Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 12185–12194.
DOI: 10.3934/math.2023614
Received: 27 October 2022
Revised: 06 March 2023
Accepted: 08 March 2023
Published: 22 March 2023

Research article

Further generalizations of the Ishikawa algorithm

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Abstract: We provide an iterative algorithm for fixed point issues in vector spaces in the paragraphs that follow. We demonstrate that, compared to the Ishikawa technique in Banach spaces, the iterative algorithm presented in this study performs better under weaker conditions. In order to achieve this, we compare the convergence behavior of iterations, and taking into account a few offered cases, we support the major findings.

Keywords: best proximity point; fixed point; uniformly convex Banach space; iterative sequence
Mathematics Subject Classification: 90C48, 47H09, 46B20

1. Introduction and preliminaries

Mann [8] presented a new iterative technique in 1953, which approximate fixed points of nonexpansive mappings in uniformly convex Banach spaces, as follows:

$$\tau_{n+1} = (1 - \varsigma_n)\tau_n + \varsigma_n T\tau_n, \tag{1.1}$$

where $\{\varsigma_n\}$ is a sequence in $(0, 1)$ so that $\lim_{n\to\infty} \varsigma_n = 0$ and $\sum_{n=1}^{\infty} \varsigma_n = \infty$.

Following that, Ishikawa [2] established the following innovative iteration procedure in 1974 for approximating fixed points of nonexpansive mappings:

$$\begin{cases} \tau_{n+1} = (1 - \varsigma_n)\tau_n + \varsigma_n T\nu_n, \\ \nu_n = (1 - \zeta_n)\tau_n + \zeta_n T\tau_n, \quad n = 1, 2, 3, ..., \end{cases} \tag{1.2}$$

where $\{\varsigma_n\}$ and $\{\zeta_n\}$ are sequences in $[0, 1)$ which satisfy the following conditions:

(i) $0 \le \varsigma_n \le \zeta_n \le 1$, $\lim_{n\to\infty} \zeta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \varsigma_n \zeta_n = \infty$.

It is worth noting that the Mann iteration procedure is a special case of Ishikawa, where $\zeta_n = 0$ for all $n \in \mathbb{N}$.

In 2013, Khan [5] gave the concept of Picard-Mann hybrid iterative scheme. This scheme is defined as follows:

$$\begin{cases} \tau_1 = \tau \in X \\ \tau_{n+1} = T\nu_n \\ \nu_n = (1-\alpha_n)\tau_n + \alpha_n T\tau_n \end{cases} \quad \text{with } n \in \mathbb{Z}^+ \tag{1.3}$$

where $\{\alpha_n\} \in (0, 1)$. In 2019, following Khan, Okeke [9] gave the Picard-Ishikawa hybrid iterative scheme which is defined as:

$$\begin{cases} \tau_1 = \tau \in X \\ \tau_{n+1} = T\nu_n \\ \nu_n = (1-\alpha_n)\tau_n + \alpha_n Tc_n \\ c_n = (1-\beta_n)\tau_n + \beta_n T\tau_n \end{cases} \quad \text{with } n \subseteq \mathbb{Z}^+ \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$. Recently, Srivastava [10] introduced the Picard-S hybrid iterative scheme which is defined as:

$$\begin{cases} \tau_1 = a \in X \\ \tau_{n+1} = Tb_n \\ \nu_n = (1-\alpha_n)T\tau_n + \alpha_n Tc_n \\ c_n = (1-\beta_n)\tau_n + \beta_n T\tau_n \end{cases} \quad \text{with } n \in \mathbb{Z}^+ \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$. Also, Lamba and Panwar [7] introduced the Picard-S^*-iterative scheme which is defined as:

$$\begin{cases} \tau_1 = \tau \in X \\ \tau_{n+1} = T\nu_n \\ \nu_n = (1-\alpha_n)T\tau_n + \alpha_n Tc_n \\ c_n = (1-\beta_n)T\tau_n + \beta_n T\mu_n \\ \mu_n = (1-\gamma_n)\tau_n + \gamma_n T\tau_n \end{cases} \quad \text{with } n \in \mathbb{Z}^+ \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$.

Recently, Jia et al. [3] proposed the Picard-Thakur-iterative scheme which is defined as:

$$\begin{cases} \tau_1 \in X \\ \tau_{n+1} = T\nu_n \\ \nu_n = (1-\alpha_n)T\mu_n + \alpha_n Tc_n \\ c_n = (1-\beta_n)d_n + \beta_n T\mu_n \qquad \text{with } n \in \mathbb{Z}^+ \\ \mu_n = (1-\gamma_n)\tau_n + \gamma_n T\tau_n \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Assume that Banach space X has a nonempty subset A. We recall that a self-mapping $T : A \to A$ is said to be *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in A$. It was stated in [1, 6] that the nonexpansive mapping T has a fixed point if X is a uniformly convex Banach space and A is a bounded, closed, and convex subset of X.

Here, we provide some prerequisites that are required.

Consistent with [4], we denote by Θ_0 the family of functions $\theta : (0,\infty) \to (1,\infty)$ such that
(θ_1) θ is increasing;
(θ_2) for each sequence $\{\rho_n\} \subseteq (0,\infty)$, $\lim\limits_{n\to\infty} \theta(\rho_n) = 1$ iff $\lim\limits_{n\to\infty} \rho_n = 0$;
(θ_3) there are a $y \in (0,1)$ and a $\lambda \in (0,\infty]$ so that $\lim\limits_{\rho\to 0^+} \frac{\theta(\rho)-1}{\rho^y} = \lambda$.

Theorem 1.1. *[4, Corollary 2.1] Let T be a self-mapping on a complete metric space (X,d) so that*

$$x,\omega \in X, \quad d(Tx,T\omega) \neq 0 \ \Rightarrow\ \theta(d(Tx,T\omega)) \leq \theta(d(x,\omega))^{\alpha},$$

where $\theta \in \Theta_0$ and $\alpha \in (0,1)$. Then T has a unique fixed point.

It should be noted that the Banach contraction principle is a specific instance of the Theorem 1.1.

Denote by Θ the set of increasing continuous functions $\theta : (0,\infty) \to (1,\infty)$. This family is a new collection which is presented according to [4].

The following lemma is a modification of Lemma 2.1 in [11] in which $a_n = s_n$, $c_n = \alpha_n$ and $b_n = \beta_n/c_n$. In addition, the left side of suppositions (i) and (iii) are valid. It is better to know that in the following lemma, we do not need to have the right side of hypothesis (i) and (iii).

Lemma 1.2. *[11] Let $\{a_n\},\{b_n\} \subset [0,\infty)$ and $\{c_n\} \subset [0,1)$ be sequences of real numbers such that $a_{n+1} \leq (1-c_n)a_n + b_n$ for all $n \in \mathbb{N}$ and*

$$\sum_{n=1}^{\infty} c_n = \infty \ \textit{and} \ \sum_{n=1}^{\infty} b_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 1.3. *[2] If E be a convex compact subset of a Hilbert space H, T be a Lipschitzian pseudo-contractive map from E into itself and x_0 is any point in E, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each $n \in \mathbb{N}$ by*

$$y_n = \varsigma_n T x_n + (1-\varsigma_n)x_n,$$

$$x_{n+1} = \eta_n T y_n + (1-\eta_n)y_n,\ n \in \mathbb{N}, x_0 \in X,$$

where $0 \leq \varsigma_n \leq \eta_n \leq 1$ for all n, $\lim_{n\to\infty} \eta_n = 0$*, and* $\sum_{n=1}^{\infty} \varsigma_n \eta_n = \infty$.

2. Main results

The iterative algorithm for fixed point issues in a vector space is provided below. We demonstrate that the iterative technique presented in this research performs better than the Ishikawa algorithm in Banach spaces under weaker constraints.

We denote by Φ the family of functions $\phi : X \to (0,\infty)$ so that:
(i) If $\phi(x) = \phi(y)$, then $x = y$.
(ii) If for each sequence $\{x_n\} \subseteq X$, $\lim\limits_{n\to\infty} x_n = x$, then $\lim\limits_{n\to\infty} \phi(x_n) = \phi(x)$.
(iii) If for each sequence $\{x_n\} \subseteq X$, $\lim\limits_{n\to\infty} \phi(x_n) = \zeta$, then $\zeta > 0$.

Example 2.1. *Let X be a norm space. It is clear that $f(t) = e^{\|t\|}$ is an element of Φ. Other examples are*

$$f(t) = e^{-\|t\|}, \quad f(t) = \cosh\|t\|, \quad f(t) = \frac{2\cosh\|t\|}{1+\cosh\|t\|},$$

$$f(t) = 1 + \ln(1+\|t\|), \quad f(t) = \frac{2+2\ln(1+\|t\|)}{2+\ln(1+\|t\|)}, \quad f(t) = e^{\|t\|e^{\|t\|}}.$$

Let the function $\rho(.) : [0,\infty) \to [1,\infty)$ be defined as follows:

$$\rho(a) = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } 0 < a < 1, \\ 1 & \text{if } a = 0. \end{cases}$$

It is clear that $\rho(ab) \leq \rho(a)\rho(b)$ and $\rho(a^s) = \rho(a)^{|s|}$, for all $s \in \mathbb{R}$.

Definition 2.2. *A sequence $(x_n) \subseteq [0,\infty)$ is said to be ρ-convergent to a $x \in [0,\infty)$ if for all $\epsilon > 1$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\rho(x_n x^{-1}) < 1 + \epsilon$. Hence, $\rho(x_n x^{-1}) \to 1$ as $n \to \infty$ which is denoted by $x_n \mapsto x$. A sequence $(x_n) \subseteq [0,\infty)$ is said to be ρ-Cauchy in $[0,\infty)$, if for all $\epsilon > 1$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\rho(x_n x_m^{-1}) < 1 + \epsilon$. Hence $\rho(x_n x_m^{-1}) \to 1$ as $n \to \infty$.*

Also, $([0,\infty),\rho)$ is said to be ρ-complete if every ρ-Cauchy sequence be a ρ-convergent sequence. It is clear that since $\mathbb{R}$ is a complete space, hence $([0,\infty),\rho)$ is ρ-complete.

Definition 2.3. *Let X be a vector space. Then we say that $F : X \to X$ is a Φ-contraction mapping if for a $\gamma \in (0,1)$ and a $\phi \in \Phi$ we have*

$$\rho\left(\frac{\phi(Fx)}{\phi(Fy)}\right) \leq \rho\left(\frac{\phi(x)}{\phi(y)}\right)^{\gamma}, \qquad x, y \in X_{\phi},$$

where $X_{\phi} = \{x \in X : \phi(x) > 0\}$.

Theorem 2.1. *Let X be a vector space, and let $F : X \to X$ be a Φ-contraction. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have*

$$\lim_{n\to\infty} F^n(x) = u$$

with

$$\rho(\phi(F^n x)\phi(u)^{-1}) \leq \rho(\phi(x)(\phi(Fx))^{-1})^{\frac{\gamma^n}{1-\gamma}}.$$

Proof. We first show the uniqueness. Suppose that there exist $x, y \in X$ with $x = Fx$ and $y = Fy$. Then

$$1 \leq \rho(\phi(x)\phi(y)^{-1}) = \rho(\phi(Fx)(\phi(Fy))^{-1}) \leq \rho(\phi(x)\phi(y)^{-1})^{\gamma} < \rho(\phi(x)\phi(y)^{-1}),$$

which is a contradition and so $x = y$.

To show the existence, select $x \in X$. We first show that $\{\phi(F^n x)\}$ is a Cauchy sequence. Notice for $n \in \{0, 1, ...\}$ that

$$\rho(\phi(F^n x)(\phi(F^{n+1}x))^{-1}) \leq \rho(\phi(F^{n-1}x)(\phi(F^n x))^{-1})^{\gamma} \leq ... \leq \rho(\phi(x)(\phi(Fx))^{-1})^{\gamma^n}.$$

Thus for $m > n$ where $n \in \{0, 1, ...\}$,

$$\begin{aligned}\rho(\phi(F^n x)(\phi(F^m x))^{-1}) &\leq \rho(\phi(F^n x)(\phi(F^{n+1} x))^{-1})\, \rho(\phi(F^{n+1} x)\phi(F^{n+2} x))^{-1}) \\ &\quad ...\, \rho(\phi(F^{m-1} x)(\phi(F^m x))^{-1}) \\ &\leq \rho(\phi(x)(\phi(Fx))^{-1})^{\gamma^n} \,...\, \rho(\phi(x)\phi(Fx))^{-1})^{\gamma^{m-1}} \\ &\leq \rho(\phi(x)(\phi(Fx))^{-1})^{\gamma^n[1+\gamma+\gamma^2+...]} \\ &= \rho(\phi(x)(\phi(Fx))^{-1})^{\frac{\gamma^n}{1-\gamma}}.\end{aligned}$$

That is, for $m > n$, $n \in \{0, 1, ...\}$,

$$\rho(\phi(F^n x)(\phi(F^m x))^{-1}) \leq \rho(\phi(x)(\phi(Fx))^{-1})^{\frac{\gamma^n}{1-\gamma}}. \tag{2.1}$$

This shows that $\{\phi(F^n x)\}$ is a ρ-Cauchy sequence, and so there exists $\zeta \in [0, \infty)$ with $\lim_{n\to\infty} \phi(F^n x) = \zeta$. By (iii), there exists $u \in X$ with $\zeta = \phi(u)$. Moreover the continuity of ϕ and F yields

$$\phi(u) = \lim_{n\to\infty} \phi(F^{n+1} x) = \lim_{n\to\infty} \phi(F(F^n x)) = \phi(Fu).$$

Therefore, u is a fixed point of F. Finally, letting $m \to \infty$ in (2.1) yields

$$\rho(\phi(F^n(x))\phi(u^{-1})) \leq \rho(\phi(x)\phi((F(x)))^{-1})^{\frac{\gamma^n}{1-\gamma}}.$$

□

Theorem 2.2. *Let X be a vector space. Let $F : X \to X$ be a Φ-contraction mapping, $0 < \gamma \leq 1$ and $x_0 \in X$. We define a sequence $\{x_n\} \subseteq X$ by*

$$\begin{aligned}\phi(y_n) &= \phi(Fx_n)^{\varsigma_n} \phi(x_n)^{1-\varsigma_n}, \\ \phi(x_{n+1}) &= \phi(Fy_n)^{\eta_n} \phi(y_n)^{1-\eta_n},\end{aligned}$$

where $n \in \mathbb{N}$, $0 \leq \varsigma_n \leq \eta_n \leq 1$, $\lim_{n\to\infty} \eta_n = 0$ and $\sum_{n=1}^{\infty} \varsigma_n = \infty$. Then the sequence $\{x_n\}$ is convergent strongly to an element $p \in X$ such that $Fp = p$.

Proof. By Theorem 2.1, F has a unique fixed point $p \in X$. From the assumption that F is a Φ-contraction, we have the following inequalities:

$$\begin{aligned}\rho\left(\frac{\phi(x_{n+1})}{\phi(p)}\right) &= \rho\left(\frac{\phi(Fy_n)^{\eta_n}\phi(y_n)^{1-\eta_n}}{\phi(p)}\right) \\ &= \rho\left(\frac{\phi(Fy_n)^{\eta_n}\phi(y_n)^{1-\eta_n}}{\phi(p)^{\eta_n}\phi(p)^{1-\eta_n}}\right) \\ &= \rho\left(\left[\frac{\phi(Fy_n)}{\phi(p)}\right]^{\eta_n}\left[\frac{\phi(y_n)}{\phi(p)}\right]^{1-\eta_n}\right) \\ &\leq \rho\left(\frac{\phi(Fy_n)}{\phi(p)}\right)^{\eta_n} \rho\left(\frac{\phi(y_n)}{\phi(p)}\right)^{1-\eta_n} \\ &\leq \rho\left(\frac{\phi(y_n)}{\phi(p)}\right)^{\gamma\eta_n} \rho\left(\frac{\phi(y_n)}{\phi(p)}\right)^{1-\eta_n}\end{aligned}$$

$$\begin{aligned}
&= \rho(\frac{\phi((Fx_n)^{\varsigma_n} x_n^{1-\varsigma_n})}{\phi(p)})^{\gamma\eta_n}\rho(\frac{\phi(y_n)}{\phi(p)})^{1-\eta_n} \\
&= \rho(\frac{\phi(Fx_n)^{\varsigma_n}\phi(x_n)^{1-\varsigma_n}}{\phi(p)^{\varsigma_n}\phi(p)^{1-\varsigma_n}})^{\gamma\eta_n}\rho(\frac{\phi(y_n)}{\phi(p)})^{1-\eta_n} \\
&\le [\rho(\frac{\phi(Fx_n)}{\phi(p)})^{\varsigma_n}\rho(\frac{\phi(x_n)}{\phi(p)})^{1-\varsigma_n}]^{\gamma\eta_n}\rho(\frac{\phi(y_n)}{\phi(p)})^{1-\eta_n} \\
&\le [\rho(\frac{\phi(x_n)}{\phi(p)})^{\gamma\varsigma_n}\rho(\frac{\phi(x_n)}{\phi(p)})^{1-\varsigma_n}]^{\gamma\eta_n}\rho(\frac{\phi(y_n)}{\phi(p)})^{1-\eta_n} \\
&= \rho(\frac{\phi(x_n)}{\phi(p)})^{\varsigma_n\gamma^2\eta_n+(1-\varsigma_n)\gamma\eta_n+1-\eta_n}.
\end{aligned}$$

Put $\mu_n = \eta_n - \varsigma_n\gamma^2\eta_n - (1-\varsigma_n)\gamma\eta_n$. Since

$$\mu_n = \eta_n(1-\gamma) + \varsigma_n\eta_n\gamma(1-\gamma) \ge \eta_n(1-\gamma),$$

$\sum_{n=1}^{\infty}\mu_n = \infty$. If we set $a_n = \rho(\frac{\phi(x_n)}{\phi(p)})$, then $a_{n+1} \le a_n^{(1-\mu_n)}$ and so $\ln a_{n+1} \le (1-\mu_n)\ln a_n$. Therefore, by Lemma 1.2, we have $\lim \ln a_n = 0$ and so $\lim a_n = 1$. Therefore, $\lim_{n\to\infty}\rho(\frac{\phi(x_n)}{\phi(p)}) = 1$ and so $\lim_{n\to\infty}\phi(x_n) = \phi(p)$. Hence, we have $x_n \to p$. □

Theorem 2.3. *Let X be a vector space. Let $F : X \to X$ be a Φ-contraction mapping, $0 < \gamma \le 1$ and $x_0 \in X$ such that*

$$\rho(\frac{\phi(Fx)}{\phi(Fy)}) \le \rho(\frac{\phi(x)}{\phi(y)})^{\alpha}\rho(\frac{Fx}{x})^{\beta}\rho(\frac{Fy}{y})^{\beta} \quad x, y \in X_\phi, \tag{2.2}$$

where $X_\phi = \{x \in X : \phi(x) > 0\}$, $\alpha, \beta \ge 0$ and $\alpha + 2\beta < 1$. Then F has a unique fixed point.

Proof. Suppose that $x_0 \in X_\phi$. We define $x_{n+1} = Fx_n$. Now,

$$\begin{aligned}
\rho(\frac{\phi(x_{n+2})}{\phi(x_{n+1})}) &= \rho(\frac{\phi(Fx_{n+1})}{\phi(Fx_n)}) \\
&\le \rho(\frac{\phi(x_{n+1})}{\phi(x_n)})^{\alpha}\rho(\frac{\phi(Fx_{n+1})}{\phi(x_{n+1})})^{\beta}\rho(\frac{\phi(Fx_n)}{\phi(x_n)})^{\beta} \\
&= \rho(\frac{\phi(x_{n+1})}{\phi(x_n)})^{\alpha}\rho(\frac{\phi(x_{n+2})}{\phi(x_{n+1})})^{\beta}\rho(\frac{\phi(x_{n+1})}{\phi(x_n)})^{\beta}.
\end{aligned}$$

So,

$$\rho(\frac{\phi(x_{n+2})}{\phi(x_{n+1})}) \le \rho(\frac{\phi(x_{n+1})}{\phi(x_n)})^{\frac{\alpha+\beta}{1-\beta}}.$$

Let $\gamma = \frac{\alpha+\beta}{1-\beta} < 1$. Hence, inductively we have

$$\rho(\frac{\phi(x_{n+2})}{\phi(x_{n+1})}) \le \rho(\frac{\phi(x_1)}{\phi(x_0)})^{\gamma^{n+1}},$$

and so

$$\rho(\frac{\phi(x_{n+1})}{\phi(x_n)}) \to 1.$$

Therefore, similar to the proof of Theorem 2.1, the sequence $\{x_n\}$ is ρ-convergent to an $x \in X_\phi$ such that $Fx = x$. □

Remark 2.1. *Under weaker conditions, Theorem 2.2 produces stronger results than Theorem 1.3. Because Theorem 2.2 is in the framework of a vector space, but Theorem 1.3 is in the framework of a Hilbert space. Also, the condition $\sum_{n=1}^{\infty} \varsigma_n \eta_n = \infty$ is replaced by $\sum_{n=1}^{\infty} \varsigma_n = \infty$.*

On the other hand, the algorithm of Theorem 2.2 has a better rate of convergence than the Ishikawa algorithm, as we will demonstrate in the following section.

Ishikawa algorithm which is defined by

$$y_n = \varsigma_n F x_n + (1 - \varsigma_n) x_n,$$

$$x_{n+1} = \eta_n F y_n + (1 - \eta_n) y_n,$$

is algorithm I.

The algorithm of Theorem 2.2 was defined by

$$\phi(y_n) = \phi(F x_n)^{\varsigma_n} \phi(x_n)^{1-\varsigma_n},$$

$$\phi(x_{n+1}) = \phi(F y_n)^{\eta_n} \phi(y_n)^{1-\eta_n},$$

where $n \in \mathbb{N}$, $x_0 \in X$, $0 \le \varsigma_n \le \eta_n \le 1$, $\lim_{n\to\infty} \eta_n = 0$, and $\sum_{n=1}^{\infty} \varsigma_n = \infty$. Suppose that $\phi_1(t) = e^{\|t\|}$. Hence we have

$$e^{\|y_n\|} = e^{\|F x_n\| \varsigma_n} e^{\|x_n\|(1-\varsigma_n)},$$

$$e^{\|x_{n+1}\|} = e^{\|F y_n\| \eta_n} e^{\|y_n\|(1-\eta_n)},$$

and so we have

$$\|y_n\| = \varsigma_n \|F x_n\| + (1 - \varsigma_n)\|x_n\|,$$

$$\|x_{n+1}\| = \eta_n \|F y_n\| + (1 - \eta_n)\|y_n\|,$$

which is algorithm II.

Suppose that $\phi(t) = 1 + \ln(1 + \|t\|)$. Hence we have

$$1 + \ln(1 + \|y_n\|) = [1 + \ln(1 + \|F x_n\|)]^{\varsigma_n} [1 + \ln(1 + \|x_n\|)]^{(1-\varsigma_n)},$$

$$1 + \ln(1 + \|x_{n+1}\|) = [1 + \ln(1 + \|F y_n\|)]^{\eta_n} [1 + \ln(1 + \|y_n\|)]^{(1-\eta_n)},$$

which we call it algorithm III.

In the following, we consider and compare the algorithms I–III with some examples.

Example 2.4. *Let $X = \mathbb{R}^2$ and $F : (\mathbb{R}^2, \|.\|_\infty) \to (\mathbb{R}^2, \|.\|_\infty)$ be defined by $F(x, y) = (\frac{x}{2}, \frac{y}{2})$. We have*

$$\max\{|y_{1n}|, |y_{2n}|\} = \varsigma_n \max\{|\frac{x_{1n}}{2}|, |\frac{x_{2n}}{2}|\} + (1 - \varsigma_n) \max\{|x_{1n}|, |x_{2n}|\},$$

$$\max\{|x_{1(n+1)}|, |x_{2(n+1)}|\} = \eta_n \max\{|\frac{y_{1n}}{2}|, |\frac{y_{2n}}{2}|\} + (1 - \eta_n) \max\{|y_{1n}|, |y_{2n}|\},$$

which is algorithm II.

$$1 + \ln(1 + \max\{|y_{1n}|, |y_{2n}|\}) = [1 + \ln(1 + \max\{|\frac{x_{1n}}{2}|, |\frac{x_{2n}}{2}|\})]^{\varsigma_n} [1 + \ln(1 + \max\{|x_{1n}|, |x_{2n}|\})]^{(1-\varsigma_n)},$$

$$1 + \ln(1 + \max\{|x_{1(n+1)}|, |x_{2(n+1)}|\}) = [1 + \ln(1 + \max\{|\frac{y_{1n}}{2}|, |\frac{y_{2n}}{2}|\})]^{\eta_n} [1 + \ln(1 + \max\{|y_{1n}|, |y_{2n}|\})]^{(1-\eta_n)},$$

that is algorithm III.

Example 2.5. *Let $X = \mathbb{R}^2$, $F : (\mathbb{R}^2, \|.\|_2) \to (\mathbb{R}^2, \|.\|_2)$ be defined by $F(x, y) = (\frac{x}{2}, \frac{y}{2})$. We have*

$$\sqrt{y_{1n}^2 + y_{2n}^2} = \varsigma_n \sqrt{\frac{x_{1n}^2}{2} + \frac{x_{2n}^2}{2}} + (1 - \varsigma_n) \sqrt{x_{1n}^2 + x_{2n}^2},$$

$$\sqrt{x_{1(n+1)}^2 + x_{2(n+1)}^2} = \eta_n \sqrt{\frac{y_{1n}^2}{2} + \frac{y_{2n}^2}{2}} + (1 - \eta_n) \sqrt{y_{1n}^2 + y_{2n}^2},$$

which is algorithm II.

$$1 + \ln(1 + \sqrt{y_{1n}^2 + y_{2n}^2}) = [1 + \ln(1 + \sqrt{\frac{x_{1n}^2}{2} + \frac{x_{2n}^2}{2}})]^{\varsigma_n}[1 + \ln(1 + \sqrt{x_{1n}^2 + x_{2n}^2})]^{(1-\varsigma_n)},$$

$$1 + \ln(1 + \sqrt{x_{1(n+1)}^2 + x_{2(n+1)}^2}) = [1 + \ln(1 + \sqrt{\frac{y_{1n}^2}{2} + \frac{y_{2n}^2}{2}})]^{\eta_n}[1 + \ln(1 + \sqrt{y_{1n}^2 + y_{2n}^2})]^{(1-\eta_n)},$$

which is algorithm III. We illustrate the results of the example in Table 1. In Table 1, we perform some tests for the convergence behavior of an iterative scheme for the initial point $(2, 2)$*.*

In Figure 1, we perform the convergence. For the initial point $(2, 2)$*, we see that the iterative scheme (III) reaches the fixed point faster.*

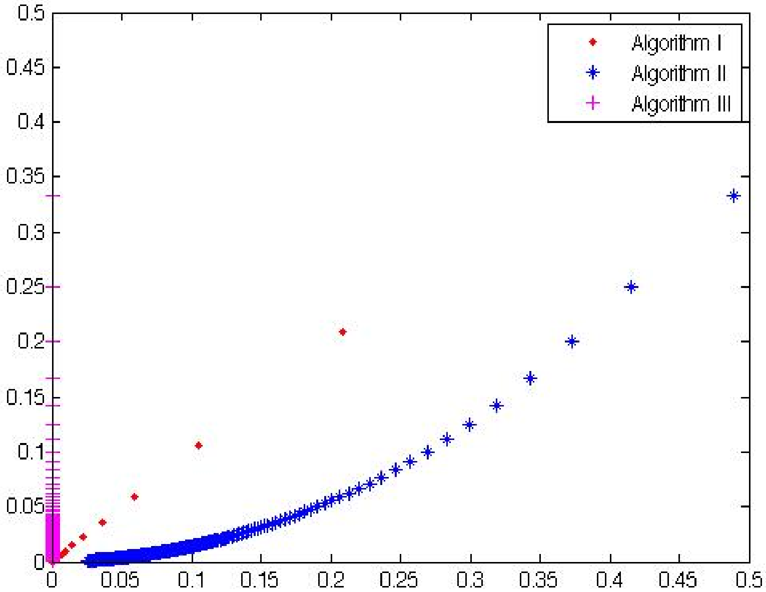

Figure 1. The comparison of the rate of convergence for iterations I–III.

Table 1. The comparison of the rate of convergence for iterations I–III.

Initial point	Iteration Processes	n=100	
	I (Ishikawa)	II	III
$(2, 2)$	(0.0009, 0.0009)	(0.00260, 0.0010)	(0.00003, 0.0009)

3. Conclusions

Ishikawa iteration is widely used in the solution of fixed point equations which takes the shape $Fx = x$, where $F : X \to X$ is a nonexpansive mapping and X is a non-empty, closed and convex subset of a Banach space. This algorithm converges weakly to the fixed point of F provided that the underlying space is a Hilbert space. It is interesting to address the apparent deficiency of the previous algorithm by building an algorithm that converges to the fixed point of F in a vector space. Also, we demonstrated that in comparison to the Ishikawa technique in Banach spaces, the iterative algorithm presented in this study performs better under weaker conditions. In order to achieve this, we compared the convergence behavior of iterations, and taking into account a few offered cases, we support the major findings.

Acknowledgements

The publication of this article was partially supported by the 2022 Development Fund of the Babes-Bolyai University.

Conflict of interest

The authors declare that there is not any competing interest regarding the publication of this manuscript.

References

1. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
2. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/S0002-9939-1974-0336469-5
3. J. Jia, K. Shabbir, K. Ahmad, N. A. Shah, T. Botmart, Strong convergence of a new hybrid iterative scheme for nonexpensive mappings and applications, *J. Funct. Spaces*, **2022** (2022), 4855173. https://doi.org/10.1155/2022/4855173
4. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. https://doi.org/10.1186/1029-242X-2014-38
5. S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory Appl.*, **2013** (2013), 69. https://doi.org/10.1186/1687-1812-2013-69
6. W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Mon.*, **72** (1965), 1004–1006. https://doi.org/10.2307/2313345
7. P. Lamba, A. Panwar, A Picard-S^* iterative algorithm for approximating fixed points of generalized α-nonexpansive mappings, *J. Math. Comput. Sci.*, **11** (2021), 2874–2892.
8. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162

9. G. A. Okeke, Convergence analysis of the Picard-Ishikawa hybrid iterative process with applications, *Afr. Mat.*, **30** (2019), 817–835. https://doi.org/10.1007/s13370-019-00686-z

10. J. Srivastava, Introduction of new Picard-S hybrid iteration with application and some results for nonexpansive mappings, *Arab J. Math. Sci.*, **28** (2022), 61–76. https://doi.org/10.1108/AJMS-08-2020-0044

11. H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.*, **116** (2003), 659–678. https://doi.org/10.1023/A:1023073621589