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Research article

Benefiting from statistical modeling in the analysis of current health expenditure to gross domestic product

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Abstract: In this article, we provide a novel criterion for decision making by addressing the statistical analysis and modeling of health protection expenditures relative to health system of gross domestic product in a comparative study of four different countries, namely the United States, Malaysia, Egypt, and kingdom of Saudi Arabia. Researchers examined the issue of spending on health protection expenditures in relation to gross domestic product from a variety of angles, including social and statistical. Previous statistical studies also addressed the study of statistical modeling through regression approach. Here we study this issue from a different perspective, namely modeling with statistical distributions. In the statistical modeling of the data, we use an extended heavy-tailed updated version of Weibull distribution named the generalized Weibull distribution Weibull (GWD-W) model, which has good statistical properties in terms of flexibility and goodness of fit. Some distributional properties and statistical functions, including the Renyi entropy, skewness, kurtosis, the heavy-tailed behavior, regular variation, and identifiable property are given. Two important actuarial risk measures are derived. A simulation study is conducted to prove the usefulness of the two actuarial measures in finance. The estimation of the model parameters via the maximum likelihood approach is discussed. Comparison study vs some competitive statistical models is performed using the Kolmogorov-Smirnov test for a sample and some detection criteria. The discussion shows that proposed statistical modeling of health care expenditure as a percentage of gross domestic product (GDP) for health care compares well with their peers.

Keywords: current health expenditure; GDP; Weibull distribution; maximum likelihood estimation; statistical modeling
Mathematics Subject Classification: 62N02, 62E10, 62N01

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1. Introduction

The importance of the study lies in the fact that the decline in government support for the health sectors leads to a decline in social well-being indicators, which requires work to increase social expenditure on members of society, which is reflected in economic growth. Therefore, the importance of the study by explaining the nature and extent of the relationship between aspects of social government expenditure, especially in the field of health and GDP. Many countries in the world generally suffer from a lack of social programs and policies that affect the health of individuals and society, leading to a decline in social welfare and, at the same time, a decline in economic growth rates.

In this article, we address the statistical modeling of health protection expenditures relative to health system GDP for four different countries in terms of social policies, social welfare levels, credentials, and economic growth rates. The countries are Saudi Arabia, Egypt, Malaysia, and the United States, which provides a future vision for policy makers.

The researchers examined the issue of spending on health protection in relation to gross domestic product from different angles, including political, social, and statistical. Sardar et al. [1] examined the issue of creating a measure of social welfare based on unadjusted GDP, as the researchers relied on the benefit-cost analysis method for economic growth. The researchers found that there is a significant difference between GDP per capita and adjusted GDP per capita, so GDP can be used as a measure of social welfare. Sardar and Matthew [2] used a modern methodology to analyse the relationships between economic growth, health outcomes for individuals, and social well-being in both developing and developed countries. The researchers concluded that economic growth can improve health outcomes and expand the health sector and social well-being, but that its impact is limited due to biological laws. In addition, achieving economic growth may have a negative external effect that reduces health outcomes. Therefore, researchers used the health-to-GDP indicator to show the relationship between economic growth, health outcomes, and social well-being of individuals from the perspective of social choice. Therefore, the importance of improving health to social well-being is significant. Brent [3] has developed a number of alternative applications for measuring well-being that have emerged in recent decades. These applications are linked to specific indicators to assess the application of well-being when economic progress is achieved. Romina et al. [4] has concluded that there are several indicators to measure the specific social indicators related to well-being. In the countries of the Organisation for Economic Cooperation, the values of most of these social indicators are significantly related to average GDP per capita, but the changes in these indicators over time are not significant.

Malin [5] examined the relationship between economic growth and happiness from intellectual, political, and economic perspectives with what is happening in the Western world today. Gabriel [6] analyzed firstly, the expenditure on social protection spending in the Greek social system and second estimated the relationship between this expenditure on social protection spending and economic growth. The researcher found that the relationship between social protection spending and GDP is low compared to other European Union countries, such as Spain and Portugal. Hong [7] examined standard tests of the impact of economic growth on public social spending using its main components: Income Support, Pensions, and Other Health Benefits. The researcher used the mutual effect model of cross-sectional data for countries of the Organisation for Economic Cooperation and Development, which showed a strong and negative correlation between the rate of social spending. Partha and Edward [8] usd two classes of nonlinear statistical models to described the health care utilisation and spending. Malehi et al. found that the gamma regression model behaved well in estimating the population mean of health care costs. The approximate results are consistent when the sample size is increased. Cuckler and Sisko [10] described the methods underlying the econometric model. Guemmegne [11] et al. examined the dynamics of national health care spending in the United States from 1960 to 2011.

Statistical methods play a crucial role in analysis of medical data [12, 13], environmental data [14, 15], engineering data [16, 17], social data [18], actuarial data [19], test data [20], reliability data [21], sports data [22], educational data [23], measurement system errors [24], risk assessment [25], robust analysis [26].

Let X be a random variable (R.V.) follows the three-parameters Weibull(β, φ, ϕ), and $x \geq \phi; \beta, \varphi, \phi > 0$, then its CDF (cumulative distribution function) is:

$$W(x; \beta, \varphi, \phi) = 1 - e^{-\left(\frac{x-\phi}{\varphi}\right)^{\beta}}. \tag{1.1}$$

The probability density function (PDF) is

$$w(x; \beta, \varphi, \phi) = \frac{\beta}{\varphi}\left(\frac{x-\phi}{\varphi}\right)^{\beta-1} e^{-\left(\frac{x-\phi}{\varphi}\right)^{\beta}}, \qquad x \geq \phi; \beta, \varphi, \phi > 0. \tag{1.2}$$

In general, actuarial are skewed positively [27, 28], unimodally shaped [29] and have heavy tails [30]. Therefore, some right-skewed and unimodal models have been utilized to modelling such data [31–34].

2. The statistical model

Cordeiro et al. [35] propose the generalized Weibull distribution (GWD-X) family. The CDF and PDF can be, respectively, written as

$$F(x; \kappa, \delta) = 1 - e^{-\kappa A^{\delta}(x)}, \qquad x \in \mathbb{R}; \kappa, \delta > 0, \tag{2.1}$$

and

$$f(x; \kappa, \delta) = \kappa\delta A^{\delta-1}(x) e^{A(x)-\kappa A^{\delta}(x)} g(x), \qquad x \in \mathbb{R}; \kappa, \delta > 0, \tag{2.2}$$

where $A(x) = -\log(1 - G(x))$. The special sub-models of the GWD-X family provide symmetric, asymmetric, density, unimodal , and bimodal shapes. By replacing $g(x)$ and $G(x)$ in Eqs (2.1) and (2.2) by $w(x; \beta, \varphi, \phi)$ and $W(x; \beta, \varphi, \phi)$ in Eq (1.2), give the CDF of the GWD-W model as

$$F(x; \kappa, \beta', \varphi, \phi) = 1 - e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}, \qquad x \geq \phi; \kappa, \beta', \varphi, \phi > 0, \tag{2.3}$$

where $\beta' = \delta\beta$. The corresponding PDF (f), hazard function (h), survival function (S), cumulative hazard function (H), reverse hazard (r), and the quantile (X_p) function are

$$f(x;\kappa,\beta',\varphi,\phi) = \frac{\kappa\beta'}{\varphi}\left(\frac{x-\phi}{\varphi}\right)^{\beta'-1} e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}, \tag{2.4}$$

$$h(x;\kappa,\beta',\varphi,\phi) = \frac{f(x;\kappa,\beta',\varphi,\phi)}{1-F(x;\kappa,\beta',\varphi,\phi)} = \frac{\kappa\beta'}{\varphi}\left(\frac{x-\phi}{\varphi}\right)^{\beta'-1}, \tag{2.5}$$

$$S(x;\kappa,\beta',\varphi,\phi) = 1-F(x;\kappa,\beta',\varphi,\phi) = e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}, \tag{2.6}$$

$$H(x;\kappa,\beta',\varphi,\phi) = -\log\left(F(x;\kappa,\beta',\varphi,\phi)\right) = -\log\left(1-e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}\right), \tag{2.7}$$

$$r(x;\kappa,\beta',\varphi,\phi) = \frac{f(x;\kappa,\beta',\varphi,\phi)}{F(x;\kappa,\beta',\varphi,\phi)} = \frac{\kappa\beta'\left(\frac{x-\phi}{\varphi}\right)^{\beta'-1}}{\varphi\left(e^{\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}-1\right)}, \tag{2.8}$$

$$X_p(p;\kappa,\beta',\varphi,\phi) = F(x;\kappa,\beta',\varphi,\phi)^{-1}(p) = \phi+\varphi\left(-\frac{\log(1-p)}{\kappa}\right)^{\frac{1}{\beta'}}, \quad p>0. \tag{2.9}$$

The GWD-W model are motivated by: (i) a convenient way to mutate the Weibull model, as well as have simple and closed forms; (ii) improve the flexibility of Weibull model, can provide right-skewed, left-skewed form, unimodal, increasing, decreasing, symmetric, asymmetric curved densities; (iii) improve the statistical properties of Weibull model; (iv) includes the Weibull distribution as a special case and it can provide adequate fit for positively skewed actuarial data; (v) offers also the heavy-tailed behavior and the regular variational property.

3. Distributional properties

Here, we derive some distribution properties of the GWD-W model. These distribution properties include linear representation, Renyi entropy, skewness, kurtosis, heavy-tailed characteristic, regular variational property, VAR (value at risk), TVAR (tail value at risk), and the identifiability property (I-P).

3.1. Linear representation

Using the real exponential function : $\mathbb{R}\to\mathbb{R}$ that commonly defined by the following power series:

$$e^x = \sum_{J=0}^{\infty}\frac{x^J}{J!}. \tag{3.1}$$

The corresponding GWD-W density can be written as

$$f(x;\kappa,\beta',\varphi,\phi) = \frac{\kappa\beta'}{\varphi}\sum_{J=0}^{\infty}\frac{(-\kappa)^J}{J!}\left(\frac{x-\phi}{\varphi}\right)^{(J'+1)\beta'-1}, \qquad x\geq\phi;\kappa,\beta',\varphi,\phi>0. \tag{3.2}$$

Also the corresponding GWD-W CDF can be expressed as

$$F(x;\kappa,\beta',\varphi,\phi) = 1-\sum_{J=0}^{\infty}\frac{(-\kappa)^J}{J!}\left(\frac{x-\phi}{\varphi}\right)^{J'\beta'}, \qquad x\geq\phi;\kappa,\beta',\varphi,\phi>0. \tag{3.3}$$

Based on Eq (3.3), if X is R.V. has the GWD-W model, then the transformed R.V. Z $(= \frac{X-\phi}{\varphi} \geq 0)$ has CDF $F(z;\kappa)$, and is defined by

$$F(z;\kappa) = 1 - \sum_{J=0}^{\infty} \frac{(-\kappa)^J}{J!} z^{J'}, \qquad z \geq 0; \kappa > 0. \tag{3.4}$$

Hence, the distribution function of the GWD-W model can be expressed as a linear combination of the transformed R.V. Z, which $Z = \frac{X-\phi}{\varphi}$. Using the same procedures, all the GWD-W mode functions Eq (2.4)–(2.9), can be derived in the form of a linear combination in terms of Z.

3.2. Renyi entropy

Let X is R.V. has the PDF $f(x)$, the Renyi entropy $(R_{\varrho}(X))$ of X is a measure of variation of uncertainty. The $R_{\varrho}(X)$ for any $\varrho > 0, \varrho \neq 1$ (see, Golomb [36]), is given by

$$R_{\varrho}(X) = \frac{1}{1-\varrho} \log\left(\int_X (f(x))^{\varrho}\, dx\right), \qquad \varrho > 0, \varrho \neq 1. \tag{3.5}$$

Now, let the R.V. $X \sim$ GWD-W model, $(\kappa, \beta', \varphi, \phi)$. By substituting (2.4) in (3.5), $R_{\varrho}(X)$ become

$$R_{\varrho}(X) = \frac{1}{1-\varrho} \log\left(\int_{\phi}^{\infty} \left(\frac{\kappa\beta'}{\varphi}\left(\frac{x-\phi}{\varphi}\right)^{\beta'-1} e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}\right)^{\varrho} dx\right), \qquad \varrho > 0, \varrho \neq 1. \tag{3.6}$$

By solving the integration and using the transformation $y = \kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}$, we have

$$R_{\varrho}(X) = \frac{1}{1-\varrho} \log\left(\left(\frac{\kappa}{\varphi}\right)^{\varrho} (\beta')^{\varrho-1} (\kappa\varrho)^{\frac{\varrho-1}{\beta'}-\varrho}\, \Gamma\left(\frac{1-\varrho}{\beta'} + \varrho\right)\right), \qquad \varrho > 0, \varrho \neq 1. \tag{3.7}$$

3.3. The skewness and kurtosis functions

The Skewness (S_k) (see, Bowley [37]) and Kurtosis (K) (see, Moor [38]) formulas are given by, respectively

$$S_k = \frac{2X_{1/2} - X_{3/4} - X_{1/4}}{X_{1/4} - X_{3/4}}, \quad \text{and } K = \frac{X_{1/8} - X_{3/8} + X_{5/8} - X_{7/8}}{X_{2/8} - X_{6/8}}.$$

And by using Eq (2.9), we have

$$S_k(\beta'') = \frac{\log\left(\frac{4}{3}\right)^{\beta''} - 2\log(2)^{\beta''} + \log(4)^{\beta''}}{-\log\left(\frac{4}{3}\right)^{\beta''} + \log(4)^{\beta''}}, \tag{3.8}$$

$$K(\beta'') = \frac{\log\left(\frac{8}{7}\right)^{\beta''} - \log\left(\frac{8}{5}\right)^{\beta''} + \log\left(\frac{8}{3}\right)^{\beta''} - \log(8)^{\beta''}}{\log\left(\frac{4}{3}\right)^{\beta''} - \log(4)^{\beta''}}, \tag{3.9}$$

where $\beta'' = \frac{1}{\beta'}$. Table 1 represents some numerical values of the X_p at the median ($p = 0.50$) when $\kappa = 0.2, \varphi = 0.2, \phi = 0.2,$), S_k and K for $\beta' = 0.1, 0.3, 0.5, 0.7, 0.9, 1, 2, 5, 10$. Figure 1 plots for the S_k and K and for different values of β'.

Table 1. Some numerical values of the X_p, S_k and K for some β' values.

β'	X_p	S_k	K
0.1	50001.9	0.998047	57.6336
0.3	12.7994	0.811157	3.59360
0.5	2.60227	0.567503	1.93857
0.7	1.38077	0.405684	1.51258
0.9	0.99579	0.300789	1.34988
1.0	0.89314	0.261860	1.30627
2.0	0.57233	0.075908	1.20397
5.0	0.45644	-0.04058	1.22495
10	0.42646	-0.07957	1.24738
100	0.40250	-0.11455	1.27465

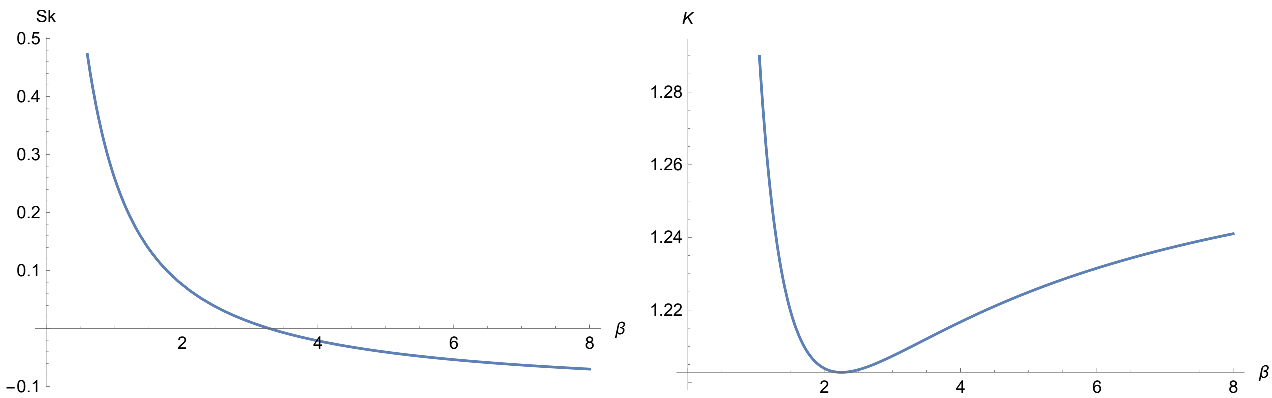

Figure 1. Plots for the S_k(left) and K (right).

3.4. The heavy-tailed behavior

The GWD-W model offers also the heavy-tailed behavior. A probability model is called a heavy tailed distribution, if it satisfies

$$\lim_{y\to\infty} e^{py}(1 - F(y; \kappa, \beta', \varphi, \phi)) \to \infty, \qquad p > 0. \tag{3.10}$$

Theorem 1. Let, $p, \kappa, \varphi, \phi > 0$ and $0 < \beta' < 1$, the PDF $f(y; \kappa, \beta', \varphi, \phi)$ that given in Eq (2.4) is heavy tailed distribution as $y \to \infty$.

Proof. Based on Eq (2.3), we can write

$$\begin{aligned}
\lim_{y\to\infty} e^{py}(1-F(y;\kappa,\beta',\varphi,\phi)) &= \lim_{y\to\infty} e^{py}\left(e^{-\kappa\left(\frac{x-\phi}{\varphi}\right)^{\beta'}}\right) \\
&= \left(\lim_{y\to\infty} e^{py}\right)\times\left(\lim_{y\to\infty} e^{-\kappa\left(\frac{y-\phi}{\varphi}\right)^{\beta'}}\right) \\
&= e^{\infty}\times\left(e^{-\frac{1}{\infty}}\right)_{0<\beta'<1} \\
&= e^{\infty}\times e^{0} \\
&= e^{\infty}\times 1 \\
&\to \infty. \qquad \square \qquad (3.11)
\end{aligned}$$

According to Seneta's [39] theorem, the GWD-W model in terms of CDF $S\left(y;\kappa,\beta',\varphi,\phi\right)$ is regularly varying, if it satisfies

$$\frac{1-F(py;\kappa,\beta',\varphi,\phi)}{1-F(y;\kappa,\beta',\varphi,\phi)} = p^{\Delta}, \qquad \forall p,\Delta>0. \qquad (3.12)$$

deqnarray where Δ represents an index of regular variation.
Theorem 2. Let, $p,\kappa,\varphi>0$, $\phi=0$ and $0<\beta'<1$, the PDF $f(y;\kappa,\beta',\varphi,\phi)$ that given in Eq (2.4), the PDF $f(y;\kappa,\beta',\varphi,\phi)$ that given in Eq (2.4) is regularly varying model.
Proof. Using Eq (2.3), we have

$$\frac{1-F(x;\kappa,\beta',\varphi,\phi)(\text{p y})}{1-F(x;\kappa,\beta',\varphi,\phi)(y)} = e^{\kappa\left(\frac{y}{\varphi}\right)^{\beta'}\left(1-p^{\beta'}\right)}. \qquad (3.13)$$

Using Eq (3.13), the PDF $f(y;\kappa,\beta',\varphi,\phi)$ that given in Eq (2.4), the PDF $f(y;\kappa,\beta',\varphi,\phi)$ that given in Eq (2.4) with index of regular variation $\Delta=\frac{\kappa}{\log(p)}\left(1-p^{\beta'}\right)\left(\frac{y}{\varphi}\right)^{\beta'}$ is regularly varying model. $\square$

3.5. The VAR and TVAR

Based on the Monte Carlo simulation, the actuarial measures VAR and TVAR are the empirical approaches to find heavy-tailed models. Mathematically, VAR ($VAR_q(X)$) can be specified with a certain confidence level (C.L.) $q\in(0,1)$, which is typically 99%, 95%, or 90%, see, [40]. Explicit expressions of the VAR and TVAR can be produced, as

$$VAR_q(X) = -inf[x\in\mathbb{R}: F_X(x)>q]. \qquad (3.14)$$

deqnarray Let X follow the GWD-W model, based on Eq (2.9), then

$$VAR_q(X) = \phi+\varphi\left(-\frac{\log\left(1-q\right)}{\kappa}\right)^{\frac{1}{\beta'}}. \qquad (3.15)$$

The TVAR is given by

$$TVAR_q(X) = E[X|X\leq -VAR_q(X)]. \qquad (3.16)$$

Let $\phi=0$, and using Eqs (3.14) and (3.16), we have

$$\begin{aligned} TVAR_q(X) &= \frac{1}{1-q}\int_{VAR_q(X)}^{\infty} x f(x;\kappa,\beta',\varphi,\phi)dx \\ &= \frac{1}{1-q}\int_{VAR_q(X)}^{\infty} x\frac{\kappa\beta'}{\varphi}\left(\frac{x}{\varphi}\right)^{\beta'-1} e^{-\kappa\left(\frac{x}{\varphi}\right)^{\beta'}} dx \\ &= \frac{1}{1-q}\kappa\beta' \int_{VAR_q(X)}^{\infty} \left(\frac{x}{\varphi}\right)^{\beta'} e^{-\kappa\left(\frac{x}{\varphi}\right)^{\beta'}} dx, \end{aligned} \tag{3.17}$$

and by using the transformation $z = \left(\frac{x}{\varphi}\right)^{\beta'}$, we have

$$TVAR_q(X) = \frac{1}{1-q}\kappa\varphi \int_{\left(\frac{VAR_q(X)}{\varphi}\right)^{\beta'}}^{\infty} e^{-\kappa z}\, z^{\frac{1}{\beta'}} dz. \tag{3.18}$$

On solving, we get

$$TVAR_q(X) = \frac{\varphi\kappa^{-1/\beta'}}{1-q}\Gamma\left(1+\frac{1}{\beta'},\kappa\left(\frac{VAR_q(X)}{\varphi}\right)^{\beta'}\right), \tag{3.19}$$

where $\Gamma(.,.)$ is incomplete gamma constant.

Using Eqs (3.16) and (3.19), a numerical simulation study of VAR and TVAR to show empirically the heaviness of the GWD-W model tail is provided. The simulation algorithm:

(1) Using Eq (2.9), we generating random samples of size n = 100 from both GWD-W and Weibull models.

(2) Using the maximum likelihood estimation (MLE) for estimating the parameters of both models.

(3) Based on Eqs (3.16) and (3.19), we calculate the VAR and TVAR of both models.

(4) Table 2, report the numerical simulation results of VAR and TVAR for GWD-W ($\kappa = 3.5, \beta' = 2.5, \varphi = 2.5, \phi = 0$) and Weibull ($\beta = 2.5, \varphi = 2.5, \phi = 0$).

(5) Tables 3, report the numerical simulation results of VAR and TVAR for GWD-W ($\kappa = 0.5, \beta' = 2.5, \varphi = 0.5, \phi = 0$) and Weibull ($\beta = 0.5, \varphi = 0.5, \phi = 0$).

(6) For visual comparison and based on the numerical simulation results of Tables 2 and 3, Figures 2 and 3, respectively, show the shapes of the proposed two risk measures of both models.

The numerical simulation results of Figures 2 and 3 illustrate that, the VAR and TVAR results for the GWD-W models are higher than those of the Weibull models, indicating that the GWD-W models can better capture extreme events. The results also show that the GWD-W models have a higher risk measure than the Weibull models, which is beneficial for actuarial applications.

Table 2. Numerical simulation results of VAR and TVAR for n = 100 at different confidence level.

Model	Parameters	C.L.	VAR	TVAR
Weibull	$(\beta = 2.5, \varphi = 2.5, \phi = 0)$	0.700	0.6797	0.8517
		0.750	0.7198	0.8822
		0.800	0.7648	0.9173
		0.850	0.8176	0.9596
		0.900	0.8845	1.0147
		0.950	0.9843	1.0994
		0.990	1.1721	1.2653
GWD-W	$(\kappa = 3.5, \beta' = 2.5, \varphi = 2.5, \phi = 0)$	0.700	1.6724	2.1490
		0.750	1.7858	2.2388
		0.800	1.9085	2.3357
		0.850	2.0534	2.4530
		0.900	2.2409	2.6098
		0.950	2.5081	2.8354
		0.990	3.0540	3.3240

Table 3. Numerical simulation results of VAR and TVAR for n = 100 at different confidence level.

Model	Parameters	C.L.	VAR	TVAR
Weibull	$(\beta = 0.5, \varphi = 0.5, \phi = 0)$	0.700	6.79560	22.7162
		0.750	8.73790	25.7156
		0.800	11.4016	29.6459
		0.850	15.2861	35.1248
		0.900	21.5909	43.6137
		0.950	34.5163	60.1338
		0.990	74.2942	107.758
		0.700	14.9913	44.6265
GWD-W	$(\kappa = 0.5, \beta' = 2.5, \varphi = 0.5, \phi = 0)$	0.750	18.8542	50.0525
		0.800	24.2379	57.5652
		0.850	31.9053	67.8062
		0.900	43.0760	81.5283
		0.950	67.3053	111.495
		0.990	136.093	191.053

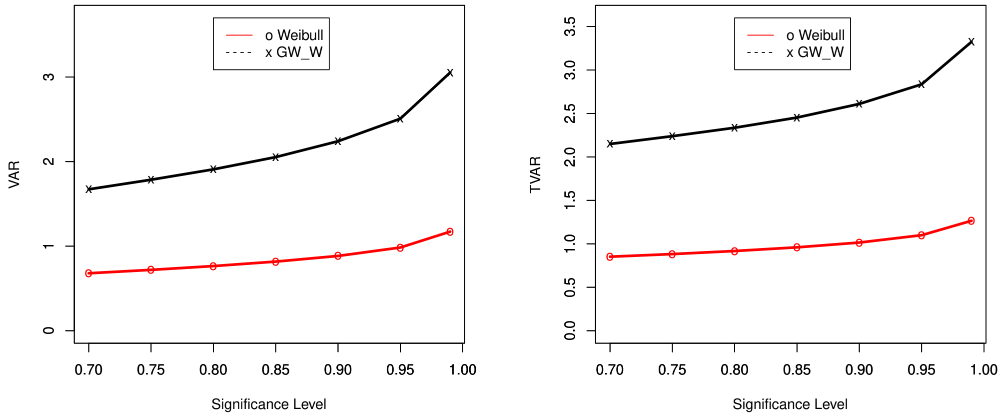

Figure 2. Shapes of the VAR and TVAR of the NEHTW and Weibull distributions based on Table 1.

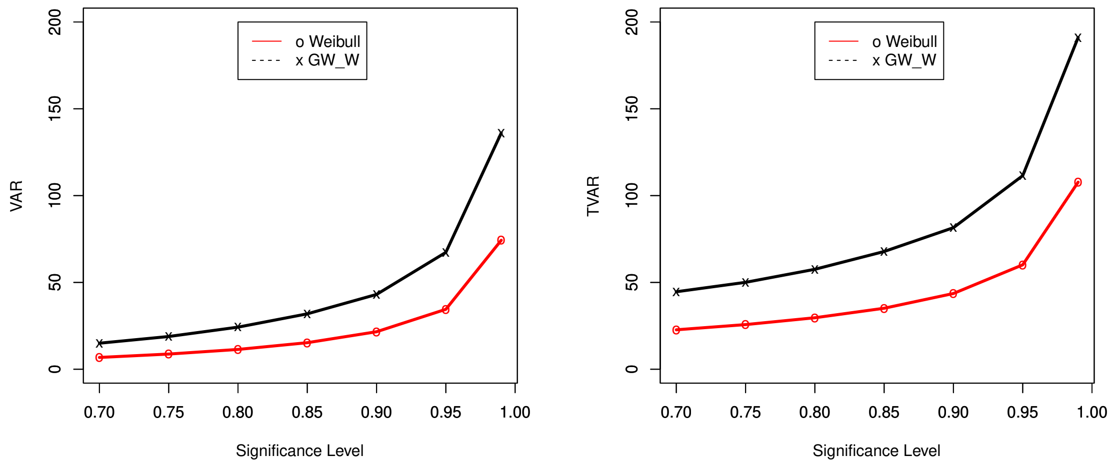

Figure 3. Shapes of the VAR and TVAR of the NEHTW and Weibull distributions based on Table 2.

3.6. The I-P

This subsection offers proof of the I-P of the GWD-W model for the parameters κ, β', φ, and ϕ.

3.6.1. The I-P using κ

Let, $F(x; \kappa_1, \beta', \varphi, \phi) = F(x; \kappa_2, \beta', \varphi, \phi)$, the parameter κ of the GWD-W model is called identifiable, if $\kappa_1 = \kappa_2$. To prove the I-P property of the GWD-W model for κ, we start with

$$\begin{aligned} F(x; \kappa_1, \beta', \varphi, \phi) &= F(x; \kappa_2, \beta', \varphi, \phi) \\ 1 - e^{-\kappa_1 \left(\frac{y-\phi}{\varphi}\right)^{\beta'}} &= 1 - e^{-\kappa_2 \left(\frac{y-\phi}{\varphi}\right)^{\beta'}} \end{aligned}$$

$$
\begin{aligned}
\kappa_1\left(\frac{y-\phi}{\varphi}\right)^{\beta'} &= \kappa_2\left(\frac{y-\phi}{\varphi}\right)^{\beta'} \\
\kappa_1 &= \kappa_2.
\end{aligned}
$$

3.6.2. The I-P using β'

Let, $F(x;\kappa,\beta'_1,\varphi,\phi) = F(x;\kappa,\beta'_2,\varphi,\phi)$, the parameter β' of the GWD-W model is identifiable, such that

$$
\begin{aligned}
F(x;\kappa,\beta'_1,\varphi,\phi) &= F(x;\kappa,\beta'_2,\varphi,\phi) \\
1-e^{-\kappa\left(\frac{y-\phi}{\varphi}\right)^{\beta'_1}} &= 1-e^{-\kappa\left(\frac{y-\phi}{\varphi}\right)^{\beta'_2}} \\
A^{\beta'_1} &= A^{\beta'_2} \\
\beta'_1 &= \beta'_2.
\end{aligned} \tag{3.20}
$$

3.6.3. The I-P using φ

Let, $F(x;\kappa,\beta',\varphi_1,\phi) = F(x;\kappa,\beta',\varphi_2,\phi)$, the parameter φ of the GWD-W model is identifiable, such that

$$
\begin{aligned}
F(x;\kappa,\beta'_1,\varphi,\phi) &= F(x;\kappa,\beta'_2,\varphi,\phi) \\
1-e^{-\kappa\left(\frac{y-\phi}{\varphi_1}\right)^{\beta'}} &= 1-e^{-\kappa\left(\frac{y-\phi}{\varphi_2}\right)^{\beta'}} \\
\left(\frac{y-\phi}{\varphi_1}\right)^{\beta'} &= \left(\frac{y-\phi}{\varphi_1}\right)^{\beta'} \\
\frac{\varphi_1}{y-\phi} &= \frac{\varphi_2}{y-\phi} \\
\varphi_1 &= \varphi_2.
\end{aligned} \tag{3.21}
$$

3.6.4. The I-P using ϕ

Let, $F(x;\kappa,\beta',\varphi,\phi_1) = F(x;\kappa,\beta',\varphi,\phi_2)$, the parameter ϕ of the GWD-W model is identifiable, such that

$$
\begin{aligned}
F(x;\kappa,\beta'_1,\varphi,\phi) &= F(x;\kappa,\beta'_2,\varphi,\phi) \\
1-e^{-\kappa\left(\frac{y-\phi_1}{\varphi}\right)^{\beta'}} &= 1-e^{-\kappa\left(\frac{y-\phi_2}{\varphi}\right)^{\beta'}} \\
\left(\frac{y-\phi_1}{\varphi}\right)^{\beta'} &= \left(\frac{y-\phi_2}{\varphi}\right)^{\beta'} \\
\frac{y-\phi_1}{\varphi} &= \frac{y-\phi_2}{\varphi} \\
\phi_1 &= \phi_2.
\end{aligned} \tag{3.22}
$$

4. Estimations

This section assigns the MLEs to estimate the parameters of the GWD-W model. Suppose $X_{1:n}, X_{2:n}, \ldots X_{r:n}$ is R.V.s from the GWD-W$(\kappa,\beta',\varphi,\phi)$ model that given in (2.4). The GWD-

W(κ,β', φ, ϕ) likelihood is

$$l = \left(\frac{\kappa\beta'}{\varphi^{\beta'}}\right)^n e^{-\kappa\sum_{i=1}^{n}\left(\frac{x-\phi}{\varphi}\right)^{\beta'}} \left(\prod_{i=1}^{n}(x_i-\phi)^{\beta'-1}\right), \tag{4.1}$$

and the log-likelihood function $log(l)$ is

$$L = n\log\left(\frac{\kappa\beta'}{\varphi^{\beta'}}\right) - \kappa\sum_{i=1}^{n}\left(\frac{x_i-\phi}{\varphi}\right)^{\beta'} + (\beta'-1)\sum_{i=1}^{n}\log\left[x_i-\phi\right]. \tag{4.2}$$

The first partial derivatives of (4.2) w.r.t κ,β', φ, ϕ are

$$\frac{\partial log(l)}{\partial\kappa} = \frac{n}{\kappa} - \sum_{i=1}^{n}\left(\frac{x_i-\phi}{\varphi}\right)^{\beta'}, \tag{4.3}$$

$$\frac{\partial log(l)}{\partial\beta'} = \frac{n}{\beta'} - n\log[\varphi] + \delta\sum_{i=1}^{n}\left(\log\left[x_i-\phi\right] - \kappa\left(\frac{x_i-\phi}{\varphi}\right)^{\beta'}\log\left[\frac{-\phi+x_i}{\varphi}\right]\right), \tag{4.4}$$

$$\frac{\partial log(l)}{\partial\varphi} = \frac{\beta'}{\varphi}\left(-n + \frac{\kappa}{\varphi}\sum_{i=1}^{n}(x_i-\phi)\left(\frac{x_i-\phi}{\varphi}\right)^{\beta'-1}\right), \tag{4.5}$$

$$\frac{\partial log(l)}{\partial\phi} = (\beta'-1)\sum_{i=1}^{n}\left(\frac{\kappa\beta'}{\varphi}\left(\frac{x_i-\phi}{\varphi}\right)^{\beta'-1} - \frac{1}{x_i-\phi}\right). \tag{4.6}$$

The MLEs $\hat{\kappa}_{ML}$,$\hat{\beta}'_{ML}$,$\hat{\varphi}_{ML}$, and $\hat{\phi}_{ML}$ of the GWD-W(κ,β', φ, ϕ) parameters are the solutions of the Eqs (4.3)–(4.6) after equating each of them by zero. One can solve them numerically to obtain the MLEs. By using any statistical software, these nonlinear system of equations can be solved.

5. Simulation study

The numerical simulation findings is executed for the GWD-W model by R software (version 4.1.3) with the optim() function, see, Appendix A, and the argument method = "L-BFGS-B". The simulation algorithm is:

(1) We generate sample of sizes n = 25, 50, ..., 1000 from the GWD-W for two different sets of initial parameters.
set 1: $\kappa^{(0)} = 1.7, \beta'^{(0)} = 2.4, \varphi^{(0)} = 0.01$, and $\phi^{(0)} = 3.4$,
set 2: $\kappa^{(0)} = 0.8, \beta'^{(0)} = 1.5, \varphi^{(0)} = 0.02$, and $\phi^{(0)} = 1.2$.

(2) Use sets 1 and 2 for conducting the MLEs of each parameter κ,β'_1, φ, and ϕ.

(3) Repeated the steps (1) and (2) M=1000 times.

(4) Obtain the estimates, then calculate the Bias and the Estimated Risk (ER).

(5) The Biases and ERs for the parameter λ(= κ,β', φ, ϕ) are given by, respectively

$$Bias\left(\hat{\lambda}\right) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\lambda}_i - \lambda\right),$$

and

$$ER\left(\hat{\lambda}\right) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\lambda}_i - \lambda\right)^2.$$

(6) Table 4 presents the estimates, Bias and ER, respectively, for set 1.

(7) Table 5 presents the estimates, Bias and ER, respectively, for set 2.

(8) Figure 4 shows graphically the simulation results of Table 4.

(9) Figure 5 shows graphically the simulation results of Table 5.

Table 4. Simulation results of the GWD-W model for set 1.

	n	$\hat{\kappa}$	$\hat{\phi}$	$\hat{\varphi}$	$\hat{\beta}'$
Estimate	25	1.934058	2.070890	0.3822123	3.160707
	100	1.813316	2.206425	0.1868472	3.289071
	200	1.781590	2.252850	0.1438054	3.321575
	400	1.760950	2.282059	0.1180102	3.342488
	600	1.753156	2.295761	0.1091235	3.350055
	800	1.744710	2.314527	0.1041973	3.357933
	1000	1.743711	2.314436	0.1036497	3.359054
ER	25	0.104241	0.353084	0.2418436	0.115640
	100	0.025832	0.088620	0.0505954	0.025839
	200	0.012392	0.045711	0.0243883	0.011988
	400	0.006234	0.025515	0.0136085	0.005797
	600	0.004440	0.018176	0.0105188	0.004084
	800	0.003378	0.013216	0.0091346	0.003109
	1000	0.003117	0.013031	0.0089548	0.002836
Bias	25	0.234057	-0.32910	0.3722122	-0.23929
	100	0.113316	-0.19357	0.1768471	-0.11092
	200	0.081589	-0.14714	0.1338053	-0.07842
	400	0.060949	-0.11794	0.1080101	-0.05751
	600	0.053156	-0.10423	0.0991234	-0.04994
	800	0.044710	-0.08547	0.0941973	-0.04206
	1000	0.043710	-0.08556	0.0936496	-0.04094

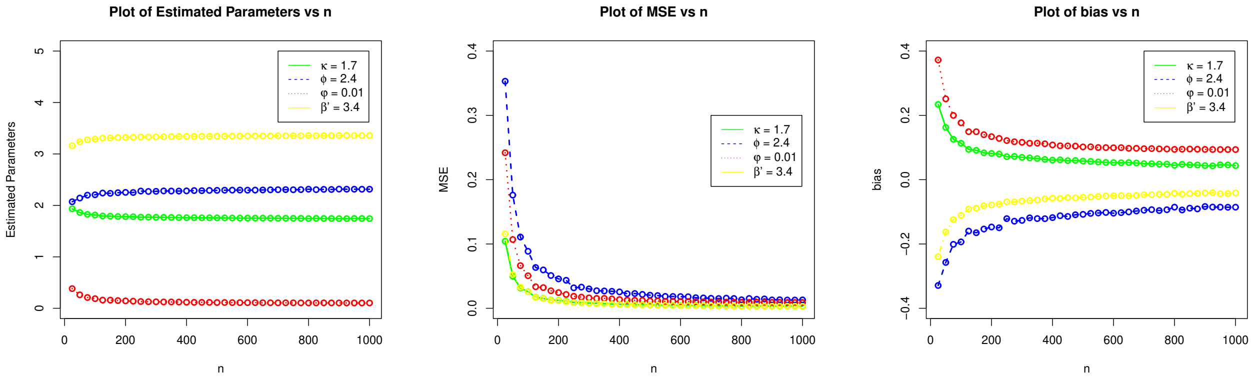

Figure 4. Plot of the simulation results of the GWD-W model.

Table 5. Simulation results of the GWD-W model for set 2.

	n	$\hat{\kappa}$	$\hat{\phi}$	$\hat{\varphi}$	$\hat{\beta}'$
	25	1.0827323	1.438229	0.2050052	2.334154
	100	0.9054890	1.459133	0.1167855	2.304047
	200	0.8321919	1.487253	0.1024533	2.301241
Estimate	400	0.8075830	1.496414	0.1001413	2.300303
	600	0.8012212	1.499789	0.1000194	2.300012
	800	0.8	1.5	0.1	2.3
	1000	0.8	1.5	0.1	2.3
	25	0.282732345	-6.18e-02	0.18500519	0.2341538
	100	0.105488959	-4.09e-02	0.09678553	0.2040466
	200	0.032191938	-1.27e-02	0.08245334	0.2012406
ER	400	0.007583002	-3.59e-03	0.08014134	0.2003029
	600	0.001221231	-2.11e-04	0.08001936	0.200012
	800	0	0	0.08	0.2
	1000	0	0	0.08	0.2
	25	1.56e-01	3.78e-02	0.056146880	0.06017765
	100	3.06e-02	8.54e-03	0.011109625	0.04171349
	200	7.53e-03	2.14e-03	0.006917182	0.04051953
Bias	400	1.63e-03	4.35e-04	0.006424866	0.04012519
	600	2.17e-04	2.91e-05	0.006403419	0.04000489
	800	0	0	0.0064	0.04
	1000	0	0	0.0064	0.04

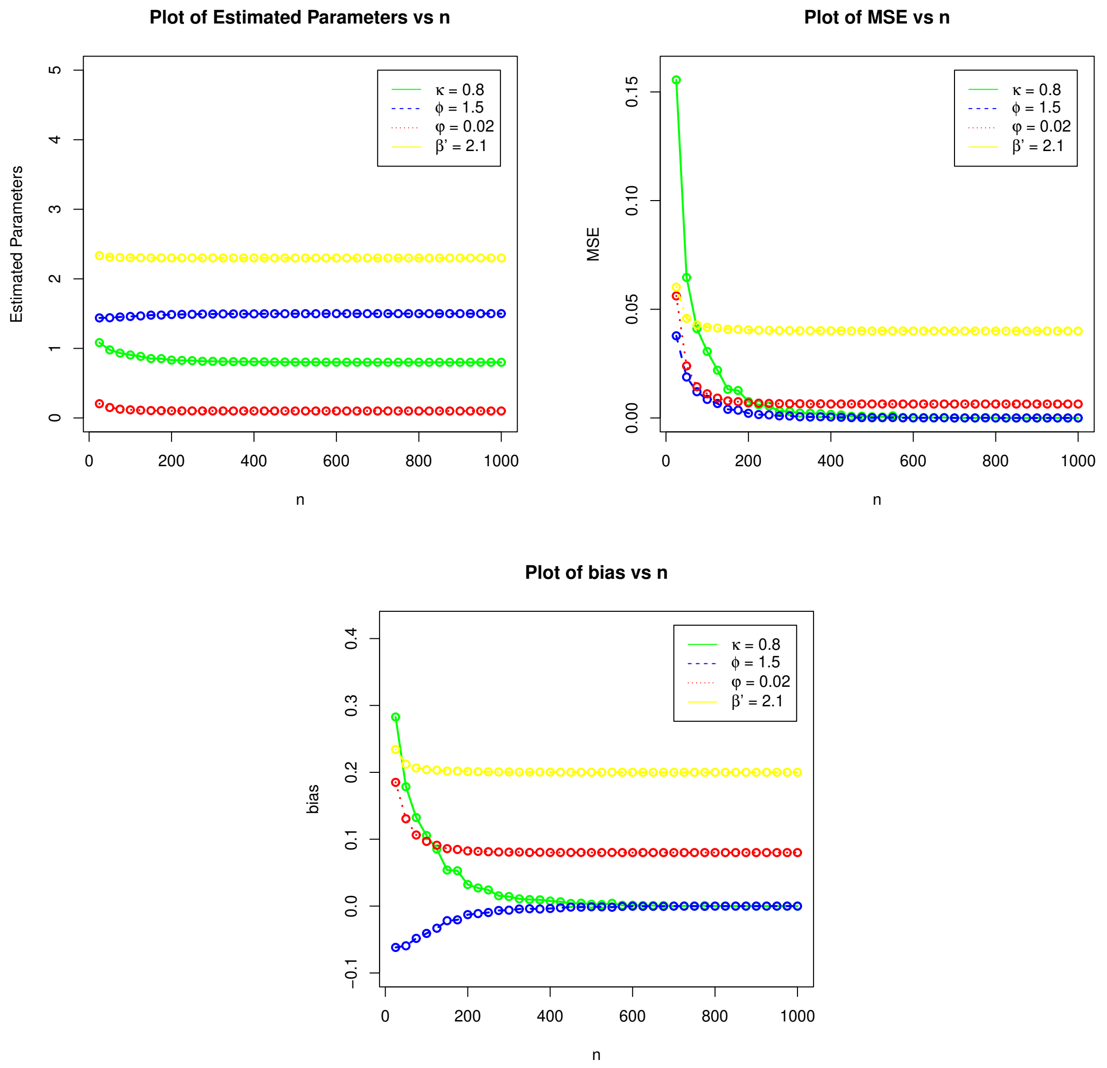

Figure 5. Plot of the simulation results of the GWD-W model.

The simulation results in Tables 4 and 5 and Figures 4 and 5 can be explained though the following steps:

(1) The MLE estimates of κ and φ are overestimated for both sets 1 and 2.

(2) The MLE estimates of ϕ and β' are underestimated for both set 1 , while the MLE estimates of ϕ and β' are overestimated set 2.

(3) The performance of MLE was good even when the small sample sizes.

(4) Increasing the sample size n leads to a decrease in the estimated biases and ERs, which approach zero as n increases. These results demonstrate both the efficiency and consistency properties of the MLEs.

6. Application of the GWD-W model

Probability distributions are widely used in various real data modeling applications. Heavy tailed distributions play an important role in data analysis and modeling in various application areas of life such as economics, financial mathematics, actuarial science, risk management, banking, etc. The quality of statistical procedures mainly depends on the probability distributions given for the application data in question. Based on the data were collected by the World Bank (accessed on 15 September 2021), we address the statistical modeling of current health expenditure (% of GDP) for four different countries in terms of social policies, social welfare levels, credentials, and economic growth rates. The countries are United States, Malaysia, Egypt, and Saudi Arabia. The data are analysed and the maximum likelihood estimates of the model parameters are obtained, see, Appendix B. The data sets for the last fifty years are provided in Table 6.

The goodness-of-fit results of the GWD-W model are compared with some other models. The comparison of the GWD-W model is made with some important distributions including generalized Weibull two-parameter Weibull distribution (GW-OWD), exponentiated distribution (EXP-WD) and two-parameter Weibull distribution (TW-D). The CDF of the competing probability models are, respectively, given by

$$F(x;\tau,\delta,\sigma) = 1 - e^{-\tau\left(\frac{x}{\sigma}\right)^{\delta}}, \qquad x \geq 0; \tau,\delta,\sigma, > 0, \tag{6.1}$$

$$F(x;\tau,\delta,\sigma) = \left(1 - e^{-(\delta x)^{\sigma}}\right)^{\tau}, \qquad x \geq 0; \tau,\delta,\sigma > 0. \tag{6.2}$$

$$F(x;\delta,\sigma) = 1 - e^{-\delta x^{\sigma}}, \qquad x \geq 0; \delta,\sigma > 0. \tag{6.3}$$

Table 7 shows the descriptive statistics of the proposed current health expenditure data sets. Table 8 shows the result of the estimates as well as the one-sample Kolmogorov-Smirnov test. Table 9 compare the GWD-W model via some recognition criterion, such as, Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and consistent Akaike information Criterion (CAIC). The results in Tables 7 and 8 suggest that the GWD-W model provides a better fit than other competing models and could be chosen as a suitable model for analyzing all data sets. The boxplot, the simulated PDF, simulated CDF, the Kaplan-Meier survival function, the PP plot and Q-Q plot of the proposed current health expenditure data of United States, Malaysia, Egypt, and Saudi Arabia are shown in Figures 6–9, respectively. These figures confirm the best fitting of the GWD-W model for the statistical modeling of current health expenditure for all four different countries. The proposed simulated PDF fits the histogram plot very well. The simulated CDF fits the empirical plot very well. The proposed model fits the Kaplan-Meier survival plot very well. The PP plot and Q-Q plot fits the symmetric line very well.

Table 6. The current health expenditure (% of GDP) data sets.

USA	12.48431969, 13.16362572, 13.98185062, 14.49863911, 14.59480953, 14.60504532, 14.71834183, 14.93882084, 15.26701450, 16.23350334, 16.25922203, 16.19850540, 16.17543411, 16.06451797, 16.25333214, 16.52407265, 16.84432411, 16.80583572, 16.68710518, 16.76706314.
MYS	2.51463366, 2.67539740, 2.66876554, 2.92417097, 2.86016941, 2.78692698, 3.10897732, 3.07041144, 3.00692534, 3.26105690, 3.16474295, 3.31451941, 3.45786357, 3.51275206, 3.71036983, 3.81819606, 3.68569684, 3.70413661, 3.74833083, 3.82514191.
EGY	4.92244291, 5.40030336, 5.51073790, 5.22425270, 4.85732365, 4.92229176, 4.84190083, 4.44493246, 4.46634817, 4.37870121, 4.15319490, 4.35706949, 4.71025944, 4.91730928, 5.02518845, 5.33657265, 5.36399841, 5.63305616, 4.94757700, 4.73997355.
KSA	4.21159410, 4.46173573, 4.24937630, 3.97909737, 3.58400607, 3.41867280, 3.61922503, 3.56228733, 2.97100425, 4.29041958, 3.64785600, 3.71177721, 4.01962376, 4.46568298, 5.22795486, 5.99834490, 5.83562946, 6.26256323, 5.74845695, 5.68828773.

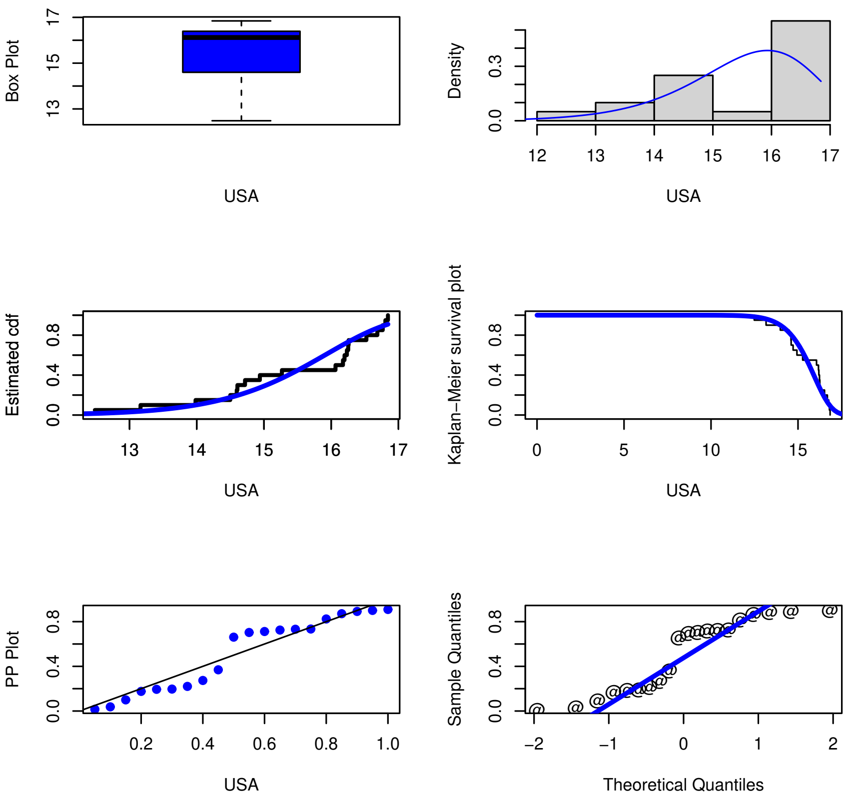

Figure 6. The box plot, fitted PDF, CDF, Kaplan-Meier survival, PP, and QQ plots of the GWD-W model for USA data.

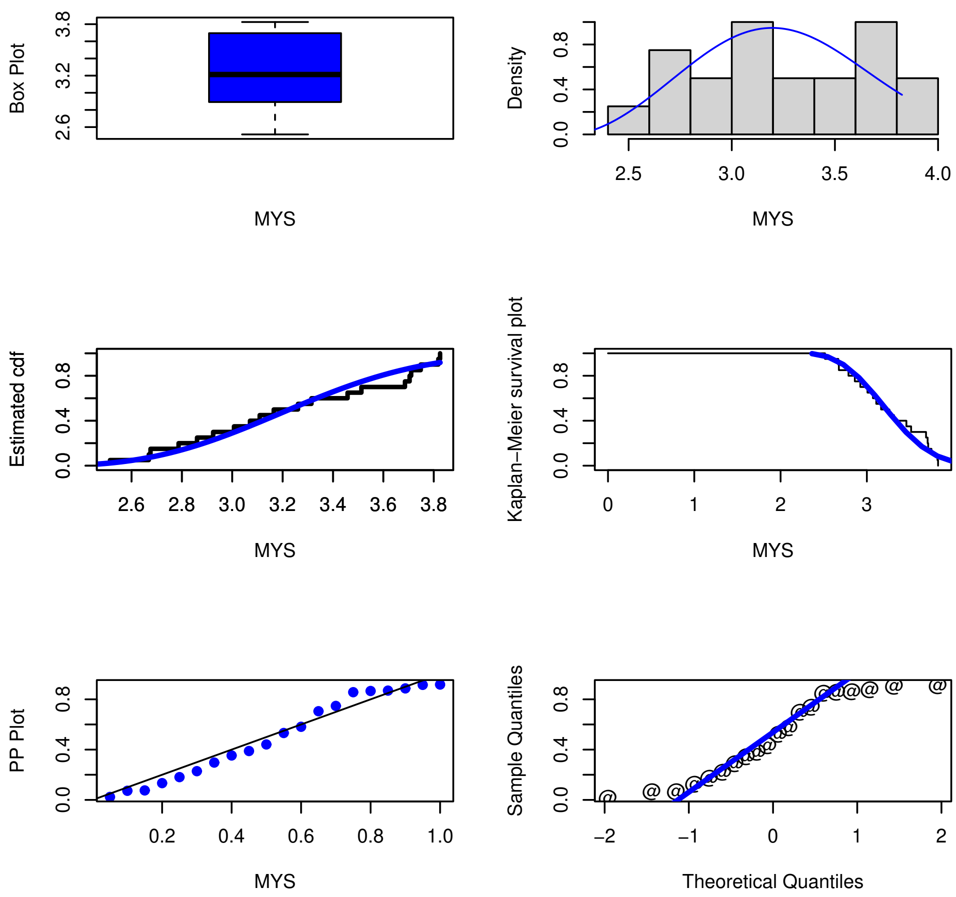

Figure 7. The box plot, fitted PDF, CDF, Kaplan-Meier survival, PP, and QQ plots of the GWD-W model for MYS data.

Table 7. Descriptive statistics of the current health expenditure data sets.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max
USA	12.48	14.6	16.12	15.45	16.33	16.84
MYS	2.515	2.908	3.213	3.241	3.69	3.825
EGY	4.153	4.649	4.92	4.908	5.252	5.633
KSA	2.971	3.641	4.23	4.448	5.343	6.263

Table 8. The estimate results and the one-sample Kolmogorov-Smirnov test.

MYS	Parameters Estimates	KS	p-value
GWD-W	(3.0885, 2.7478, 2.2195, 1.7353)	0.15679	0.65300
EWD	(0.3098, 1.9974, 1.7480)	0.59093	3.48e-07
MEWD	(0.3972 ,1.9670 ,1.7881)	0.44434	0.000406
WD	(2.6520 ,2.6447)	0.38901	0.003092
EGY			
GWD-W	(3.1871 ,2.9054 ,3.8478 1.7742)	0.11145	0.94170
EWD	(0.2401 ,1.7714 ,1.4968)	0.50134	3.56e-05
MEWD	(0.2829 ,1.9714 ,1.9902)	0.55960	1.98e-06
WD	(1.4864 ,1.6676)	0.9793	2.20e-16
KSA			
GWD-W	(2.8248, 1.6712, 2.8632, 3.2990)	0.13693	0.799600
EWD	(0.2406, 1.8339 ,1.5996)	0.28297	0.065580
MEWD	(0.2994, 1.9216 ,1.8336)	0.40230	0.001955
WD	(1.5026, 2.9303)	0.6665	2.81e-09
USA			
GWD-W	(0.01817365, 16.78765860, 0.0000, 12.59339566)	0.21116	0.2915
EWD	(0.0772, 1.2013, 1.1032)	0.58612	4.59e-07
MEWD	(0.0818, 1.7145, 1.5862)	0.49980	3.82e-05
WD	(0.4367, 2.8966)	0.84937	2.20e-16

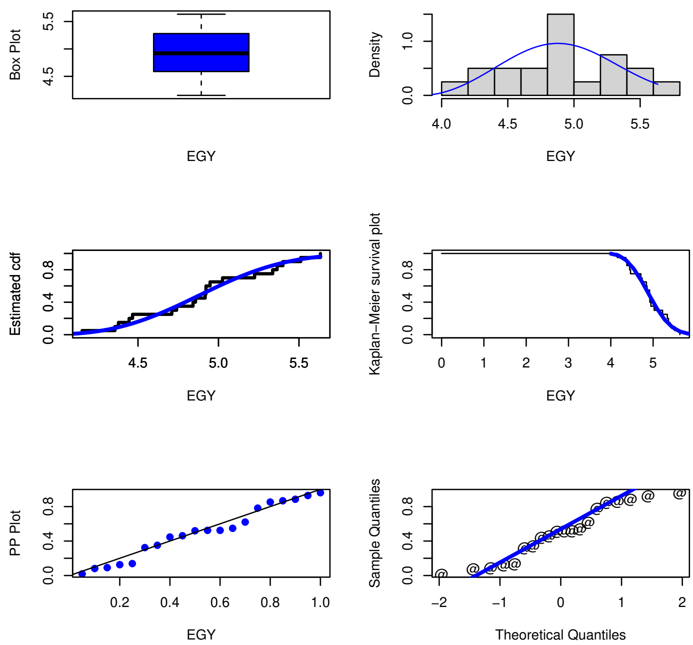

Figure 8. The box plot, fitted PDF, CDF, Kaplan-Meier survival, PP, and QQ plots of the GWD-W model for EGY data.

Table 9. Relative quality of the GWD-W distribution Vs competings.

USA	GWD-W	EWD	MEWD	WD
AIC	67.73391	146.7170	125.6622	200.3585
CAIC	69.23391	148.2170	127.1622	201.0644
BIC	70.7211	149.7042	128.6494	202.3500
HQIC	68.31704	147.3001	126.2454	200.7473
MYS				
AIC	28.62533	62.02928	57.32898	58.21176
CAIC	31.29199	62.73516	58.82898	59.71176
BIC	32.60826	64.02074	60.31618	61.19896
HQIC	29.40284	62.41803	57.91211	58.79489
EGY				
AIC	27.96922	79.04032	72.75102	187.1658
CAIC	30.63589	80.54032	74.25102	187.8717
BIC	31.95215	82.02751	75.73821	189.1573
HQIC	28.74673	79.62345	73.33415	187.5546
KSA				
AIC	59.80316	75.86634	74.51765	98.97521
CAIC	62.46983	77.36634	76.01765	99.68109
BIC	63.78609	78.85354	77.50485	100.9667
HQIC	60.58067	76.44947	75.10078	99.36397

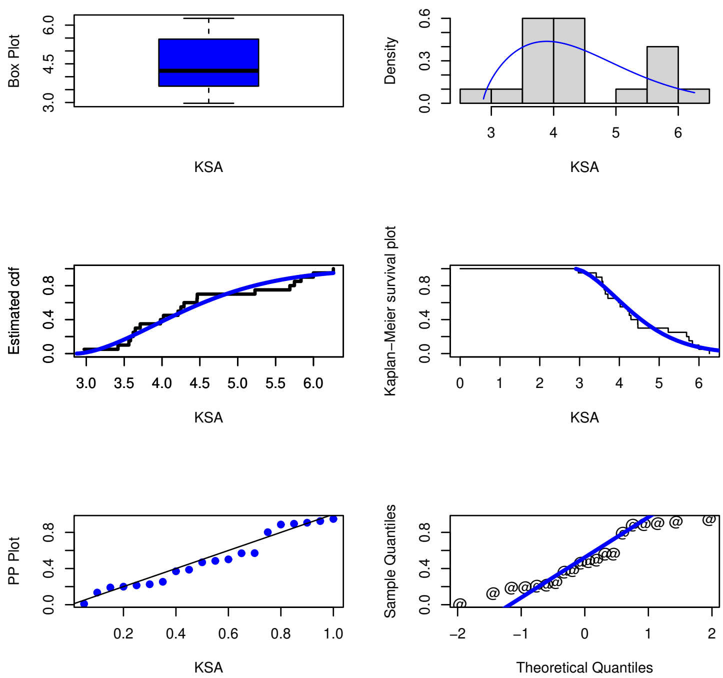

Figure 9. The box plot, fitted PDF, CDF, Kaplan-Meier survival, PP, and QQ plots of the GWD-W model for KSA data.

Table 10. One-sample Kolmogorov-Smirnov test.

Model	KS	p-value
GWD-W model	0.15252	0.9476
GW-OWD	0.21753	0.6559
EXP-CD	0.22573	0.6116
TW-D	0.26447	0.4142

7. Discussion and future works

We have presented a new application for modeling health expenditures as a percentage of GDP in a comparative study for four different countries, namely the United States, Malaysia, Egypt, and the Kingdom of Saudi Arabia. We have presented a model with flexible properties and proven efficiency in statistical modeling of a new application that represents health spending as a percentage of GDP for four different countries in terms of social policies, welfare levels, credentials, and economic growth rates.

The Weibull distribution is a versatile tool that can be used to model a variety of phenomena. It has been used to model downtime in reliability engineering, survival times in medical research, and other applications. The generalized Weibull family of distributions provides additional flexibility for modeling data with more complex shapes. The new distributions generated from the generalized Weibull family can be used to better describe data that does not fit the traditional Weibull distribution.

In this work, we used an extended, updated version of the Weibull distribution, called a generalized Weibull distribution model, which has good statistical properties in terms of flexibility and goodness of fit. Some distributional properties and statistical functions were derived in closed forms, including Renyi entropy, skewness, kurtosis, highly fluctuating behavior, regular variation, and identifiable property.

Based on Monte Carlo simulation, the empirical studies of the actuarial measures provide evidence of the severity of the GWD-W model tail. The results of VAR and TVAR for the GWD-W models in Tables 2 and 3 and Figures 2 and 3 are higher than those of the Weibull models, indicating that the GWD-W models better represent extreme events. The results also show that the GWD-W models have a higher risk measure than the Weibull models, which is beneficial for actuarial applications.

The results of the statistical analysis show that the GWD-W model is a better fit to the data than the traditional Weibull distribution and other competing models. Our study provides useful information for decision makers with regard to allocating resources for health protection expenditures. AIC, BIC, HQC, and CAIC are all criteria used to compare different models and select the best model for a given data set. These criteria measure how well a model fits the data and penalize models with more parameters to prevent overfitting. Both the goodness of fit of the model and its complexity are considered when calculating the criteria, with more complex models being penalized more heavily. From the results in Table 8, the GWD-W model could be selected as the best model among the fitted models.

8. Conclusions

In this article, we address statistical analysis and modeling of World Bank data to compare health protection expenditure as a percentage of GDP in four countries. In statistically modeling the data, we used an extended, updated version of the Weibull distribution with heavy tails, the generalized Weibull distribution (Weibull model). Some of the properties and characteristics of the distribution were discussed, and some of the properties were proved. The model presented is very flexible and can be used effectively for modeling data with heavy tails. The empirical studies of the actuarial measures show the tail heaviness of the generalized Weibull distribution of the Weibull model and can better capture extreme events. The results also show the efficiency and consistency properties of the maximum likelihood approaches based on the proposed model. The results show that the proposed model fits better than other competing models and could be chosen for statistical analysis and modeling of health protection expenditures relative to gross domestic product in a comparative study of the United States, Malaysia, Egypt, and the Kingdom of Saudi Arabia.

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Conflict of interest

The authors declare no conflict of interest.

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