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Research article

Oscillation results for a nonlinear fractional differential equation

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Abstract: In this paper, the authors work with a general formulation of the fractional derivative of Caputo type. They study oscillatory solutions of differential equations involving these general fractional derivatives. In particular, they extend the Kamenev-type oscillation criterion given by Baleanu et al. in 2015. In addition, we prove results on the existence and uniqueness of solutions for many of the equations considered. Also, they complete their study with some examples.

Keywords: fractional derivatives and integrals; fractional differential equations; oscillatory differential equations
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1. Introduction

Fractional calculus has its foundations in classical calculus; however, its accelerated development in recent years has created its own space within pure and applied mathematics. For an epistemological study of fractional calculus and its theoretical-practical applications see [1–5]. Also, the interested reader may consult [6–10] for an illustrative description of boundary layers problems within the scope of fractional calculus, the formulation of the notion of fractional derivatives with general analytic kernels (e.g., the AB fractional model), a study about categorization for systems of fractional differential algebraic equations, the solvability of such systems, and others related approaches.

Fractional differential operators and, in particular, fractional differential equations is an area of research of growing interest mainly because of its theoretical scope and its applications to real-world problems.

In the same direction, this work relies on the use of a Caputo-type differential operator, depending on a general kernel function, which includes well-studied and well known derivatives. We will use a kernel F in order to define general fractional derivatives of Caputo type

$$^{C}D^{\alpha}_{F,a}.$$

Moreover, we study oscillatory solutions of differential equations involving these general fractional derivatives. One of the virtues of this approach is that it allows results to be obtained in a unified way for many fractional derivatives, e.g., the classical Caputo derivative, and Caputo-Fabricio and Atangana-Baleanu extensions. Theorem 10 extends the Kamenev-type oscillation criterion given by Baleanu et al. in [11, Theorem 1] to a very general framework. In particular, Theorem 17 has the same conclusion of [11, Theorem 1] with weaker hypothesis. Besides, we obtain Theorem 13, which is a new result of Kamenev-type for the case of ordinary differential equations of order n. Also, we prove results on existence and uniqueness of solutions for the equations considered. Finally, we complete our study with some numerical examples.

2. Caputo type operator derivative

Michele Caputo proposes a new fractional derivative in [12]. This definition has an important property associated with the resolution of differential equations, since it is not necessary to define the initial conditions of fractional order. Multiple applications of the so-called Caputo differential operator can be found in [13].

The *Caputo derivative* of a differentiable function f of order $0 < \alpha < 1$ is defined as

$$^{C}D^{\alpha}_{a}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds. \tag{2.1}$$

An extension of $^{C}D^{\alpha}$ is the so-called *Caputo-Fabricio derivative* (see [14, 15]), given by:

$$^{CF}D^{\alpha}_{a}f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s)e^{-\frac{\alpha(t-s)}{1-\alpha}}\, ds, \tag{2.2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. A more recent extension is the *Atangana-Baleanu derivative*, defined in [2] by

$$^{ABC}D^{\alpha}_{a}f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s)E_{\alpha}\Big(-\frac{\alpha(t-s)^{\alpha}}{1-\alpha}\Big)ds, \tag{2.3}$$

where,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

is the *Mittag-Leffler function*. Note that $E_1(z) = e^z$.

Other recent and relevant research to be taken into account are [16–20].

Definition 1. *We say that F is an* admissible kernel *for the interval* $[a, b]$ *if* $F : [0, b-a]\times(0, 1) \to [0, \infty)$ *is a non-negative continuous function such that*

$$\int_0^{b-a} \frac{ds}{F(s, \alpha)} < \infty$$

for each $\alpha \in (0, 1)$*. Also, F is an* admissible kernel *for* $[a, \infty)$ *if it is admissible for* $[a, b]$ *for every* $b > a$*.*

Next, we present our definition of the generalized Caputo derivative.

Definition 2. *Let* $\alpha \in (0, 1)$*, F be an admissible kernel for* $[a, b]$*,* $f : [a, b] \to \mathbb{R}$ *be a differentiable function and t in* $[a, b]$*. The* generalized Caputo derivative *of order α of the function f at the point t is*

$$ {}^{C}D^{\alpha}_{F,a} f(t) = \int_a^t \frac{f'(s)}{F(t-s, \alpha)}\, ds. \tag{2.4}$$

Remark 3. *If* $F(x, \alpha) = \Gamma(1-\alpha)\, x^{\alpha}$*, then we obtain the classical Caputo derivative. Similarly, we can obtain the kernels for Caputo-Fabricio and Atangana-Baleanu extensions. Hence, each result for* ${}^{C}D^{\alpha}_{F,a}$ *has as a consequence the same result for the classical Caputo derivative, and Caputo-Fabricio and Atangana-Baleanu extensions. This shows that the generalized Caputo derivative is quite general and unifying.*

The following integral operator is associated to the generalized Caputo derivative in a natural way.

Definition 4. *Let* $\alpha \in (0, 1)$*, F be an admissible kernel for* $[a, b]$*,* $f : [a, b] \to \mathbb{R}$ *be a differentiable function and t in* $[a, b]$*. The* generalized Caputo integral operator *of order α of the function f at the point t is*

$$ {}^{C}J^{\alpha}_{F,a} f(t) = \int_a^t \frac{f(s)}{F(t-s, \alpha)}\, ds.$$

Hence,

$$ {}^{C}D^{\alpha}_{F,a} f(t) = {}^{C}J^{\alpha}_{F,a} f'(t).$$

Next, we will state some properties of the generalized Caputo derivative and its associated integral operator.

First of all, it is clear that they are non-local linear operators. Also, the following bounds hold.

Proposition 5. *Let* $\alpha \in (0, 1)$*,* $x \in [a, b]$*, and F be an admissible kernel for* $[a, b]$*. If f is a differentiable function on* $[a, x]$ *and* $\mathbb{F}_\alpha = \min_{y\in[0,x-a]} F(y, \alpha) > 0$*, then*

$$\left\| {}^{C}J^{\alpha}_{F,a} f \right\|_{L^\infty[a,x]} \le \frac{1}{\mathbb{F}_\alpha} \|f\|_{L^1[a,x]} ,$$

$$\left\| {}^{C}D^{\alpha}_{F,a} f \right\|_{L^\infty[a,x]} \le \frac{1}{\mathbb{F}_\alpha} \|f'\|_{L^1[a,x]} .$$

Proof. Fix $t \in [a, x]$. Since $F \ge 0$, we have

$$\begin{aligned} \left| {}^{C}J^{\alpha}_{F,a} f(t) \right| &= \left| \int_a^t \frac{f(s)}{F(t-s, \alpha)}\, ds \right| \le \int_a^t \frac{|f(s)|}{\min_{y\in[0,t-a]} F(y, \alpha)}\, ds \\ &\le \int_a^x \frac{|f(s)|}{\min_{y\in[0,x-a]} F(y, \alpha)}\, ds = \int_a^x \frac{|f(s)|}{\mathbb{F}_\alpha}\, ds = \frac{1}{\mathbb{F}_\alpha} \|f\|_{L^1[a,x]} . \end{aligned}$$

Since the inequality holds for every $t \in [a,x]$, we conclude

$$\left\| {}^{C}J^{\alpha}_{F,a} f \right\|_{L^\infty[a,x]} \leq \frac{1}{\mathbb{F}_\alpha} \|f\|_{L^1[a,x]} .$$

The second inequality follows from the first one and the equality

$${}^{C}D^{\alpha}_{F,a} f(t) = {}^{C}J^{\alpha}_{F,a} f'(t).$$

□

By applying Proposition 5 to the function $f - g$, we obtain the following result.

Proposition 6. *Let $\alpha \in (0,1)$, $x \in [a,b]$, and F be an admissible kernel for $[a,b]$. If f, g are differentiable functions on $[a,x]$ and $\mathbb{F}_\alpha = \min_{y\in[0,x-a]} F(y,\alpha) > 0$, then*

$$\left\| {}^{C}J^{\alpha}_{F,a} f - {}^{C}J^{\alpha}_{F,a} g \right\|_{L^\infty[a,x]} \leq \frac{1}{\mathbb{F}_\alpha} \|f - g\|_{L^1[a,x]} ,$$
$$\left\| {}^{C}D^{\alpha}_{F,a} f - {}^{C}D^{\alpha}_{F,a} g \right\|_{L^\infty[a,x]} \leq \frac{1}{\mathbb{F}_\alpha} \|f' - g'\|_{L^1[a,x]} .$$

Definition 7. *Let $n \in \mathbb{Z}^+$, $\alpha \in (n-1,n)$, $t \in [a,b]$, and F be an admissible kernel for $[a,b]$. For a n times differentiable function $f : [a,b] \to \mathbb{R}$, the* generalized Caputo derivative *of f of order α at t is*

$${}^{C}D^{\alpha}_{F,a} f(t) = \int_a^t \frac{f^{(n)}(s)}{F(t-s,\alpha+1-n)}\, ds. \tag{2.5}$$

The following interesting composition property follows from Definition 7.

Proposition 8. *Let $\alpha \in (0,1)$, $n \in \mathbb{Z}^+$, and F be an admissible kernel for $[a,b]$. If f is $(n+1)$-differentiable function on $[a,b]$, then*

$${}^{C}D^{\alpha+n}_{F,a} f(t) = {}^{C}D^{\alpha}_{F,a} f^{(n)}(t),$$

for every $t \in [a,b]$.

Proof. Since $\alpha \in (0,1)$ and $n \in \mathbb{Z}^+$, we have $\alpha + n \in (n, n+1)$ and

$${}^{C}D^{\alpha+n}_{F,a} f(t) = \int_a^t \frac{f^{(n+1)}(s)}{F(t-s,\alpha+n-n)}\, ds = \int_a^t \frac{(f^{(n)})'(s)}{F(t-s,\alpha)}\, ds = {}^{C}D^{\alpha}_{F,a} f^{(n)}(t).$$

□

Note that the equality in Proposition 8 is interesting, since we write ${}^{C}D^{\alpha+n}_{F,a}$ as a composition of a local operator and a non-local operator.

3. On the Kamenev type oscillation for a nonlinear generalized differential equation

Probably the beginning of the oscillation theory as an independent mathematical area, can be located almost 90 years ago. In 1931 Andronov settled down in Gorky, far away from Moscow, where a small institute of radiophysics was leading this field. The varied tools, highly theoretical, used and developed by Andronov and his collaborators (specific applications, recurrences, bifurcations, critical cases, stability, and the famous notion of "systémes grossiers") were all intended for applications. Most of these results, together with their context of use, were collected in the book [21]. It is worth mentioning another school which was founded in Kiev, in the 1930s, and whose most reputed representatives were Nikolai M. Krylov and Nikolai M. Bogoliubov. From these years, with the increase in military and space applications, oscillation theory became a field of maximum interest and development. The study of the existence of oscillatory solutions and the oscillation of all solutions of a differential equation, or system became central research problems.

We say that a solution of a differential equation on the interval $[t_0, \infty)$ is *oscillatory* if its zeros in $[t_0, \infty)$ are an unbounded set. If every non-trivial solution of the differential equation is oscillatory, we say that this equation is oscillatory.

One of the best known results in oscillation theory is the Kamenev criterion for the second order linear differential equation

$$y'' + q(t)y = 0, \qquad t > 0. \tag{3.1}$$

Kamenev states (see [22]), generalizing some previous results, that if

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty, \tag{3.2}$$

for $n > 2$ and $t_0 > 0$, then (3.1) is oscillatory.

There are some results on oscillation of fractional differential equations with Caputo derivative with $0 < \alpha < 1$ (see [23–25] and the references cited therein).

Therefore, it is of theoretical importance, to extend known results for the ordinary case to fractional differential equations of Caputo type of any order.

Our goal is to present a Kamenev type oscillation criterion in the framework of generalized fractional differential equations.

We will consider the generalized non-linear differential equation

$$\left({}^{C}D^{\alpha}_{F,a} y(t)\right)' + q(t)\, r(y(t)) = P(t, y(t), {}^{C}D^{\alpha}_{F,a} y(t)), \tag{3.3}$$

where P, q and r are appropriate functions (see the statement of Theorem 10 below).

In [11] there is a Kamenev type result for the hybrid differential equation of order $(1 + \alpha)$

$$(y^{(\alpha)})' + q(t)y = 0, \qquad t > 0, \tag{3.4}$$

with $\alpha \in (0, 1)$, in the framework of the classic Caputo derivative.

Theorem 9. *[11, Theorem 1] Fix* $\alpha \in (0, 1)$. *If*

$$\limsup_{t\to\infty} \frac{1}{t^{\varepsilon}} \int_{t_0}^{t} (t-s)^{\varepsilon}\, q(s)\, ds = \infty,$$

for some $\varepsilon > 2$ *and* $t_0 > 0$*, then any solution of* (3.4) *either oscillates or satisfies the inequality*

$$\liminf_{t\to\infty} y^{(\alpha)}(t)\left(y'(t) - y^{(\alpha)}(t)\right) \leq 0. \tag{3.5}$$

More precisely, if (3.5) *holds, then there exists an increasing sequence* $\{t_n\}$ *in* $[t_0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and*

$$y^{(\alpha)}(t_n)\left(y'(t_n) - y^{(\alpha)}(t_n)\right) < 0$$

for every n.

Note that the [25] studies oscillatory solutions of non-hybrid Caputo differential equations. In [23,24] appear hybrid Caputo differential equations, but these equations are very different from (3.4) and (3.7) below. The papers [26,27] study oscillatory solutions of non-hybrid conformable and Caputo delay differential equations, respectively. In [28] appear hybrid Liouville differential equations. See also the survey [29].

Here, we generalize Theorem 9 to the framework of the Caputo generalized derivative ${}^{C}D^{\alpha}_{F,a}$. In particular, Theorem 10 below improves Theorem 9 in the following ways:

(1) We consider more general operators, which include the Caputo generalized derivative.

(2) We replace the hypothesis $\alpha \in (0, 1)$ by $\alpha \in (0, \infty)$.

(3) We replace the function t^{ε} with $\varepsilon > 2$ by a function $B(t)$ in a large class of functions.

(4) This class of functions for $B(t)$ includes the functions t^{ε} with $\varepsilon > 1$ and $t\big(\log(1+t)\big)^{\sigma}$ with $\sigma > 0$.

(5) We replace $q(t)y$ by $q(t)r(y(t))$, where r belongs to a large class of functions.

(6) We allow an appropriate non-homogeneous term.

We say that a function $B : [0, \infty) \to \mathbb{R}$ is *proper* if it is strictly increasing, absolutely continuous, $B(0) = 0$, and for each T large enough there exists M such that

$$\frac{1}{B(t)} \int_T^t \frac{B'(t-s)^2}{B(t-s)}\, ds \leq M$$

for every $t \geq T + 1$.

Let us state the main result of this paper, in a very general framework.

Theorem 10. *Let* $S, S_1, \ldots, S_k$ *be (linear or non-linear) operators such that* $S f, S_1 f, \ldots, S_k f$ *are functions defined on* $[t_0, \infty)$ *for each smooth enough function* f *on* $[t_0, \infty)$*. Let B be a proper function and let* $q : [t_0, \infty) \to \mathbb{R}$ *be a continuous function satisfying*

$$\limsup_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^t B(t-s)\, q(s)\, ds = \infty. \tag{3.6}$$

Let $r : \mathbb{R} \to \mathbb{R}$ *be a differentiable function with* $r(x) \neq 0$ *if* $x \neq 0$*, and let* $P : [t_0, \infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ *be a continuous function satisfying either: (a)* $P \equiv 0$ *or (b)* $r \geq 0$ *and* $P \leq 0$*.*

Then each non-trivial differentiable solution of

$$(S y(t))' + q(t)\, r(y(t)) = P(t, y(t), S_1 y(t), \ldots, S_k y(t)), \tag{3.7}$$

on $[t_0, \infty)$ *either oscillates or satisfies*

$$\liminf_{t\to\infty} S y(t)\left((r(y(t)))' - S y(t)\right) \leq 0. \tag{3.8}$$

More precisely, if (3.8) *holds, then there exists an increasing sequence* $\{t_n\}$ *in* $[t_0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and*

$$S\,y(t_n)\Big((r(y(t_n)))' - S\,y(t_n)\Big) < 0$$

for every n.

Remark 11. *Obviously, if* S *or* S_j *is the derivative of order n or some fractional derivative of order* $\alpha \in (n-1, n)$*, then* f *must be n times differentiable in order to consider* $S\,f$ *or* $S_j f$*.*

Proof. Let us consider a non-trivial solution $y(t)$ of (3.7) on $[t_0,\infty)$, and assume that $y(t)$ is nonoscillatory, say $y(t) > 0$ for $t \ge t_0$.

For the sake of a contradiction, assume that there exists $T \ge t_0$ with

$$y(t) \ne 0 \qquad \text{and} \qquad S\,y(t)\Big((r(y(t)))' - S\,y(t)\Big) \ge 0$$

for every $t \ge T$.

Since $r(x) \ne 0$ if $x \ne 0$, we can define for any $t \ge T$ the function

$$w(t) = \frac{S\,y(t)}{r(y(t))}\,.$$

If $P \equiv 0$, then (3.7) gives

$$\frac{(S\,y(t))'}{r(y(t))} = -q(t).$$

If $r \ge 0$ and $P \le 0$, then

$$(S\,y(t))' = -q(t)\,r(y(t)) + P(t, y(t), S_1 y(t), \dots, S_k y(t)) \le -q(t)\,r(y(t)),$$

$$\frac{(S\,y(t))'}{r(y(t))} \le -q(t), \tag{3.9}$$

for every $t \ge T$. Hence, (3.9) holds for every $t \ge T$ in both cases. Since r, y and $S\,y$ are differentiable functions, we have

$$w'(t) = \frac{r(y(t))(S\,y(t))' - (r(y(t)))'\,S\,y(t)}{r(y(t))^2} \le -q(t) - \frac{(r(y(t)))'\,S\,y(t)}{r(y(t))^2}$$

and

$$\begin{aligned} w'(t) + w(t)^2 + q(t) &\le \frac{-(r(y(t)))'\,S\,y(t)}{r(y(t))^2} + \frac{(S\,y(t))^2}{r(y(t))^2} \\ &= \frac{-S\,y(t)[(r(y(t)))' - S\,y(t)]}{r(y(t))^2} \le 0 \end{aligned}$$

for every $t \ge T$. Since $B(0) = 0$, $B(x) > 0$ for any $x > 0$, and $B'(x) \ge 0$ for almost every $x \ge 0$, the previous inequality and an integration by parts gives

$$\begin{aligned} \int_T^t B(t-s)\,q(s)\,ds &\le -\int_T^t B(t-s)\,w'(s)\,ds - \int_T^t B(t-s)\,w(s)^2\,ds \\ &= w(T)B(t-T) - \int_T^t B'(t-s)\,w(s)\,ds - \int_T^t B(t-s)\,w(s)^2\,ds \\ &\le |w(T)|B(t-T) + \int_T^t B'(t-s)\,|w(s)|\,ds - \int_T^t B(t-s)\,w(s)^2\,ds, \end{aligned}$$

for $t \geq T$. This inequality and

$$\begin{aligned} B'(t-s)\,|w(s)| - B(t-s)\,w(s)^2 &= -B(t-s)\Big(|w(s)| - \frac{B'(t-s)}{2B(t-s)}\Big)^2 + \frac{B'(t-s)^2}{4B(t-s)} \\ &\leq \frac{B'(t-s)^2}{4B(t-s)}, \end{aligned}$$

give

$$\begin{aligned} \int_T^t B(t-s)\,q(s)\,ds \leq |w(T)|B(t-T) & \\ &+ \int_T^t B'(t-s)\,|w(s)|\,ds - \int_T^t B(t-s)\,w(s)^2\,ds \\ &\leq |w(T)|B(t-T) + \int_T^t \frac{B'(t-s)^2}{4B(t-s)}\,ds. \end{aligned}$$

Since B is admissible, without loss of generality we can assume that T is large enough and there exists a constant M such that

$$\frac{1}{B(t)} \int_T^t \frac{B'(t-s)^2}{B(t-s)}\,ds \leq M$$

for every $t \geq T+1$. Since B is an increasing function, we conclude

$$\begin{aligned} \frac{1}{B(t)} \int_T^t B(t-s)\,q(s)\,ds &\leq |w(T)|\,\frac{B(t-T)}{B(t)} + \frac{1}{B(t)} \int_T^t \frac{B'(t-s)^2}{4B(t-s)}\,ds \\ &\leq |w(T)| + \frac{1}{4}M \end{aligned}$$

for every $t \geq T+1$. Note that

$$\begin{aligned} \Big|\frac{1}{B(t)} \int_{t_0}^t B(t-s)\,q(s)\,ds - \frac{1}{B(t)} \int_T^t B(t-s)\,q(s)\,ds\Big| &= \Big|\frac{1}{B(t)} \int_{t_0}^T B(t-s)\,q(s)\,ds\Big| \\ &\leq \int_{t_0}^T \frac{B(t-s)}{B(t)}\,|q(s)|\,ds \leq \int_{t_0}^T |q(s)|\,ds < \infty \end{aligned}$$

for every $t \geq T$. Hence,

$$\frac{1}{B(t)} \int_{t_0}^t B(t-s)\,q(s)\,ds \leq |w(T)| + \frac{1}{4}M + \int_{t_0}^T |q(s)|\,ds,$$

for every $t \geq T+1$, and this contradicts (3.6). □

We can obtain now the result for the Caputo generalized derivative.

Theorem 12. *Let $a, t_0 \in \mathbb{R}$ with $a \leq t_0$, $\alpha > 0$, F be an admissible kernel for $[a, \infty)$, B be a proper function, and $q : [t_0, \infty) \to \mathbb{R}$ be a continuous function satisfying*

$$\limsup_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^t B(t-s)\,q(s)\,ds = \infty. \tag{3.10}$$

Let $S_1, \ldots, S_k$ be linear operators that are compositions of classical derivatives (of any order) and/or generalized Caputo derivatives (of any order). Let $r : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $r(x) \neq 0$ if $x \neq 0$, and let $P : [t_0, \infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function satisfying either $P \equiv 0$ or $r \geq 0$ and $P \leq 0$.

Then each solution of

$$\left({}^{C}D^{\alpha}_{F,a} y(t)\right)' + q(t)\, r(y(t)) = P(t, y(t), S_1 y(t), \ldots, S_k y(t)), \tag{3.11}$$

on $[t_0, \infty)$ either oscillates or satisfies the inequality

$$\liminf_{t\to\infty} {}^{C}D^{\alpha}_{F,a} y(t) \left((r(y(t)))' - {}^{C}D^{\alpha}_{F,a} y(t)\right) \leq 0. \tag{3.12}$$

More precisely, if (3.12) holds, then there exists an increasing sequence $\{t_n\}$ in $[t_0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$${}^{C}D^{\alpha}_{F,a} y(t_n) \left((r(y(t_n)))' - {}^{C}D^{\alpha}_{F,a} y(t_n)\right) < 0$$

for every n.

Proof. Let us consider a solution $y(t)$ of (3.11) on $[t_0, \infty)$. Since $y \equiv 0$ satisfies (3.12), we can assume that y is not the function $y \equiv 0$. Then it suffices to apply Theorem 10 with $S = {}^{C}D^{\alpha}_{F,a}$. □

Theorem 10 also gives the following result which is new even in the context of ordinary differential equations.

Theorem 13. *Let $t_0 \in \mathbb{R}$, B be a proper function, and $q : [t_0, \infty) \to \mathbb{R}$ be a continuous function satisfying*

$$\limsup_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^{t} B(t-s)\, q(s)\, ds = \infty. \tag{3.13}$$

Let $r : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $r(x) \neq 0$ if $x \neq 0$, and let $P : [t_0, \infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function satisfying either $P \equiv 0$ or $r \geq 0$ and $P \leq 0$.

Then each solution of

$$y^{(n)}(t) + q(t)\, r(y(t)) = P(t, y(t), y'(t), \ldots, y^{(k)}(t)),$$

on $[t_0, \infty)$ either oscillates or satisfies the inequality

$$\liminf_{t\to\infty} y^{(n-1)}(t) \left((r(y(t)))' - y^{(n-1)}(t)\right) \leq 0. \tag{3.14}$$

More precisely, if (3.14) holds, then there exists an increasing sequence $\{t_m\}$ in $[t_0, \infty)$ such that $\lim_{m\to\infty} t_m = \infty$ and

$$y^{(n-1)}(t_m) \left((r(y(t_m)))' - y^{(n-1)}(t_m)\right) < 0$$

for every m.

Note that in Theorem 13 there is no relation between n and k.

Next, we show some examples of proper functions.

Proposition 14. *The function $B(t) = t^{\varepsilon}\big(\log(1+t)\big)^{\sigma}$ is proper for each $\varepsilon \geq 1$ and $\sigma \geq 0$ with $\varepsilon + \sigma > 1$.*

Proof. It is clear that B is a strictly increasing absolutely continuous function. We have

$$\begin{aligned} B'(t) &= \varepsilon\, t^{\varepsilon-1}\big(\log(1+t)\big)^{\sigma} + t^{\varepsilon}\sigma\big(\log(1+t)\big)^{\sigma-1}\frac{1}{t+1} \\ &= t^{\varepsilon-1}\big(\log(1+t)\big)^{\sigma}\Big(\varepsilon+\sigma\,\frac{t}{(t+1)\log(1+t)}\Big). \end{aligned}$$

Since the function on $(0,\infty)$

$$R(t) = \frac{t}{(t+1)\log(1+t)}$$

is continuous and satisfies

$$\lim_{t\to 0^+} R(t) = 1, \qquad \lim_{t\to\infty} R(t) = 0,$$

R is positive and bounded. Hence, there exists a positive constant c such that

$$\varepsilon\, t^{\varepsilon-1}\big(\log(1+t)\big)^{\sigma} \le B'(t) \le c\, t^{\varepsilon-1}\big(\log(1+t)\big)^{\sigma}$$

for every $t \ge 0$.

Let us consider $t \ge T+1$. If $\varepsilon > 1$ and $\sigma \ge 0$, then

$$\begin{aligned} \frac{1}{B(t)}\int_T^t \frac{B'(t-s)^2}{B(t-s)}\,ds &\le \frac{c^2}{t^{\varepsilon}\big(\log(1+t)\big)^{\sigma}}\int_T^t \frac{(t-s)^{2\varepsilon-2}\big(\log(1+t-s)\big)^{2\sigma}}{(t-s)^{\varepsilon}\big(\log(1+t-s)\big)^{\sigma}}\,ds \\ &= \frac{c^2}{t^{\varepsilon}}\int_T^t (t-s)^{\varepsilon-2}\frac{\big(\log(1+t-s)\big)^{\sigma}}{\big(\log(1+t)\big)^{\sigma}}\,ds \\ &\le \frac{c^2}{t^{\varepsilon}}\int_T^t (t-s)^{\varepsilon-2}\,ds = \frac{c^2}{\varepsilon-1}\,\frac{(t-T)^{\varepsilon-1}}{t^{\varepsilon}} \\ &\le \frac{c^2}{\varepsilon-1}\,\frac{1}{t} \le \frac{c^2}{(\varepsilon-1)(T+1)}\,. \end{aligned}$$

If $\varepsilon = 1$ and $\sigma > 0$, then

$$\begin{aligned} \frac{1}{B(t)}\int_T^t \frac{B'(t-s)^2}{B(t-s)}\,ds &\le \frac{c^2}{t\big(\log(1+t)\big)^{\sigma}}\int_T^t (t-s)^{-1}\big(\log(1+t-s)\big)^{\sigma}\,ds \\ &= \frac{c^2}{\sigma+1}\,\frac{\big(\log(1+t-T)\big)^{\sigma+1}}{t\big(\log(1+t)\big)^{\sigma}} \\ &\le \frac{c^2}{\sigma+1}\,\frac{\log(1+t-T)}{t} \le \frac{c^2}{\sigma+1}\,. \end{aligned}$$

□

Proposition 15. *For each $\varepsilon > 1$, let us consider the function*

$$B(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ t\,(-\log t)^{-\varepsilon} & \text{if } 0 < t \le e^{-1}, \\ t & \text{if } t > e^{-1}. \end{cases}$$

Then $B(t)$ is a proper function.

Proof. It is clear that B is a strictly increasing absolutely continuous function.

For $t \geq T+1$, we have

$$\begin{aligned}\int_T^t \frac{B'(t-s)^2}{B(t-s)}\,ds &= \int_0^{t-T} \frac{B'(x)^2}{B(x)}\,dx = \int_0^{e^{-1}} \frac{B'(x)^2}{B(x)}\,dx + \int_{e^{-1}}^{t-T} \frac{B'(x)^2}{B(x)}\,dx \\ &= \int_0^{e^{-1}} \frac{\big((-\log x)^{-\varepsilon} + \varepsilon(-\log x)^{-\varepsilon-1}\big)^2}{x(-\log x)^{-\varepsilon}}\,dx + \int_{e^{-1}}^{t-T} \frac{1}{x}\,dx \\ &= \int_1^\infty \frac{\big(u^{-\varepsilon} + \varepsilon u^{-\varepsilon-1}\big)^2}{u^{-\varepsilon}}\,du + \log(t-T) + 1 \\ &= \int_1^\infty \big(u^{-\varepsilon} + 2\varepsilon u^{-\varepsilon-1} + \varepsilon^2 u^{-\varepsilon-2}\big)\,du + \log(t-T) + 1 \\ &= c_\varepsilon + \log(t-T),\end{aligned}$$

since $\varepsilon > 1$. Hence,

$$\frac{1}{B(t)} \int_T^t \frac{B'(t-s)^2}{B(t-s)}\,ds = \frac{c_\varepsilon + \log(t-T)}{t}$$

is a bounded function on $[T+1, \infty)$. □

Finally, we show some examples of functions q satisfying (3.6).

Proposition 16. *Let us consider $\varepsilon, \sigma, \beta, c, t_0, T \in \mathbb{R}$ with $\varepsilon \geq 1, \sigma \geq 0, c > 0, \beta > -1$ and $T \geq t_0$, and let $B(t) = t^\varepsilon(\log(1+t))^\sigma$ and $q : [t_0, \infty) \to \mathbb{R}$ be a continuous function satisfying $q(t) \geq c\, t^\beta$ for every $t \geq T$. Then*

$$\lim_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^t B(t-s)\, q(s)\,ds = \infty.$$

Proof. We have for $t \geq T$

$$\begin{aligned}\frac{1}{B(t)} \int_{t_0}^t B(t-s)\,q(s)\,ds &= \frac{1}{t^\varepsilon\big(\log(1+t)\big)^\sigma} \int_{t_0}^t (t-s)^\varepsilon\big(\log(1+t-s)\big)^\sigma q(s)\,ds \\ &\geq \frac{1}{t^\varepsilon\big(\log(1+t)\big)^\sigma} \int_{t_0}^T (t-s)^\varepsilon\big(\log(1+t-s)\big)^\sigma q(s)\,ds \\ &\quad + \frac{c}{t^\varepsilon(\log(1+t))^\sigma} \int_T^t (t-s)^\varepsilon(\log(1+t-s))^\sigma s^\beta\,ds \\ &\geq \frac{c}{(\log(1+t))^\sigma} \int_T^t \left(1-\frac{s}{t}\right)^\varepsilon (\log(1+t-s))^\sigma s^\beta\,ds \\ &\quad - \int_{t_0}^T \left(1-\frac{s}{t}\right)^\varepsilon \frac{(\log(1+t-s))^\sigma}{(\log(1+t))^\sigma} |q(s)|\,ds \\ &\geq \frac{ct^{\beta+1}}{(\log(1+t))^\sigma} \int_{T/t}^1 (1-x)^\varepsilon(\log(1+t-tx))^\sigma x^\beta dx - \int_{t_0}^T |q(s)|\,ds\,.\end{aligned}$$

If $t \geq T + e - 1$, then

$$\frac{T}{t} \leq 1 - \frac{e-1}{t} \leq 1.$$

If

$$\frac{T}{t} \le x \le 1 - \frac{e-1}{t} \qquad \Rightarrow \qquad \log(1 + t - tx) \ge 1$$

and so,

$$\begin{aligned}
&\frac{1}{B(t)} \int_{t_0}^{t} B(t-s)\, q(s)\, ds \\
&\quad \ge \frac{ct^{\beta+1}}{(\log(1+t))^{\sigma}} \int_{T/t}^{1} (1-x)^{\ell} (\log(1+t-tx))^{\sigma} x^{\beta} dx - \int_{t_0}^{T} |q(s)|\, ds \\
&\quad \ge \frac{ct^{\beta+1}}{(\log(1+t))^{\sigma}} \int_{T/t}^{1-(e-1)/t} (1-x)^{\varepsilon} (\log(1+t-tx))^{\sigma} x^{\beta}\, dx - \int_{t_0}^{T} |q(s)|\, ds \\
&\quad \ge \frac{ct^{\beta+1}}{(\log(1+t))^{\sigma}} \int_{T/t}^{1-(e-1)/t} (1-x)^{\varepsilon} x^{\beta}\, dx - \int_{t_0}^{T} |q(s)|\, ds\,.
\end{aligned}$$

Since

$$\lim_{t\to\infty} \int_{T/t}^{1-(e-1)/t} (1-x)^{\varepsilon} x^{\beta}\, dx = \int_{0}^{1} (1-x)^{\varepsilon} x^{\beta}\, dx > 0, \qquad \int_{t_0}^{T} |q(s)|\, ds < \infty,$$

and $\beta > -1$, we conclude

$$\lim_{t\to\infty} \frac{1}{B(t)} \int_{t_0}^{t} B(t-s)\, q(s)\, ds = \infty.$$

□

Theorem 12 and Proposition 14 also improve the result for the Caputo derivative in [11] in that we allow $\alpha > 0$ and $\varepsilon > 1$ instead of $\alpha \in (0,1)$ and $\varepsilon > 2$.

Theorem 17. *Fix $\alpha > 0$. If*

$$\limsup_{t\to\infty} \frac{1}{t^{\varepsilon}} \int_{t_0}^{t} (t-s)^{\varepsilon}\, q(s)\, ds = \infty$$

for some $\varepsilon > 1$ and $t_0 > 0$, then any solution of (3.4) *either oscillates or satisfies the inequality*

$$\liminf_{t\to\infty} y^{(\alpha)}(t) \left(y'(t) - y^{(\alpha)}(t) \right) \le 0. \tag{3.15}$$

More precisely, if (3.15) *holds, then there exists an increasing sequence $\{t_n\}$ in $[t_0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and*

$$y^{(\alpha)}(t_n) \left(y'(t_n) - y^{(\alpha)}(t_n) \right) < 0$$

for every n.

4. Existence and uniqueness

For the sake of completeness, although it is not possible to obtain an existence theorem for an equation as general as (3.7) in Theorem 10, we will prove now an existence and uniqueness result for the Eq (3.3) for the usual Caputo derivative. In fact, we start with a more general result.

In the Euclidean space $\mathbb{R}^k$, we are going to consider the infinity norm $|\cdot|_{\infty}$.

Theorem 18. *Consider $\alpha \in (0,1)$, $r, L_1, L_2 > 0$, $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$, $A_r = [a, a+r] \times \overline{B((x_0,y_0),r)}$ and continuous functions $f_1 : A_r \to \mathbb{R}^n$, $f_2 : A_r \to \mathbb{R}^m$ satisfying*

$$|f_j(t,x,y) - f_j(t,\tilde{x},\tilde{y})|_\infty \le L_j |(x,y) - (\tilde{x},\tilde{y})|_\infty$$

for $j = 1,2$ and every $(t,x,y), (t,\tilde{x},\tilde{y}) \in A_r$. Then there exists a unique solution of

$$\begin{aligned} x'(t) &= f_1(t,x(t),y(t)), \\ {}^C D_a^\alpha y(t) &= f_2(t,x(t),y(t)), \\ x(a) &= x_0, \quad y(a) = y_0. \end{aligned} \tag{4.1}$$

Proof. Since f_j is a continuous function, $|f_j(t,x,y)|_\infty \le M_j$ for every $(t,x,y) \in A_r$ and some constants M_j for $j = 1,2$. Let us consider the Banach space X of continuous functions $(x,y) : [a, a+T] \to \mathbb{R}^{n+m}$ with the usual infinity norm $\|\cdot\|_\infty$, where the constant $0 < T \le r$ will be appropriately chosen later.

It is well-known that

$$x'(t) = f_1(t,x(t),y(t)), \qquad x(a) = x_0,$$

is equivalent to the integral equation

$$x(t) = x_0 + \int_a^t f_1(s,x(s),y(s))\,ds =: R_1(x,y)(t).$$

Also, [30, Theorem 3.24] gives that

$${}^C D_{F,a}^\alpha y(t) = f_2(t,x(t),y(t)), \qquad y(a) = y_0,$$

is equivalent to the integral equation

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f_2(s,x(s),y(s))\,ds =: R_2(x,y)(t).$$

Consider the sequences of functions defined recursively as

$$\begin{aligned} x_{n+1}(t) &= R_1(x_n,y_n)(t), \\ y_{n+1}(t) &= R_2(x_n,y_n)(t), \end{aligned}$$

for $n \ge 0$, and let $R = (R_1, R_2)$. If

$$T \le \min\left\{ r, \frac{r}{M_1}, \left(\frac{r\Gamma(\alpha+1)}{M_2} \right)^{1/\alpha} \right\},$$

then

$$\begin{aligned} |x_1(t) - x_0|_\infty &= \left| \int_a^t f_1(s,x_0,y_0)\,ds \right|_\infty \le \int_a^t M_1\,ds \le M_1 T \le r, \\ |y_1(t) - y_0|_\infty &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f_2(s,x_0,y_0)\,ds \right|_\infty \\ &\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} M_2\,ds \le \frac{M_2 T^\alpha}{\Gamma(\alpha+1)} \le r, \end{aligned}$$

and so, $|(x_1(t), y_1(t)) - (x_0, y_0)|_\infty \le r$ for every $t \in [a, a+T]$ and $f_j(t, x_1(t), y_1(t))$ is defined for every $j = 1, 2$ and $t \in [a, a+T]$. Thus, (x_2, y_2) is well defined on $[a, a+T]$. If we assume that $|(x_n(t), y_n(t)) - (x_0, y_0)|_\infty \le r$ for every $t \in [a, a+T]$, then the previous argument gives that (x_{n+1}, y_{n+1}) is well defined on $[a, a+T]$. Hence, $(x_n, y_n) \in X$ for every $n \ge 0$.

Fix $0 < k < 1$ and define

$$T := \min\left\{r,\ \frac{r}{M_1}, \left(\frac{r\Gamma(\alpha+1)}{M_2}\right)^{1/\alpha}, \frac{k}{L_1}, \left(\frac{k\Gamma(\alpha+1)}{L_2}\right)^{1/\alpha}\right\}.$$

Thus,

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)|_\infty &= \left|\int_a^t \left(f_1(s, x_n(s), y_n(s)) - f_1(s, x_{n-1}(s), y_{n-1}(s))\right) ds\right|_\infty \\
&\le \int_a^t L_1 \left|(x_n(s), y_n(s)) - (x_{n-1}(s), y_{n-1}(s))\right|_\infty ds \\
&\le L_1 T \left\|(x_n, y_n) - (x_{n-1}, y_{n-1})\right\|_\infty \\
&\le k \left\|(x_n, y_n) - (x_{n-1}, y_{n-1})\right\|_\infty, \\
|y_{n+1}(t) - y_n(t)|_\infty &= \left|\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(f_2(s, x_n(s), y_n(s)) - f_2(s, x_{n-1}(s), y_{n-1}(s))\right) ds\right|_\infty \\
&\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} L_2 \left|(x_n(s), y_n(s)) - (x_{n-1}(s), y_{n-1}(s))\right|_\infty ds \\
&\le \frac{L_2 T^\alpha}{\Gamma(\alpha+1)} \left\|(x_n, y_n) - (x_{n-1}, y_{n-1})\right\|_\infty \\
&\le k \left\|(x_n, y_n) - (x_{n-1}, y_{n-1})\right\|_\infty,
\end{aligned}$$

and so,

$$\begin{aligned}
\left\|R(x_n, y_n) - R(x_{n-1}, y_{n-1})\right\|_\infty &= \left\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\right\|_\infty \\
&\le k \left\|(x_n, y_n) - (x_{n-1}, y_{n-1})\right\|_\infty
\end{aligned}$$

for every $n \ge 1$. Since $0 < k < 1$, the contractive mapping theorem gives that R has a unique fixed point, i.e., there exists a unique solution of (4.1). □

If $n = m = 1$ and $x(t) = {}^C D_a^\alpha y(t)$, Theorem 18 has the following consequences.

Theorem 19. *Consider $\alpha \in (0, 1)$, $r, L > 0$, $y_0, y_0^* \in \mathbb{R}$, $A_r = [a, a+r] \times \overline{B((y_0, y_0^*), r)}$ and a continuous function $f : A_r \to \mathbb{R}$ satisfying*

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \le L|(x, y) - (\tilde{x}, \tilde{y})|_\infty$$

for every $(t, x, y), (t, \tilde{x}, \tilde{y}) \in A_r$. Then there exists a unique solution of

$$\left({}^C D_a^\alpha y(t)\right)' = f(t, y(t), {}^C D_a^\alpha y(t)), \quad y(a) = y_0, \quad {}^C D_a^\alpha y(a) = y_0^*. \tag{4.2}$$

Theorem 20. *Consider $\alpha \in (0, 1)$, $r_0, L > 0$, $y_0, y_0^* \in \mathbb{R}$, $A_{r_0} = [a, a+r_0] \times \overline{B((y_0, y_0^*), r_0)}$ and functions $r \in C[a, a+r_0]$, $q \in C^1[y_0 - r_0, y_0 + r_0]$, and $P \in C(A_{r_0})$ satisfying*

$$|P(t, x, y) - P(t, \tilde{x}, \tilde{y})| \le L|(x, y) - (\tilde{x}, \tilde{y})|_\infty$$

for every $(t, x, y), (t, \tilde{x}, \tilde{y}) \in A_{r_0}$. *Then there exists a unique solution of*

$$\left({}^{C}D_{a}^{\alpha}y(t)\right)' + q(t)r(y(t)) = P(t, y(t), {}^{C}D_{a}^{\alpha}y(t)), \quad y(a) = y_0, \quad {}^{C}D_{a}^{\alpha}y(a) = y_0^*. \tag{4.3}$$

5. Numerical calculus

Numerical methods for the solution of the initial value problem for fractional differential equations (FDEs) with Caputo's derivative, formulated as

$$\begin{cases} {}^{C}D_{a}^{\alpha}y(t) = f(t, y(t)), \\ y(a) = y_0,\ y'(a) = y_0^{(1)}, \ldots, y^{(m-1)}(a) = y_0^{(m-1)}, \end{cases} \tag{5.1}$$

can be found in [31]. In particular, it establishes the explicit multi-step Euler method

$$Y_n = T_{m-1}[y; a](t_n) + h^{\alpha} \sum_{j=0}^{n-1} b_{n-j-1}^{(\alpha)} f(t_j, Y_j), \tag{5.2}$$

where

$$\begin{aligned} b_n^{(\alpha)} &= \frac{(n+1)^{\alpha} - n^{\alpha}}{\Gamma(\alpha+1)}, \\ T_{m-1}[y; a](t) &= \sum_{k=0}^{m-1} \frac{(t-a)^k}{k!} y^{(k)}(a). \end{aligned}$$

In this section, we extend the numerical methods for a mixed differential equation of the type (4.3), by using the standard procedure of making a change of variable to transform the differential equation of second order into a system of two differential equations of first order. In our case, we define $z(t) = {}^{C}D_{a}^{\alpha}y(t)$ and we obtain the following system of mixed differential equations:

$$\begin{cases} {}^{C}D_{a}^{\alpha}y(t) = z(t), \\ \quad z'(t) = P(t, y(t), z(t)) - q(t)r(y(t)), \\ \quad y(a) = y_0, \qquad z(a) = y_0^*. \end{cases} \tag{5.3}$$

Note that this system has the same structure as system (4.1). To numerically solve the system (5.3), we use a one-step method: the Runge Kutta method of order 4, for an auxiliary system of linear differential equations; and after each iteration at the n-th step, update the value of the function $y(t_n) = Y_n$ by using the multi-step method given by (5.2) for the Caputo derivative. Because of the persisting memory of fractional-order operators, multi-step methods are a natural choice, the number of steps involved in the computation increases as the proceeds forward, and the whole history of the solution is involved in the computation of each step. On the other hand, if the value of the parameter m is 1, then $T_{m-1}[y; a](t) = y(a) = y_0$.

The numerical algorithm is summarized as:

$$\begin{aligned} Z_0 &= y_0^*, \\ Y_0 &= y_0, \end{aligned}$$

for $n = 1, 2, \ldots,$

$Z_n = Z_{n-1} + \dfrac{h}{6}(A_1 + 2A_2 + 2A_3 + A_4),$

with

$$\left[\begin{array}{l} A_1 = P(t_{n-1}, Y_{n-1}, Z_{n-1}) - q(t_{n-1})r(Y_{n-1}), \\ B_1 = Z_{n-1}, \\ A_2 = P\left(t_{n-1} + \frac{h}{2}, Y_{n-1} + \frac{h}{2}B_1, Z_{n-1} + \frac{h}{2}A_1\right) - q(t_{n-1} + \frac{h}{2})r(Y_{n-1} + \frac{h}{2}B_1), \\ B_2 = Z_{n-1} + \frac{h}{2}A_1, \\ A_3 = P\left(t_{n-1} + \frac{h}{2}, Y_{n-1} + \frac{h}{2}B_2, Z_{n-1} + \frac{h}{2}A_2\right) - q(t_{n-1} + \frac{h}{2})r(Y_{n-1} + \frac{h}{2}B_2), \\ B_3 = Z_{n-1} + \frac{h}{2}A_2, \\ A_4 = P(t_{n-1} + h, Y_{n-1} + hB_3, Z_{n-1} + hA_3) - q(t_{n-1} + h)r(Y_{n-1} + hB_3), \\ B_4 = Z_{n-1} + A_3, \end{array}\right.$$

$$Y_n = Y_0 + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1}^{(\alpha)} Z_j.$$

end

Next we show the numerical results for the homogeneous equation

$$\left({}^C D_a^\alpha y(t)\right)' + t^\beta y(t) = 0, \quad y(0) = 0, \quad {}^C D_0^\alpha y(0) = 1,$$

and compare it with the numerical solution of the ordinary differential equation $y''(t) + t^\beta y(t) = 0$, with the same initial conditions. The graph in Figure 1 shows the result for the value of the parameter $\beta = 2$.

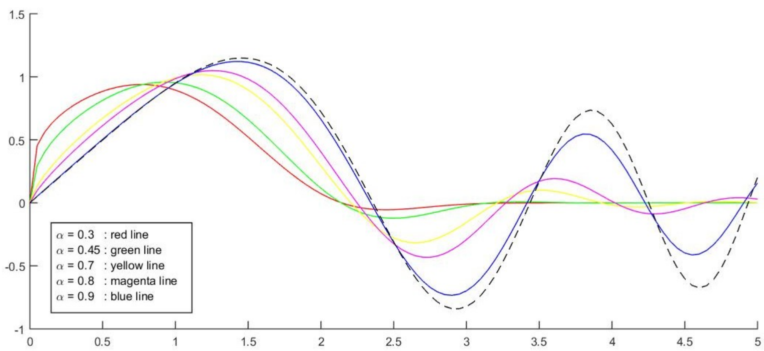
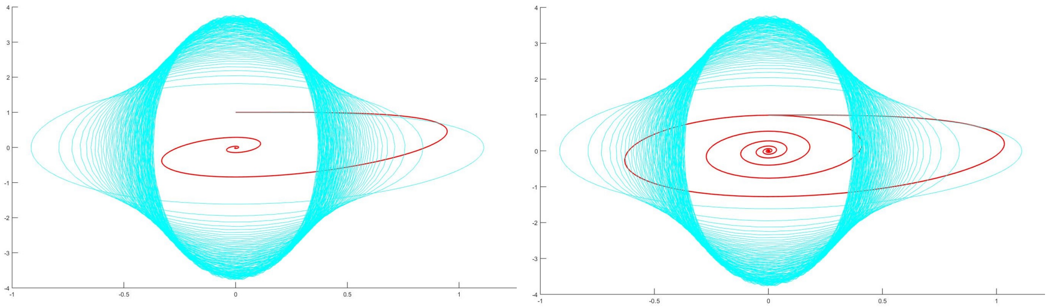


Figure 1. Numerical results for the homogeneous fractional order differential equation (β = 2).

In this graph, the function with dashed lines represents the solution of the standard ordinary differential equation. The solid lines are the numerical solutions of the differential equations for different values of the fractional order (the parameter α). In particular, it is observed that the blue line $\alpha = 0.9$ is very close to the standard solution, and for smaller values of the parameter α, the solution moves away. From the physical point of view, the alpha parameter can be interpreted in this context, as a shift and damping parameter.

To reinforce this damping idea, we perform another numerical experiment with $\beta = 1$ and the same initial conditions. This experiment is shown in the following two figures in the phase plane. In both cases, the trajectories of the ordinary differential equation (cyan line) are shown next to the trajectory given by the fractional order differential equation. A substantial difference is observed in terms of the speed of convergence to the attractor.



(a) For $\alpha = 0.7$ **(b)** For $\alpha = 0.9$

Figure 2. Phase plane with $\beta = 1$.

The following Figure 3 shows the same differential equation but with the value of the parameter $\beta = -0.6$ and the initial conditions $y(1) = 0.5$ and ${}^{C}D_{0}^{\alpha}y(1) = 1$.

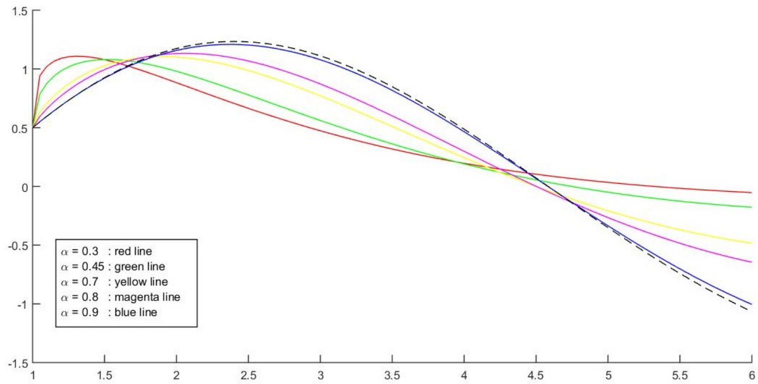


Figure 3. Numerical results for the homogeneous fractional order differential equation ($\beta = -0.6$).

As can be seen, the behavior of the numerical solutions of the fractional order differential equations remains the same as the previous case: while the fractional order approaches one, the graph of the solution of the fractional differential equation approaches the solution of the ordinary differential equation. On the other hand, it is interesting to note how the β parameter amplifies the oscillation frequency of the solutions.

Finally, for the case of the differential equation of fractional order

$$\left({}^{C}D_{a}^{\alpha}y(t)\right)' + t^{\beta}y^{2}(t) = 0, \quad y(0.5) = 0, \quad {}^{C}D_{0}^{\alpha}y(0.5) = 1,$$

of order $\alpha = 0.8$, Figure 4 shows how the numerical solutions vary for the different values of the β parameter.

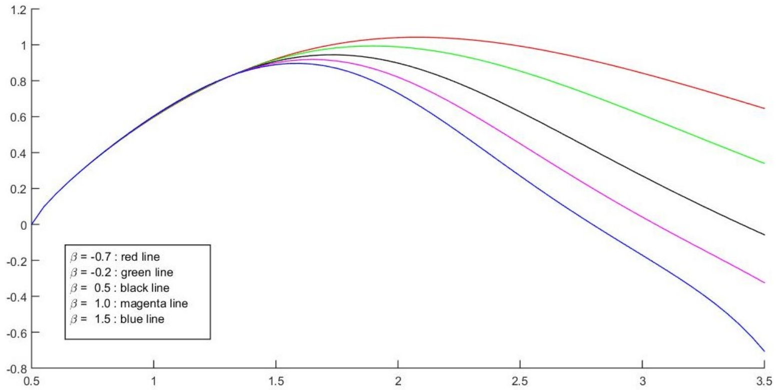


Figure 4. Numerical results for the homogeneous fractional order differential equation ($\alpha = 0.8$).

6. Conclusions

In this work, we present a generalized version of the Caputo fractional derivative and, on this basis, we obtain a Kamenev-type criterion for a very general hybrid differential equation, improving previous results.

In particular, Theorem 10 improves [11, Theorem 1] in the following ways: We consider more general operators, which include the Caputo generalized derivative. We replace the hypothesis $\alpha \in (0, 1)$ by $\alpha \in (0, \infty)$. We replace the function t^{ε} with $\varepsilon > 2$ by a function $B(t)$ in a large class of functions. This class of functions for $B(t)$ includes the functions t^{ε} with $\varepsilon > 1$ and $t\big(\log(1 + t)\big)^{\sigma}$ with $\sigma > 0$. We replace $q(t)y$ by $q(t)r(y(t))$, where r belongs to a large class of functions. We allow an appropriate non-homogeneous term.

As a consequence of Theorem 10, we obtain a new oscillation result for the case of ordinary differential equations of order n in Theorem 13.

Also, we prove results on the existence and uniqueness of solutions in many of the equations considered.

Finally, we complete our study with some numerical calculations that show the oscillation of some solutions.

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Conflicts of Interest:

The authors declare no conflicts of interest.

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