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Research article

Parameter estimation for discretely observed Cox–Ingersoll–Ross model driven by fractional Lévy processes

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Abstract: This paper deals with least squares estimation for the Cox–Ingersoll–Ross model with fractional Lévy noise from discrete observations. The contrast function is given to obtain the least squares estimators. The consistency and asymptotic distribution of estimators are derived when a small dispersion coefficient $\varepsilon \to 0$, $n \to \infty$, $\varepsilon n^{\frac{1}{2}-d} \to 0$, and $n\varepsilon \to \infty$ simultaneously.

Keywords: least squares estimation; Cox–Ingersoll–Ross model; fractional Lévy noise; consistency; asymptotic distribution
Mathematics Subject Classification: 62E20, 62F03, 62F12, 62P05, 62P20

1. Introduction

Stochastic differential equations are an important tool to describe random phenomena, and parameter estimation is a non-negligible problem in the study of stochastic differential equations. In the past few decades, some scholars have studied the parameter estimation problem of stochastic models. For example, Bishwa [1] presented the estimation of the unknown parameters in stochastic differential models based on continuous and discrete observations and examined asymptotic properties of several estimators under maximum likelihood, minimum contrast and Bayesian methods. Genon-Catalot [2] studied the asymptotic distribution of the maximum contrast estimator of a one-dimensional diffusion process based on the sample path and gave the condition that the maximum contrast estimator is asymptotically equivalent to the maximum likelihood estimator. Pasemann and Stannat [3] used the maximum likelihood method to study the parameter estimation problem of a class of semi-linear stochastic evolution equations and gave the conditions of consistency and asymptotic normality according to the growth and continuity of the nonlinear part. Cialenco et al. [4] studied the asymptotic properties of maximum likelihood estimators for stochastic partial differential equations. Neuenkirch and Tindel [5] derived the least squares estimator for stochastic differential equations with additive fractional noise and verified its strong consistency. Fei [6] studied the consistency of least squares estimation of parameters of stochastic differential equations under distributional uncertainty. Karimi and McAuley [7] developed a Bayesian algorithm for estimating measurement noise variance, disturbance strength, and model parameters in nonlinear stochastic differential equation models. Kan et al. [8] studied the parameter estimation of linear stochastic systems using Bayesian methods. When the system is observed discretely, Wei et al. [9] studied the problem of parameter estimation for square radial Ornstein-Uhlenbeck processes driven by α-stable noise from discrete observations. Long [10] performed parameter estimation for stochastic differential equations driven by small stable noise under discrete observations. Wei [11] focused on the parameter estimation problem of the stochastic Lotka-Volterra model driven by small Lévy noise, verifying the consistency of the estimator and the asymptotic distribution of the estimation error.

Among them, the Lévy process is a natural noise model in the random process, but it has long-term dependence. It is inappropriate to describe random fluctuations in many aspects only by the Lévy process or even the general Markov process without aftereffect. Fractional Lévy noise, as an important non-Gaussian noise, can more accurately reflect actual fluctuations. Because of this, more and more scholars have devoted themselves to the qualitative analysis of stochastic differential equations driven by fractional Lévy processes. Laskin et al. [12] explored fractional Lévy process probability density functions and applied them in network traffic modeling. Lin et al. [13] studied the existence and joint continuity of local time for multiparameter fractional Lévy processes. Bender et al. [14,15] discussed a certain class of stochastic integrals of real-valued fractional Lévy processes and the finite variation of fractional Lévy processes. Lu and Dai [16] explored the stochastic integration of pure jump 0-mean square-integrable fractional Lévy processes and their driven stochastic differential equations. Bender et al. [17] listed maximum inequalities for fractional Lévy and related processes. Shen et al. [18] studied the problem of parameter estimation for Ornstein-Uhlenbeck processes driven by fractional Lévy processes. Bishwal [19] incorporated jumps and long memory into the volatility process driven by fractional Lévy processes to obtain refined inference results like confidence interval. Boniece et al. [20] studied sample path properties and stochastic integration with respect to fractional Lévy processes. Rao [21] studied the problem of nonparametric estimation of linear multipliers of stochastic differential equations driven by small noise fractional Lévy processes. Prakasa [22] described and predicted trends in nonparametric estimation of stochastic differential equations driven by fractional Lévy processes.

The Cox–Ingersoll–Ross (CIR) model, proposed in 1985 [23,24], is an extension of the Vasicek model and solves the problem of possible negative interest rates. However, due to the observational discontinuity and heavy tails of financial samples, the CIR model cannot capture these characteristics [25], and it is necessary to replace the Brownian motion in the CIR model with other processes. There have been some studies, such as Ma et al. [26], who studied the central limit theorem of the least squares estimator of the CIR model driven by α-stable noise. Li and Ma [27] derived the consistency of weighted least squares estimators in the CIR model. Yang [28] studied the maximum likelihood estimation of discrete-observation CIR models with small α-stable noise. There are few studies on building CIR models driven by fractional Lévy processes and making statistical inferences, and the equations in the existing literature contain only one unknown parameter. Inspired by the above results, in this paper, we consider the model with two unknown parameters. Parameter estimation and statistical inference are performed on stochastic financial models driven by fractional Lévy processes. Research on the consistency of estimators, the asymptotic distribution of estimation errors, hypothesis testing and other asymptotic properties can reflect the effectiveness of estimators and estimation methods and can more accurately grasp the dynamic changes of assets. Therefore, this study has certain practical value and significance.

2. Problem formulation and preliminaries

Definition 1. (Marquardt [29]) Let $L=\{L(t), t\in R\}$ be a zero-mean two-sided Lévy process with $E[L(1)^2]<\infty$ and without a Brownian component. For fractional integration parameter $d\in(0,\frac{1}{2})$, a stochastic process

$$L_t^d := \frac{1}{\Gamma(d+1)}\int_{-\infty}^{\infty}[(t-s)_+^d-(-s)_-^d]L(ds), t\in \mathbf{R}$$

is called a fractional Lévy process, where $x_+ = x\vee 0$.

In this paper, we study the parametric estimation problem for the Cox–Ingersoll–Ross model driven by fractional Lévy processes described by the following stochastic differential equation:

$$\begin{cases} dX_t = (\alpha-\beta X_t)dt+\varepsilon\sqrt{X_t}dL_t^d, \quad t\in[0,1], d\in(0,\frac{1}{2}) \\ X_0 = x_0 \end{cases} \tag{1}$$

where α and β are unknown parameters, and L_t^d is a fractional Lévy process. Assume that this process is observed at n regularly spaced time points $\left\{t_i=\frac{i}{n}, i=1,2,\ldots,n\right\}$, $\varepsilon\in[0,1]$.

To get the least squares estimators, we introduce the following contrast function:

$$\rho_{n,\varepsilon}(\alpha,\beta)=\sum_{i=1}^{n}\left|X_{t_i}-X_{t_{i-1}}-(\alpha-\beta X_{t_{i-1}})\Delta t_{i-1}\right|^2 \tag{2}$$

where $\Delta t_{i-1}=t_i-t_{i-1}=\frac{1}{n}$. Then, the least squares estimators $\hat{\alpha}_{n,\varepsilon}$ and $\hat{\beta}_{n,\varepsilon}$ are defined as follows:

$$\rho_{n,\varepsilon}(\hat{\alpha}_{n,\varepsilon},\hat{\beta}_{n,\varepsilon})=\min\rho_{n,\varepsilon}(\alpha,\beta).$$

It is easy to obtain the least square estimators:

$$\begin{cases} \hat{\alpha}_{n,\varepsilon} = \dfrac{n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}}\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} - \dfrac{n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ \hat{\beta}_{n,\varepsilon} = \dfrac{n^2\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}} - n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} \end{cases}. \tag{3}$$

3. Main results and proofs

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true values of the parameters:

$$dX_t^0 = (\alpha - \beta X_t^0)dt, X_0^0 = x_0. \tag{4}$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\alpha - \beta\int_{t_{i-1}}^{t_i} X_s ds + \varepsilon\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d. \tag{5}$$

Bring the formula into (3), and a more explicit decomposition for $\hat{\alpha}_{n,\varepsilon}$ and $\hat{\beta}_{n,\varepsilon}$ can be obtained:

$$\begin{aligned} \hat{\alpha}_{n,\varepsilon} &= \alpha + \frac{n\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} - \frac{n\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ &+ \frac{n\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} - \frac{n\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ &= \alpha + \frac{\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} - \frac{\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} - \frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}, \end{aligned}$$

$$\hat{\beta}_{n,\varepsilon} = \frac{n\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\sum_{i=1}^{n}X_{t_{i-1}}}{n^2 - \sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}} - \frac{n^2\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds}{n^2 - \sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}$$

$$+\frac{n^2\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d}{n^2-\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{n\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\sum_{i=1}^{n}X_{t_{i-1}}}{n^2-\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}$$

$$=\frac{\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_sds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}-\frac{\beta\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_sds}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}$$

$$+\frac{\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}-\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}.$$

Before giving the theorems, we need to establish some preliminary results.

Lemma 1. ([29]) Let $|f|,|g|\in H$, where H is the completion of $L^1(\mathbf{R})\cap L^2(\mathbf{R})$ with respect to the norm $\|g\|_H^2=\mathbf{E}[L^2(1)]\int_R(I_{-g}^d)^2(u)du$. Then,

$$\mathbf{E}[\int_R f(s)dL_s^d\int_R g(s)dL_s^d]=\frac{\Gamma(1-2d)\mathbf{E}[L^2(1)]}{\Gamma(d)\Gamma(1-d)}\int_R\int_R f(t)g(s)|t-s|^{2d-1}dsdt.$$

Lemma 2. ([29]) For any $0\le b_2\le b_1\le a_1,\ 0\le b_2\le a_2\le a_1,\ b_1-b_2=a_1-a_2$, there exists a constant C that only depends on r and d such that

$$\left|\int_{b_2}^{b_1}\int_{a_2}^{a_1}e^{r(u+v)}|u-v|^{2d-1}dudv\right|\le\begin{cases}C\left|e^{r(a_1+b_1)}-e^{r(a_2+b_2)}\right||a_1-b_2|^{2d}, r\neq 0\\ C|a_1-b_2|^{2d}, r=0\end{cases}$$

where r denotes a constant, and d is the fractional integration parameter of the fractional Lévy process.

Lemma 3. When $\varepsilon\to 0, n\to\infty$, we have

$$\sup_{0\le t\le 1}\left|X_t-X_t^0\right|\xrightarrow{P}0.$$

Proof. Observe that

$$X_t-X_t^0=-\beta\int_0^t(X_s-X_s^0)ds+\varepsilon\int_0^t\sqrt{X_s}dL_s^d. \tag{6}$$

By using the Cauchy-Schwarz inequality, we obtain

$$|X_t-X_t^0|^2\le 2\beta^2\left|\int_0^t(X_s-X_s^0)ds\right|^2+2\varepsilon^2\left|\int_0^t\sqrt{X_s}dL_s^d\right|^2$$

$$\leq 2\beta^2 t\int_0^t |X_s - X_s^0|\, ds + 2\varepsilon^2 \sup_{0\leq t\leq 1}\left|\int_0^t \sqrt{X_s} dL_s^d\right|^2.$$

According to Gronwall's inequality, we get

$$\left|X_t - X_t^0\right|^2 \leq 2\varepsilon^2 e^{2\beta^2 t^2} \sup_{0\leq t\leq 1}\left|\int_0^t \sqrt{X_s}\, dL_s^d\right|^2. \tag{7}$$

Then,

$$\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right| \leq \sqrt{2}\varepsilon e^{\beta^2} \sup_{0\leq t\leq 1}\left|\int_0^t \sqrt{X_s}\, dL_s^d\right|. \tag{8}$$

By the Markov inequality and the results in Lemma 1 and Lemma 2, when $f(s) = g(s) = 1$, $r = 0$, for any given $\delta > 0$, when $\varepsilon \to 0$, we have

$$\begin{aligned}
&\mathrm{P}\left(\sqrt{2}\varepsilon e^{\beta^2} \sup_{0\leq t\leq 1}\left|\int_0^t \sqrt{X_s}\, dL_s^d\right| > \delta\right) \\
&\leq 2\delta^{-2}\varepsilon^2 e^{2\beta^2}\, \mathrm{E}\left[\sup_{0\leq t\leq 1}\left|\int_0^t \sqrt{X_s}\, dL_s^d\right|\right]^2 \\
&\leq 2\delta^{-2}\varepsilon^2 e^{2\beta^2}\, \mathrm{E}\left[\left|\int_0^t \sqrt{X_s}\, dL_s^d\right|\right]^2 \\
&\leq 2C\delta^{-2}\varepsilon^2 e^{2\beta^2} \int_0^1\int_0^1 |t-s|^{2d-1}\, dsdt \\
&\leq 2C\delta^{-2}\varepsilon^2 e^{2\beta^2} \\
&\to 0
\end{aligned}$$

where C is a constant.

Therefore, it is easy to check that

$$\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right| \xrightarrow{P} 0. \tag{9}$$

The proof is complete.

Lemma 4. When $\varepsilon \to 0, n \to \infty$, we have

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2 \xrightarrow{p} \int_0^1 (X_t^0)^2\, dt.$$

Proof. Since

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^{n} (X_{t_{i-1}}^0)^2 + \frac{1}{n}\sum_{i=1}^{n} (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2), \tag{10}$$

it is clear that

$$\frac{1}{n}\sum_{i=1}^{n}(X_{t_{i-1}}^{0})^{2}\xrightarrow{P}\int_{0}^{1}(X_{t}^{0})^{2}dt\,. \tag{11}$$

For $\frac{1}{n}\sum_{i=1}^{n}(X_{t_{i-1}}^{2}-(X_{t_{i-1}}^{0})^{2})$, according to Lemma 3 and the fact that $\frac{1}{n}\sum_{i=1}^{n}\left|X_{t_{i-1}}^{0}\right|\xrightarrow{P}\int_{0}^{1}\left|X_{t}^{0}\right|dt$, when $\varepsilon\to 0, n\to\infty$, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}(X_{t_{i-1}}^{2}-(X_{t_{i-1}}^{0})^{2})\right| \\
&=\left|\frac{1}{n}\sum_{i=1}^{n}(X_{t_{i-1}}+X_{t_{i-1}}^{0})(X_{t_{i-1}}-X_{t_{i-1}}^{0})\right| \\
&\le\frac{1}{n}\sum_{i=1}^{n}\left|X_{t_{i-1}}-X_{t_{i-1}}^{0}\right|(\left|X_{t_{i-1}}\right|+\left|X_{t_{i-1}}^{0}\right|) \\
&\le\frac{1}{n}\sum_{i=1}^{n}(\left|X_{t_{i-1}}-X_{t_{i-1}}^{0}\right|^{2}+2\left|X_{t_{i-1}}^{0}\right|\left|X_{t_{i-1}}-X_{t_{i-1}}^{0}\right|) \\
&=\frac{1}{n}\sum_{i=1}^{n}\left|X_{t_{i-1}}-X_{t_{i-1}}^{0}\right|^{2}+2\frac{1}{n}\sum_{i=1}^{n}\left|X_{t_{i-1}}^{0}\right|\left|X_{t_{i-1}}-X_{t_{i-1}}^{0}\right| \\
&\le(\sup_{0\le t\le 1}\left|X_{t}-X_{t}^{0}\right|)^{2}+2\sup_{0\le t\le 1}\left|X_{t}-X_{t}^{0}\right|\frac{1}{n}\sum_{i=1}^{n}\left|X_{t_{i-1}}^{0}\right| \\
&\xrightarrow{P}0
\end{aligned}$$

and therefore obtain

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^{2}\xrightarrow{p}\int_{0}^{1}\left(X_{t}^{0}\right)^{2}dt\,. \tag{12}$$

The proof is complete.

Lemma 5. When $\varepsilon\to 0, n\to\infty$, we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\xrightarrow{p}\int_{0}^{1}\frac{1}{X_{t}^{0}}dt\,.$$

Proof.

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^{0}}+\frac{1}{n}\sum_{i=1}^{n}(\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^{0}}),$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}^{0}}\xrightarrow{P}\int_{0}^{1}\frac{1}{X_{t}^{0}}dt\,.$$

Assume $\inf_{0\le t\le 1}\{X_{t}\}=X_{N}>0$. According to Lemma 1, when $\varepsilon\to 0, n\to\infty$, it can be checked that

$$\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^{0}}\right)\right|=\left|\frac{1}{n}\sum_{i=1}^{n}\frac{X_{t_{i-1}}^{0}-X_{t_{i-1}}}{X_{t_{i-1}}X_{t_{i-1}}^{0}}\right|\leq\frac{1}{n}\sum_{i=1}^{n}\frac{\left|X_{t_{i-1}}^{0}-X_{t_{i-1}}\right|}{\left|X_{t_{i-1}}X_{t_{i-1}}^{0}\right|}$$
$$\leq\sup_{0\leq t\leq 1}\frac{\left|X_{t}-X_{t}^{0}\right|}{\left|X_{t}X_{t}^{0}\right|}\leq\sup_{0\leq t\leq 1}\frac{\left|X_{t}-X_{t}^{0}\right|}{X_{N}^{2}}\xrightarrow{P}0 .$$

Therefore, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\xrightarrow{p}\int_{0}^{1}\frac{1}{X_{t}^{0}}dt .$$

The proof is complete.

In the following theorem, the consistency of the least squares estimators is proved.

Theorem 1. When $\varepsilon\to 0, n\to\infty, \varepsilon n^{\frac{1}{2}-d}\to 0$, the least squares estimators $\hat{\alpha},\hat{\beta}$ are consistent, namely,

$$\hat{\alpha}_{n,\varepsilon}\xrightarrow{P}\alpha,\ \hat{\beta}_{n,\varepsilon}\xrightarrow{P}\beta .$$

Proof. According to Lemma 4 and Lemma 5, it is clear that

$$1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\xrightarrow{p}1-\int_{0}^{1}X_{t}^{0}dt\int_{0}^{1}\frac{1}{X_{t}^{0}}dt . \tag{13}$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^{2}-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^{2}\xrightarrow{p}\left(\int_{0}^{1}X_{t}^{0}dt\right)^{2}-\int_{0}^{1}\left(X_{t}^{0}\right)^{2}dt . \tag{14}$$

When $\varepsilon\to 0, n\to\infty$, it can be verified that

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}\beta\int_{0}^{1}\frac{X_t}{X_t^{0}}dt\int_{0}^{1}X_{t}^{0}dt \tag{15}$$

and

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\xrightarrow{P}\beta\int_{0}^{1}X_t dt . \tag{16}$$

At the same time, according to Lemma 3, the limits of items 1 and 2 of the detailed decomposition formula $\hat{\alpha}$ can be obtained.

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}-\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\xrightarrow{P}0 . \tag{17}$$

Since

$$P\left(\left|\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\right|>\delta\right)\le P\left(\varepsilon\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\right|>\delta\right),$$

by the Markov inequality,

$$\begin{aligned}
&P\left(\varepsilon\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\right|>\delta\right)\\
&\le \delta^{-2}\varepsilon^2\mathbf{E}\left(\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\right|\right)^2\\
&\le \delta^{-2}\varepsilon^2 n\sum_{i=1}^{n}\mathbf{E}\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\right|^2\\
&\le C\delta^{-2}\varepsilon^2 n^2\sup_{0\le t\le 1}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\mathbf{E}[\sqrt{X_s}\sqrt{X_t}]\left|t-s\right|^{2d-1}dsdt\\
&\le C\delta^{-2}(\varepsilon n^{\frac{1}{2}-d})^2\mathbf{E}[X_M]\\
&\to 0
\end{aligned}.$$

When $\varepsilon\to 0, n\to\infty, \varepsilon n^{1-d}\to 0$, we obtain

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\xrightarrow{P}0. \tag{18}$$

According to Lemma 5 and (18), it is obvious that

$$\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\xrightarrow{P}0. \tag{19}$$

Then, we have

$$\left|\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}d\,L_s^d\right|$$

$$\begin{aligned}
&=\left|\varepsilon\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\int_{t_{i-1}}^{t_i}\sqrt{X_s}d\,L_s^d\right|\le\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}d\,L_s^d\right|\\
&\le\varepsilon\sum_{i=1}^{n}\left(\left|\frac{1}{X_{t_{i-1}}^0}\right|+\left|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0}\right|\right)\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}d\,L_s^d\right|\\
&\le\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}d\,L_s^d\right|+\varepsilon\sup_{0\le t\le 1}\left|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}d\,L_s^d\right|.
\end{aligned}$$

By the Markov inequality, we obtain

$$
\begin{aligned}
&P(\left|\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^{0}}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}\,dL_s^d\right|\right|>\delta)\\
&\le \delta^{-2}\varepsilon^{2}(\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^{0}}\right|)^{2}\mathbf{E}\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}\,dL_s^d\right|^{2}\\
&\le C\delta^{-2}\varepsilon^{2}(\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^{0}}\right|)^{2}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}\sqrt{X_s}\sqrt{X_t}\,|t-s|^{2d-1}\,dsdt,\\
&\le C\delta^{-2}(\varepsilon n^{1-d})^{2}(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^{0}}\right|)^{2}\\
&\to 0
\end{aligned}
$$

which implies that $\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^{0}}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}\,dL_s^d\right|\xrightarrow{P}0$ as $\varepsilon\to 0, n\to\infty, \varepsilon n^{\frac{1}{2}-d}\to 0$.

According to Lemma 3, when $\varepsilon\to 0, n\to\infty$, it is obvious that

$$
\varepsilon\sup_{0\le t\le 1}\left|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^{0}}\right|\left|\int_{t_{i-1}}^{t_i}\sqrt{X_s}\,dL_s^d\right|\xrightarrow{P}0. \tag{20}
$$

Then, we have

$$
\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d\,\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}0. \tag{21}
$$

Therefore, with the results of (17), (19) and (21), when $\varepsilon\to 0, n\to\infty, \varepsilon n^{\frac{1}{2}-d}\to 0$, we have

$$
\hat{\alpha}_{n,\varepsilon}\xrightarrow{P}\alpha.
$$

Using the same methods, it can be easily checked that

$$
\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\xrightarrow{P}\beta. \tag{22}
$$

$$
\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\xrightarrow{P}\beta\int_0^1 X_t^0 dt\int_0^1\frac{1}{X_t^0}dt. \tag{23}
$$

$$
\frac{\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\xrightarrow{P}\beta. \tag{24}
$$

Moreover,

$$\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\xrightarrow{P}0. \tag{25}$$

$$\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\xrightarrow{P}0. \tag{26}$$

Therefore, when $\varepsilon\to 0, n\to\infty, \varepsilon n^{\frac{1}{2}-d}\to 0$, we have $\hat{\beta}_{n,\varepsilon}\xrightarrow{P}\beta$.

The proof is complete.

Theorem 2. When $\varepsilon\to 0,\ n\to\infty, \varepsilon n^{\frac{1}{2}-d}\to 0,\ n\varepsilon\to\infty$,

$$\varepsilon^{-1}(\hat{\alpha}_{n,\varepsilon}-\alpha)\xrightarrow{d}\frac{\int_0^1\sqrt{X_s^0}dL_s^d-\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d\int_0^1 X_s^0 ds}{1-\int_0^1 X_s^0 ds\int_0^1\frac{1}{X_s^0}ds}$$

$$\varepsilon^{-1}(\hat{\beta}_{n,\varepsilon}-\beta)\xrightarrow{d}\frac{\int_0^1\sqrt{X_s^0}dL_s^d\int_0^1\frac{1}{X_s^0}ds-\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d}{1-\int_0^1 X_s^0 ds\int_0^1\frac{1}{X_s^0}ds}.$$

Proof. According to the explicit decomposition for $\hat{\alpha}_{n,\varepsilon}$, it is obvious that

$$\varepsilon^{-1}(\hat{\alpha}_{n,\varepsilon}-\alpha)=\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}$$
$$+\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}.$$

From Lemma 3, when $\varepsilon\to 0, n\to\infty, n\varepsilon\to\infty$,

$$\left|\varepsilon^{-1}\beta\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}\int_{t_{i-1}}^{t_i}X_s ds\right|\le\varepsilon^{-1}\beta\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}}\right|\left|\int_{t_{i-1}}^{t_i}X_s ds\right|$$
$$\le\varepsilon^{-1}n^{-1}\beta\sum_{i=1}^{n}\left(\left|\frac{1}{X_{t_{i-1}}}-\frac{1}{X_{t_{i-1}}^0}\right|+\left|\frac{1}{X_{t_{i-1}}^0}\right|\right)\sup_{t_{i-1}\le t\le t_i}|X_t|\xrightarrow{P}0.$$

Then, it is easy to check that

$$\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} 0.$$

Combining with Lemma 4, we have

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}} \xrightarrow{P} 0, \tag{27}$$

and

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}} \xrightarrow{P} 0. \tag{28}$$

Since

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s^0}dL_s^d+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sqrt{X_s}-\sqrt{X_s^0})dL_s^d\ , \tag{29}$$

using Markov's inequality, for any given $\delta > 0$, we have

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sqrt{X_s}-\sqrt{X_s^0})dL_s^d\right|>\delta\right)\\
&\le \delta^{-2}\mathbb{E}\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\sqrt{X_s}-\sqrt{X_s^0})dL_s^d\right|^2\\
&\le \delta^{-2}\sum_{i=1}^{n}\mathbb{E}\left|\int_{t_{i-1}}^{t_i}(\sqrt{X_s}-\sqrt{X_s^0})dL_s^d\right|^2\\
&\le C\delta^{-2}n\sup_{0\le t\le 1}\mathbb{E}\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i}(\sqrt{X_s}-\sqrt{X_s^0})(\sqrt{X_t}-\sqrt{X_t^0})|t-s|^{2d-1}dsdt\\
&\to 0.
\end{aligned}$$

Moreover,

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s^0}dL_s^d=\int_0^1\sqrt{X_s^0}dL_s^d\ ,$$

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d \xrightarrow{P} \int_0^1\sqrt{X_s^0}dL_s^d\ . \tag{30}$$

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d\int_0^1 X_s^0ds\,. \tag{31}$$

It is obvious that

$$\varepsilon^{-1}(\hat{\alpha}_{n,\varepsilon}-\alpha)\xrightarrow{d}\frac{\int_0^1\sqrt{X_s^0}dL_s^d-\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d\int_0^1X_s^0ds}{1-\int_0^1X_s^0ds\int_0^1\frac{1}{X_s^0}ds}\,. \tag{32}$$

According to the detailed decomposition formula of $\hat{\beta}_{n,\varepsilon}$, we get

$$\varepsilon^{-1}(\hat{\beta}_{n,\varepsilon}-\beta)=\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_sds\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\varepsilon^{-1}\beta$$
$$+\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}\,.$$

Then, we have

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}ds}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_sds\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\varepsilon^{-1}\beta\xrightarrow{P}0 \tag{33}$$

and

$$\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sqrt{X_s}dL_s^d\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\sqrt{X_s}}{X_{t_{i-1}}}dL_s^d}{1-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_{t_{i-1}}}}$$
$$\xrightarrow{d}\frac{\int_0^1\sqrt{X_s^0}dL_s^d\int_0^1\frac{1}{X_s^0}ds-\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d}{1-\int_0^1X_s^0ds\int_0^1\frac{1}{X_s^0}ds}\,. \tag{34}$$

Then, we have

$$\varepsilon^{-1}(\hat{\beta}_{n,\varepsilon}-\beta)\xrightarrow{d}\frac{\int_0^1\sqrt{X_s^0}dL_s^d\int_0^1\frac{1}{X_s^0}ds-\int_0^1\frac{1}{\sqrt{X_s^0}}dL_s^d}{1-\int_0^1 X_s^0 ds\int_0^1\frac{1}{X_s^0}ds}. \tag{35}$$

The proof is complete.

4. Simulation

In this experiment, we use an iterative approach to generate a discrete sample $(X_{t_{i-1}})_{i=1,\dots,n}$ and compute $\hat{\alpha}_{n,\varepsilon}$ and $\hat{\beta}_{n,\varepsilon}$ from the sample. We let $x_0 = 0.01$ and $d = 0.02$. The first column of the table is the true value of the parameter (α,β). The size of the sample is represented as "Size n" and given in the table. In Table 1, $\varepsilon = 0.1$, and the size is increasing from 1000 to 5000. In Table 2, $\varepsilon = 0.01$, and the size is increasing from 10000 to 50000. The table lists the values of "$\hat{\alpha}_{n,\varepsilon}$", "$\hat{\beta}_{n,\varepsilon}$" and the absolute errors (AE) of least squares estimators.

The two tables indicate that the absolute error between the estimator and the true value depends on the size of the true value samples for any given parameter. According to the simulation results, when n is large enough, and ε is small enough, the estimator is very close to the true parameter value. If we let n go to infinity and ε converge to zero, the estimator will converge to the true value.

Table 1. Least squares estimator simulation results of α and β.

True		Aver		AE	
(α,β)	size n	$\hat{\alpha}_{n,\varepsilon}$	$\hat{\beta}_{n,\varepsilon}$	$\lvert\hat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\hat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	1000	1.2162	1.2061	0.2162	0.2061
	2000	1.0823	1.1071	0.0823	0.1071
	5000	1.0421	1.0529	0.0421	0.0529
(2,3)	1000	2.2377	3.1907	0.2377	0.1907
	2000	2.1193	3.1249	0.1193	0.1249
	5000	2.0524	3.0693	0.0524	0.0693
(4,5)	1000	4.2556	5.2294	0.2556	0.2294
	2000	4.1372	5.1291	0.1372	0.1291
	5000	4.0583	5.0487	0.0583	0.0487

Table 2. Least squares estimator simulation results of α and β.

True		Aver		AE	
(α,β)	size n	$\hat{\alpha}_{n,\varepsilon}$	$\hat{\beta}_{n,\varepsilon}$	$\lvert\hat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\hat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,1)	10000	1.1265	1.1182	0.1265	0.1182
	20000	1.0372	1.0525	0.0372	0.0525
	50000	1.0017	1.0012	0.0017	0.0012
(2,3)	10000	2.1373	3.1264	0.1373	0.1264
	20000	1.9432	3.0473	0.0568	0.0473
	50000	2.0026	3.0037	0.0026	0.0037
(4,5)	10000	4.1775	5.1643	0.1775	0.1643
	20000	4.0413	5.0518	0.0413	0.0518
	50000	4.0041	5.0032	0.0041	0.0032

5. Discussion

Fractional Lévy noise, as an important non-Gaussian noise, can more accurately reflect actual fluctuations. Because of this, more and more scholars have devoted themselves to the qualitative analysis of stochastic differential equations driven by fractional Lévy processes. Due to the observational discontinuity and heavy tails of financial samples, the CIR model cannot capture these characteristics, and it is necessary to replace the Brownian motion in the CIR model with fractional Lévy noise.

6. Conclusions

The purpose of this paper is to estimate the parameters of the CIR model driven by a fractional Lévy process with discrete observations. First, the comparison function is introduced to obtain the explicit expression of the least square estimator. Then, the consistency and asymptotic distribution of the estimator are derived according to the Markov inequality, Gronwall inequality and Cauchy-Schwarz inequality. The research topic can be extended to the parameter estimation problem for other stochastic models driven by fractional Lévy process.

Conflict of interest

The authors declare that there are no conflicts of interest.

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