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Research article

Decision-making algorithm based on Pythagorean fuzzy environment with probabilistic hesitant fuzzy set and Choquet integral

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Abstract: The Pythagorean Probabilistic Hesitant Fuzzy (PyPHF) Environment is an amalgamation of the Pythagorean fuzzy set and the probabilistic hesitant fuzzy set that is intended for some unsatisfactory, ambiguous, and conflicting situations where each element has a few different values created by the reality of the situation membership hesitant function and the falsity membership hesitant function with probability. The decision-maker can efficiently gather and analyze the information with the use of a strategic decision-making technique. In contrast, ambiguity will be a major factor in our daily lives while gathering information. We describe a decision-making technique in the PyPHF environment to deal with such data uncertainty. The fundamental operating principles for PyPHF information under Choquet Integral were initially established in this study. Then, we put up a set of new aggregation operator names, including Pythagorean probabilistic hesitant fuzzy Choquet integral average and Pythagorean probabilistic hesitant fuzzy Choquet integral geometric aggregation operators. Finally, we explore a multi-attribute decision-making (MADM) algorithm based on the suggested operators to address the issues in the PyPHF environment. To demonstrate the work and contrast the findings with those of previous studies, a numerical example is provided. Additionally, the paper provides sensitivity analysis and the benefits of the stated method to support and reinforce the research.

Keywords: Pythagorean fuzzy information; aggregation operators; decision-making
Mathematics Subject Classification: 03B52, 03E72

List of abbreviation and symbols

Name	Abbreviation	Name	Abbreviation
Membership grade	MG	MG	$\Re$
Non membership grade	NMG	NMG	$\Im$
Fuzzy Set	FS	Decision Making	DM
Pythagorean fuzzy set	PyFS	Aggregation Operators	AOs
intuitionistic fuzzy set	IFS	Probability in MG	$\wp$
intuitionistic fuzzy number	IFN	Probability in NMG	¥
Pythagorean fuzzy number	PyFN	Hesitation	$\hbar$
Hesitant Fuzzy Set	HFS	Hesitation part in number	$\flat$
Probabilistic Hesitant Fuzzy Set	PHFS	Multiple criteria decision making	MCDM

1. Introduction

Teams in the surgery room include the surgeon(s), anesthesiologist, circulation nurse, and surgical technologists [1]. Unless expressly prohibited by state law or institutional policy, medical technicians are subordinate to and under the surgeon's direct control. Surgical technicians are largely responsible for the initial scrub job. Prior to an operation, they help set up sterile medical tools, supplies, and apparatus such as drapes, gowns, gloves, and suction tubing. They also take medications and solutions from the circulator. Surgical technologists build the sterile equipment, test it for performance, and make any necessary adjustments [2]. They help the surgeon prepare for surgery by putting on his or her uniform and equipment and by covering the patient with sterilized drapes to create a sterile field. In order to prevent the patient from developing a surgical site infection, the surgical technologist must anticipate the needs of the surgeon during the methodology by passing equipment and offering supplies like sponges, counting sponges, sharps, and instruments, giving the surgeon solutions and medications, receiving tissue samples to be passed off to the circulator, and making sure there are no breaks in sterile technique. Surgical technologists may carry out an assistant circulator's responsibilities in accordance with state law and/or hospital policy. As part of this task, a technician also assists in situating the patient on the operating table and does skin prep at the incision site. They also assist with patient transportation to the surgery room (s). The surgical technologist will provide clean bandages, aid in counts, change empty suction containers, and help move the patient from the operating table to the stretcher for transportation to the recovery area after the procedure [3]. They will also assist in getting the first scrub surgical technician any additional supplies, such as sponges or sutures, that they may want. There are several forms of surgical technology. Due to the advantages and disadvantages of different technologies, the surgical technologists are unable to select the one that would best suit their system and produce the best results. Fuzzy set theory is the only technique for making a decision based on expert data [4].

The classical logic has been expanded to fuzzy logic since Zadeh's innovative research [5], which is characterized by a membership grade (MG) in [0, 1] and offers a potent substitute to probability theory to represent imprecision, uncertainty, and obscureness in a variety of domains. The complexity of the data and the ambiguity of the human mind have led to the gradual realization that sometimes the MG of the fuzzy set is not sufficient to expose the characteristics of objects [6]. In order to fix this issue with the fuzzy set, Atanassov [7] introduced a non-membership function and a hesitation function, converting the fuzzy set into an intuitionistic fuzzy set (IFS). An IFS can be used to depict any of the three views of supremacy, inferiority, and reluctance, which are commonly represented as intuitionistic fuzzy numbers (IFNs) [8]. The Pythagorean fuzzy set (PyFS) was recently developed by Yager [9–11] as a novel assessment format to capture more important information in ambiguous and imprecise scenarios. The PyFS is characterised by the MG and non-membership grade (NMG) satisfying the condition that their square sum is less than 1. Pythagorean fuzzy numbers and the detailed mathematical expression for PyFS were introduced by Zhang and Xu [12]. (PyFN). Because the PyFS MG space is larger than the IFS MG space, the PyFS is more generic than the IFS. It is evident that the IFN does not address this issue, for instance, if a decision-maker presents evaluation data with MG of 0.5 and NMG of 0.9. This is because $0.4 + 0.9 > 1$. However, the PyFN can record this assessment data because $(0.5)^2 + (0.9)^2 < 1$. The PyFS demonstrates a greater range of applications than the IFS in this situation. PyFSs, a novel evaluation, have been successfully used in a variety of industries, including online stock investment [13], domestic airline service quality [12], and the choice of the governor of the Asian Infrastructure Investment Bank [14]. Torra [15] acknowledged the idea of FSs with hesitation in order to overcome the hesitating. Numerous authors detected problems by combining the group decision-making operators utilizing the hesitant fuzzy set (HFS) [16, 17]. The use of probabilistic hesitant fuzzy rough sets in decision support systems was covered by Khan et al. in 2020 [18]. In 2017, Xu and Zhou [19] identified a novel concept known as probabilistic hesitant fuzzy sets (PHFSs). A novel choice strategy for decision support algorithms was developed by the author [20, 21].

Motivation and novelty

It is generally known that multiple criteria decision making (MCDM) is an essential tool for resolving more complicated problems in the real world [22–24]. Various issues or ideologies have inspired the development of a variety of MCDM approaches. For instance, [25, 26] explored the challenges brought about by the MCDM problem's multiple criteria and the decision-makers' varied risk preferences. In [27, 28], a framework for selecting the best MCDA techniques for a specific decision scenario was put forward. PHFSs models have recently been employed by many academics to examine decision-making (DM) challenges [29, 30]. For particular, in [31, 32] applied a form of pythagorean probabilistic hesitant fuzzy sets (PyPHFSs) model using DM problems. In [33], a method for DM issues employing a specific type of PyFS model was described. Uncertainty affects the bulk of decision-making processes in real life. As a result, the more information a decision-making process contains, the more dependable it becomes. Despite the success of the aforementioned work, none of the studies listed above looked at how confident the traits were [34]. In other words, every researcher who has reviewed a study has done so with the assumption that the decision-makers are familiar with the studied objects.

Aggregation operators plays a vital rule in decision making. All of the current Pythagorean fuzzy aggregation operators only consider cases in which each PyFS component is independent, i.e., they only consider increasing the importance of specific elements. Grabisch [35] and Torra [36] provided a classic example: Assume we are evaluating a group of students in relation to three subjects: "chemistry, mathematics, and literature, and we want to give more importance to science-related subjects than to literature, but on the other hand, we want to give some advantage to students that are good both in literature and in a science subject". As a result, we must devise some novel strategies to deal with situations in which the decision data in question are correlated. The Choquet integral [37] is a very useful tool for estimating the expected utility of an unknown event and can be used to highlight the interplay of the factors under consideration.

Pythagorean fuzzy models offer greater spaces between membership and non-membership grades, which can be used to represent imperfect knowledge, they are more practical and valuable than intuitionistic fuzzy models. The general parameter Choquet integral operators are quite flexible. Since Choquet integral operators have not yet been applied to PyPHFSs, we have presented Choquet integral operations on PyPHFNs in this study. These operations were inspired by these operators. The Pythagorean probabilistic hesitant fuzzy environment has led us to introduce new aggregation operators, such as the probabilistic hesitant fuzzy Choquet integral averaging operator and the Pythagorean probabilistic hesitant fuzzy Choquet integral geometric operator. Aggregation operators are mathematical tools that are essential for turning a set of values into a unique value by utilising all of these concepts. In order to develop these operators, various important properties, including idempotency, monotonicity, boundedness, reducibility, and commutativity, have also been explored.

In this study, based on PyPHFSs, we suggest several novel AOs, such as Pythagorean probabilistic hesitant fuzzy Choquet integral averaging (PyPHF-CIA) and Pythagorean probabilistic hesitant fuzzy Choquet integral geometric (PyPHF-CIG) operations, for the following reasons:

(1) Because PyPHFSs integrate the ideas of Py and PHFSs, they anticipate giving decision-makers more flexibility.

(2) The advance condition that the total membership and non membership must be within the range [0, 1] is used by PyPHFSs.

(3) As opposed to PyHF aggregation operators, PyPHF-CIA and PyPHF-CIG AOs can incorporate experts' level of knowledge with examined items for initial assessment.

(4) This article seeks to cover more sophisticated and complex data, taking into mind that PyPHF-CIA and PyPHF-CIG operators are simple and cover the decision-making technique.

(5) All current drawbacks are limited by the suggested work. The data that is used to make decisions is smoothed by the Choquet integral, which is not possible with existing techniques. The current techniques give more flexibility due to probability in data.

Consequently, the following are the research's findings:

i. To undertake the development of new AOs like PyPHF-CIA and PyPHF-CIG.

ii. We've defined attributes for the suggested aggregating operations.

iii. Multi-criteria decision-making (MCDM) is a technique developed to handle the increasingly complicated data.

iv. Additionally, a PyPHFS-based new technology that uses machine learning to improve surgical operations has been shown, and a practical use for the method is provided.

The structure of this article is as follows. Section 2 examines the fundamental ideas that underlie FS, PyFS, IFS, PHFS, as well as a few fundamental operational laws. We introduce two new aggregation operators, PyPHF-CIA and PyPHF-CIG, in Section 3. In Section 4, a decision-making approach based on the proposed AOs is built, along with a solution to a numerical problem and numerical examples. We compared some of the current practises with the advised ones in Section 5. The conclusion is reached in Section 6.

2. Preliminaries

In this section we recall some basic definition and basic operators from the literature.

Definition 1. *For a fixed set F. A FS [5]* $\mathfrak{I}$ *in* β *is denoted as*

$$A = \{\langle \xi_\flat, \mathfrak{R}_\mathfrak{I}(\xi_\flat)\rangle \,|\xi_\flat \in F\},$$

for each $\xi_\flat \in F$*, the MG* $\mathfrak{R}_\mathfrak{I} : F \to \Delta$ *specifies the degree to which the element* $\xi_\flat \in A$*, where* $\Delta = [0, 1]$ *be the interval.*

Definition 2. *For a fixed set F. An IFS [7] A in* β *is mathematically described as*

$$A = \{\langle \xi_\flat, \mathfrak{R}_A(\xi_\flat), \mathfrak{I}_A(\xi_\flat)\rangle \,|\xi_\flat \in F\},$$

for each $\xi_\flat \in F$*, the MG* $\mathfrak{R}_A : F \to \Delta$ *and the non membership grade (NMG)* $\mathfrak{I}_A : F \to \Delta$ *specifies the MG and NMG of* $\xi_\flat$ *to the Intuitionistic fuzzy set A, respectively, where* $\Delta = [0, 1]$ *be the unit interval. Moreover, it is required that* $0 \le \mathfrak{R}_A(\xi_\flat) + \mathfrak{I}_A(\xi_\flat) \le 1,$ *for each* $\xi_\flat \in F$.

Definition 3. *See ([38]) For a fixed set F. An PyFS A in F is mathematically described as*

$$\mathfrak{I} = \{\langle \xi_\flat, \mathfrak{R}_A(\xi_\flat), \mathfrak{I}_A(\xi_\flat)\rangle \,|\xi_\flat \in F\},$$

for each $\xi_\flat \in F$*, the MG* $\mathfrak{R}_A : F \to \Delta$ *and the NMG* $\mathfrak{I}_A : \beta \to \Delta$ *specifies the MG and NMG of* $\xi_\flat$ *to the PyFS A, respectively, where* $\Delta = [0, 1]$ *be the unit interval. Moreover, it is required that* $0 \le \mathfrak{R}_A^2(\xi_\flat) + \mathfrak{I}_A^2(\xi_\flat) \le 1,$ *for each* $\xi_\flat \in F$.

Definition 4. *see ([39]) Let F be a fixed set. The mathematically representation of HFS D is defined as:*

$$D = \{\langle \daleth, \mathfrak{R}_{h_D}(\daleth)\rangle | \daleth \in F\}$$

where $\mathfrak{R}_{h_D}(\daleth)$ *is a set of some values in* $[0, 1]$*, indicate the MG of the element* $\daleth \in F$ *to the set D.*

Definition 5. *[40] Let F be a fixed set. The mathematically representation probabilistic HF set (PHFS)* $\mathfrak{R}$ *is defined as:*

$$= \left\{\langle \daleth, \mathfrak{R}_{h_\mathfrak{R}}(\daleth)/\partial_{h(\daleth)}\rangle | \daleth \in F\right\}$$

where $\mathfrak{R}_{h_\mathfrak{R}}(\daleth)$ *is a subset of* $[0, 1]$*, and* $\mathfrak{R}_{h_\mathfrak{R}}(\daleth)/\partial_\daleth$ *shows a MG of the element* $\daleth \in F$ *to the set* $\mathfrak{R}$*. And* $\partial_\daleth$ *shows the possibilities with the property that* $\oplus_{\perp=1}^{s}\partial_{h_\perp} = 1$.

Pythagorean hesitant fuzzy sets

Definition 6. *Let F be a fixed set [41]. The structure of Pythagorean hesitant fuzzy set (PyHFS) in F is the form.*

$$Py = \left\{\left\langle \mathfrak{R}_{Py}(\varkappa), \mathfrak{I}_{Py}(\varkappa) | \varkappa \in F\right\rangle\right\}$$

where $\mathfrak{R}_{Py}(\varkappa)$ and $\mathfrak{I}_{Py}(\varkappa)$ are mappings F to interval $[0, 1]$, denoting a MG and NMG of element $\varkappa \in F$ in PyHFS, respectively, and for every element

$$\varkappa \in F, \forall \hbar_{Py}(\varkappa) \in \mathfrak{R}_{Py}(\varkappa), \exists \hbar_{Py}(\varkappa) \in \mathfrak{I}_{Py}(\varkappa)$$

such that

$$0 \leq \hbar^2_{Py}(\varkappa) + \hbar^2_{Py}(\varkappa) \leq 1,$$

and

$$\forall \hbar^{'}_{Py}(\varkappa) \in \mathfrak{I}_{Py}(\varkappa), \exists \hbar_{Py}(\varkappa) \in \mathfrak{R}_{Py}(\varkappa)$$

such that

$$0 \leq \hbar^2_{Py}(\varkappa) + \hbar^{'2}_{Py}(\varkappa) \leq 1.$$

Additionally, PyHFS(F) designates the collection of all PyHFSs elements. If F has element, $\left\langle \varkappa, \mathfrak{R}_{Py}(\varkappa), \mathfrak{I}_{Py}(\varkappa)\right\rangle$ is known to be Pythagorean hesitant fuzzy number and is described as $\flat = \langle \mathfrak{R}_\flat, \mathfrak{I}_\flat \rangle$ for convenience.

Definition 7. *Let*

$$\flat = \langle \mathfrak{R}_\flat, \mathfrak{I}_\flat \rangle,\ \flat_1 = \langle \mathfrak{R}_{\flat_1}, \mathfrak{I}_{\flat_1} \rangle, \flat_2 = \langle \mathfrak{R}_{\flat_2}, \mathfrak{I}_{\flat_2} \rangle$$

be three PyHFNs, and $\lambda > 0$, *then their operations laws [42] can be defined below:*

(1) $\flat_1 \cup \flat_2 = \langle \max\{\mathfrak{R}_{\flat_1}, \mathfrak{R}_{\flat_2}\}, \min\{\mathfrak{I}_{\flat_1}, \mathfrak{I}_{\flat_1}\}\rangle$,

(2) $\flat_1 \cap \flat_2 = \langle \max\{\mathfrak{R}_{\flat_1}, \mathfrak{R}_{\flat_2}\}, \min\{\mathfrak{I}_{\flat_1}, \mathfrak{I}_{\flat_1}\}\rangle$,

(3) $\flat^c = \langle \mathfrak{R}_\flat, \mathfrak{I}_\flat \rangle$,

(4) $\flat_1 \oplus \flat_2 = \left\langle \cup_{\hbar_{\flat_1} \in \mathfrak{R}_{\flat_1}, \hbar_{\flat_2} \in \mathfrak{R}_{\flat_2}} \left\{ \sqrt{\hbar^2_{\flat_1} + \hbar^2_{\flat_2} - \hbar^2_{\flat_1} \hbar^2_{\flat_2}} \right\}, \cup_{\hbar^{'}_{\flat_1} \in \mathfrak{I}_{\flat_1}, \hbar^{'}_{\flat_2} \in \mathfrak{I}_{\flat_2}} \left\{ \hbar^{'}_{\flat_1} \hbar^{'}_{\flat_2} \right\} \right\rangle$,

(5) $\flat_1 \otimes \flat_2 = \left\langle \cup_{\hbar_{\flat_1} \in \mathfrak{R}_{\flat_1}, \hbar_{\flat_2} \in \mathfrak{R}_{\flat_2}} \{\hbar_{\flat_1} \hbar_{\flat_2}\}, \cup_{\hbar^{'}_{\flat_1} \in \mathfrak{I}_{\flat_1}, \hbar^{'}_{\flat_2} \in \mathfrak{I}_{\flat_2}} \left\{ \sqrt{\hbar^{'2}_{\flat_1} + \hbar^{'2}_{\flat_2} - \hbar^{'2}_{\flat_1} \hbar^{'2}_{\flat_2}} \right\} \right\rangle$,

(6) $\lambda \flat = \left\langle \cup_{\hbar_\flat \in \mathfrak{R}_\flat} \left\{ \sqrt{1 - (1 - (\hbar_\flat)^2)^\lambda} \right\}, \cup_{\hbar^{'}_\flat \in \mathfrak{I}_\flat} \left\{ (\hbar^{'}_\flat) \right\} \right\rangle$,

(7) $\flat^\lambda = \left\langle \cup_{\hbar_\flat \in \mathfrak{R}_\flat} \left\{ \hbar^k_\flat \right\}, \cup_{\hbar^{'}_\flat \in \mathfrak{I}_\flat} \left\{ \sqrt{1 - (1 - (\hbar^{'}_\flat)^2)^\lambda} \right\} \right\rangle$.

The authors compare and rank among PFNs as follows:

Definition 8. *Let*

$$\flat_\perp = \langle \mathfrak{R}_{\flat_\perp}, \beta_{\flat_\perp} \rangle (\perp = 1, 2)$$

be two PyHFNs, $S(\flat_1), S(\flat_2)$ *be the score of* $\flat_1, \flat_2$, *respectively, defined by*

$$S(\flat_1)=\left(\frac{1}{l_{\hbar_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar_{\flat_1}\in\Re_{\flat_1}}\hbar_{\flat_1}\right)^2-\left(\frac{1}{l_{\hbar'_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar'_{\flat_1}\in\Im_{\flat_1}}\hbar'_{\flat_1}\right)^2,$$

$$S(\flat_2)=\left(\frac{1}{l_{\hbar_{\flat_2}\in\Re_{\flat_2}}}\sum_{\hbar_{\flat_2}\in\Re_{\flat_2}}\hbar_{\flat_2}\right)^2-\left(\frac{1}{l_{\hbar'_{\flat_2}\in\Re_{\flat_2}}}\sum_{\hbar'_{\flat_2}\in\Im_{\flat_2}}\hbar'_{\flat_2}\right)^2$$

and $\overline{\ell}(\flat_1),\overline{\ell}(\flat_2)$ be the deviation degree of $\flat_1,\flat_2$, respectively, defined by

$$\overline{\ell}(\flat_1)=\left(\frac{1}{l_{\hbar_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar_{\flat_1}\in\Re_{\flat_1}}\hbar_{\flat_1}-S(\flat_1)\right)^2+\left(\frac{1}{l_{\hbar'_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar'_{\flat_1}\in\Im_{\flat_1}}\hbar'_{\flat_1}-S(\flat_1)\right)^2,$$

$$\overline{\ell}(\flat_2)=\left(\frac{1}{l_{\hbar_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar_{\flat_1}\in\Re_{\flat_1}}\hbar_{\flat}-S(\flat_1)\right)^2+\left(\frac{1}{l_{\hbar'_{\flat_1}\in\Re_{\flat_1}}}\sum_{\hbar'_{\flat_1}\in\Im_{\flat_1}}\hbar'_{\flat_1}-S(\flat_1)\right)^2$$

where $l_{\hbar_{\flat_1}},l_{\hbar_{\flat_2}}$ represent the number of elements in $\flat_1,\flat_2$, respectively.
Then,

(1) If $S(\flat_1)<S(\flat_2)$, then $\flat_1<\flat_2$,

(2) If $S(\flat_1)>S(\flat_2)$, then $\flat_1>\flat_2$,

(3) If $S(\flat_1)=S(\flat_2)$, then $\flat_1\sim\flat_2$.

For the derivation

(1) If $\overline{\ell}(\flat_1)<\overline{\ell}(\flat_2)$, then $\flat_1<\flat_2$,

(2) If $\overline{\ell}(\flat_1)>\overline{\ell}(\flat_2)$, then $\flat_1>\flat_2$,

(3) If $\overline{\ell}(\flat_1)=\overline{\ell}(\flat_2)$, then $\flat_1\sim\flat_2$.

Definition 9. *Let* $\flat_\perp=\langle\Re_{\flat_\perp},\Im_{\flat_\perp}\rangle\,(\perp=1,2,3,....,n)$ *be a collection of all PyHFNs and* $\nabla=(\nabla_1,\nabla_2,...,\nabla_n)^T$ *be the weight vector of* $\flat_\perp\,(\perp=1,2,3,....,n)$ *with* $\nabla_\perp\geq 0\,(\perp=1,2,3,....,n)$ *where* $\nabla_\perp\in[0,1]$ *and* $\sum_{\perp=1}^n\nabla_\perp=1$.

Then, the aggregation result using PyHFWA operator is also a PyHFN and PHFWA

$$(\flat_1,\flat_2,...,\flat_n)=\left\{\begin{array}{c}\left(\cup_{\hbar_{\flat_1}\in\Re'_{\flat_1},\hbar_{\flat_2}\in\Re_{\flat_2},...\hbar_{\flat_n}\in\Re_{\flat_n}}\left\{\sqrt{1-\prod_{\perp=1}^{n}\left(1-\hbar_{\flat_\perp}^2\right)^{\nabla_\perp}}\right\}\right),\\ \left(\cup_{\hbar'_{\flat_1}\in\Im_{\flat_1},\hbar'_{\flat_2}\in\Im_{\flat_2},...\hbar_{\flat_n}\in\Im_{\flat_n}}\left\{\prod_{\perp=1}^{n}\left(\hbar'_{\flat_\perp}\right)^{\nabla_\perp}\right\}\right)\end{array}\right\}.$$

Definition 10. *Let* $\flat_\perp=\langle\Re_{\flat_\perp},\Im_{\flat_\perp}\rangle\,(\perp=1,2,3,....,n)$ *be a collection of all PyHFNs and* $\nabla=(\nabla_1,\nabla_2,...,\nabla_n)^T$ *be the weight vector of* $\flat_\perp\,(\perp=1,2,3,....,n)$ *with* $\nabla_\perp\geq 0\,(\perp=1,2,3,....,n)$ *where* $\nabla_\perp\in[0,1]$ *and* $\sum_{\perp=1}^n\nabla_\perp=1$.

Then, the aggregation result using PyHFWG operator is also a PyHFN, and PyHFWG

$$(\flat_1, \flat_2, ..., \flat_n) = \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_1} \in \Re'_{\flat_1}, \hbar_{\flat_2} \in \Re_{\flat_2}, ...\hbar_{\flat_n} \in \Re_{\flat_n}} \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_\perp} \right)^{\nabla_\perp} \right\} \right), \\ \left(\cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}, \hbar'_{\flat_2} \in \Im_{\flat_2}, ...\hbar_{\flat_n} \in \Im_{\flat_n}} \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar^2_{\flat_\perp} \right)^{\nabla_\perp}} \right\} \right) \end{array} \right\}.$$

Definition 11. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}, \Im_{\flat_\perp} \rangle \, (\perp = 1, 2, 3, ..., n)$$

be a collection of all PyHFNs and $\nabla = (\nabla_1, \nabla_2, ..., \nabla_n)^T$ *be the weight vector of* $\flat_\perp \, (\perp = 1, 2, 3, ..., n)$ *with* $\nabla_\perp \geq 0 \, (\perp = 1, 2, 3, ..., n)$

where $\nabla_\perp \in [0, 1]$ and $\sum_{\perp=1}^{n} \nabla_\perp = 1$.

Then, the aggregation result using PyHFOWA operator is also a PyHFN, and PyHFOWA

$$(\flat_1, \flat_2, ..., \flat_n) = \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(1)}} \in \Re'_{\flat_{\ell(1)}}, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}, ...\hbar_{\flat_{\ell(n)}} \in \Re_{\flat_{\ell(n)}}} \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar^2_{\flat_{\ell(\perp)}} \right)^{\nabla_\perp}} \right\} \right), \\ \left(\cup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}, ...\hbar_{\flat_{\ell(n)}} \in \Im_{\flat_{\ell(n)}}} \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\ell(\perp)}} \right)^{\nabla_\perp} \right\} \right) \end{array} \right\}.$$

Definition 12. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}, \Im_{\flat_\perp} \rangle \, (\perp = 1, 2, 3, ..., n)$$

be a collection of all PyHFNs and $\nabla = (\nabla_1, \nabla_2, ..., \nabla_n)^T$ *be the weight vector of* $\flat_\perp \, (\perp = 1, 2, 3, ..., n)$ *with* $\nabla_\perp \geq 0 \, (\perp = 1, 2, 3, ..., n)$ *where* $\nabla_\perp \in [0, 1]$ *and* $\sum_{\perp=1}^{n} \nabla_\perp = 1$.

Then, the aggregation result using PyHFOWG operator is also a PyHFN, and PyHFOWG

$$(\flat_1, \flat_2, ..., \flat_n) = \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(1)}} \in \Re'_{\flat_{\ell(1)}}, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}, ...\hbar_{\flat_{\ell(n)}} \in \Re_{\flat_{\ell(n)}}} \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\ell(\perp)}} \right)^{\nabla_\perp} \right\} \right), \\ \left(\cup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}, ...\hbar_{\flat_{\ell(n)}} \in \Im_{\flat_{\ell(n)}}} \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar^2_{\flat_{\ell(\perp)}} \right)^{\nabla_\perp}} \right\} \right) \end{array} \right\}.$$

3. Pythagorean probabilistic hesitant fuzzy Choquet integral aggregation operators

We construct some aggregation operators for Pythagorean probabilistic hesitant fuzzy numbers in this part and look into some of their characteristics.

Definition 13. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}/\wp, \Im_{\flat_\perp}/¥ \rangle \, (\perp = 1, 2, 3, ..., n)$$

be a collection of all PyHFN's and $\aleph$ *be a fuzzy measure on X.*

Then, Pythagorean probabilistic hesitant fuzzy Choquet integral average (PyPHF-CIA) operator of dimension n is a mapping PyPHF-CIA: $F^n \to F$ such that PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n) = \left\{ \begin{array}{c} \{(\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)}))\,(\flat_{\ell(1)})\} / \wp_1 \oplus \\ \{(\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)}))\,(\flat_{\ell(2)})\} / \wp_2 \oplus \\ ... \oplus \{(\aleph(A_{\ell(n)}) - \aleph(A_{\ell(n-1)}))\,(\flat_{\ell(n)})\} / \wp_n \end{array} \right\}, \tag{3.1}$$

where $\{\ell(1), \ell(2), ..., \ell(n)\}$ is a permutation of $(1, 2, 3, ..., n)$ such that

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$, $\sum_{\perp=1}^{n} \wp_{\perp} = 1$ and $\flat_{\ell(0)} = \phi$.

Four cases can be inferred from the definition above.

i. If Eq (3.1) satisties, then

$$\aleph \{\varkappa_{\ell(\perp)}\} = \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)}) \, (\perp = 1, 2, 3, ..., n)$$

which denoted that Eq (3.1) reduces to PyPHFWA.

ii. If

$$\aleph(A) = \sum_{\perp=1}^{|A|} \nabla_{\perp},$$

$\forall A \in X$ where $|A|$ is the number of elements in A,

$$\nabla_{\perp} = \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)}) \, (\perp = 1, 2, 3, ..., n)$$

where $\nabla = (\nabla_1, \nabla_2, ..., \nabla_n)^T$, with $\sum_{\perp=1}^{n} \nabla_{\perp} = 1$. This shows that Eq (3.1) reduces to PyPHFOWA operator.

iii. If $\aleph(A) = 1$, for all $A \in X$, then PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n) = \max(\flat_1, \flat_2, ..., \flat_n) = \flat_{\ell(1)}.$$

iv. If $\aleph(A) = 0$, for all $A \in X$, then PyHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n) = \min(\flat_1, \flat_2, ..., \flat_n) = \flat_{\ell(n)}.$$

Theorem 1. *Let $\flat_{\perp} = \langle \Re_{\flat_{\perp}} / \wp, \Im_{\flat_{\perp}} / ¥ \rangle \, (\perp = 1, 2, 3, ..., n)$ be a set of all PyPHFNs. The aggregate outcome employing the PyPHF-CIA operator is also a PyPHFN and PyPHF-CIA at that point.*

$$\begin{aligned} & (\flat_1, \flat_2, ..., \flat_n) \\ = & \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1, \hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2, ... \hbar_{\flat_n} \in \Re_{\flat_n}/\wp_n} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar_{\flat_{\perp}}^2\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/¥_1, \hbar'_{\flat_2} \in \Im_{\flat_2}/¥_2, ... \hbar_{\flat_n} \in \Im_{\flat_n}/¥_n} \\ \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\perp}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\} \end{aligned} \tag{3.2}$$

where permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$, of $(1, 2, 3, ..., n)$ such that

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ and $\flat_{\ell(0)} = \phi$.

Proof. From the definition of PyPHFS, the first part of the theorem follows immediately. Next, by using mathematical induction method, we prove that Eq (3.2) holds for all n. For this, first we show that Eq (3.2) holds for $n = 2$. □

Since

$$(\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)}))\,\flat_1 = \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1} \\ \left\{ \sqrt{1 - \left(1 - \left(\hbar_{\flat_{\ell(1)}}\right)^2\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/\yen_1} \\ \left\{ \left(\hbar'_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})} \right\} \end{array} \right), \end{array} \right\}$$

and

$$(\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)}))\,\flat_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2} \\ \left\{ \sqrt{1 - \left(1 - \left(\hbar_{\flat_{\ell(2)}}\right)^2\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_2} \in \Im_{\flat_2}/\yen_2} \\ \left\{ \left(\hbar'_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})} \right\} \end{array} \right) \end{array} \right\},$$

so PyPHF-CIA

$$(\flat_1, \flat_2) = (\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)}))\,\flat_1 \oplus (\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)}))\,\flat_2$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1} \\ \left\{ \sqrt{1 - \left(1 - \left(\hbar_{\flat_{\ell(1)}}\right)^2\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/\yen_1} \\ \left\{ \left(\hbar'_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})} \right\} \end{array} \right) \end{array} \right\} \oplus \\ \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2} \\ \left\{ \sqrt{1 - \left(1 - \left(\hbar_{\flat_{\ell(2)}}\right)^2\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_2} \in \Im_{\flat_2}/\yen_2} \\ \left\{ \left(\hbar'_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})} \right\} \end{array} \right) \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{ \sqrt{ \begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1/\wp_1}, \hbar_{\flat_2} \in \Re_{\flat_2/\wp_2}} \\ \left\{ \begin{array}{c} \left(1 - \left(1 - \hbar^2_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})}\right) + \\ \left(1 - \left(1 - \hbar^2_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})}\right) - \\ \left(\begin{array}{c} \left(1 - \hbar^2_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})} \\ \left(1 - \hbar^2_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})} \end{array} \right) \end{array} \right\} \end{array} } \right\}, \\ \left\{ \begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1/¥_1}, \hbar'_{\flat_2/¥_2}} \\ \left\{ \begin{array}{c} \left(\hbar'_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})} \\ \left(\hbar'_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})} \end{array} \right\} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1, \hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{2} \left(1 - \hbar_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/¥_1, \hbar'_{\flat_2} \in \Im_{\flat_2}/¥_2} \\ \left\{ \prod_{\perp=1}^{2} \left(\hbar'_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\}.$$

Thus, Eq (3.2) is true for $n = 2$. suppose Eq (3.2) holds at $n = k$, i.e., PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_k)$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1/\wp_1}, \hbar_{\flat_2} \in \Re_{\flat_2/\wp_2}, ... \hbar_{\flat_k} \in \Re_{\flat_k/\wp_k}} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{k} \left(1 - \hbar^2_{\flat_\perp}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1/¥_1}, \hbar'_{\flat_2} \in \Im_{\flat_2/¥_2}, ... \hbar_{\flat_k} \in \Im_{\flat_k/¥_k}} \\ \left\{ \prod_{\perp=1}^{k} \left(\hbar'_{\flat_\perp}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\}$$

We prove that Eq (3.2) holds for $n = k + 1$, i.e., PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_{k+1})$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1}\in\Re_{\flat_1}/\wp_1,\hbar_{\flat_2}\in\Re_{\flat_2}/\wp_2,\ldots\hbar_{\flat_k}\in\Re_{\flat_k}/\wp_k} \\ \left\{ \sqrt{1-\prod\limits_{\perp=1}^{k}\left(1-\hbar_{\flat_\perp}^2\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1}\in\Im_{\flat_1/\yen_1},\hbar'_{\flat_2}\in\Im_{\flat_2/\yen_2},\ldots\hbar_{\flat_k}\in\Im_{\flat_k/\yen_k}} \\ \left\{ \prod\limits_{\perp=1}^{k}\left(\hbar'_{\flat_\perp}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\} \oplus \\ \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{k+1}}\in\Re_{\flat_{k+1}/\wp_{k+1}}} \\ \left\{ \sqrt{1-\left(\hbar_{\flat_{\ell(k+1)}}^2\right)^{\aleph(A_{\ell(k+1)})-\aleph(A_{\ell(k)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{k+1}}\in\Im_{\flat_{k+1}/\yen_{k+1}}} \\ \left\{ \left(\hbar'_{\flat_{\ell(k+1)}}\right)^{\aleph(A_{\ell(k+1)})-\aleph(A_{\ell(k)})} \right\} \end{array} \right) \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\{ \sqrt{ \left\{ \begin{array}{c} \left(1-\prod\limits_{\perp=1}^{k}\left(1-\hbar_{\flat_{\ell(\perp)}}^2\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right) + \\ \left(1-\left(\hbar_{\flat_{\ell(k+1)}}^2\right)^{\aleph(A_{\ell(k+1)})-\aleph(A_{\ell(k)})}\right) - \\ \left\{ \begin{array}{c} \left(1-\prod\limits_{\perp=1}^{k}\left(1-\hbar_{\flat_{\ell(\perp)}}^2\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right) \\ \left(\begin{array}{c} 1-\left(1-\hbar_{\flat_{\ell(k+1)}}^2\right)^{\aleph(A_{\ell(k+1)})-\aleph(A_{\ell(k)})} \\ \left(\hbar'_{\flat_{\ell(k+1)}}\right)^{\aleph(A_{\ell(k+1)})-\aleph(A_{\ell(k)})} \end{array} \right) \end{array} \right\} \end{array} \right\} }^{\cup_{\hbar_{\flat_1}\in\Re_{\flat_1}/\wp_1,\hbar_{\flat_2}\in\Re_{\flat_2}/\wp_2,\ldots\hbar_{\flat_k}\in\Re_{\flat_k}/\wp_k,\ \cup_{\hbar_{\flat_{k+1}}}\in\Re_{\flat_{k+1}/\wp_{k+1}}} \right\}, \\ \left\{ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1}\in\Im_{\flat_1/\yen_1},\hbar'_{\flat_2}\in\Im_{\flat_2/\yen_2},\ldots\hbar_{\flat_k}\in\Im_{\flat_k/\yen_k},\ \cup_{\hbar'_{\flat_{k+1}}}\in\Im_{\flat_{k+1}/\yen_{k+1}}} \\ \left\{ \prod\limits_{\perp=1}^{k}\left(\hbar'_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1}\in\Re_{\flat_1/\wp_1},\hbar_{\flat_2}\in\Re_{\flat_2/\wp_2},\ldots\hbar_{\flat_{k+1}}\in\Re_{\flat_{k+1}/\wp_{k+1}}} \\ \left\{ \sqrt{1-\prod\limits_{\perp=1}^{k+1}\left(1-\hbar_{\flat_{(\perp)}}^2\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1}\in\Im_{\flat_1/\yen_1},\hbar'_{\flat_2}\in\Im_{\flat_2/\yen_2},\ldots\hbar_{\flat_{k+1}}\in\Im_{\flat_{k+1}/\yen_{k+1}}} \\ \left\{ \prod\limits_{\perp=1}^{k+1}\left(\hbar'_{\flat_{(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\}.$$

Thus, Eq (3.2) is true for $n = k + 1$. Hence it is true for all n.

The PyPHF-CIA operator's characteristics are explained as above.

Theorem 2. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/\yen_\perp \rangle\, (\perp = 1, 2, 3, ..., n)$$

be a set of PyPHFN's, and permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ *of* $(1, 2, 3, ..., n)$ *such that*

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1, \Sigma_{\perp=1}^{n}(\wp_\perp) = 1, \Sigma_{\perp=1}^{n}(\yen_\perp) = 1$ *and* $\flat_{\ell(0)} = \phi$.

Proof. **(1) Idempotency** If all $\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp \rangle (\perp = 1,2,3,....,n)$ are equal, i.e., $\flat_\perp = (1,2,3,....,n) = \flat$, then PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n) = \flat. \tag{3.3}$$

(2) Boundedness

$$\flat^- \leq PyPHFC\perp A(\flat_1, \flat_2, ..., \flat_n) \leq \flat^+ \tag{3.4}$$

□

where

$$\flat^- = \left\langle \hbar^-, \hbar^{+'} \right\rangle, \flat^+ = \left\langle \hbar^+, \hbar^{-'} \right\rangle, \hbar^- = \cup_{\hbar_\perp \in \Re_{\flat_\perp}} \min_\perp \{\hbar_\perp\},$$

$$\hbar^+ = \cup_{\hbar_\perp \in \Re_{\flat_\perp}} \max_\perp \{\hbar_\perp\}, \hbar^{-'} = \cup_{\hbar'_\perp \in \Im_{\flat_\perp}} \min_\perp \left\{\hbar'_\perp\right\}, \hbar^{+'} = \cup_{\hbar'_\perp \in \Im_{\flat_\perp}} \max_\perp \left\{\hbar'_\perp\right\}.$$

(3) Monotonicity If $\flat_\perp > \flat^*_\perp$, then, PyPHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n) \leq PHFC\perp A(\flat^*_1, \flat^*_2, ..., \flat^*_n). \tag{3.5}$$

i. By Theorem 1, we have PyPHF-CIA

$$\begin{aligned}
&(\flat_1, \flat_2, ..., \flat_n) \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}/\wp_\perp}} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar^2_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}/¥_\perp}} \\ \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}/\wp_\perp}} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar^2_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}/¥_\perp}} \\ \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}/\wp_\perp}} \\ \left\{ \sqrt{1 - \left(1 - \hbar^2_{\flat_{\ell(\perp)}}\right)^{\sum_{\perp=1}^{n} \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}/¥_\perp}} \\ \left\{ \left(\hbar'_{\flat_{\ell(\perp)}}\right)^{\sum_{\perp=1}^{n} \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right) \end{array} \right\}.
\end{aligned}$$

Since $\sum_{\perp=1}^{n} \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)}) = 1$, we have PyPHF-CIA $(\flat_1, \flat_2, ..., \flat_n)$

$$= \left\{ \begin{array}{c} \left(\cup_{\hbar_\flat \in \Re_\flat / \wp_\perp} \left\{ \sqrt{1 - \left(1 - \hbar_\flat^2\right)} \right\} \right), \\ \left(\cup_{\hbar'_\flat \in \Im_\flat / \yen_\perp} \left\{ \hbar'_\flat \right\} \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\cup_{\hbar_\flat \in \Re_\flat / \wp_\perp} \{\hbar_\flat\} \right), \\ \left(\cup_{\hbar'_\flat \in \Im_\flat / \yen_\perp} \left\{ \hbar'_\flat \right\} \right) \end{array} \right\} = \flat.$$

ii. Since

$$\left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \min_\perp \left\{ \hbar_{\ell(\perp)} \right\} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \left\{ \hbar_{\ell(\perp)} \right\} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \max_\perp \left\{ \hbar_{\ell(\perp)} \right\} \right) \end{array} \right\} \tag{3.6}$$

and

$$\left\{ \begin{array}{c} \left(\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}} / \yen_\perp} \min_\perp \left\{ \hbar'_{\ell(\perp)} \right\} \right) \\ \leq \left(\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}} / \yen_\perp} \left\{ \hbar'_{\ell(\perp)} \right\} \right) \\ \leq \left(\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}} / \yen_\perp} \max_\perp \left\{ \hbar'_{\ell(\perp)} \right\} \right) \end{array} \right\} \tag{3.7}$$

from Eq (3.6) we have

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{\min_\perp \left\{ \hbar_{\ell(\perp)}^2 \right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{\left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{\max_\perp \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\}} \right) \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{1 - \max_\perp \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{1 - \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}} / \wp_\perp} \sqrt{1 - \min_\perp \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\}} \right) \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{\left(1 - \max_\perp \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{\left(1 - \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{\left(1 - \min_\perp \left\{ \left(\hbar_{\ell(\perp)} \right)^2 \right\} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right) \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\prod_{\perp=1}^{n}\left(1-\max_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \\ \leq \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \\ \leq \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\prod_{\perp=1}^{n}\left(1-\min_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\left(1-\max_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\sum_{\perp=1}^{n}\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \\ \leq \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \\ \leq \left(\frac{\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\left(1-\min_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\sum_{\perp=1}^{n}\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}} \right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{1-\max_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{1-\min_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}} \right) \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{-1+\min_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{-\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \right) \\ \leq \left(\cup_{\hbar_{\flat_{\ell(\perp)}} \in \Re_{\flat_{\ell(\perp)}}} \sqrt{-1+\max_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}} \right) \end{array} \right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{1-1+\min\limits_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}}}\right)\\ \leq\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{1-\prod\limits_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}}\right)\\ \leq\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{1-1+\max\limits_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}}}\right)\end{array}\right\}$$

$$\Longleftrightarrow\left\{\begin{array}{c}\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\min\limits_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}}}\right)\\ \leq\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{1-\prod\limits_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}}\right)\\ \leq\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{\max\limits_{\perp}\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}}}\right)\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\begin{array}{c}\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}\\ \min\limits_{\perp}\left\{\hbar_{\ell(\perp)}\right\}\end{array}\right)\\ \leq\left(\dfrac{\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}}{\sqrt{1-\prod\limits_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^{2}\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}}\right)\\ \leq\left(\begin{array}{c}\cup_{\hbar_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_{\perp}}\\ \max\limits_{\perp}\left\{\hbar_{\ell(\perp)}\right\}\end{array}\right)\end{array}\right\}.$$

Now, from Eq (3.7) we have

$$\left\{\begin{array}{c}\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\min\limits_{\perp}\left\{\hbar'_{\ell(\perp)}\right\}\right)\\ \leq\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\left\{\hbar'_{\ell(\perp)}\right\}\right)\\ \leq\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\max\limits_{\perp}\left\{\hbar'_{\ell(\perp)}\right\}\right)\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\min\limits_{\perp}\left\{\left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\right)\\ \leq\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\left\{\left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\right)\\ \leq\left(\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_{\perp}}\max\limits_{\perp}\left\{\left(\hbar'_{(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\right)\end{array}\right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\prod_{\perp=1}^{n} \min_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\prod_{\perp=1}^{n} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\prod_{\perp=1}^{n} \max_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\min_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\sum_{\perp=1}^{n} \aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\prod_{\perp=1}^{n} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\max_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\sum_{\perp=1}^{n} \aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \left(\min_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right) \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\prod_{\perp=1}^{n} \left\{ \left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \\ \leq \left(\max_{\perp} \left\{ \left(\hbar'_{\ell(\perp)}\right) \right\} \right)^{\cup_{\hbar'_{\flat_{\ell(\perp)}} \in \Im_{\flat_{\ell(\perp)}}/¥_{\perp}}} \end{array} \right\}.$$

The scoring function indicates, we have PyPHF-CIA$(\flat_1, \flat_2, ..., \flat_n) \geq \flat^-$ with equality iff the $\flat^-$ $= PyPHF - C\perp A(\flat)$.

Similarly,

$$PyPHF - C\perp A(\flat_1, \flat_2, ..., \flat_n) \leq \flat^+,$$

with equality iff the $PyPHF - C\perp A(\flat)$ is the same as $\flat^+$.

Hence,

$$\flat^- \leq PyPHF - C\perp A(\flat_1, \flat_2, ..., \flat_n) \leq \flat^+.$$

iii. If $\flat_\perp > \flat^*_\perp$, then

$$PHFC\perp A(\flat_1, \flat_2, ..., \flat_n) \leq PHFC\perp A(\flat^*_1, \flat^*_2, ..., \flat^*_n).$$

Since

$$A_{\ell(\perp)} \subseteq A_{\ell(\perp-1)}, \lambda(A_{\ell(\perp)}) - \lambda(A_{\ell(\perp-1)}) \geq 0.$$

For all $\perp$, $\mathfrak{R}_{\flat^*_{\ell(\perp)}} \geq \mathfrak{R}_{\flat_{\ell(\perp)}}$, $\mathfrak{I}_{\flat_{\ell(\perp)}} \geq \mathfrak{I}_{\flat^*_{\ell(\perp)}}$.
If $\mathfrak{R}_{\flat_{\ell(\perp)}} \leq \mathfrak{R}_{\flat^*_{\ell(\perp)}}$, then

$$\left\{\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \left\{\hbar_{\ell(\perp)}\right\}\end{array}\right)\leq\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \left\{\hbar^*_{\ell(\perp)}\right\}\end{array}\right)\right\}$$

$$\Leftrightarrow\left\{\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}\end{array}\right)\leq\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}\end{array}\right)\right\}$$

$$\Leftrightarrow\left\{\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{\left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}}\end{array}\right)\leq\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{\left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}}\end{array}\right)\right\}$$

$$\Leftrightarrow\left\{\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{1-\left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}}\end{array}\right)\leq\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{1-\left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}}\end{array}\right)\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}}\sqrt{\left(1-\left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\right)\\ \leq\left(\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}}\sqrt{\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\right)\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\end{array}\right)\\ \leq\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\end{array}\right)\end{array}\right\}$$

$$\Longleftrightarrow\left\{\begin{array}{c}\left(\begin{array}{c}\cup_{\hbar_{\ell(\perp)}\in\mathfrak{R}_{\flat_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{1-\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\end{array}\right)\\ \leq\left(\begin{array}{c}\cup_{\hbar^*_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \sqrt{\prod_{\perp=1}^{n}\left(1-\left\{\left(\hbar^*_{\ell(\perp)}\right)^2\right\}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}}\end{array}\right)\end{array}\right\}. \tag{3.8}$$

Now, if $\mathfrak{I}_{\flat_{\ell(\perp)}/\yen_\perp} \geq \mathfrak{I}_{\flat^*_{\ell(\perp)}/\yen_\perp}$, then

$$\left\{\left(\begin{array}{c}\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/\yen_\perp}\\ \left\{\left(\hbar'_{\ell(\perp)}\right)\right\}\end{array}\right)\geq\left(\begin{array}{c}\cup_{\hbar^{*\prime}_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^*_{\ell(\perp)}}/\wp_\perp}\\ \left\{\left(\hbar^{*\prime}_{\ell(\perp)}\right)\right\}\end{array}\right)\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\begin{array}{c}\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/¥_{\perp}}\\ \left\{\left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\end{array}\right)\\ \geq\left(\begin{array}{c}\cup_{\hbar^{*'}_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^{*}_{\ell(\perp)}}/\wp_{\perp}}\\ \left\{\left(\hbar^{*'}_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\end{array}\right)\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\begin{array}{c}\cup_{\hbar'_{\flat_{\ell(\perp)}}\in\mathfrak{I}_{\flat_{\ell(\perp)}}/¥_{\perp}}\\ \left\{\prod_{\perp=1}^{n}\left(\hbar'_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\end{array}\right)\\ \geq\left(\begin{array}{c}\cup_{\hbar^{*'}_{\flat_{\ell(\perp)}}\in\mathfrak{R}_{\flat^{*}_{\ell(\perp)}}/\wp_{\perp}}\\ \left\{\prod_{\perp=1}^{n}\left(\hbar^{*'}_{\ell(\perp)}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}\right\}\end{array}\right)\end{array}\right\}. \tag{3.9}$$

Let

$$\flat = PHFC\perp A(\flat_1, \flat_2, ..., \flat_n)$$

and

$$\flat^* = PHFC\perp A(\flat_1^*, \flat_2^*, ..., \flat_n^*).$$

Then, from Eqs (3.8) and (3.9) we have

$$S(\flat) \leq S(\flat^*).$$

If $S(\flat) < S(\flat^*)$, then

$$PyPHF - C\perp A(\flat_1, \flat_2, ..., \flat_n) < PyPHF - C\perp A(\flat_1^*, \flat_2^*, ..., \flat_n^*).$$

If $S(\flat) = S(\flat^*)$, then

$$\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar_{\flat}\in\mathfrak{R}_{\flat}/\wp_{\perp}}}\sum_{\hbar_{\flat}\in\mathfrak{R}_{\flat}/\wp_{\perp}}\hbar_{\flat}\right)^{2}\\ -\left(\frac{1}{l_{\hbar'_{\flat}\in\mathfrak{I}_{\flat}/¥_{\perp}}}\sum_{\hbar'_{\flat}\in\mathfrak{I}_{\flat}/¥_{\perp}}\hbar_{\flat}\right)^{2}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar^{*}_{\flat^{*}}\in\mathfrak{R}^{*}_{\flat}/\wp_{\perp}}}\sum_{\hbar^{*}_{\flat^{*}}\in\mathfrak{R}_{\flat^{*}}/\wp_{\perp}}\hbar^{*}_{\flat^{*}}\right)^{2}\\ -\left(\frac{1}{l_{\hbar^{*'}_{\flat^{*}}\in\mathfrak{I}_{\flat^{*}}/¥_{\perp}}}\sum_{\hbar^{*'}_{\flat^{*}}\in\mathfrak{I}_{\flat^{*}}/¥_{\perp}}\hbar^{*'}_{\flat^{*}}\right)^{2}\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar_{\flat}\in\mathfrak{R}_{\flat}/\wp_{\perp}}}\sum_{\hbar_{\flat}\in\mathfrak{R}_{\flat}/\wp_{\perp}}\hbar_{\flat}\right)^{2}\\ =\left(\frac{1}{l_{\hbar^{*}_{\flat^{*}}\in\mathfrak{R}^{*}_{\flat}/\wp_{\perp}}}\sum_{\hbar^{*}_{\flat^{*}}\in\mathfrak{R}_{\flat^{*}}/\wp_{\perp}}\hbar^{*}_{\flat^{*}}\right)^{2}\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}}\sum\limits_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}\hbar_{\flat}\right)^{2}\\=\left(\frac{1}{l_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}}\sum\limits_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}\hbar^{*\prime}_{\flat^{*}}\right)^{2}\end{array}\right\}$$

$$\Leftrightarrow\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar_{\flat}\in\Re_{\flat}/\wp_{\perp}}}\sum\limits_{\hbar_{\flat}\in\Re_{\flat}/\wp_{\perp}}\hbar_{\flat}\right)\\=\left(\frac{1}{l_{\hbar^{*}_{\flat^{*}}\in\Re^{*}_{\flat}/\wp_{\perp}}}\sum\limits_{\hbar^{*}_{\flat^{*}}\in\Re_{\flat^{*}}/\wp_{\perp}}\hbar^{*}_{\flat^{*}}\right)\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\left(\frac{1}{l_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}}\sum\limits_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}\hbar_{\flat}\right)\\=\left(\frac{1}{l_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}}\sum\limits_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}\hbar^{*\prime}_{\flat^{*}}\right)\end{array}\right\}.$$

Since

$$\begin{aligned}\bar{\ell}(\flat) &= \left\{\begin{array}{c}\left(\frac{1}{l_{\hbar_{\flat}\in\Re_{\flat}/\wp_{\perp}}}\sum\limits_{\hbar_{\flat}\in\Re_{\flat}/\wp_{\perp}}\hbar_{\flat}-S(\flat)\right)^{2}\\+\left(\frac{1}{l_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}}\sum\limits_{\hbar'_{\flat}\in\Im_{\flat}/¥_{\perp}}\hbar_{\flat}-S(\flat)\right)^{2}\end{array}\right\}\\&= \left\{\begin{array}{c}\left(\frac{1}{l_{\hbar^{*}_{\flat^{*}}\in\Re^{*}_{\flat}/\wp_{\perp}}}\sum\limits_{\hbar^{*}_{\flat^{*}}\in\Re_{\flat^{*}}/\wp_{\perp}}\hbar^{*}_{\flat^{*}}-S(\flat^{*})\right)^{2}\\-\left(\frac{1}{l_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}}\sum\limits_{\hbar^{*\prime}_{\flat^{*}}\in\Im_{\flat^{*}}/¥_{\perp}}\hbar^{*\prime}_{\flat^{*}}-S(\flat^{*})\right)^{2}\end{array}\right\}\\&= \left\{\bar{\ell}(\flat^{*})\right\}.\end{aligned}$$

Therefore,

$$PyPHF-C\perp A(\flat_{1},\flat_{2},...,\flat_{n})=\flat^{*}=PyPHF-C\perp A(\flat^{*}_{1},\flat^{*}_{2},...,\flat^{*}_{n}).$$

Definition 14. *Let $\flat_{\perp}=\langle\Re_{\flat_{\perp}}/\wp_{\perp},\Im_{\flat_{\perp}}/¥_{\perp}\rangle$ $(\perp=1,2,3,....,n)$ be a set of all PyPHFN's and λ be a fuzzy measure on X. As a result, the Pythagorean probabilistic hesitant fuzzy Choquet integral geometric (PyPHF-CIG) operator of dimension n is a mapping PyPHFCIG: $F^{n}\rightarrow F$ such that PyPHF-CIA,*

$$(\flat_{1},\flat_{2},...,\flat_{n}) \tag{3.10}$$

$$= \left\{ \begin{array}{c} \left((\flat_{\ell(1)})^{\left(\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})\right)} \right) \otimes \\ \left((\flat_{\ell(2)})^{\left(\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})\right)} \right) \otimes \\ \ldots \otimes \left((\flat_{\ell(n)})^{\left(\aleph(A_{\ell(n)}) - \aleph(A_{\ell(n-1)})\right)} \right) \end{array} \right\}$$

where permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ of $(1, 2, 3, ..., n)$ such that:

$$\{\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}\}$$

for $k \geq 1$ and $\flat_{\ell(0)} = \phi$.

Four cases can be inferred from the definition above.

i. If Eq (3.10) satisties, then

$$\aleph(\{\varkappa_{\ell(\perp)}|\}) = \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)}) \, (\perp = 1, 2, 3, ..., n)$$

which discribed that Eq (3.10) reduces to PyPHFWA operator.

ii. If $\aleph(A) = \sum_{\perp=1}^{|A|} \nabla_\perp$, for all $A \in X$ where $|A|$ is the number of elements in A,

$$\nabla_\perp = \aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)}) \, (\perp = 1, 2, 3, ..., n)$$

where $\nabla = (\nabla_1, \nabla_2, ..., \nabla_n)^T$, with $\sum_{\perp=1}^{n} \nabla_\perp = 1$.

This described that Eq (3.10) reduces to PyPHFOWG operator.

iii. If $\aleph(A) = 1$, for all $A \in X$, then PyPHF-CIG

$$(\flat_1, \flat_2, ..., \flat_n) = \max(\flat_1, \flat_2, ..., \flat_n) = \flat_{\ell(1)}.$$

iv. If $\aleph(A) = 0$, for all $A \in X$, then PyPHF-CIG

$$(\flat_1, \flat_2, ..., \flat_n) = \min(\flat_1, \flat_2, ..., \flat_n) = \flat_{\ell(n)}.$$

Theorem 3. *Let* $\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/\yen_\perp \rangle \, (\perp = 1, 2, 3, ..., n)$ *be a set of all PyPHFNs. The aggregate outcome employing the PyPHF-CIG operator is also a PyPHFN and PyPHF-CIG at that point.*

$$(\flat_1, \flat_2, ..., \flat_n) \tag{3.11}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2, ... \hbar_{\flat_{\ell(n)}} \in \Re_{\flat_{\ell(n)}}/\wp_n} \\ \left\{ \prod_{\perp=1}^{n} \left(\hbar'_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/\yen_1, \hbar'_{\flat_2} \in \Im_{\flat_2}/\yen_2, ... \hbar_{\flat_n} \in \Im_{\flat_n}/\yen_n} \\ \left\{ \sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar'^2_{\flat_\perp} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right\} \end{array} \right) \end{array} \right\}$$

where permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ of $(1, 2, 3, ..., n)$ such that:

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ with probability is one and $\flat_{\ell(0)} = \phi$.

Proof. First part of the theorem directly follows from basic definition. Next, by mathematical induction we prove that Eq (3.11) holds for all n. For this, first we show that Eq (3.11) holds for $n = 2$. □

Since

$$\mathfrak{b}_{\ell(1)}^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} = \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\mathfrak{b}_{\ell(1)}} \in \Re_{\mathfrak{b}_{\ell(1)}}/\wp_1} \\ \left\{ \left(\hbar'_{\mathfrak{b}_{\ell(1)}} \right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\mathfrak{b}_{\ell(1)}} \in \Im_{\mathfrak{b}_{\ell(1)}}/\yen_1} \\ \left\{ \sqrt{1-\left(1-\hbar'^2_{\mathfrak{b}_{\ell(1)}}\right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})}} \right\} \end{array} \right) \end{array} \right\}$$

and

$$\mathfrak{b}_{\ell(2)}^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} = \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\mathfrak{b}_{\ell(2)}} \in \Re_{\mathfrak{b}_{\ell(2)}}//\wp_2} \\ \left\{ \left(\hbar'_{\mathfrak{b}_{\ell(2)}} \right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\mathfrak{b}_{\ell(2)}} \in \Im_{\mathfrak{b}_{\ell(2)}}/\yen_2} \\ \left\{ \sqrt{1-\left(1-\hbar'^2_{\mathfrak{b}_{\ell(2)}}\right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})}} \right\} \end{array} \right) \end{array} \right\}$$

PyPHF-CIG

$$\begin{aligned} (\mathfrak{b}_1, \mathfrak{b}_2) &= \left\{ \mathfrak{b}_{\ell(1)}^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \otimes \mathfrak{b}_{\ell(2)}^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \\ &= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\mathfrak{b}_{\ell(1)}} \in \Re_{\mathfrak{b}_{\ell(1)}}/\wp_1} \\ \left\{ \left(\hbar'_{\mathfrak{b}_{\ell(1)}} \right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\mathfrak{b}_{\ell(1)}} \in \Im_{\mathfrak{b}_{\ell(1)}}/\yen_1} \\ \left\{ \sqrt{1-\left(1-\hbar'^2_{\mathfrak{b}_{\ell(1)}}\right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})}} \right\} \end{array} \right) \end{array} \right\} \otimes \\ \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\mathfrak{b}_{\ell(2)}} \in \Re_{\mathfrak{b}_{\ell(2)}}/\wp_2} \\ \left\{ \left(\hbar'_{\mathfrak{b}_{\ell(2)}} \right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\mathfrak{b}_{\ell(2)}} \in \Im_{\mathfrak{b}_{\ell(2)}}//\yen_2} \\ \left\{ \sqrt{1-\left(1-\hbar'^2_{\mathfrak{b}_{\ell(2)}}\right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})}} \right\} \end{array} \right) \end{array} \right\} \end{array} \right\} \end{aligned}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1}, \cup_{\hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2} \\ \left\{ \begin{array}{c} \left(\hbar'_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})} \\ \left(\hbar'_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})} \end{array} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1} \cup_{\hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2} \\ \left\{ \sqrt{\left(\begin{array}{c} \left(1-\left(1-\hbar'^2_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})}\right)+ \\ \left(1-\left(1-\hbar'^2_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})}\right)- \\ \left(\begin{array}{c} 1-\left(1-\hbar'^2_{\flat_{\ell(1)}}\right)^{\aleph(A_{\ell(1)})-\aleph(A_{\ell(0)})} \\ \left(1-\hbar'^2_{\flat_{\ell(2)}}\right)^{\aleph(A_{\ell(2)})-\aleph(A_{\ell(1)})} \end{array} \right) \end{array} \right)} \right\} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1}, \cup_{\hbar'_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2} \\ \left\{ \prod_{\perp=1}^{2} \left(\hbar_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1} \cup_{\hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2} \\ \sqrt{1-\prod_{\perp=1}^{2}\left(1-\hbar'^2_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \end{array} \right) \end{array} \right\}.$$

Hence, Eq (3.10) is true for $n = 2$.

Suppose that Eq (3.11) holds for $n = k + 1$, i.e., PyPHF-CIG

$$(\flat_1, \flat_2, ..., \flat_k)$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2, \ldots \hbar_{\flat_{\ell(k)}} \in \Re_{\flat_{\ell(k)}}/\wp_k} \\ \left\{ \prod_{\perp=1}^{k} \left(\hbar_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2, \ldots \hbar_{\flat_{\ell(k)}} \in \Im_{\flat_{\ell(k)}}/¥_k} \\ \sqrt{1-\prod_{\perp=1}^{k}\left(1-\hbar'^2_{\flat_{\ell(\perp)}}\right)^{\aleph(A_{\ell(\perp)})-\aleph(A_{\ell(\perp-1)})}} \end{array} \right) \end{array} \right\}.$$

We describe that Eq (3.11) holds for $n = k + 1$.

PyPHF-CIG

$$
(\flat_1, \flat_2, ..., \flat_{k+1}) = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left(\begin{array}{c} \bigcup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2, ... \hbar_{\flat_{\ell(k)}} \in \Re_{\flat_{\ell(k)}}/\wp_k} \\ \left\{ \prod_{\perp=1}^{k} \left(\hbar_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \bigcup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2, ... \hbar_{\flat_{\ell(k)}} \in \Im_{\flat_{\ell(k)}}/¥_k} \\ \sqrt{1 - \prod_{\perp=1}^{k} \left(1 - \hbar'^2_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \end{array} \right) \end{array} \right\} \otimes \\ \left\{ \begin{array}{c} \left(\begin{array}{c} \bigcup_{\hbar_{\flat_{\ell(k+1)}} \in \Re_{\flat_{\ell(k+1)}}/\wp_{k+1}} \\ \left\{ \left(\hbar_{\flat_{\ell(k+1)}} \right)^{\aleph(A_{\ell(k+1)}) - \aleph(A_{\ell(k)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \bigcup_{\hbar'_{\flat_{\ell(k+1)}} \in \Im_{\flat_{\ell(k+1)}}/¥_{k+1}} \\ \sqrt{1 - \left(1 - \hbar'^2_{\flat_{\ell(k+1)}} \right)^{\aleph(A_{\ell(k+1)}) - \aleph(A_{\ell(k)})}} \end{array} \right) \end{array} \right\} \end{array} \right\}
$$

$$
= \left\{ \begin{array}{c} \left(\begin{array}{c} \bigcup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2, ... \hbar_{\flat_{\ell(k)}} \in \Re_{\flat_{\ell(k)}}/\wp_k, \bigcup \hbar_{\flat_{\ell(k+1)}} \in \Re_{\flat_{\ell(k+1)}}/\wp_{k+1}} \\ \left\{ \prod_{\perp=1}^{k} \left(\begin{array}{c} \left(\left(\hbar_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right) \\ \left(\left(\hbar_{\flat_{\ell(k+1)}} \right)^{\aleph(A_{\ell(k+1)}) - \aleph(A_{\ell(k)})} \right) \end{array} \right) \right\} \end{array} \right), \\ \bigcup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2, ... \hbar_{\flat_{\ell(k)}} \in \Im_{\flat_{\ell(k)}}/¥_k, \bigcup \hbar'_{\flat_{\ell(k+1)}} \in \Im_{\flat_{\ell(k+1)}}/¥_{k+1}} \\ \sqrt{\left(\begin{array}{c} \left(1 - \prod_{\perp=1}^{k} \left(1 - \hbar'^2_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right) + \\ \left(1 - \left(1 - \hbar'^2_{\flat_{\ell(k+1)}} \right)^{\aleph(A_{\ell(k+1)}) - \aleph(A_{\ell(k)})} \right) - \\ \left(\begin{array}{c} \left(\sqrt{1 - \prod_{\perp=1}^{k} \left(1 - \hbar'^2_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right) \\ \left(1 - \left(1 - \hbar'^2_{\flat_{\ell(k+1)}} \right)^{\aleph(A_{\ell(k+1)}) - \aleph(A_{\ell(k)})} \right) \end{array} \right) \end{array} \right)} \end{array} \right\}
$$

$$
= \left\{ \begin{array}{c} \left(\begin{array}{c} \bigcup_{\hbar_{\flat_{\ell(1)}} \in \Re_{\flat_{\ell(1)}}/\wp_1, \hbar_{\flat_{\ell(2)}} \in \Re_{\flat_{\ell(2)}}/\wp_2, ... \hbar_{\flat_{\ell(k+1)}} \in \Re_{\flat_{\ell(k+1)}}/\wp_{k+1}} \\ \left\{ \prod_{\perp=1}^{k+1} \left(\hbar_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})} \right\} \end{array} \right), \\ \left(\begin{array}{c} \bigcup_{\hbar'_{\flat_{\ell(1)}} \in \Im_{\flat_{\ell(1)}}/¥_1, \hbar'_{\flat_{\ell(2)}} \in \Im_{\flat_{\ell(2)}}/¥_2, ... \hbar'_{\flat_{\ell(k+1)}} \in \Im_{\flat_{\ell(k+1)}}/¥_{k+1}} \\ \sqrt{1 - \prod_{\perp=1}^{k+1} \left(1 - \hbar'^2_{\flat_{\ell(\perp)}} \right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \end{array} \right) \end{array} \right\}.
$$

Hence, Eq (3.11) is true for $n = k + 1$. Hence, the result holds $\forall\ n$.

Theorem 4. *Let*

$$\flat_\perp = \{\Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp\} \, (\perp = 1, 2, 3, ..., n)$$

be a set of all PyPHFN's and permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ *of* $(1, 2, 3, ..., n)$ *such that*

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ *and* $\flat_{\ell(0)} = \phi$. *Then,*

(1) Idempotency If all $\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp \rangle\, (\perp = 1, 2, 3, ..., n)$ are equal, i.e., $\flat_\perp = (1, 2, 3, ..., n) = \flat$, then PyPHF-CIG

$$(\flat_1, \flat_2, ..., \flat_n) = \flat. \tag{3.12}$$

(2) Boundedness

$$\flat^- \leq PyPHF - C\perp G(\flat_1, \flat_2, ..., \flat_n) \leq \flat^+ \tag{3.13}$$

where

$$\flat^- = \left\langle \hbar^-, \hbar^{+'} \right\rangle, \flat^+ = \left\langle \hbar^+, \hbar^{-'} \right\rangle, \hbar^- = \cup_{\hbar_\perp \in \Re_{\flat_\perp}} \min_\perp \{\hbar_\perp\},$$

$$\hbar^+ = \cup_{\hbar_\perp \in \Re_{\flat_\perp}} \max_\perp \{\hbar_\perp\}, \hbar^{-'} = \cup_{\hbar'_\perp \in \Im_{\flat_\perp}} \min_\perp \left\{\hbar'_\perp\right\}, \hbar^{+'} = \cup_{\hbar'_\perp \in \Im_{\flat_\perp}} \max_\perp \left\{\hbar'_\perp\right\}.$$

(3) Monotonicity If $\flat_\perp > \flat^*_\perp$, then

$$PyPHF - C\perp G(\flat_1, \flat_2, ..., \flat_n) \leq PyPHF - C\perp G(\flat^*_1, \flat^*_2, ..., \flat^*_n). \tag{3.14}$$

Proof. Proof is same as Theorem 2.

Definition 15. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp \rangle\, (\perp = 1, 2, 3, ..., n)$$

be a collection of all PyPHFN's and λ be a fuzzy measure on X.

□

Then, Pythagorean probabilistic hesitant fuzzy Choquet integral geometric (PyPHF-CIG) operator of dimension n is a mapping

GPyPHF-CIG : $F^n \to F$ such that GPyPHF-CIG

$$(\flat_1, \flat_2, ..., \flat_n) = \left\{ \begin{array}{c} ((\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)}))\, (\flat_{\ell(1)})^\delta \otimes \\ ((\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)}))\, (\flat_{\ell(2)})^\delta \otimes \\ ... \otimes ((\aleph(A_{\ell(n)}) - \aleph(A_{\ell(n-1)}))\, (\flat_{\ell(n)})^{\frac{1}{\delta}} \end{array} \right\} \tag{3.15}$$

where $\aleph > 0$, permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ of $(1, 2, 3, ..., n)$ such that:

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ and $\flat_{\ell(0)} = \phi$.

Theorem 5. *Let $\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp \rangle\, (\perp = 1, 2, 3, ..., n)$ be a set of all PyPHFNs. Then, the aggregation outcomes using GPyPHF-CIA operator is also a PyPHFN and GPyPHF-CIA*

$$(\flat_1, \flat_2, ..., \flat_n) \tag{3.16}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1, \hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2, \ldots \hbar_{\flat_n} \in \Re_{\flat_n}/\wp_n} \\ \left\{ \left(\sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar_{\flat_\perp}^{\delta}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right)^{\frac{1}{\delta}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/¥_1, \hbar'_{\flat_1} \in \Im_{\flat_2}/¥_2, \ldots \hbar'_{\flat_n} \in \Im_{\flat_n}/¥_n} \\ \left\{ \sqrt{1 - \left(1 - \prod_{\perp=1}^{n} \left(1 - \left(1 - \hbar'^{2}_{\flat_\perp}\right)^{\delta}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}\right)^{\frac{1}{\delta}}} \right\} \end{array} \right) \end{array} \right\}$$

where $\delta > 0$, permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ of $(1, 2, 3, ..., n)$ such that:

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ and $\flat_{\ell(0)} = \phi$.

Proof. Same as above theorems. □

Theorem 6. *Let*

$$\flat_\perp = \langle \Re_{\flat_\perp}/\wp_\perp, \Im_{\flat_\perp}/¥_\perp \rangle \, (\perp = 1, 2, 3, ..., n)$$

be a collection of all PHFN's and λ be a fuzzy measure on X.

Then, Pythagorean probabilistic hesitant fuzzy Choquet integral geometric (GPyPHF-CIG) operator of dimension n is a mapping PyPHF-CIA: $F^n \to F$ such that GPy-PHF-CIA

$$(\flat_1, \flat_2, ..., \flat_n)$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \cup_{\hbar_{\flat_1} \in \Re_{\flat_1}/\wp_1, \hbar_{\flat_2} \in \Re_{\flat_2}/\wp_2, \ldots \hbar_{\flat_n} \in \Re_{\flat_n}/\wp_n} \\ \left\{ \sqrt{1 - \left(1 - \prod_{\perp=1}^{n} \left(1 - \left(1 - \hbar^{2}_{\flat_\perp}\right)^{\delta}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}\right)^{\frac{1}{\delta}}} \right\} \end{array} \right), \\ \left(\begin{array}{c} \cup_{\hbar'_{\flat_1} \in \Im_{\flat_1}/¥_1, \hbar'_{\flat_1} \in \Im_{\flat_2}/¥_2, \ldots \hbar'_{\flat_n} \in \Im_{\flat_n}/¥_n} \\ \left\{ \left(\sqrt{1 - \prod_{\perp=1}^{n} \left(1 - \hbar'^{\delta}_{\flat_\perp}\right)^{\aleph(A_{\ell(\perp)}) - \aleph(A_{\ell(\perp-1)})}} \right)^{\frac{1}{\delta}} \right\} \end{array} \right) \end{array} \right\}.$$

$$\flat_1, \flat_2, ..., \flat_n) \tag{3.17}$$

$$= \left\{ \frac{1}{\delta} \left\{ \begin{array}{c} \left((\delta(\flat_{\ell(1)})^{\aleph(A_{\ell(1)}) - \aleph(A_{\ell(0)})} \right) \otimes \\ \delta \left((\flat_{\ell(2)})^{\aleph(A_{\ell(2)}) - \aleph(A_{\ell(1)})} \right) \otimes ... \otimes \\ \delta \left((\flat_{\ell(n)})^{\aleph(A_{\ell(n)}) - \aleph(A_{\ell(n-1)})} \right) \end{array} \right\} \right\}$$

where $\delta > 0$, permutation $\{\ell(1), \ell(2), ..., \ell(n)\}$ of $(1, 2, 3, ..., n)$ such that

$$\flat_{\ell(1)} \geq \flat_{\ell(2)} \geq ... \geq \flat_{\ell(n)}, A_{\ell(k)} = \{\varkappa_{\ell(\perp)} | \perp \leq k\}$$

for $k \geq 1$ and $\flat_{\ell(0)} = \phi$.

4. Decision-making based on Pythagorean probabilistic hesitant fuzzy information

The Pythagorean probabilistic hesitant fuzzy aggregation operators are used in this section to perform multi-attribute decision-making while maintaining anonymity.

Step 1. In this step, we we construct the Pythagorean probabilistic hesitant fuzzy decision-matrices $C = h_{\perp j} m \times n$ for decision where

$h_{\perp j} = \left(\mathfrak{R}_{\perp j}, \mathfrak{I}_{\perp j}\right)(\perp = 1, 2, ..., n; j = 1, 2, ..., m)$. The Pythagorean probabilistic hesitant decision matrix can be transformed into the normalized Pythagorean probabilistic hesitant fuzzy decision matrix if the characteristic has two categories, such as cost and benefit attributes.

Step 2. Verify the fuzzy density for every attribute according to given values of experts.

Step 3. Use the defined aggregation operators to extract the PyPHFN $\flat_{\perp}$ $(\perp = 1, 2, ..., n)$ for the alternatives $X_{\perp}$, or the collective overall preference values $\flat_{\perp}$ $(\perp = 1, 2, ..., n)$ of the alternative $X_{\perp}$.

Step 4. Obtained the score values using score function definition.

Step 5. Decide which option, $X_{\perp}$ $(\perp = 1, 2, ..., n)$, is better by ranking the others.

4.1. Numerical example

We provide a numerical example to demonstrate the prospective evaluation of new technology commercialization with Pythagorean probabilistic hesitant fuzzy information in order to highlight the established method in this part. Let's say the decision-makers decide to utilize the following four criteria to assess the new technology being applied to surgical procedures:

I. $Ⓢ_1$ is 3D printing technology: Although 3D printing serves a wide range of purposes across numerous industries, it primarily serves four purposes in the medical sector. Allie Nawrat discovered how this technology could be used to reduce the need for expensive surgical instruments, speed up surgical operations, replace human organ transplants, and enhance the lives of people who depend on prosthetic limbs. Medical technology is just one of the numerous fields in which the approach has been used. The initial digital model is frequently created using medical imaging techniques like X-rays, computed tomography (CT) scans, magnetic resonance imaging (MRI) scans, and ultrasounds before being loaded into the 3D printer. Anatomical models—3D printers can create incredibly accurate and comprehensive anatomical models to aid surgeons in preparing for difficult procedures, leading to better results and at a cheaper cost. Additionally reducing surgical time is 3D technology.

II. $Ⓢ_2$ is New imaging methods: The majority of medical diagnoses are made using imaging technologies. The process of identifying a patient's illness and its symptoms is known as medical diagnosis. The medical diagnosis provides the data on the illness or condition that is required for therapy and is gathered from the patient's medical history, physical examinations, or surveys. The numerous indications and symptoms of a condition can be difficult to diagnose because they lack specificity. For instance, erythema (skin redness) is an indication of numerous disorders. As a result, many diagnostic techniques are required to identify the root causes of certain diseases and to treat or prevent them. The first medical diagnosis ever made by humans was based on observations made by ancient physicians using their eyes, ears, and occasionally by examining human specimens. For instance, testing on bodily fluids like urine and saliva was done using the earliest techniques (before 400 B.C). Ancient Egyptian and Mesopotamian physicians were able to assess issues with the gastrointestinal tract, blood flow, heartbeat, spleen, liver, monthly irregularities, etc. But sadly, only the wealthy and aristocracy had access to medical care. Numerous sophisticated techniques have been created and their working principles, applications in medical labs, and advancements in imaging techniques can all be used to describe them. Advanced medical imaging techniques include digital mammography, sonography, positron emission tomography (PET), magnetic resonance imaging (MRI), single-photon emission computed tomography (SPECT), and computed tomography (CT). In order to comprehend their benefits and uses in the diagnosis, management, and treatment of many diseases, such as cardiovascular disease, cancer, neurological conditions, and trauma, these are all listed below. Clinicians utilize these approaches frequently because they can quickly decide how to treat diseases based on photographs.

III. $\circledS_3$ **is Surgical robots:** Robotic surgery, commonly referred to as robot-assisted surgery, allows doctors to perform a range of complex procedures with more precision, adaptability, and control than is possible using conventional techniques. Minimally invasive surgery, which involves procedures done through small incisions, is usually linked to robotic surgery. The most popular clinical robotic surgical system has an arm for the camera and mechanized arms with surgical tools fastened to them. The surgeon sits and operates the arms from a computer station beside the operating table. The console shows the surgeon an expanded, high-definition 3D image of the surgery location. The other team members who assist with the procedure are guided by the surgeon. Surgeons who use the robotic system say it enhances precision, flexibility, and control during the procedure and gives them a better view of the area as compared to traditional methods. Robotic surgery enables surgeons to do delicate and sophisticated procedures that would otherwise be difficult or impossible. The risks of robotic surgery may include some that are similar to those of open surgery, such as a minuscule chance of infection and other issues.

IV. $\circledS_4$ **is Artificial Intelligence (AI):** Machine learning models are used in medicine to scan medical data and unearth insights to assist enhance patient experiences and health outcomes. Building intelligent machines that can carry out tasks that traditionally require human intelligence is the focus of the broad field of artificial intelligence (AI). Automated interfaces for speech recognition, decision-making, visual perception, and language translation are some applications of AI. AI is a multidisciplinary field of study. Machine learning models are used in medicine to scan medical data and unearth insights to assist enhance patient experiences and health outcomes. Artificial intelligence (AI) has recently made significant strides in computer science and informatics, and it is now becoming a crucial component of contemporary healthcare. Medical practitioners are supported by AI algorithms and other applications powered by AI in clinical settings and current research.

Clinical decision assistance and image analysis are currently AI's most prevalent uses in medical settings. By giving them instant access to information or research that is pertinent to their patient, clinical decision support systems assist physicians in making decisions about treatments, drugs, mental health, and other patient requirements. AI technologies are being used to analyse CT scans, x-rays, MRIs, and other pictures in the field of medical imaging in order to look for lesions or other findings that a human radiologist would overlook. Many healthcare organizations around the world have begun field-testing new Automation technologies, such as algorithms designed to support patient monitoring and AI-powered tools to screen COVID-19 patients, as a result of the difficulties that the COVID-19 pandemic formed for many medical systems. The overall standards for the use of AI in medicine are still being established, as are the research and test outcomes. However, there are more and more chances for AI to help doctors, scientists, and the patients they treat. There is currently little question that AI will play a significant role in the digital health platforms that will influence and assist modern medicine.

4.2. Algorithm steps with numerical data

Step 1. In the Table 1, we summarize the expert information in the form of benefit attributes. So we don't need to normalize the data.

Table 1. Experts information.

$\Bbbk$	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
Ⓢ$_1$	$\left(\begin{array}{c}\left\{\frac{0.1}{0.4},\frac{0.9}{0.3},\frac{0.1}{0.3}\right\},\\ \left\{\frac{0.14}{0.6},\frac{0.4}{0.4}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.81}{0.45},\frac{0.19}{0.55}\right\},\\ \left\{\frac{0.61}{0.65},\frac{0.89}{0.35}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.75}{1.0}\right\},\\ \left\{\frac{0.19}{1.0}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.2}{0.2},\frac{0.9}{0.3},\frac{0.1}{0.5}\right\},\\ \left\{\frac{0.2}{0.3},\frac{0.9}{0.3},\frac{0.3}{0.4}\right\}\end{array}\right)$
Ⓢ$_2$	$\left(\begin{array}{c}\left\{\frac{0.55}{1.0}\right\},\\ \left\{\frac{0.05}{0.45},\frac{0.95}{0.55}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.15}{0.67},\frac{0.95}{0.33}\right\},\\ \left\{\frac{0.44}{0.6},\frac{0.96}{0.4}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.1}{0.4},\frac{0.9}{0.6}\right\},\\ \left\{\frac{0.8}{1.0}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.8}{0.1},\frac{0.1}{0.5},\frac{0.1}{0.4}\right\},\\ \left\{\frac{0.3}{0.3},\frac{0.7}{0.6},\frac{0.1}{0.1}\right\}\end{array}\right)$
Ⓢ$_3$	$\left(\begin{array}{c}\left\{\frac{0.01}{0.14},\frac{0.99}{0.86}\right\},\\ \left\{\frac{0.12}{0.4},\frac{0.95}{0.3},\frac{0.11}{0.3}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.89}{1.0}\right\},\\ \left\{\frac{0.1}{0.65},\frac{0.9}{0.35}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.1}{0.24},\frac{0.9}{0.76}\right\},\\ \left\{\frac{0.34}{1.0}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.1}{0.5},\frac{0.5}{0.3},\frac{0.7}{0.2}\right\},\\ \left\{\frac{0.1}{0.2},\frac{0.6}{0.6},\frac{0.7}{0.2}\right\}\end{array}\right)$
Ⓢ$_4$	$\left(\begin{array}{c}\left\{\frac{0.51}{0.3},\frac{0.19}{0.3},\frac{0.12}{0.4}\right\},\\ \left\{\frac{0.15}{1.0}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.45}{0.8},\frac{0.6}{0.2}\right\},\\ \left\{\frac{0.9}{1.0}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.7}{0.5},\frac{0.9}{0.5}\right\},\\ \left\{\frac{0.3}{0.4},\frac{0.9}{0.7}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\frac{0.4}{0.4},\frac{0.9}{0.4},\frac{0.1}{0.2}\right\},\\ \left\{\frac{0.5}{0.8},\frac{0.6}{0.1},\frac{0.7}{0.1}\right\}\end{array}\right)$

Step 2. Now we find the density function values according to the expert information. If

$\aleph(A_{\ell_1}) = 0.23, \aleph(A_{\ell_2}) = 0.13, \aleph(A_{\ell_3}) = 0.33, \aleph(A_{\ell_4}) = 0.24$,then $\lambda = 0.11$.

Now, $\aleph(A_{\ell_1}, A_{\ell_2}) = 0.51, \aleph(A_{\ell_1}, A_{\ell_3}) = 0.62, \aleph(A_{\ell_1}, A_{\ell_4}) = 0.53, \aleph(A_{\ell_2}, A_{\ell_3}) = 0.62, \aleph(A_{\ell_2}, A_{\ell_4}) = 0.5, \aleph(A_{\ell_3}, A_{\ell_4}) = 0.54$,

$\aleph(A_{\ell_1}, A_{\ell_2}, A_{\ell_3}) = 0.81, \aleph(A_{\ell_1}, A_{\ell_2}, A_{\ell_4}) = 0.71, \aleph(A_{\ell_1}, A_{\ell_3}, A_{\ell_4}) = 0.77, \aleph(A_{\ell_2}, A_{\ell_3}, A_{\ell_4}) = 0.78, \aleph(A_{\ell_1}, A_{\ell_2}, A_{\ell_3}, A_{\ell_4}) = 0.99$.

Step 3. Using the aggregation operators PyPHF-CIA and PyPHF-CIG find out the performance of the alternatives.

Step 4. Applying the Score function to find out the score values of the alternatives. The score values are given in the Table 2.

Table 2. Score values.

Operators	Ⓢ$_1$	Ⓢ$_2$	Ⓢ$_3$	Ⓢ$_4$
PyPHF-CIA	0.3452	0.3734	0.4253	0.2439
PyPHF-CIG	0.3843	0.2955	0.3965	0.1442

Step 5. Rank the alternatives $Ⓢ_k (k = 1, 2, ..., 4)$ is enclosed in Table 3.

Table 3. Ranking of the alternatives

Operators	Ranking	Best option
PyPHF-CIA	$Ⓢ_3 > Ⓢ_2 > Ⓢ_1 > Ⓢ_4$	$Ⓢ_3$
PyPHF-CIG	$Ⓢ_3 > Ⓢ_1 > Ⓢ_2 > Ⓢ_4$	$Ⓢ_3$

4.3. Comparison analysis

The practicality of the proposed process, the adaptability of its aggregation to handle certain inputs and outputs, the influence of score functions, sensitivity analysis, superiority, and finally the comparison of the proposed methodology with current methods are all covered in this section. Because the Choquet Integral improves and smoothes the data and converts it to simple pythagorean numbers, the suggested solution is accurate and appropriate for all types of input data. The framework that was developed effectively manages uncertainties. We come across a variety of components and input parameters under the appropriate conditions in various MADM problems. The suggested PyPHFS is easy to use, quick to comprehend, and flexible to a wide range of features and options. The offered approaches can be used to a range of input and output circumstances with confidence and practicality. Due to the multiple scoring functions, there is no distinction in the classification of the offered algorithms. Since comparison criteria vary dependent on the MADM system circumstances and raise grade space, this methodology is more accurate than others.

In this section, we compare our results with other defined operators for validation of our method. Using the information of Table 1,

We find the following results as shown in Table 4. We compared our results using Generalized PyPHF-CIA (GPyPHF-CIA), Generalized PyPHF-CIG (GPyPHF-CIG), PyPHF weighted Averaging (PyPHFWA) [43] and PyPHF weighted geometric (PyPHFWG) operators [43]. We can see in the table that the ordering in the current method is correct and other methods can give some other options but the best option is accurate.

Table 4. Ranking of the alternatives.

Sr	Operators	Ranking	Best option	Worst option
1	PyPHF-CIA	$Ⓢ_3 > Ⓢ_2 > Ⓢ_1 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$
2	PyPHF-CIG	$Ⓢ_3 > Ⓢ_1 > Ⓢ_2 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$
3	GPyPHF-CIA	$Ⓢ_3 > Ⓢ_2 > Ⓢ_1 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$
4	GPyPHF-CIG	$Ⓢ_3 > Ⓢ_2 > Ⓢ_1 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$
5	PyPHFWA	$Ⓢ_3 > Ⓢ_1 > Ⓢ_2 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$
6	PyPHFWG	$Ⓢ_3 > Ⓢ_1 > Ⓢ_2 > Ⓢ_4$	$Ⓢ_3$	$Ⓢ_4$

From the comparison analysis, we can see that the best option is $Ⓢ_3$ and the worst option is $Ⓢ_4$. As shown in the Figure 1 that the gray line is the best option in each operator and the yellow line shows the worst option. The blue line is the linear line for operators.

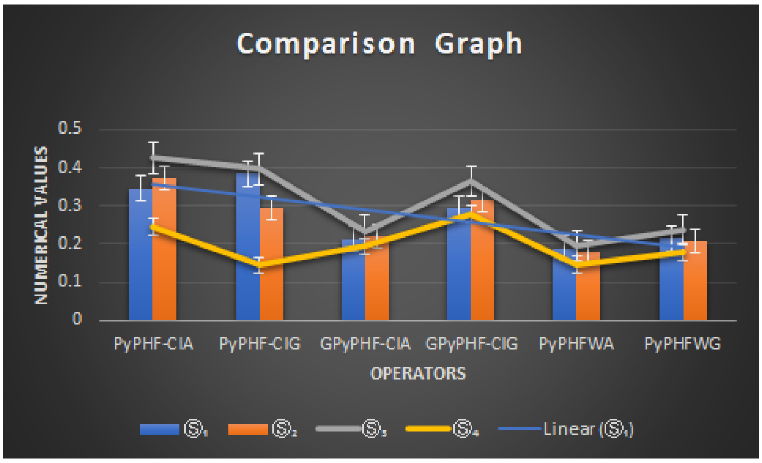

Figure 1. Graphical comparison analysis.

5. Conclusions

We deal with sophisticated and complex data every day. To work more efficiently and compute thorough information, we developed methodologies and tools for this type of data. Aggregation entails inherent costs to reduce the volume of data to a single value. The PyPHFS was created as a potent amalgam of a PyFS and PHFS for situations where each item has a range of probable values that are dictated by MG and NMG. On this page, operators for PyPHF-CIA and PyPHF-CIG are recommended. Additionally, a novel MCDM approach based on the PyPHF-CIA and PyPHF-CIG operators was proposed. More information about these techniques' advantages is provided below.

(1) Idempotency, commutativity, boundedness, and monotonicity of the PyPHF-CIA and PyPHF-CIG operators are first explored.

(2) Second, it compares the flexibility of the suggested AOs to the earlier AOs and shows that our suggested operators are more adaptable than the earlier operators.

(3) Third, when compared to other existing techniques for MCDM problems in a PyPHF setting, the results produced by the PyPHF-CIA and PyPHF-CIG operators are accurate and dependable, indicating their usefulness in real-world scenarios.

(4) The MCDM techniques proposed in this paper are also capable of recognizing more correlation between attributes and alternatives, demonstrating that they have a higher accuracy and a larger set-point than the existing methodologies, which are unable to take into account the inter relationships of attributes in practical uses. This shows that by using the MCDM procedures outlined in this paper, many additional links between features may be discovered.

(5) The suggested aggregation operators are also utilized in practise to look at analysis in order to determine a useful option.

(6) The suggested AOs could be applied to future studies on personalized individual stability handling consensus difficulties, consensus reaching with non-cooperative behavior management decision-making issues, and two-sided perfectly matched decision-making with multi-granular and incomplete criteria weight information. This analysis of the restrictions imposed by recommended AOs disregards the levels of participation, abstention, and nonmembership. On this side of the anticipated AOs, a novel hybrid structure of prioritized, interactive AOs is being implemented.

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Conflict of interest

The authors declare no conflict of interest.

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