, \mu_2)\}$, then, from (3.12) and the triangle inequality, we get that

$$\delta(\mu_1, \mu_2) \leq \frac{b_1 + b_2}{1 - b_2}\delta(\mu_0, \mu_1). \tag{3.15}$$

Let us define

$$\beta := \max\left\{(b_1 + b_2), \left(\frac{b_1}{1 - b_2}\right), \left(\frac{b_1 + b_2}{1 - b_2}\right)\right\} < 1,$$

where $(b_1 + 2b_2) < 1$; then, from (3.13)–(3.15), we have that

$$\delta(\mu_1, \mu_2) \leq \beta\delta(\mu_0, \mu_1). \tag{3.16}$$

Again from (3.11), we have

$$\left(b_1\delta(\mu_2, \mu_1) + b_2 \max\left\{ \begin{array}{c} \delta(\mu_2, S\mu_2), \delta(\mu_1, T\mu_1), \\ \delta(\mu_1, S\mu_2), \delta(\mu_2, T\mu_1) \end{array} \right\} \right) \in s(S\mu_2, T\mu_1).$$

Since $\mu_2 \in T\mu_1$, and by Lemma 2.5(iii), we have

$$\left(b_1\delta(\mu_1, \mu_2) + b_2 \max\left\{ \begin{array}{c} \delta(\mu_2, S\mu_2), \delta(\mu_1, \mu_2), \\ \delta(\mu_1, S\mu_2), \delta(\mu_2, \mu_2) \end{array} \right\} \right) \in s(\mu_2, S\mu_2).$$

Then there exists $\mu_3 \in S\mu_2$ such that

$$\left(b_1\delta(\mu_1, \mu_2) + b_2 \max\{\delta(\mu_2, \mu_3), \delta(\mu_1, \mu_2), \delta(\mu_1, \mu_3)\} \right) \in s(\delta(\mu_3, \mu_2)).$$

This implies that

$$\delta(\mu_2, \mu_3) \leq b_1\delta(\mu_1, \mu_2) + b_2 \max\{\delta(\mu_1, \mu_2), \delta(\mu_2, \mu_3), \delta(\mu_1, \mu_3)\}. \tag{3.17}$$

Then, we may have the following three cases:

(a) If $\delta(\mu_1, \mu_2)$ is the maximum term of $\{\delta(\mu_1, \mu_2), \delta(\mu_2, \mu_3), \delta(\mu_1, \mu_3)\}$, then, from (3.17), we get that

$$\delta(\mu_2, \mu_3) \leq (b_1 + b_2)\delta(\mu_1, \mu_2). \tag{3.18}$$

(b) If $\delta(\mu_2, \mu_3)$ is the maximum term of $\{\delta(\mu_1, \mu_2), \delta(\mu_2, \mu_3), \delta(\mu_1, \mu_3)\}$, then, from (3.17), we have

$$\delta(\mu_2, \mu_3) \leq \frac{b_1}{1 - b_2}\delta(\mu_1, \mu_2). \tag{3.19}$$

(c) If $\delta(\mu_1, \mu_3)$ is the maximum term of $\{\delta(\mu_1, \mu_2), \delta(\mu_2, \mu_3), \delta(\mu_1, \mu_3)\}$, then, from (3.17) and the triangle inequality, we get that

$$\delta(\mu_2, \mu_3) \leq \frac{b_1 + b_2}{1 - b_2}\delta(\mu_1, \mu_2). \tag{3.20}$$

Then from (3.18)–(3.20), we find that

$$\delta(\mu_2, \mu_3) \leq \beta\delta(\mu_1, \mu_2), \tag{3.21}$$

where β is as in (3.16). From (3.16) and (3.21), we have

$$\delta(\mu_2, \mu_3) \leq \beta\delta(\mu_2, \mu_1) \leq \beta^2\delta(\mu_0, \mu_1).$$

By repeatedly applying the above arguments we construct a sequence $\{\mu_n\}$ in U such that

$$\mu_{2n+1} \in S\mu_{2n}, \text{ and } \mu_{2n+2} \in T\mu_{2n+1}, \ \forall\, n \in \mathrm{N}.$$

And

$$\delta(\mu_n, \mu_{n+1}) \leq \beta\delta(\mu_{n-1}, \mu_n), \tag{3.22}$$

where β is as in (3.16).

Thus, by induction, we obtain

$$\delta(\mu_n, \mu_{n+1}) \leq \beta^n\delta(\mu_0, \mu_1), \ \forall\, n \in \mathrm{N}. \tag{3.23}$$

Now, we have to show that $\{\mu_n\}$ is a Cauchy sequence. Let $m > n$; then, by the triangular inequality and from (3.23), we have

$$\begin{aligned}
\delta(\mu_n, \mu_m) &\leq \delta(\mu_n, \mu_{n+1}) + \delta(\mu_{n+1}, \mu_{n+2}) + \cdots + \delta(\mu_{m-1}, \mu_m)\\
&\leq \beta^n\delta(\mu_0, \mu_1) + \beta^{n+1}\delta(\mu_0, \mu_1) + \cdots + \beta^{m-1}\delta(\mu_0, \mu_1)\\
&\leq \beta^n(1 + \beta + \beta^2 + \cdots + \beta^{m-n-1} + \cdots)\delta(\mu_0, \mu_1)\\
&\leq \frac{\beta^n}{1-\beta}\delta(\mu_0, \mu_1) \to \theta \quad \text{as } n \to +\infty.
\end{aligned}$$

By Lemma 2.4(ii), $\{\mu_n\}$ is a Cauchy sequence in (U, δ). Since (U, δ) is complete, there exists $\omega_1 \in U$ such that $\mu_n \to \omega_1$ as $n \to +\infty$. Therefore,

$$\lim_{n\to+\infty}\mu_{2n+1} = \lim_{n\to+\infty}\mu_{2n+2} = \omega_1. \tag{3.24}$$

Now, we have to prove that $\omega_1 \in S\omega_1$. From (3.11), we have

$$\left(b_1\delta(\omega_1, \mu_{2n+1}) + b_2 \max\left\{ \begin{array}{c} \delta(\omega_1, S\omega_1), \delta(\mu_{2n+1}, T\mu_{2n+1}),\\ \delta(\omega_1, T\mu_{2n+1}), \delta(\mu_{2n+1}, S\omega_1) \end{array} \right\} \right) \in s(S\omega_1, T\mu_{2n+1}).$$

Since $\mu_{2n+2} \in T\mu_{2n+1}$ and by Lemma 2.5(iii), we have

$$\left(b_1\delta(\omega_1, \mu_{2n+1}) + b_2 \max\left\{ \begin{array}{c} \delta(\omega_1, S\omega_1), \delta(\mu_{2n+1}, \mu_{2n+2}),\\ \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, S\omega_1) \end{array} \right\} \right) \in s(\mu_{2n+2}, S\omega_1).$$

Then, there exists $v_n \in S\omega_1$ such that

$$\left(b_1\delta(\omega_1, \mu_{2n+1}) + b_2 \max\left\{ \begin{array}{c} \delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \\ \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n) \end{array} \right\} \right) \in s(\delta(\mu_{2n+2}, v_n)).$$

This implies that

$$\delta(\mu_{2n+2}, v_n) \leq b_1\delta(\omega_1, \mu_{2n+1}) + b_2 \max\left\{ \begin{array}{c} \delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \\ \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n) \end{array} \right\}. \tag{3.25}$$

Then, we may have the following four cases:

(a) If $\delta(\omega_1, v_n)$ is the maximum term of $\{\delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n)\}$, then, from (3.25) and the triangle inequality, we get that

$$\delta(\mu_{2n+2}, v_n) \leq \frac{b_1}{1-b_2}\delta(\omega_1, \mu_{2n+1}) + \frac{b_2}{1-b_2}\delta(\omega_1, \mu_{2n+2}). \tag{3.26}$$

(b) If $\delta(\mu_{2n+1}, \mu_{2n+2})$ is the maximum term of $\{\delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n)\}$, then, from (3.25) and the triangle inequality, we get that

$$\delta(\mu_{2n+2}, v_n) \leq (b_1 + b_2)\delta(\omega_1, \mu_{2n+1}) + b_2\delta(\omega_1, \mu_{2n+2}). \tag{3.27}$$

(c) If $\delta(\omega_1, \mu_{2n+2})$ is the maximum term of $\{\delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n)\}$, then, from (3.25), we get that

$$\delta(\mu_{2n+2}, v_n) \leq b_1\delta(\omega_1, \mu_{2n+1}) + b_2\delta(\omega_1, \mu_{2n+2}). \tag{3.28}$$

(d) If $\delta(\mu_{2n+1}, v_n)$ is the maximum term of $\{\delta(\omega_1, v_n), \delta(\mu_{2n+1}, \mu_{2n+2}), \delta(\omega_1, \mu_{2n+2}), \delta(\mu_{2n+1}, v_n)\}$, then, from (3.25) and the triangle inequality, we get that

$$\delta(\mu_{2n+2}, v_n) \leq \frac{b_1 + b_2}{1-b_2}\delta(\omega_1, \mu_{2n+1}) + \frac{b_2}{1-b_2}\delta(\omega_1, \mu_{2n+2}). \tag{3.29}$$

Then, we define

$$\lambda_1 := \max\left\{\frac{b_1}{1-b_2}, (b_1 + b_2), b_1, \frac{b_1 + b_2}{1-b_2}\right\}$$

and

$$\lambda_2 := \max\left\{\frac{b_2}{1-b_2}, b_2\right\}.$$

Then, from (3.26)–(3.29), we have that

$$\delta(\mu_{2n+2}, v_n) \leq \lambda_1\delta(\omega_1, \mu_{2n+1}) + \lambda_2\delta(\omega_1, \mu_{2n+2}).$$

Now, by taking the limit as $n \to +\infty$, we get that

$$\lim_{n\to+\infty} \delta(\mu_{2n+2}, v_n) = \theta.$$

As in the proof of Theorem (3.1), this implies that

$$\lim_{n\to+\infty} \nu_n = \omega_1.$$

Since $S\omega_1$ is closed, sequentially we deduce that $\omega_1 \in S\omega_1$. Similarly, we can prove that $\omega_1 \in T\omega_1$. Hence, it is proved that the mappings S and T have a CFP in U, that is, $\omega_1 \in S\omega_1 \cap T\omega_1$.

By reducing the maximum term in Theorem 3.7, we get the following corollaries:

Corollary 3.8. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings satisfying*

$$b_1\delta(\mu, \nu) + b_2 \max\{\delta(\mu, S\mu), \delta(\nu, T\nu)\} \in s(S\mu, T\nu) \tag{3.30}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2 \geq 0$ with $(b_1 + b_2) < 1$. Then S and T have a CFP in U.

Corollary 3.9. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings satisfying*

$$b_1\delta(\mu, \nu) + b_2 \max\{\delta(\nu, S\mu), \delta(\mu, T\nu)\} \in s(S\mu, T\nu) \tag{3.31}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2 \geq 0$ with $(b_1 + 2b_2) < 1$. Then S and T have a CFP in U.

If we put $S = T$ in Theorem 3.7, we get the following corollary:

Corollary 3.10. *Let (U, δ) be a complete CM-space. Let $S\colon U \to CB(U)$ be a multi-valued mapping such that*

$$\left(b_1\delta(\mu, \nu) + b_2 \max\left\{ \begin{array}{c} \delta(\mu, S\mu), \delta(\nu, S\nu), \\ \delta(\nu, S\mu), \delta(\mu, S\nu) \end{array} \right\} \right) \in s(S\mu, S\nu) \tag{3.32}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2 \geq 0$ with $(b_1 + 2b_2) < 1$. Then S has an FP in U.

In the sense of Nadler's multi-valued concept [26], Theorem 3.7 can be stated as follows:

Corollary 3.11. *Let (U, δ) be a complete CM-space. Let $S, T\colon U \to CB(U)$ be a pair of multi-valued mappings so that*

$$H_\delta(S\mu, T\nu) \leq b_1\delta(\mu, \nu) + b_2 \max\{\delta(\mu, S\mu), \delta(\nu, T\nu), \delta(\nu, S\mu), \delta(\mu, T\nu)\} \tag{3.33}$$

for all $\mu, \nu \in U$, $b_1 \in (0, 1)$ and $b_2 \geq 0$ with $(b_1 + 2b_2) < 1$. Then S and T have a CFP in U.

Example 3.12. Let $U = [0, 1]$ and the cone

$$\mathbb{P} := \{u \in \mathbb{E} : u(t) \geq 0, \text{ for } t \in [0, 1]\}$$

on $\mathbb{E}$ where

$$\mathbb{E} = C([0, 1], \mathbb{R})$$

denoting continuous functions on [0,1]. Then $\mathbb{P}$ is a normal cone with respect to the norm of the space $\mathbb{E}$ with the constant $K = 1$. A cone metric $\delta\colon U \times U \to \mathbb{E}$ is defined as

$$\delta(u_1, u_2) = |u_1 - u_2|$$

for all $u_1, u_2 \in U$. Let $\mathfrak{B}$ be a family of nonempty closed and bounded subsets of U of the form

$$\mathfrak{B} = \{[0, u] : u \in U\}.$$

Now, we define a pair of multi-valued mappings $S, T : U \to \mathfrak{B}$ by

$$Su = Tu = \left[0, \frac{2u}{7}\right].$$

Moreover, for $u_1, u_2 \in U (u_1 \neq u_2)$ and $u_1, u_2 \neq 0$, let

$$b_1 = \frac{2}{7} \text{ and } b_2 = b_3 = \frac{2}{21}.$$

Then, we have that

$$\begin{aligned}
\begin{pmatrix} \frac{2}{7}\delta(\mu, \nu) \\ +\frac{2}{21}[\delta(\mu, S\mu) + \delta(\nu, T\nu)] \\ +\frac{2}{21}[\delta(\nu, S\mu) + \delta(\mu, T\nu)] \end{pmatrix} \in s(S\mu, T\nu) &\Leftrightarrow \frac{62}{147}(\mu + \nu) \in s(S\mu, T\nu) \\
&\Leftrightarrow \frac{62}{147}(\mu + \nu) \in \left(\bigcap_{x \in S\mu} \bigcup_{y \in T\nu} s(\delta(x, y))\right) \cap \left(\bigcap_{y \in T\nu} \bigcup_{x \in S\mu} s(\delta(x, y))\right) \\
&\Leftrightarrow (\exists x \in S\mu)(\exists y \in T\nu) \frac{62}{147}(\mu + \nu) \in s(\delta(x, y)) \\
&\Leftrightarrow s(\delta(x, y)) \leq \frac{62}{147}(\mu + \nu) = \begin{pmatrix} b_1\delta(\mu, \nu) \\ +b_2[\delta(\mu, S\mu) + \delta(\nu, T\nu)] \\ +b_3[\delta(\nu, S\mu) + \delta(\mu, T\nu)] \end{pmatrix}.
\end{aligned}$$

Now, by taking

$$x = \frac{2}{7}\mu, \; y = \frac{2}{7}\nu$$

and

$$(b_1 + 2b_2 + 2b_3) = \frac{2}{3} < 1,$$

all hypothesis of Theorem 3.1 are satisfied, and the pair of multi-valued mappings S and T have a CFP in U, that is, "0".

4. Application

In this section, we present a supportive application of integral equations for this new theory. A number of researchers have used various applications in differential and integral equations in the context of M-spaces for FP results. Some of their works can be found in [4, 41–43] and the references therein. Here in this section, we develop an approach for solving the nonlinear integral type problems represented by the following integral equations:

$$\mu(\xi) = \int_0^a K_1(\xi, s, \mu(s))ds, \ \text{ and } \ \nu(\xi) = \int_0^a K_2(\xi, s, \nu(s))ds, \tag{4.1}$$

where K_1, K_2: $[0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ are continuous with $a > 0$. Let $U = C([0, a], \mathbb{R})$ be the Banach space of all continuous functions defined on $[0, a]$ and endowed with the usual supremum norm:

$$\|\mu\|_\infty = \max_{\xi \in [0,a]} |\mu(\xi)|, \ \text{ where } \mu \in C([0, a], \mathbb{R}),$$

and the induced metric (U, δ) is defined by

$$\delta(\mu, \nu) = \|\mu - \nu\|_\infty$$

for all $\mu, \nu \in U$. Now, we are in the position to present the integral type application to support our work.

Theorem 4.1. *Suppose that the following hypotheses are satisfied:*

(1) Let K_1, K_2: $[0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ be continuous; for $\mu, \nu \in U$ let $B_\mu, B_\nu \in U$ be defined as

$$B_\mu(\xi) = \int_0^a K_1(\xi, s, \mu(s))ds \quad \textit{and} \quad B_\nu(\xi) = \int_0^a K_2(\xi, s, \nu(s))ds. \tag{4.2}$$

Suppose that there exists a mapping

$$\Gamma : [0, a] \times [0, a] \to [0, +\infty) \textit{ with } \Gamma(\xi, \cdot) \in L^1([0, a])$$

for all $\xi \in [0, a]$ such that

$$|K_1(\xi, s, \mu(s)) - K_2(\xi, s, \nu(s))| \le \Gamma(\xi, s)N^*(\mu, \nu), \quad \forall \mu, \nu \in U, \textit{ and } \xi, s \in [0, a],$$

where

$$N^*(\mu(s), \nu(s)) = N^*(\mu, \nu) = \min\left\{\|\mu - \nu\|_\infty, \max\left\{\begin{array}{c} \|B_\mu - \mu\|_\infty, \|B_\nu - \nu\|_\infty, \\ \|B_\mu - \nu\|_\infty, \|B_\nu - \mu\|_\infty \end{array}\right\}\right\}. \tag{4.3}$$

(2) Suppose also that

$$|K_\mu(\xi, s, \mu(s))| \le \Gamma(\xi, s)|\mu(s)|, \textit{ and } |K_\nu(\xi, s, \nu(s))| \le \Gamma(\xi, s)|\nu(s)|, \ \forall \mu, \nu \in U.$$

(3) Suppose further that there exists $\beta \in (0, 1)$ such that

$$\beta N^*(\mu, \nu) \in s(A, B) \quad \textit{for } \mu \in A, \ \nu \in B, \quad \textit{and } A, B \subseteq CB(U) \tag{4.4}$$

where $\sup_{\xi \in [0,a]} \int_0^\xi \Gamma(\xi, s)ds = \beta < 1.$

(4) Finally, suppose that there exists $\mu_0 \in U$ such that

$$\mu_0 \leq \int_0^a K_1(\xi, s, \mu_0(s))ds, \ \forall \ \xi \in [0, a].$$

Then the integral equations in (4.1) *have a common solution in U.*

Proof. Define the integral operators $S, T\colon U \to CB(U)$ by

$$B_\mu(\xi) \in S\mu(\xi) = A \ \text{ and } \ B_\nu(\xi) \in T\nu(\xi) = B, \tag{4.5}$$

for $\mu(\xi) \in A, \ \nu(\xi) \in B$ and $A, B \subseteq CB(U)$. Notice that S and T are well defined and the equations of (4.1) have a common solution if and only if S and T have a common solution, that is the CFP of the mappings S and T in U. Precisely, we have to prove that Theorem 3.7 is applicable to the operators defined in (4.5). Then, we may have the following two main cases:

(1) If $\|\mu - \nu\|_\infty$ is the minimum term in (4.3), then $N^*(\mu, \nu) = \|\mu - \nu\|_\infty$. Now, from (4.4) and (4.5), we have

$$\beta\|\mu - \nu\|_\infty = \beta\delta(\mu, \nu) \in s(A, B) = s(S\mu, T\nu) \ \text{ for } \mu \in A, \ \nu \in B \text{ and } A, B \subseteq CB(U). \tag{4.6}$$

The integral operators defined in (4.5) satisfy all of the hypotheses of Theorem 3.7 with $\beta = b_1$ and $b_2 = 0$ in (3.11). Thus, the integral equations in (4.1) have a common solution in U.

(2) If $\max\left\{\|B_\mu - \mu\|_\infty, \|B_\nu - \nu\|_\infty, \|B_\mu - \nu\|_\infty, \|B_\nu - \mu\|_\infty\right\}$ is the minimum term in (4.3), then

$$N^*(\mu, \nu) = \max\left\{\|B_\mu - \mu\|_\infty, \|B_\nu - \nu\|_\infty, \|B_\mu - \nu\|_\infty, \|B_\nu - \mu\|_\infty\right\}. \tag{4.7}$$

Then again we may have the following four subcases:

(i) If $\|B_\mu - \mu\|_\infty$ is the maximum term in (4.7), then $N^*(\mu, \nu) = \|B_\mu - \mu\|_\infty$. Now, from (4.4) and (4.5), we have

$$\beta\|B_\mu - \mu\|_\infty \in s\left(\delta(\mu, A)\right) \in s(A, B) = s(S\mu, T\nu) \ \text{ for } \mu \in A, \ \nu \in B \text{ and } A, B \subseteq CB(U). \tag{4.8}$$

(ii) If $\|B_\nu - \nu\|_\infty$ is the maximum term in (4.7), then $N^*(\mu, \nu) = \|B_\nu - \nu\|_\infty$. Now, from (4.4) and (4.5), we have

$$\beta\|B_\nu - \nu\|_\infty \in s\left(\delta(\nu, B)\right) \in s(A, B) = s(S\mu, T\nu) \ \text{ for } \mu \in A, \ \nu \in B \text{ and } A, B \subseteq CB(U). \tag{4.9}$$

(iii) If $\|B_\mu - \nu\|_\infty$ is the maximum term in (4.7), then $N^*(\mu, \nu) = \|B_\mu - \nu\|_\infty$. Now, from (4.4) and (4.5), we have

$$\beta\|B_\mu - \nu\|_\infty \in s\left(\delta(\nu, A)\right) \in s(A, B) = s(S\mu, T\nu) \ \text{ for } \mu \in A, \ \nu \in B \text{ and } A, B \subseteq CB(U). \tag{4.10}$$

(iv) If $\|B_\nu - \mu\|_\infty$ is the maximum term in (4.7), then $N^*(\mu, \nu) = \|B_\nu - \mu\|_\infty$. Now, from (4.4) and (4.5), we have

$$\beta\|B_\nu - \mu\|_\infty \in s\left(\delta(\mu, A)\right) \in s(A, B) = s(S\mu, T\nu) \ \text{ for } \mu \in A, \ \nu \in B \text{ and } A, B \subseteq CB(U). \tag{4.11}$$

Hence, from (4.8)–(4.11), the integral operators S and T, satisfy all of the hypotheses of Theorem 3.7 with $\beta = b_2$ and $b_1 = 0$ in (3.11). Thus, the integral equations in (4.1) have a common solution in U.

5. Conclusions

In this paper, we have proved some new types of multi-valued contraction results for a pair of multi-valued mappings on CM-spaces. In support of our work, we presented an illustrative example. Our main results improved and modified many results published in the last few decades. In addition, we established a supportive application of nonlinear integral equations to unify our work. This new theory will play a very good role in the theory of FPs. This new concept has a potency to modify in different directions and prove different types of multi-valued contraction results for FPs, CFPs and coincidence points in the context of different types of M-spaces with different types of nonlinear integral equations and differential equations. Furthermore, we shall present a problem, i.e., whether the said theory in this paper is applicable or not to the theory of fractional derivatives (especially in the sense of Abu-Shady and Kaabar [44, 45]).

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Conflict of interest

The authors declare that they have no conflicts of interest.

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