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Research article

Stability of the 3D MHD equations without vertical dissipation near an equilibrium

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Abstract: Important progress has been made on the standard Laplacian case with mixed partial dissipation and diffusion. The stability problem of the 3D incompressible magnetohydrodynamic (MHD) equations without vertical dissipation but with the fractional velocity dissipation $(-\Delta)^{\alpha}u$ and magnetic diffusion $(-\Delta)^{\beta}b$ is unfortunately not often well understood for many ranges of fractional powers. This paper discovers that there are new phenomena with the case $\alpha, \beta \leq 1$. We establish that, if an initial datum (u_0, b_0) in the Sobolev space $H^3(\mathbb{R}^3)$ is close enough to the equilibrium state, and we replace the terms $(-\Delta)^{\alpha}u$ and $(-\Delta)^{\beta}b$ by $(-\Delta_h)^{\alpha}u$ and $(-\Delta_h)^{\beta}b$, respectively, the resulting equations with $\alpha, \beta \in (\frac{1}{2}, 1]$ then always lead to a steady solution, where $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$.

Keywords: magnetohydrodynamic equations; background magnetic field; partial dissipation; stability
Mathematics Subject Classification: 35A05, 35Q35, 76D03

1. Introduction

The MHD equations describe the evolution in time of velocity field u and magnetic field b of some electrically conducting fluids such as plasmas, liquid metals and salt water or electrolytes [1, 2]. The set of equations that describe MHD is a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electro-magnetism. The field of MHD was initiated by Hannes Alfvén [3], for which he received the Nobel Prize in physics in 1970.

The MHD equations are also of great interest in mathematics. In recent years, the stability of the MHD equations has attracted considerable interest, and one focus has been on the MHD equations with partial or fractional dissipation and diffusion. Elegant works have been made (see, e.g., [4–12]).

For the 3D incompressible generalized MHD equations with fractional dissipation and diffusion,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nu (-\Delta)^\alpha u - \nabla P + B \cdot \nabla B, & x \in \mathbb{R}^3,\ t > 0, \\ \partial_t B + u \cdot \nabla B = -\eta (-\Delta)^\beta B + B \cdot \nabla u, & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, & x \in \mathbb{R}^3,\ t > 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta > 0$ are real parameters, $u = u(x, t) \in \mathbb{R}^3$ represents the velocity field, $B = B(x, t) \in \mathbb{R}^3$ represents the magnetic field, $P = P(x, t) \in \mathbb{R}$ represents the pressure, $\nu > 0$ denotes the kinematic viscosity, and $\eta > 0$ denotes the magnetic diffusivity. For notational convenience, we write ∂_i for the partial derivatives $\partial_{x_i}(i$=1,2,3). The fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi)$$

for

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$

The MHD equations with fractional dissipation given by (1.1) have recently attracted considerable interest, due to their mathematical importance and physical applications. The justification for the study of this fractionally dissipated system can be made from several different perspectives. First, (1.1) represents a two-parameter family of systems and contains the MHD systems with standard Laplacian dissipation as special cases. (1.1) allows us to simultaneously examine a whole family of equations and potentially reveals how the properties of its solutions are related to the sizes of α and β. Second, the fractional diffusion operators can model the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., [13, 14]). Third, fractional dissipation has been widely used in turbulence modeling to control the effective range of the non-local dissipation and to make numerical resolutions more efficient (see, e.g., [15]).

A range of global well-posedness results on (1.1) have been obtained. When $\alpha = \beta = 1$, (1.1) reduces to the standard MHD equations, which is well-known possessing global L^2 weak solutions; in two dimensions, it is also unique [7, 16]. When combined with the already established global weak solutions [8], these bounds allow us to conclude that MHD equations possess a global classical solution if α and β satisfy

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{d}{2}.$$

Wu [9] was able to sharpen this result by replacing the fractional Laplacian operators by general Fourier multiplier operators. In particular, (1.1) with a $\frac{(-\Delta)^\alpha}{\log(I-\Delta)} u$ and $\frac{(-\Delta)^\beta}{\log(I-\Delta)} b$ for α and β satisfying the bounds above is also globally well posed [6]. Yamazaki obtained the global regularity for the case when $\alpha = 2$ and $\beta = 0$ and for a logarithmically reduced fractional dissipation [17]. Logarithmic refinement of these fractional powers is contained in [18]. Ye and Xu [19] considered the global existence of the 2D generalized incompressible MHD system with velocity dissipation exponent $\alpha > \frac{1}{4}$ and magnetic diffusion exponent $\beta = 1$. After that, Ye established the global regularity solutions to the 2D incompressible MHD equations with almost Laplacian magnetic diffusion in the whole space [20].

Recently, Dai and Ji [5] established the local existence and uniqueness in inhomogeneous Besov spaces when

$$\alpha > \frac{1}{2}, \quad \beta \geq 0, \quad \alpha + \beta \geq 1.$$

Since (1.1) was proposed in [8], there have been considerable activities, and the global well-posedness problem on (1.1) is now much better understood (see, e.g., [21–26]).

The nonlinear stability for the ideal MHD equations was established in several beautiful papers [27–30]. Nevertheless, the stability of (1.1) remains unknown. The focus of this paper is the stability of perturbation near a background magnetic field which is

$$u^{(0)} \equiv (0,0,0), \quad B^{(0)} \equiv (0,1,0).$$

The perturbation (u, b) with

$$b := B - B^{(0)}$$

solves the MHD system

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nu (-\Delta)^{\alpha} u - \nabla P + b \cdot \nabla b + \partial_2 b, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t b + u \cdot \nabla b = -\eta (-\Delta)^{\beta} b + b \cdot \nabla u + \partial_2 u, \quad x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \quad x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

We now address the problem as to whether the steady weak solution of the MHD equations (1.1) does in fact depend continuously on the perturbation of $(u^{(0)}, B^{(0)})$ given in the problem. In this paper, we consider (1.1) with only horizontal fractional dissipation and lacking vertical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nu (-\Delta_h)^{\alpha} u - \nabla P + b \cdot \nabla b + \partial_2 b, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t b + u \cdot \nabla b = -\eta (-\Delta_h)^{\beta} b + b \cdot \nabla u + \partial_2 u, \quad x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \quad x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{1.3}$$

with $\alpha, \beta \in (\frac{1}{2}, 1]$. The concept of horizontal dissipation comes from geophysical fluid dynamics (see [31]), and meteorologists model the turbulent diffusion with anisotropic viscosity $-\nu_h \Delta_h - \nu_3 \partial_3^2$, where the horizontal kinetic viscosity coefficient ν_h and the vertical kinetic viscosity coefficient ν_3 are empirical constants and satisfy $0 < \nu_3 \ll \nu_h$. In this paper, we take the limit case $\nu_h = \nu$ and $\nu_3 = 0$. To give a complete view of current studies on the stability problem concerning the MHD equations with partial dissipation, we mention some of the encouraging results in [4, 10, 16, 22, 32–40] and the references therein.

A natural consideration is how the parameters α and β are determined, and this is what we choose to do in our tentative estimation work, based primarily on energy estimate method. When we bound the $\dot{H}^3$-norm of (u, b), we plan to utilize a series of anisotropic inequalities derived from a Sobolev embedding inequality and Gagliardo-Nirenberg (G-N) interpolation inequality [41–43]. Based on the relationship between the parameters of these inequalities, we ended up choosing $\alpha, \beta \in (\frac{1}{2}, 1]$ in (1.3) to study the stability of the MHD equations with fractional horizontal dissipation.

To construct a steady solution of (1.3), we make use of a bootstrap argument by anisotropic estimates

$$\begin{aligned} E(t) = &\sup_{0\le\tau\le t}\{\|u(\tau)\|_{H^3}^2+\|b(\tau)\|_{H^3}^2\} \\ &+2\nu\int_0^t \|\Lambda_h^{\alpha}u(\tau)\|_{H^3}^2\,d\tau + 2\eta\int_0^t \|\Lambda_h^{\beta}b(\tau)\|_{H^3}^2\,d\tau. \end{aligned} \tag{1.4}$$

Here $\Lambda_h = (-\Delta_h)^{\frac{1}{2}}$ denotes the Zygmund operator. Our precise result is stated in the following theorem.

Theorem 1.1. *Consider (1.3) with initial data* $(u_0, b_0) \in H^3(\mathbb{R}^3)$ *satisfying* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ *and* $\alpha,\beta \in (\frac{1}{2}, 1]$*. Then, there exists a constant* $\delta = \delta(\nu,\eta) > 0$ *such that, if*

$$\|(u_0, b_0)\|_{H^3} \le \delta, \tag{1.5}$$

then (1.3) has a unique global classical solution satisfying

$$\sup_{0\le\tau\le t}(\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2) + 2\nu\int_0^t \|\Lambda_h^{\alpha}u(\tau)\|_{H^3}^2\,d\tau + 2\eta\int_0^t \|\Lambda_h^{\beta}b(\tau)\|_{H^3}^2\,d\tau \le C\delta^2,$$

for any $t > 0$*, and* $C = C(\nu,\eta)$ *is a constant.*

A natural starting point is to bound $\|u(t)\|_{H^3} + \|b(t)\|_{H^3}$ via energy estimate. We are able to derive the following energy inequality:

$$E(t) \le E(0) + CE(t)^{\frac{3}{2}}. \tag{1.6}$$

Combined with the bootstrapping argument (see [44]), we can prove Theorem 1.1. However, the proof of Theorem 1.1 is not superficial. Due to the lack of the vertical dissipation and vertical magnetic diffusion, some nonlinear terms are not easy to control in terms of $\|u(t)\|_{H^3} + \|b(t)\|_{H^3}$ or the dissipation parts $\|\Lambda_h^{\alpha}u\|_{H^3}$ and $\|\Lambda_h^{\beta}b\|_{H^3}$. One of the most difficult terms is

$$\begin{aligned} &-\int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2 b\cdot\partial_3^3 b\,dx \\ &\lesssim \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^{\alpha}u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^{\beta}b\|_{L^2}^{\frac{1}{2\beta}}\|\nabla_h\partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\nabla_h\partial_3^2\Lambda_3^{\alpha}b\|_{L^2}^{\frac{1}{2\beta}}. \end{aligned}$$

Clearly, it does not appear possible to bound the subterms

$$\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \tag{1.7}$$

directly in terms of $\|\Lambda_h^{\alpha}u\|_{H^3}^2\|\Lambda_h^{\beta}b\|_{H^3}^2$, but in terms of $\|u(t)\|_{H^3}^2\|b(t)\|_{H^3}^2$. Therefore, we hope the sum of the corresponding exponents of the two subterms to be less than or equal to 1 for all given α and β, which is

$$1 - \frac{1}{2\alpha} + 1 - \frac{1}{2\beta} \le 1. \tag{1.8}$$

To establish the inequality of (1.6), we choose $\alpha, \beta \in (\frac{1}{2}, 1]$. In the case of

$$1 - \frac{1}{2\alpha} + 1 - \frac{1}{2\beta} = 1,$$

the subterms of (1.7) can be estimated directly by

$$\|u(t)\|_{H^3} + \|b(t)\|_{H^3}.$$

The other case is

$$1 - \frac{1}{2\alpha} + 1 - \frac{1}{2\beta} < 1.$$

Our strategy is to extract part from the rest subterms

$$\|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla_h \partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}}$$

to fill the subterms of (1.7) by G-N interpolation inequality. One reason which cannot be ignored is that $\|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}$ could be bounded by either $\|u\|_{H^3}$ or $\|\Lambda_h^\alpha u\|_{H^3}$, and $\|\nabla_h \partial_3^2 b\|_{L^2}$ could be bounded by either $\|b\|_{H^3}$ or $\|\Lambda_h^\alpha b\|_{H^3}$. In the last section of our paper, we have successfully used this method to solve all similar difficulties in proving stability and obtain inequality (1.6).

Lemma 1.2. *Assume that $\alpha, \beta, \gamma \in (\frac{1}{2}, 1]$, $f, g, h, \Lambda_h^\alpha f, \Lambda_h^\beta g, \Lambda_h^\gamma h$ and $\partial_3 h$ are all in $L^2(\mathbb{R}^3)$. Then,*

$$\int_{\mathbb{R}^3} |fgh|\, dx \lesssim \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_h^\alpha f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_h^\beta g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$
$$\int_{\mathbb{R}^3} |fgh|\, dx \lesssim \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_h^\alpha f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_h^\beta g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{1-\frac{1}{2\gamma}} \|\Lambda_3^\gamma h\|_{L^2}^{\frac{1}{2\gamma}}.$$

Here, we write $A \lesssim B$ to mean that $A \leq CB$ for some constant C and $\Lambda_3 = (-\partial_{33})^{\frac{1}{2}}$.

These anisotropic inequalities are greatly powerful in the study of global regularity and stability problems on partial differential equations with only partial dissipation. Similar inequalities have previously been used in the investigation of partially dissipated MHD systems and related equations (see, e.g., [45, 46])

The rest of this paper is divided into two sections. Section 2 provides the proofs of Theorem 1.1 and Lemma 1.2. Section 3 derives the energy inequality (1.6).

2. Proof of theorem

This section proves Theorem 1.1 and Lemma 1.2.

2.1. Proof of Theorem 1.1

Roughly speaking, the bootstrap argument starts with an ansatz that $E(t)$ is bounded, say,

$$E(t) \leq M,$$

and shows that $E(t)$ actually admits a smaller bound, say,

$$E(t) \leq \frac{1}{2}M,$$

when the initial condition is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book [44].

It follows that

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}, \tag{2.1}$$

for some pure constants C. To initiate the bootstrapping argument, we make the ansatz

$$E(t) \leq M := \frac{1}{4C^2}. \tag{2.2}$$

We then show that (2.1) allows us to conclude that $E(t)$ actually admits an even smaller bound by taking the initial H^3-norm $E(0)$ sufficiently small. In fact, when (2.2) holds, (2.1) implies

$$E(t) \leq E(0) + \frac{1}{2}E(t)$$

or

$$E(t) \leq 2E(0). \tag{2.3}$$

Therefore, if we choose $\delta > 0$ sufficiently small such that

$$\delta^2 \leq \frac{1}{4}M, \tag{2.4}$$

then

$$E(t) \leq \frac{1}{2}M. \tag{2.5}$$

$E(t)$ actually admits a smaller bound in (2.3) than the one in the ansatz (2.2). The bootstrapping argument then assesses that (2.2) holds for all times when $E(0)$ obeys (2.4). This completes the proof.

2.2. *Proof of Lemma 1.2*

The proof makes use of the version of Minkowski's inequality

$$\left\| \|f\|_{L^q_y(\mathbb{R}^n)} \right\|_{L^p_x(\mathbb{R}^m)} \leq \left\| \|f\|_{L^p_x(\mathbb{R}^m)} \right\|_{L^q_y(\mathbb{R}^n)},$$

for any $1 \leq q \leq p \leq \infty$, where $f = f(x, y)$ with $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^n$ is a measurable function on $\mathbb{R}^m \times \mathbb{R}^n$, and the following basic one-dimensional Sobolev embedding inequality [12], for $f \in H^s(\mathbb{R})$,

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{2s}} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{2s}},$$

where $s > \frac{1}{2}$. By the above inequality and Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\,dx \leq & \|f\|_{L^\infty_{x_1}L^2_{x_2}L^2_{x_3}} \|g\|_{L^2_{x_1}L^\infty_{x_2}L^2_{x_3}} \|h\|_{L^2_{x_1}L^2_{x_2}L^\infty_{x_3}} \\ \leq & C \left\| \|f\|_{L^2_{x_1}}^{1-\frac{1}{2s_1}} \|\Lambda_1^{s_1} f\|_{L^2_{x_1}}^{\frac{1}{2s_1}} \right\|_{L^2_{x_2 x_3}} \left\| \|g\|_{L^2_{x_2}}^{1-\frac{1}{2s_2}} \|\Lambda_2^{s_2} g\|_{L^2_{x_2}}^{\frac{1}{2s_2}} \right\|_{L^2_{x_1 x_3}} \\ & \times \left\| \|h\|_{L^2_{x_3}}^{1-\frac{1}{2s_3}} \|\Lambda_3^{s_3} h\|_{L^2_{x_3}}^{\frac{1}{2s_3}} \right\|_{L^2_{x_1 x_2}} \\ \leq & C \|f\|_{L^2}^{1-\frac{1}{2s_1}} \|\Lambda_1^{s_1} f\|_{L^2}^{\frac{1}{2s_1}} \|g\|_{L^2}^{1-\frac{1}{2s_2}} \|\Lambda_2^{s_2} g\|_{L^2}^{\frac{1}{2s_2}} \|h\|_{L^2}^{1-\frac{1}{2s_3}} \|\Lambda_3^{s_3} h\|_{L^2}^{\frac{1}{2s_3}}.\end{aligned}$$

Let $s_1 = \alpha, s_2 = \beta, s_3 = 1$, and we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\,dx \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_1^{\alpha} f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_2^{\beta} g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}} \\ \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_h^{\alpha} f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_h^{\beta} g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

Let $s_1 = \alpha, s_2 = \beta, s_3 = \gamma$, and we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\,dx \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_1^{\alpha} f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_2^{\beta} g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{1-\frac{1}{2\gamma}} \|\Lambda_3^{\gamma} h\|_{L^2}^{\frac{1}{2\gamma}} \\ \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_h^{\alpha} f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_h^{\beta} g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{1-\frac{1}{2\gamma}} \|\Lambda_3^{\gamma} h\|_{L^2}^{\frac{1}{2\gamma}}.\end{aligned}$$

Here, $\|f\|_{L^\infty_{x_1}L^2_{x_2}L^2_{x_3}}$ represents the L^∞-norm in the x_1-variable, followed by the L^2-norm in x_2 and the L^2-norm in x_3. This finishes the proof of Lemma 1.2.

3. The H^3-stability

Due to the equivalence of $\|(u,b)\|_{H^3}$ with $\|(u,b)\|_{L^2} + \|(u,b)\|_{\dot{H}^3}$, it suffices to bound the L^2-norm and the $\dot{H}^3$-norm of (u,b). By a simple energy estimate and $\nabla \cdot u = \nabla \cdot b = 0$, we find that the L^2-norm of (u,b) obeys

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda_h^{\alpha} u(\tau)\|_{L^2}^2\,d\tau \\ &+ 2\eta \int_0^t \|\Lambda_h^{\beta} b(\tau)\|_{L^2}^2\,d\tau = \|u(0)\|_{L^2}^2 + \|b(0)\|_{L^2}^2. \end{aligned} \tag{3.1}$$

The rest of the proof focuses on the $\dot{H}^3$-norm. Applying ∂_i^3 to (1.3) and then dotting by $(\partial_i^3 u, \partial_i^3 b)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}(\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 b\|_{L^2}^2) + \nu\|\partial_i^3 \Lambda_h^{\alpha} u\|_{L^2}^2 + \eta\|\partial_i^3 \Lambda_h^{\beta} b\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.2}$$

where

$$I_1 = \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 \partial_2 b \cdot \partial_3^3 u + \partial_i^3 \partial_2 u \cdot \partial_3^3 b\,dx,$$

$$I_2 = -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx,$$
$$I_3 = \sum_{i=1}^{3}\int_{\mathbb{R}^3} [\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b]\cdot\partial_i^3 u\,dx,$$
$$I_4 = -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\,dx,$$
$$I_5 = \sum_{i=1}^{3}\int_{\mathbb{R}^3} [\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u]\cdot\partial_i^3 b\,dx.$$

Note that, by integration by parts,

$$I_1 = 0,$$

and

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3} b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,dx + \int_{\mathbb{R}^3} b\cdot\nabla\partial_i^3 u\cdot\partial_i^3 b\,dx = 0.$$

To bound I_2, we decompose it into three pieces,

$$\begin{aligned}
I_2 =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{3}\Big(\int_{\mathbb{R}^3} \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx + 3\int_{\mathbb{R}^3} \partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx + 3\int_{\mathbb{R}^3} \partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx\Big)\\
=& I_{21} + 3I_{22} + 3I_{23},
\end{aligned}\tag{3.3}$$

where we have used the fact that $\int_{\mathbb{R}^3} u\cdot\nabla\partial_3^3 u\cdot\partial_3^3 u\,dx = 0$. I_{21} is naturally split into three parts,

$$\begin{aligned}
I_{21} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx - \int_{\mathbb{R}^3} \partial_3^3 u_h\cdot\nabla_h u\cdot\partial_3^3 u\,dx - \int_{\mathbb{R}^3} \partial_3^3 u_3\partial_3 u\cdot\partial_3^3 u dx\\
=& I_{211} + I_{212} + I_{213}.
\end{aligned}$$

By Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{211}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3} \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx|\\
\lesssim& \sum_{i=1}^{2} \|\partial_i^3 u\|_{L^2}^{2-\frac{1}{\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim& \sum_{i=1}^{2} \|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(2-\frac{1}{\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(2-\frac{1}{\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2, \tag{3.4}$$

where we have applied inequality

$$\|\partial_i u\|_{L^2} \le \|\Lambda_h^\alpha u\|_{L^2}^{\gamma}\|\Lambda_h^{1+\alpha} u\|_{L^2}^{1-\gamma} \quad (i=1,2).$$

We now turn to I_{212}, by Lemma 1.2,

$$\begin{aligned}
|I_{212}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_h \cdot \nabla_h u \cdot \partial_3^3 u \, dx| \\
\lesssim& \|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
&\times \|\nabla_h u\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha u\|_{L^2}^{\gamma(\frac{1}{1-2\alpha})} \|\Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(\frac{1}{1-2\alpha})} \\
&\times \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha \partial_3 u\|_{L^2}^{\gamma(\frac{1}{1-2\alpha})} \|\Lambda_h^{1+\alpha} \partial_3 u\|_{L^2}^{(1-\gamma)(\frac{1}{1-2\alpha})} \\
\lesssim& \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned} \tag{3.5}$$

In fact, we take $\|\nabla_h u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\nabla_h \partial_3 u\|_{L^2}^{1-\frac{1}{2\alpha}}$ from $\|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}}$ and combine it with $\|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}$ to reach our desired bound. In addition, by G-N interpolation inequality, we get

$$\|\nabla_h u\|_{L^2} \le \|\Lambda_h^\alpha u\|_{L^2}^{\gamma} \|\Lambda_h^{1+\alpha} u\|_{L^2}^{1-\gamma}.$$

We next consider the term I_{213}, and we have

$$\begin{aligned}
|I_{213}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_3 \partial_3 u \cdot \partial_3^3 u \, dx| \\
=& |\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3^2 \partial_j u_j \partial_3 u \cdot \partial_3^3 u \, dx| \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 \partial_j u_j\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3^2 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}} \\
&\times \|\partial_3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}-1} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
\lesssim& \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2,
\end{aligned} \tag{3.6}$$

where we used

$$\|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2} \le \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\gamma} \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{1-\gamma}, \quad \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}-1} \le \|u\|_{H^3}^{\frac{1}{\alpha}-1}$$

and $\nabla \cdot u = 0$. To deal with I_{22}, this term is split into three parts,

$$I_{22} = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i u \cdot \partial_i^3 u \, dx$$

$$
\begin{aligned}
&=-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2 u\cdot\partial_3^3 u\,dx\\
&=I_{221}+I_{222}+I_{223}.
\end{aligned}
$$

Similarly to (3.4),

$$
\begin{aligned}
|I_{221}|=&|-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx|\\
\lesssim&\sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i u\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 u\|_{L^2}^{\frac12}\\
\lesssim&\sum_{i=1}^{2}\|\partial_i\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i u\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 u\|_{L^2}^{\frac12}\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2. \qquad (3.7)
\end{aligned}
$$

Applying Lemma 1.2 and G-N interpolation inequality, we obtain

$$
\begin{aligned}
|I_{222}|=&|-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 u\,dx|\\
\lesssim&\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac12}\\
\lesssim&\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\nabla_h\partial_3 u\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\|\Lambda_h^\alpha\partial_3 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
&\times\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\|\Lambda_h^\alpha\partial_3^2 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3^2 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2. \qquad (3.8)
\end{aligned}
$$

Note that we separate out a part of $\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac12}$ and make it controlled by $\|\Lambda_h u\|_{H^3}$. Similarly,

$$
\begin{aligned}
|I_{223}|=&|-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
=&|\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3\partial_j u_j\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3\partial_j u_j\|_{L^2}^{\frac12}\|\partial_3^2\partial_j u_j\|_{L^2}^{\frac12}\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\|\partial_3\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|\partial_3^2\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}}\|\partial_3^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}
\tag{3.9}
$$

We deal with I_{23} in the same method, as I_{23} is naturally split into three parts,

$$
\begin{aligned}
I_{23} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx - \int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2 u\cdot\partial_3^3 u\,dx - \int_{\mathbb{R}^3}\partial_3 u_3\partial_3^3 u\cdot\partial_3^3 u\,dx\\
=&I_{231}+I_{232}+I_{233}.
\end{aligned}
$$

By Lemma 1.2 and G-N interpolation inequality, we have

$$
\begin{aligned}
|I_{231}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx|\\
\lesssim& \sum_{i=1}^{2}\|\nabla\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\nabla\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i u\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim& \sum_{i=1}^{2}\|\nabla\partial_i\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\nabla\partial_i\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\nabla\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i u\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}
\tag{3.10}
$$

We estimate I_{232} similarly as I_{213}, which yields

$$
\begin{aligned}
|I_{232}| =& |-\int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
\lesssim&\|\nabla_h\partial_3^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\nabla_h\partial_3^2\Lambda_3^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
\lesssim&\|\Lambda_h^\alpha\partial_3^2 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3^2 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^{\alpha+1}u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}}\\
&\times\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{\alpha}-1}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}
\tag{3.11}
$$

We next consider the term I_{233}, and utilizing the incompressible condition again, we have

$$
\begin{aligned}
|I_{233}| =& |-\int_{\mathbb{R}^3}\partial_3 u_3\partial_3^3 u\cdot\partial_3^3 u\,dx|\\
=&|\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_j\partial_3^3 u\cdot\partial_3^3 u\,dx|\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^3 u\|_{L^2}^{2-\frac{1}{\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\partial_j u_j\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{j=1}^{2} \|\partial_3^3 u\|_{L^2}^{2-\frac{1}{\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}} \|\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \\
&\quad \times \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\partial_3 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \\
&\lesssim \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned} \tag{3.12}
$$

Combined with (3.3)–(3.12), we obtain

$$
I_2(\tau) \lesssim \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2.
$$

To bound I_3, we can refer to the way which is used in I_2 and then divide it into three terms,

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{3} \int_{\mathbb{R}^3} [\partial_i^3 (b \cdot \nabla b) - b \cdot \nabla \partial_i^3 b] \cdot \partial_i^3 u \, dx, \\
&= \sum_{i=1}^{3} \Big(\int_{\mathbb{R}^3} \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u \, dx + 3 \int_{\mathbb{R}^3} \partial_i^2 b \cdot \nabla \partial_i b \cdot \partial_i^3 u \, dx + 3 \int_{\mathbb{R}^3} \partial_i b \cdot \nabla \partial_i^2 b \cdot \partial_i^3 u \, dx \Big) \\
&= I_{31} + 3 I_{32} + 3 I_{33}.
\end{aligned} \tag{3.13}
$$

I_{31} can be further decomposed into three parts,

$$
\begin{aligned}
I_{31} &= \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u \, dx \\
&= \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u \, dx + \int_{\mathbb{R}^3} \partial_3^3 b_h \cdot \nabla_h b \cdot \partial_3^3 u \, dx + \int_{\mathbb{R}^3} \partial_3^3 b_3 \partial_3 b \cdot \partial_3^3 u \, dx \\
&= I_{311} + I_{312} + I_{313}.
\end{aligned}
$$

By Lemma 1.2 and G-N interpolation inequality,

$$
\begin{aligned}
|I_{311}| &= \Big| \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u \, dx \Big| \\
&\lesssim \sum_{i=1}^{2} \|\partial_i^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_i^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \sum_{i=1}^{2} \|\partial_i^2 \Lambda_h^\beta b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_i^2 \Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_i^2 \Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_i^2 \Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \\
&\quad \times \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3},
\end{aligned} \tag{3.14}
$$

where we have applied inequality

$$
\|\partial_i b\|_{L^2} \leq \|\Lambda_h^\beta b\|_{L^2}^{\gamma} \|\Lambda_h^{1+\beta} b\|_{L^2}^{1-\gamma} \quad (i = 1, 2).
$$

By divergence-free condition $\nabla \cdot b = 0$ and Lemma 1.2, we have

$$
\begin{aligned}
|I_{312}| =& |\int_{\mathbb{R}^3} \partial_3^3 b_h \cdot \nabla_h b \cdot \partial_3^3 u \, dx| \\
\lesssim & \|\partial_3^3 b_h\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b_h\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
\lesssim & \|\partial_3^3 b_h\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b_h\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
& \times \|\Lambda_h^\beta b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\nabla_h b\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
& \times \|\Lambda_h^\beta \partial_3 b\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\Lambda_h^{1+\beta} \partial_3 b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \|\nabla_h b\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \\
\lesssim & \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}.
\end{aligned}
\tag{3.15}
$$

Significantly, we have used G-N interpolation inequality

$$
\|\nabla_h b\|_{L^2} \leq \|\Lambda_h^\beta b\|_{L^2}^{\gamma} \|\Lambda_h^{1+\beta} b\|_{L^2}^{1-\gamma}.
$$

We simplify I_{313} by the same way as I_{213}, that is,

$$
\begin{aligned}
|I_{313}| =& |\int_{\mathbb{R}^3} \partial_3^3 b_3 \partial_3 b \cdot \partial_3^3 u \, dx| \\
=& |-\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3^2 \partial_j b_j \partial_3 b \cdot \partial_3^3 u \, dx| \\
\lesssim & \sum_{j=1}^{2} \|\partial_3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^2 \partial_j b_j\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^2 \Lambda_3^\beta \partial_j b_j\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim & \sum_{j=1}^{2} \|\partial_3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}(\frac{1}{2\alpha}+\frac{1}{2\beta}-1)} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}(2-\frac{1}{2\alpha}-\frac{1}{2\beta})} \\
& \times \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^2 \partial_j b_j\|_{L^2}^{(1-\frac{1}{2\beta})(\frac{1}{2\alpha}+\frac{1}{2\beta}-1)} \\
& \times \|\partial_3^2 \Lambda_h^\beta b_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})(2-\frac{1}{2\alpha}-\frac{1}{2\beta})} \|\partial_3^2 \Lambda_h^{1+\beta} b_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})(2-\frac{1}{2\alpha}-\frac{1}{2\beta})} \\
& \times \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\gamma\frac{1}{2\beta}} \|\partial_3^2 \Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)\frac{1}{2\beta}} \\
\lesssim & \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}},
\end{aligned}
\tag{3.16}
$$

where we have used

$$
\|\partial_3^2 \Lambda_3^\beta \partial_j b_j\|_{L^2} \leq \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\gamma} \|\partial_3^2 \Lambda_h^{\beta+1} b\|_{L^2}^{1-\gamma}
$$

and

$$
\|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^2 \partial_j b_j\|_{L^2}^{(1-\frac{1}{2\beta})} \leq \|b\|_{H^3}.
$$

To deal with I_{32}, we also split it into three parts,

$$
I_{32} = \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 b \cdot \nabla \partial_i b \cdot \partial_i^3 u \, dx
$$

$$
\begin{aligned}
&=\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 b\cdot\nabla\partial_i b\cdot\partial_i^3 u\,dx+\int_{\mathbb{R}^3}\partial_3^2 b_h\cdot\nabla_h\partial_3 b\cdot\partial_3^3 u+\int_{\mathbb{R}^3}\partial_3^2 b_3\partial_3^2 b\cdot\partial_3^3 u\,dx\\
&=I_{321}+I_{322}+I_{323}.
\end{aligned}
$$

By Lemma 1.2 and G-N interpolation inequality,

$$
\begin{aligned}
|I_{321}|=&|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 b\cdot\nabla\partial_i b\cdot\partial_i^3 u\,dx|\\
\lesssim&\sum_{i=1}^{2}\|\partial_i^2 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_i^2\Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i b\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 b\|_{L^2}^{\frac12}\\
\lesssim&\sum_{i=1}^{2}\|\partial_i\Lambda_h^\beta b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\partial_i\Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_i^2\Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}}\\
&\times\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i b\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 b\|_{L^2}^{\frac12}\\
\lesssim&\|b\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}.
\end{aligned}\tag{3.17}
$$

Naturally,

$$
\begin{aligned}
|I_{322}|=&|\int_{\mathbb{R}^3}\partial_3^2 b_h\cdot\nabla_h\partial_3 b\cdot\partial_3^3 u\,dx|\\
\lesssim&\|\partial_3^2 b_h\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^\beta b_h\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla_h\partial_3 b\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 b\|_{L^2}^{\frac12}\\
\lesssim&\|\partial_3^2 b_h\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^\beta b_h\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\Lambda_h^\beta\partial_3 b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\Lambda_h^{1+\beta}\partial_3 b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\nabla_h\partial_3 b\|_{L^2}^{\frac{1}{2\beta}-\frac12}\\
&\times\|\Lambda_h^\beta\partial_3^2 b\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\beta}\partial_3^2 b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\nabla_h\partial_3^2 b\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\\
\lesssim&\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}
\end{aligned}\tag{3.18}
$$

and

$$
\begin{aligned}
|I_{323}|=&|\int_{\mathbb{R}^3}\partial_3^2 b_3\partial_3^2 b\cdot\partial_3^3 u\,dx|\\
=&|-\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3\partial_j b_j\partial_3^2 b\cdot\partial_3^3 u\,dx|\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3\partial_j b_j\|_{L^2}^{\frac12}\|\partial_3^2\partial_j b_j\|_{L^2}^{\frac12}\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_3\Lambda_h^\beta b_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\partial_3\Lambda_h^{1+\beta} b_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_3\partial_j b_j\|_{L^2}^{\frac{1}{2\beta}-\frac12}\\
&\times\|\partial_3^2\Lambda_h^\beta b_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3^2\Lambda_h^{1+\beta} b_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_3^2\partial_j b_j\|_{L^2}^{\frac{1}{2\alpha}-\frac12}
\end{aligned}
$$

$$\lesssim\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\alpha}u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\beta}b\|_{H^3}^{2-\frac{1}{2\alpha}}. \tag{3.19}$$

In the same way, I_{33} is split into three parts,

$$\begin{aligned}I_{33}=&\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 b\cdot\partial_i^3 u\,dx\\ =&\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 b\cdot\partial_i^3 u\,dx+\int_{\mathbb{R}^3}\partial_3 b_h\cdot\nabla_h\partial_3^2 b\cdot\partial_3^3 u\,dx+\int_{\mathbb{R}^3}\partial_3 b_3\partial_3^3 b\cdot\partial_3^3 u\,dx\\ =&I_{331}+I_{332}+I_{333}.\end{aligned}$$

Lemma 1.2 and G-N interpolation inequality imply

$$\begin{aligned}|I_{331}|=&|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 b\cdot\partial_i^3 u\,dx|\\ \lesssim&\sum_{i=1}^{2}\|\nabla\partial_i^2 b\|_{L^2}^{1-\frac{1}{2\beta}}\|\nabla\partial_i^2\Lambda_h^{\beta}b\|_{L^2}^{\frac{1}{2\beta}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^{\alpha}u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i b\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 b\|_{L^2}^{\frac{1}{2}}\\ \lesssim&\sum_{i=1}^{2}\|\nabla\partial_i\Lambda_h^{\beta}b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\nabla\partial_i\Lambda_h^{1+\beta}b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_i^2\Lambda_h^{\beta}b\|_{L^2}^{\frac{1}{2\beta}}\\ &\times\|\partial_i^2\Lambda_h^{\alpha}u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^{\alpha}u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i b\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 b\|_{L^2}^{\frac{1}{2}}\\ \lesssim&\|b\|_{H^3}\|\Lambda_h^{\alpha}u\|_{H^3}\|\Lambda_h^{\beta}b\|_{H^3}.\end{aligned} \tag{3.20}$$

We estimate I_{332} similarly as I_{313}, which yields

$$\begin{aligned}|I_{332}|=&|\int_{\mathbb{R}^3}\partial_3 b_h\cdot\nabla_h\partial_3^2 b\cdot\partial_3^3 u\,dx|\\ \lesssim&\|\partial_3 b_h\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3\Lambda_h^{\beta}b_h\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^{\alpha}u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^2\nabla_h b\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\nabla_h\Lambda_3^{\beta}b\|_{L^2}^{\frac{1}{2\beta}}\\ \lesssim&\|\partial_3 b_h\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3\Lambda_h^{\beta}b_h\|_{L^2}^{\frac{1}{2\beta}(\frac{1}{2\alpha}+\frac{1}{2\beta}-1)}\|\partial_3\Lambda_h^{\beta}b_h\|_{L^2}^{\frac{1}{2\beta}(2-\frac{1}{2\alpha}-\frac{1}{2\beta})}\\ &\times\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^{\alpha}u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^2\nabla_h b\|_{L^2}^{(1-\frac{1}{2\beta})(\frac{1}{2\alpha}+\frac{1}{2\beta}-1)}\\ &\times\|\partial_3^2\Lambda_h^{\beta}b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})(2-\frac{1}{2\alpha}-\frac{1}{2\beta})}\|\partial_3^2\Lambda_h^{1+\beta}b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})(2-\frac{1}{2\alpha}-\frac{1}{2\beta})}\\ &\times\|\partial_3^3\Lambda_h^{\beta}b\|_{L^2}^{\gamma\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^{1+\beta}b\|_{L^2}^{(1-\gamma)\frac{1}{2\beta}}\\ \lesssim&\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\alpha}u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\beta}b\|_{H^3}^{2-\frac{1}{2\alpha}}.\end{aligned} \tag{3.21}$$

Now we turn to the next term I_{333}, by $\nabla\cdot b=0$,

$$\begin{aligned}|I_{333}|=&|\int_{\mathbb{R}^3}\partial_3 b_3\partial_3^3 b\cdot\partial_3^3 u\,dx|\\ =&|-\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j b_j\partial_3^3 b\cdot\partial_3^3 u\,dx|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{j=1}^{2} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_j b_j\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_j b_j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^{2} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
&\quad \times \|\Lambda_h^\beta b_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\Lambda_h^{1+\beta} b_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_j b_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
&\quad \times \|\partial_3 \Lambda_h^\beta b_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3 \Lambda_h^{1+\beta} b_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \|\partial_3 \partial_j b_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \\
&\lesssim \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}. \qquad (3.22)
\end{aligned}
$$

Utilizing Young's inequality, combining with (3.13)–(3.22), we have

$$
\begin{aligned}
I_3(\tau) \lesssim &\|b\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3} \\
&+ \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}} \\
\lesssim &(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_h^\alpha u\|_{H^3}^2 + \|\Lambda_h^\beta b\|_{H^3}^2).
\end{aligned}
$$

Now, we try to bound I_4, and we split it into three parts,

$$
\begin{aligned}
I_4 &= -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \, dx \\
&= -\sum_{i=1}^{3} \Big(\int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla b \cdot \partial_i^3 b \, dx + 3 \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i b \cdot \partial_i^3 b \, dx + 3 \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2 b \cdot \partial_i^3 b \, dx \Big) \\
&= I_{41} + 3I_{42} + 3I_{43}. \qquad (3.23)
\end{aligned}
$$

Similarly as I_{31}, I_{41} can be divided directly into three parts,

$$
\begin{aligned}
I_{41} &= -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla b \cdot \partial_i^3 b \, dx \\
&= -\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla b \cdot \partial_i^3 b \, dx - \int_{\mathbb{R}^3} \partial_3^3 u_h \cdot \nabla_h b \cdot \partial_3^3 b \, dx - \int_{\mathbb{R}^3} \partial_3^3 u_3 \partial_3 b \cdot \partial_3^3 b \, dx \\
&= I_{411} + I_{412} + I_{413},
\end{aligned}
$$

and each term can be bounded by Lemma 1.2 and G-N interpolation inequality. Same as (3.14)–(3.15), we have

$$
\begin{aligned}
|I_{411}| &= \Big| -\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla b \cdot \partial_i^3 b \, dx \Big| \\
&\lesssim \sum_{i=1}^{2} \|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_i^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_i^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3} \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{412}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_h \cdot \nabla_h b \cdot \partial_3^3 b \, dx| \\
\lesssim & \|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
\lesssim & \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}.
\end{aligned} \tag{3.25}
$$

By $\nabla \cdot u = 0$ and $\|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{\beta}-1} \leq \|b\|_{H^3}^{\frac{1}{\beta}-1}$,

$$
\begin{aligned}
|I_{413}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_3 \partial_3 b \cdot \partial_3^3 b \, dx| \\
=& |\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3^2 \partial_j u_j \partial_3 b \cdot \partial_3^3 b \, dx| \\
\lesssim & \sum_{j=1}^{2} \|\partial_3^2 \partial_j u_j\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim & \sum_{j=1}^{2} \|\partial_3^2 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3^2 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}} \\
& \times \|\partial_3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{\beta}-1} \|\partial_3 \Lambda_h^\beta b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\alpha b\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim & \|b\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3}.
\end{aligned} \tag{3.26}
$$

Similarity, I_{42} can also be divided directly into three terms,

$$
\begin{aligned}
I_{42} =& -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i b \cdot \partial_i^3 b \, dx \\
=& -\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i b \cdot \partial_i^3 b \, dx - \int_{\mathbb{R}^3} \partial_3^2 u_h \cdot \nabla_h \partial_3 b \cdot \partial_3^3 b \, dx - \int_{\mathbb{R}^3} \partial_3^2 u_3 \partial_3^2 b \cdot \partial_3^3 b \, dx \\
=& I_{421} + I_{422} + I_{423}.
\end{aligned}
$$

Then, Lemma 1.2 and (3.17)–(3.18) imply

$$
\begin{aligned}
|I_{421}| =& |-\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i b \cdot \partial_i^3 b \, dx| \\
\lesssim & \sum_{i=1}^{2} \|\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_i^2 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_i^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla \partial_i b\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_i \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
\lesssim & \|b\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3}
\end{aligned} \tag{3.27}
$$

and

$$
|I_{422}| = |-\int_{\mathbb{R}^3} \partial_3^2 u_h \cdot \nabla_h \partial_3 b \cdot \partial_3^3 b \, dx|
$$

$$\lesssim \|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h \partial_3 b\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3^2 b\|_{L^2}^{\frac{1}{2}}$$
$$\lesssim \|u\|_{H^3}^{1-\frac{1}{2\alpha}} \|b\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}. \tag{3.28}$$

I_{423} can also be bounded via $\nabla \cdot u = 0$, Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{423}| =& |-\int_{\mathbb{R}^3} \partial_3^2 u_3 \partial_3^2 b \cdot \partial_3^3 b \, dx| \\
=& |\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3 \partial_j u_j \partial_3^2 b \cdot \partial_3^3 b \, dx| \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^2 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 \partial_j u_j\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^2 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \\
&\times \|\partial_3 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_3 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
&\times \|\partial_3^2 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_3^2 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_3^2 \partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
\lesssim& \|u\|_{H^3}^{\frac{1}{\beta}-1} \|b\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\beta b\|_{H^3}^{\frac{1}{\beta}}.
\end{aligned} \tag{3.29}$$

To deal with I_{43}, we rewrite it as

$$\begin{aligned}
I_{43} =& -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2 b \cdot \partial_i^3 b \, dx \\
=& -\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2 b \cdot \partial_i^3 b \, dx - \int_{\mathbb{R}^3} \partial_3 u_h \cdot \nabla_h \partial_3^2 b \cdot \partial_3^3 b \, dx - \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3^3 b \cdot \partial_3^3 b \, dx \\
=& I_{431} + I_{432} + I_{433}.
\end{aligned}$$

Again, by Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{431}| =& |-\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2 b \cdot \partial_i^3 b \, dx| \\
\lesssim& \sum_{i=1}^{2} \|\nabla \partial_i^2 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\nabla \partial_i^2 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\partial_i \partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \sum_{i=1}^{2} \|\nabla \partial_i \Lambda_h^\beta b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\nabla \partial_i \Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_i^2 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \\
&\times \|\partial_i^2 \Lambda_h^\beta b\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_i^2 \Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_i^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\partial_i \partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \|u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3}^2.
\end{aligned} \tag{3.30}$$

The estimate for I_{432} is more complex, and utilizing Lemma 1.2 and G-N interpolation inequality, we have

$$\begin{aligned}
|I_{432}| =& |-\int_{\mathbb{R}^3} \partial_3 u_h \cdot \nabla_h \partial_3^2 b \cdot \partial_3^3 b \, dx| \\
\lesssim& \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h \partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\nabla_h \partial_3^2 \Lambda_3^\alpha b\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim& \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h \partial_3^2 \Lambda_3^\alpha b\|_{L^2}^{\frac{1}{2\beta}} \\
&\times \|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\theta\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{(1-\theta)\frac{1}{2\alpha}} \|\nabla_h \partial_3^2 b\|_{L^2}^{\theta(1-\frac{1}{2\beta})} \|\nabla_h \partial_3^2 b\|_{L^2}^{(1-\theta)(1-\frac{1}{2\beta})} \\
\lesssim& \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 b\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\gamma\frac{1}{2\beta}} \|\partial_3^2 \Lambda_h^{1+\beta} b\|_{L^2}^{(1-\gamma)\frac{1}{2\beta}} \\
&\times \|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\theta\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{(1-\theta)\frac{1}{2\alpha}} \\
&\times \|\nabla_h \partial_3^2 b\|_{L^2}^{\theta(1-\frac{1}{2\beta})} \|\Lambda_h^\beta \partial_3^2 b\|_{L^2}^{\gamma(1-\theta)(1-\frac{1}{2\beta})} \|\Lambda_h^{1+\beta} \partial_3^2 b\|_{L^2}^{(1-\gamma)(1-\theta)(1-\frac{1}{2\beta})} \\
\lesssim& \|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\theta\frac{1}{2\alpha}} \|b\|_{H^3}^{(1-\frac{1}{2\beta})+\theta(1-\frac{1}{2\beta})} \|\Lambda_h^\alpha u\|_{H^3}^{(1-\theta)\frac{1}{2\alpha}} \|\Lambda_h^\beta b\|_{H^3}^{(1-\theta)(1-\frac{1}{2\beta})+\frac{1}{\beta}},
\end{aligned} \tag{3.31}$$

where $\theta = \frac{\frac{1}{2\alpha}+\frac{1}{2\beta}-1}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}$, $1-\theta = \frac{2-\frac{1}{\beta}}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}$. It is worth noting that

$$\|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla_h \partial_3^2 b\|_{L^2}^{1-\frac{1}{2\beta}}$$

allows us to extract part of

$$\|\partial_3 \Lambda_h^\alpha u_h\|_{L^2}^{\theta\frac{1}{2\alpha}} \|\nabla_h \partial_3^2 b\|_{L^2}^{\theta(1-\frac{1}{2\beta})}$$

which can be bounded by $\|u\|_{H^3}^{\theta\frac{1}{2\alpha}}$ and $\|b\|_{H^3}^{\theta(1-\frac{1}{2\beta})}$, and this brings us the hope of controlling I_{432} suitably. We estimate I_{433} by the same way as I_{423}, which is

$$\begin{aligned}
|I_{433}| =& |-\int_{\mathbb{R}^3} \partial_3 u_3 \partial_3^3 b \cdot \partial_3^3 b \, dx \, dx| \\
=& |\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_j u_j \partial_3^3 b \cdot \partial_3^3 b \, dx| \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^3 b\|_{L^2}^{2-\frac{1}{\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{\beta}} \|\partial_j u_j\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^3 b\|_{L^2}^{2-\frac{1}{\beta}} \|\partial_3^3 \Lambda_h^\beta b\|_{L^2}^{\frac{1}{\beta}} \|\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
&\times \|\partial_3 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_3 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
\lesssim& \|u\|_{H^3}^{\frac{1}{\beta}-1} \|b\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\beta b\|_{H^3}^{\frac{1}{\beta}}.
\end{aligned} \tag{3.32}$$

Combining with (3.23)–(3.32), we obtain

$$I_4(\tau) \lesssim \|u\|_{H^3} \|\Lambda_h^\beta b\|_{H^3}^2$$

$$
\begin{aligned}
&+\|b\|_{H^3}\|\Lambda_h^{\alpha}u\|_{H^3}\|\Lambda_h^{\beta}b\|_{H^3}\\
&+\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\alpha}u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^{\beta}b\|_{H^3}^{2-\frac{1}{2\alpha}}\\
&+\|u\|_{H^3}^{\frac{1}{\beta}-1}\|b\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^{\alpha}u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^{\beta}b\|_{H^3}^{\frac{1}{\beta}}\\
&+\|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\theta\frac{1}{2\alpha}}\|b\|_{H^3}^{(1-\frac{1}{2\beta})+\theta(1-\frac{1}{2\beta})}\|\Lambda_h^{\alpha}u\|_{H^3}^{(1-\theta)\frac{1}{2\alpha}}\|\Lambda_h^{\beta}b\|_{H^3}^{(1-\theta)(1-\frac{1}{2\beta})+\frac{1}{\beta}}\\
\lesssim&(\|u\|_{H^3}+\|b\|_{H^3})(\|\Lambda_h^{\alpha}u\|_{H^3}^2+\|\Lambda_h^{\beta}b\|_{H^3}^2).
\end{aligned}
$$

It remains to estimate I_5, and since the estimation in I_5 is similar to what is done in $I_2 - I_4$, we will omit the specific calculation process for I_5.

$$
\begin{aligned}
I_5=&\sum_{i=1}^{3}\int_{\mathbb{R}^3}[\partial_i^3(b\cdot\nabla u)-b\cdot\nabla\partial_i^3u]\cdot\partial_i^3b\,dx\\
=&\sum_{i=1}^{3}(\int_{\mathbb{R}^3}\partial_i^3b\cdot\nabla u\cdot\partial_i^3b\,dx+3\int_{\mathbb{R}^3}\partial_i^2b\cdot\nabla\partial_iu\cdot\partial_i^3b\,dx+3\int_{\mathbb{R}^3}\partial_ib\cdot\nabla\partial_i^2u\cdot\partial_i^3b\,dx)\\
=&I_{51}+3I_{52}+3I_{53}.
\end{aligned}\tag{3.33}
$$

We turn to estimate I_{51},

$$
\begin{aligned}
I_{51}=&\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3b\cdot\nabla u\cdot\partial_i^3b\,dx\\
=&\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^3b\cdot\nabla u\cdot\partial_i^3b\,dx+\int_{\mathbb{R}^3}\partial_3^3b_h\cdot\nabla_hu\cdot\partial_3^3b\,dx+\int_{\mathbb{R}^3}\partial_3^3b_3\partial_3u\cdot\partial_3^3b\,dx\\
=&I_{511}+I_{512}+I_{513}.
\end{aligned}
$$

By Lemma 1.2 and G-N interpolation inequality,

$$
\begin{aligned}
|I_{511}|=&|\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^3b\cdot\nabla u\cdot\partial_i^3b\,dx|\\
\lesssim&\|u\|_{H^3}\|\Lambda_h^{\beta}b\|_{H^3}^2,
\end{aligned}\tag{3.34}
$$

$$
\begin{aligned}
|I_{512}|=&|\int_{\mathbb{R}^3}\partial_3^3b_h\cdot\nabla_hu\cdot\partial_3^3b\,dx|\\
\lesssim&\|u\|_{H^3}^{\frac{1}{\beta}-1}\|b\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^{\alpha}u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^{\beta}b\|_{H^3}^{\frac{1}{\beta}}
\end{aligned}\tag{3.35}
$$

and

$$
\begin{aligned}
|I_{513}|=&|\int_{\mathbb{R}^3}\partial_3^3b_3\partial_3u\cdot\partial_3^3b\,dx|\\
=&|-\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3^2\partial_jb_j\partial_3u\cdot\partial_3^3b\,dx|
\end{aligned}
$$

$$\lesssim \|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\theta\frac{1}{2\alpha}}\|b\|_{H^3}^{(1-\frac{1}{2\beta})+\theta(1-\frac{1}{2\beta})}\|\Lambda_h^\alpha u\|_{H^3}^{(1-\theta)\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{(1-\theta)(1-\frac{1}{2\beta})+\frac{1}{\beta}}, \tag{3.36}$$

where $\theta = \frac{\frac{1}{2\alpha}+\frac{1}{2\beta}-1}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}$, $1-\theta = \frac{2-\frac{1}{\beta}}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}$. Now, we focus on I_{52} and set

$$\begin{aligned}
I_{52} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2 b\cdot\nabla\partial_i u\cdot\partial_i^3 b\,dx\\
&= \sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 b\cdot\nabla\partial_i u\cdot\partial_i^3 b\,dx+\int_{\mathbb{R}^3}\partial_3^2 b_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 b\,dx+\int_{\mathbb{R}^3}\partial_3^2 b_3\partial_3^2 u\cdot\partial_3^3 b\,dx\\
&= I_{521}+I_{522}+I_{523}.
\end{aligned}$$

By Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{521}| &= |\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 b\cdot\nabla\partial_i u\cdot\partial_i^3 b\,dx|\\
&\lesssim \|u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}^2,
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
|I_{522}| &= |\int_{\mathbb{R}^3}\partial_3^2 b_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 b\,dx|\\
&\lesssim \|u\|_{H^3}^{\frac{1}{\beta}-1}\|b\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\beta b\|_{H^3}^{\frac{1}{\beta}}
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
|I_{523}| &= |\int_{\mathbb{R}^3}\partial_3^2 b_3\partial_3^2 u\cdot\partial_3^3 b\,dx|\\
&= |-\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3\partial_j b_j\partial_3^2 u\cdot\partial_3^3 b\,dx|\\
&\lesssim \|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}.
\end{aligned} \tag{3.39}$$

We try to bound I_{53},

$$\begin{aligned}
I_{53} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 u\cdot\partial_i^3 b\,dx\\
&= \sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 u\cdot\partial_i^3 b\,dx+\int_{\mathbb{R}^3}\partial_3 b_h\cdot\nabla_h\partial_3^2 u\cdot\partial_3^3 b\,dx+\int_{\mathbb{R}^3}\partial_3 b_3\partial_3^3 u\cdot\partial_3^3 b\,dx\\
&= I_{531}+I_{532}+I_{533}.
\end{aligned}$$

Again, by Lemma 1.2 and G-N interpolation inequality,

$$|I_{531}| = |\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i b\cdot\nabla\partial_i^2 u\cdot\partial_i^3 b\,dx|$$

$$\lesssim \|b\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}, \tag{3.40}$$

$$\begin{aligned}|I_{532}| =& \Big|\int_{\mathbb{R}^3} \partial_3 b_h \cdot \nabla_h \partial_3^2 u \cdot \partial_3^3 b\, dx\Big|\\ \lesssim& \|b\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}|I_{533}| =& \Big|\int_{\mathbb{R}^3} \partial_3 b_3 \partial_3^3 u \cdot \partial_3^3 b\, dx\Big|\\ =& \Big| -\sum_{j=1}^{2}\int_{\mathbb{R}^3} \partial_j b_j \partial_3^3 u \cdot \partial_3^3 b\, dx\Big|\\ \lesssim& \|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}.\end{aligned} \tag{3.42}$$

Combined with (3.33)–(3.42), we obtain

$$\begin{aligned}I_5(\tau) \lesssim& \|u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}^2\\ &+ \|b\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta b\|_{H^3}\\ &+ \|u\|_{H^3}^{\frac{1}{\beta}-1}\|b\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\beta b\|_{H^3}^{\frac{1}{\beta}}\\ &+ \|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|b\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{2-\frac{1}{2\alpha}}\\ &+ \|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\theta\frac{1}{2\alpha}}\|b\|_{H^3}^{(1-\frac{1}{2\beta})+\theta(1-\frac{1}{2\beta})}\|\Lambda_h^\alpha u\|_{H^3}^{(1-\theta)\frac{1}{2\alpha}}\|\Lambda_h^\beta b\|_{H^3}^{(1-\theta)(1-\frac{1}{2\beta})+\frac{1}{\beta}}\\ \lesssim& (\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_h^\alpha u\|_{H^3}^2 + \|\Lambda_h^\beta b\|_{H^3}^2).\end{aligned}$$

Adding (3.1), (3.2) and integrating in time,

$$E(t) \lesssim E(0) + \int_0^t I_2(\tau) + I_3(\tau) + I_4(\tau) + I_5(\tau)\, d\tau,$$

and inserting all the bounds obtained above for I_2 through I_5, we obtain (1.6). For example, the bounds for I_2 yield

$$\begin{aligned}\int_0^t |I_2(\tau)| d\tau \lesssim& \int_0^t \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2\, d\tau\\ \lesssim& \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^3}\int_0^t \|\Lambda_h^\alpha u\|_{H^3}^2\, d\tau\\ \lesssim& E(t)^{\frac{3}{2}}.\end{aligned}$$

The time integrals of $I_3 - I_5$ are similarly bounded, which completes the proof of (1.6).

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Conflict of interest

The authors declare no conflict of interest.

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