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Research article

On $\mathscr{ABC}$ coupled Langevin fractional differential equations constrained by Perov's fixed point in generalized Banach spaces

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Abstract: Nonlinear differential equations are widely used in everyday scientific and engineering dynamics. Problems involving differential equations of fractional order with initial and phase changes are often employed. Using a novel norm that is comfortable for fractional and non-singular differential equations containing Atangana-Baleanu-Caputo fractional derivatives, we examined a new class of initial values issues in this study. The Perov fixed point theorems that are utilized in generalized Banach spaces form the foundation for the new findings. Examples of the numerical analysis are provided in order to safeguard and effectively present the key findings.

Keywords: coupled system; existence and uniqueness; Perov's fixed point theorem; $\mathscr{ABC}$-fractional operator; generalized Banach space
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1. Introduction

Numerous studies in science and engineering have focused on the numerical simulation of dynamical systems and the mathematical modeling of those systems. A valuable and well-researched technique for expanding the application of the classical models is the application of fractional order operators [1–5]. The history of fractional order operators covers singular and non-singular kernels as well as local and non-local kernels. Recently, these issues have been investigated by utilizing some fascinating results. Readers can peruse the stunning monographs [6–10].

Experts looked at the broad classes of fractional differential equations (FDEs), which include sequential, hybrid and mixed FDEs, and a large number of other classes that are yet understudied in this field [11–13]. The use of perturbation methods substantially facilitates the understanding of system dynamics that are described by various mathematical methods in the area of nonlinear analysis [14–16]. Even if a differential equation describing a specific dynamical system may occasionally be difficult to solve or assess, by perturbing the system in some way, it is feasible to investigate it by using methods for a range of the outcomes' features [17–20].

The illustration of fractional order hybrid DEs was then studied by several authors for different fractional order derivatives [21–25]. For instance, Zhao et al. considered the second-order kind quadratic perturbation problem for the existence and uniqueness of solution in the Riemann-Liouville (R-L) sense of derivative [26]. Sitho et al. studied fractional integro-differential equations for the existence and uniqueness of solution with their applications in R-L sense [27]. Awadalla and Abuasbeh [28] studied a second class perturbed sequential FDE for the existence and uniqueness of solution for Caputo-Hadamard operators. Gul et al. studied a system of hybrid FDEs with the application of their results to the dynamical problems where the operator they used was Caputo's derivative [29].

The authors of [30] obtained estimations of the $\mathcal{ABC}$-fractional derivative at the extreme points comparison results and the local and global existence of a solution and extremal solution for non-linear $\mathcal{ABC}$-fractional differential equations of the form

$$
{}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) = \chi(\mathrm{z}, \mathfrak{u}), \qquad \forall\, \mathrm{z} \in [0, z_0],
$$

under initial value $\mathfrak{u} = \mathfrak{u}_0$, where $0 < \beta < 1$, ${}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}$ is the $\mathcal{ABC}$-fractional derivative operator, ${}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) \in C([0, z_0])$ and $\chi \in C([0, z_0] \times \mathbb{R})$ is a continuous nonlinear function. In 2022, Amiri et al. considered the Atangana-Baleanu fractional integral inclusions system as follows

$$
\begin{cases}
\mathfrak{u}_1(\mathrm{z}) \in \left\{\mathrm{a}_1(\mathrm{z}) + {}^{\mathrm{AB}}I_{z_0}^{\beta}(\chi_{j_k}(\mathrm{z}, \mathfrak{s}, \mathfrak{u}_1(\mathfrak{s})))\right\}, \\
\mathfrak{u}_2(\mathrm{z}) \in \left\{\mathrm{a}_2(\mathrm{z}) + {}^{\mathrm{AB}}I_{z_0}^{\beta}(\chi_{r_k}(\mathrm{z}, \mathfrak{s}, \mathfrak{u}_2(\mathfrak{s})))\right\},
\end{cases} \tag{1.1}
$$

where $\mathfrak{u}_1, \mathfrak{u}_2, \mathrm{a}_1, \mathrm{a}_2 \in \mathrm{C}^2([z_0, z_1]), \chi_{j_k}, \chi_{r_k} \in \mathrm{C}^2([z_0, z_1]^2 \times \mathbb{R})$ are bounded functions for $k = 1, 2$, and ${}^{\mathrm{AB}}I_{z_0}^{\beta}$ is Atangana-Baleanu fractional integral of order $0 < \beta \leq 1$ [31]. For more related works, see [32–36] and references therein.

In the current study, we implement the existence analysis on the following nonlinear fractional differential equations:

$$
\begin{cases}
{}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta_1}\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_1} + \gamma_1\right)\mathfrak{u}_1(\mathrm{z}) = \chi_1(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})), \\
{}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta_2}\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_2} + \gamma_2\right)\mathfrak{u}_2(\mathrm{z}) = \chi_2(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})), \\
\mathfrak{u}_i^{(k)}(0) = \mu_{i,k}, & 0 \leq k < m_i, \\
\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i} + \gamma_i\right)^{(k)}\mathfrak{u}_i(0) = \nu_{i,k}, & 0 \leq k < n_i,
\end{cases} \tag{1.2}
$$

for $0 < \mathrm{z} < 1$, where $\gamma_i \in \mathbb{R}$, $m_i < \sigma_i \le m_i + 1$, $n_i < \beta_i \le n_i + 1$, $m_i, n_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, ${}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta_i}$ and ${}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i}$ represent the $\mathcal{ABC}$-fractional derivatives and $\chi_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, $i = 1, 2$.

The rest of the paper is organized as follows: In Section 2, we mainly introduces the basic concepts, definitions and initial results presupposed to prove our key findings. In Section 3, we shall present the existence and uniqueness results for the problem (1.2). The result relies on the Perov's fixed point theorems in generalized Banach spaces. Section 4 promotes our outcomes to the problem (1.2) by giving an illustrative example to support and justify the acquired results.

2. Preliminaries

We first recall some notions regarding the fractional integrals and derivatives that are required throughout the manuscript. For more details, we refer the reader to [2, 37].

Let $\mathfrak{u} \in L^1(0, z_\circ)$, $z_\circ > 0$; then, the $\mathcal{AB}$-fractional integral ($\mathcal{AB}$-FI) and $\mathcal{ABC}$-fractional derivative ($\mathcal{ABC}$-FD) of order $\beta > 0$ and $\beta \in [0, 1]$ for function $\mathfrak{u}$ are defined as

$$
{}^{\mathcal{AB}}\mathbf{D}\,\mathbf{I}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) = \frac{1-\beta}{\mathtt{M}(\beta)}\mathfrak{u}(\mathrm{z}) + \frac{\beta}{\mathtt{M}(\beta)}\mathbf{I}^{\beta}\mathfrak{u}(\mathrm{z}),
$$

where $\mathbf{I}^{\beta}\mathfrak{u}(\mathrm{z}) = \frac{1}{\Gamma(\beta)}\int_0^{\mathrm{z}} \mathfrak{u}(q)(\mathrm{z}-q)^{\beta-1}\,\mathrm{d}q$ is the R-L fractional integral and

$$
{}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) = \frac{\mathtt{M}(\beta)}{1-\beta}\int_0^{\mathrm{z}} \mathfrak{u}'(q) E_{\beta}\left(\frac{-\beta(\mathrm{z}-q)^{\beta}}{1-\beta}\right)\mathrm{d}q,
$$

respectively where

$$
\mathtt{M}(\beta) = (1-\beta) + \frac{\beta}{\Gamma(\beta)} > 0,
$$

is a normalization function, $\mathtt{M}(0) = \mathtt{M}(1) = 1$ and E_α represents the well-known Mittag-Leffler function [2]. Note that the integral and derivative of the power function in R-L sense are given as follows for β, $k > 0$:

$$
\begin{cases}
\mathbf{I}^{\beta}\mathrm{z}^{k} = \dfrac{k!}{\Gamma(\beta+k+1)}\mathrm{z}^{\beta+k}, \\
\mathbf{D}^{\beta}\mathrm{z}^{k} = \dfrac{k!}{(k-j)!}\mathrm{z}^{k-j}.
\end{cases}
$$

Definition 2.1. *Let $\mathfrak{u}$ be a function such that $\mathfrak{u}^{(n)} \in H^1(0, z_\circ)$ and $n < \beta \le n+1$, $n \in \mathbb{W} = \{0, 1, 2, \dots\}$. Then the $\mathcal{ABC}$-FD satisfies the following formula*

$$
\begin{cases}
{}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) = {}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta-n}\mathfrak{u}^{(n)}(\mathrm{z}), \\
{}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z}) = \mathbf{I}^{n}\left({}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\beta-n}\mathfrak{u}(\mathrm{z})\right).
\end{cases}
$$

Lemma 2.2. *For $\beta \in (n, n+1]$, $n \in \mathbb{W}$, the following outcome holds*

$$
{}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\beta}\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta}\mathfrak{u}(\mathrm{z})\right) = \mathfrak{u}(\mathrm{z}) + c_0 + c_1\mathrm{z} + c_2\mathrm{z}^2 + \cdots + c_n\mathrm{z}^n,
$$

for arbitrary constants c_i with $i \in \{0\} \cup \mathbb{N}_d$, where $\mathbb{N}_n = \{1, 2, \dots, n\}$.

Let $\mathfrak{u}, \acute{\mathfrak{u}} \in \mathbb{R}^d$, $d \in \mathbb{N}$ with $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_d)$ and $\acute{\mathfrak{u}} = (\acute{\mathfrak{u}}_1, \acute{\mathfrak{u}}_2, \dots, \acute{\mathfrak{u}}_d)$. By $\mathfrak{u} \leq \acute{\mathfrak{u}}$, we mean $\mathfrak{u}_i \leq \acute{\mathfrak{u}}_i$, $i \in \mathbb{N}_d$. Also,

$$\begin{cases} |\mathfrak{u}| = (|\mathfrak{u}_1|, |\mathfrak{u}_2|, \dots, |\mathfrak{u}_d|), \\ \max(\mathfrak{u}, \acute{\mathfrak{u}}) = (\max(\mathfrak{u}_1, \acute{\mathfrak{u}}_1), \max(\mathfrak{u}_2, \acute{\mathfrak{u}}_2), \dots, \max(\mathfrak{u}_d, \acute{\mathfrak{u}}_d)), \end{cases}$$

and $\mathbb{R}^d_+ = \left\{\mathfrak{u} \in \mathbb{R}^d \,|\, \mathfrak{u}_i \in \mathbb{R}_+,\ i \in \mathbb{N}_d\right\}$. If $\mathring{r} \in \mathbb{R}$, then $\mathfrak{u} \leq \mathring{r}$ means $\mathfrak{u}_i \leq \mathring{r}$, $i \in \mathbb{N}_d$. Some preliminaries about generalized metric spaces and their related topics are given below. For more details, see [19, 32, 38] and the references therein. Let $\mathcal{W}$ be a nonempty subset in $\mathbb{R}^d$. A vector-valued metric on $\mathcal{W}$ is a map $\rho : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^d$ with the following properties:

(i) $\rho(\mathfrak{u}, \acute{\mathfrak{u}}) \geq 0$, for each $\mathfrak{u}, \acute{\mathfrak{u}} \in \mathcal{W}$; $\rho(\mathfrak{u}, \acute{\mathfrak{u}}) = 0$ iff $\mathfrak{u} = \acute{\mathfrak{u}}$.
(ii) $\rho(\mathfrak{u}, \acute{\mathfrak{u}}) = \rho(\acute{\mathfrak{u}}, \mathfrak{u})$, for each $\mathfrak{u}, \acute{\mathfrak{u}} \in \mathcal{W}$.
(iii) $\rho(\mathfrak{u}, \acute{\mathfrak{u}}) \leq \rho(\mathfrak{u}, \mathfrak{u}_\circ) + \rho(\mathfrak{u}_\circ, \acute{\mathfrak{u}})$, for each $\mathfrak{u}, \acute{\mathfrak{u}}$ and $\mathfrak{u}_\circ \in \mathcal{W}$.

The pair $(\mathcal{W}, \rho)$ is called a generalized metric space. The convergence and completeness in $(\mathcal{W}, \rho)$ are similar to those in the usual metric space. Assume that $0 < \beta < 1$ and $1 \leq p < \infty$. We say that a measurable function $\mathfrak{u} : [0, 1] \to \mathbb{R}^d$ belongs to $L_{p,\beta}\left([0, 1], \mathbb{R}^d\right)$ if and only if

$$\|\mathfrak{u}\|_{p,\beta} := \sup_{0 \leq z \leq 1} \left(\int_0^z \frac{|\mathfrak{u}(z)|^p}{(z - q)^\beta} dq\right)^{\frac{1}{p}} < \infty.$$

Lemma 2.3. *Suppose that* $0 < \beta, \bar{\beta} < 1$ *and* $1 \leq p < \infty$. *If* $\beta < \bar{\beta}$, *then* $\|\mathfrak{u}\|_{p,\beta} \leq \|\mathfrak{u}\|_{p,\bar{\beta}}$, *and clearly* $L_{p,\bar{\beta}} \subseteq L_{p,\beta}$.

Proof. Since $0 \leq (z - q) \leq 1$, for any $z \geq q \geq 0$, we have that $(z - q)^{\bar{\beta}} \leq (z - q)^\beta$. So, it follows immediately that

$$\left(\int_0^z \frac{|\mathfrak{u}(q)|^p}{(z - q)^\beta}\, dq\right)^{\frac{1}{p}} \leq \left(\int_0^z \frac{|\mathfrak{u}(q)|^p}{(z - q)^{\bar{\beta}}}\, dq\right)^{\frac{1}{p}}.$$

The result follows. □

Lemma 2.4. *If* $0 < \beta < 1$ *and* $1 \leq p < \infty$, $\|\mathfrak{u}\|_p \leq \|\mathfrak{u}\|_{p,\beta}$, *and clearly* $L_{p,\beta} \subseteq L_p$.

Proof. The proof is evident because this $(z - q)^\beta \leq 1$. □

For $0 < \beta < 1$ and $1 \leq p < \infty$, $L_{p,\beta}$ is a complete metric space [38]. A square matrix M is said to be convergent to zero if $M^k \to 0$ as $k \to \infty$. For example, the matrix $M \in M_{2\times2}(\mathbb{R})$ defined by

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

converges to zero in the following cases. (I) $b = c = 0$, $a, d > 0$ and $\max\{a, d\} < 1$; (II) $c = 0$, $a, d > 0$, $a + d < 1$ and $-1 < b < 0$; (III) $a + b = c + d = 0$, $a > 1, c > 0$ and $|a - c| < 1$.

Lemma 2.5. *[19] If M is a square matrix that converges to zero and the elements of another square matrix $\acute{M}$ are small enough, then $M + \acute{M}$ also converges to zero.*

A square matrix M of real numbers is said to be convergent to zero if and only if f its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{d\times d}(\mathbb{R})$. Let $(\mathcal{W}, \rho)$ be the generalized metric space; the map $\top : \mathcal{W} \to \mathcal{W}$ is called a contractive map if there exists a convergent to zero matrix M such that $\rho(\top(\mathfrak{u}), \top(\acute{\mathfrak{u}})) \leq M\rho(\mathfrak{u}, \acute{\mathfrak{u}})$, for $\mathfrak{u}, \acute{\mathfrak{u}} \in \mathcal{W}$. In this case, M is called $\top'$s Lipschitz matrix.

Theorem 2.6. *For any nonnegative square matrix M, the following properties are equivalent*

(i) *M is convergent to zero;*

(ii) $\rho(M) < 1$*;*

(iii) *The matrix $I - M$ is non-singular and $(I - M)^{-1} = I + M + M^2 + \cdots$;*

(iv) *$I - M$ is non-singular and $(I - M)^{-1}$ is a nonnegative matrix.*

Theorem 2.7. *[32] (Perov's fixed point theorem) Let $(\mathcal{W}, \rho)$ be a complete generalized metric space and $\top : \mathcal{W} \to \mathcal{W}$ a contractive map with a Lipschitz matrix M. Then $\top$ has a unique fixed point $\mathfrak{u}^*$ if*

$$\rho\left(\top^k(\mathfrak{u}), \mathfrak{u}^*\right) \leq M^k (I - M)^{-1} \rho(\mathfrak{u}, \top(\mathfrak{u})), \qquad \forall\, \mathfrak{u} \in \mathcal{W},\ k \in \mathbb{N}.$$

3. Main results

The existence of a unique solution of the problem (1.2) can be investigated by using Perov's fixed point theorem. The next lemma involves a linear-type of the problem (1.2), by which we can find an integral solution for the problem (1.2).

Lemma 3.1. *Let $m_i < \sigma_i \leq m_i + 1$, $n_i < \beta_i \leq n_i + 1$, for $i = 1, 2$, $n_i, m_i \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\gimel_i$ be a Lebesgue measurable function. Then, the linear fractional problem*

$$\begin{cases} {}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\beta_i}\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i} + \gamma_i\right)\mathfrak{u}_i(\mathrm{z}) = \gimel_i(\mathrm{z}), & \\ \mathfrak{u}_i^{(j)}(0) = \mu_{i,j}, & j = 0, 1, \ldots, m_i, \\ \left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i} + \gamma_i\right)^{(j)}\mathfrak{u}_i(0) = \nu_{i,j}, & j = 0, 1, \ldots, n_i, \end{cases} \tag{3.1}$$

for $0 < \mathrm{z} < 1$ has an integral solution

$$\begin{aligned} \mathfrak{u}_i(\mathrm{z}) = {} & \sum_{k=0}^{m_i} \mu_{i,k}\mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)}\Bigg(\sum_{k=0}^{n_i} \frac{\nu_{i,k}\mathrm{z}^{m_i+k}}{(m_i + k)!} + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)}\mathbb{I}^{m_i+n_i}\gimel_i(\mathrm{z}) \\ & + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)}\mathbf{I}^{m_i+\beta_i}\gimel_i(\mathrm{z}) - \gamma_i \mathbf{I}^{m_i}\mathfrak{u}_i(\mathrm{z})\Bigg) + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)}\Bigg(\sum_{k=0}^{n_i} \frac{\nu_{i,k}\mathrm{z}^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)} \\ & + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)}\mathbb{I}^{\beta_i+n_i}\gimel_i(\mathrm{z}) + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)}\mathbf{I}^{\sigma_i+\beta_i}\gimel_i(\mathrm{z}) - \gamma_i\mathbf{I}^{\sigma_i}\mathfrak{u}_i(\mathrm{z})\Bigg). \end{aligned} \tag{3.2}$$

Proof. Let $\mathfrak{u}_i$ be a solution of the linear fractional problem (3.1); hence, in virtue of Definition 2.1, and Lemma 2.2, we have

$$\begin{aligned} \left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i} + \gamma_i\right)\mathfrak{u}_i(\mathrm{z}) & = \sum_{k=0}^{n_i} a_{i,k}\mathrm{z}^k + \mathbf{I}^{n_i}\left({}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\beta_i - n_i}\gimel_i(\mathrm{z})\right) \\ & = \sum_{k=0}^{n_i} a_{i,k}\mathrm{z}^k + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)}\mathbf{I}^{n_i}\gimel_i(\mathrm{z}) + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)}\mathbf{I}^{\beta_i}\gimel_i(\mathrm{z}). \end{aligned}$$

The jth derivative of the last equation leads to

$$\frac{\mathrm{d}^j}{\mathrm{dz}^j}\left(\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i}+\gamma_i\right)\mathfrak{u}_i(\mathrm{z})\right)=\sum_{k=j}^{n_i}\frac{a_{i,k}k!}{(k-j)!}\mathrm{z}^{k-j}+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{n_i-j}\gimel_i(\mathrm{z})+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\beta_i-j}\gimel_i(\mathrm{z}).$$

The given condition

$$\left({}^{\mathcal{ABC}}\mathbf{D}_{\mathrm{z}}^{\sigma_i}+\gamma_i\right)^{(j)}\mathfrak{u}_i(0)=\nu_{i,j},$$

implies that $a_{i,j}=\frac{\nu_{i,j}}{j!},\quad j=0,1,\dots,n_i,\ i=1,2.$ Another application of Definition 2.1, and Lemma 2.2 leads to

$$\begin{aligned}\mathfrak{u}_i(\mathrm{z})&=\sum_{k=0}^{m_i}b_{i,k}\mathrm{z}^k+{}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\sigma_i}\Bigg(\sum_{k=0}^{n_i}a_{i,k}\mathrm{z}^k+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{n_i}\gimel_i(\mathrm{z})\\&\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\beta_i}\gimel_i(\mathrm{z})-\gamma_i\mathfrak{u}_i(\mathrm{z})\Bigg)\\&=\sum_{k=0}^{m_i}b_{i,k}t^k+\mathbf{I}^{m_i}\left({}^{\mathcal{AB}}\mathbf{I}_{\mathrm{z}}^{\sigma_i-m_i}\right)\Bigg(\sum_{k=0}^{n_i}a_{i,k}\mathrm{z}^k+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{n_i}\gimel_i(\mathrm{z})\\&\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\beta_i}\gimel_i(\mathrm{z})-\gamma_i\mathfrak{u}_i(\mathrm{z})\Bigg)\\&=\sum_{k=0}^{m_i}b_{i,k}\mathrm{z}^k+\frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\Bigg(\sum_{k=0}^{n_i}a_{i,k}\mathbf{I}^{m_i}\mathrm{z}^k+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+n_i}\gimel_i(\mathrm{z})\\&\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+\beta_i}\gimel_i(\mathrm{z})-\gamma_i\mathbf{I}^{m_i}\mathfrak{u}_i(\mathrm{z})\Bigg)\\&\quad+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Bigg(\sum_{k=0}^{n_i}a_{i,k}\mathbf{I}^{\sigma_i}\mathrm{z}^k+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+n_i}\gimel_i(\mathrm{z})\\&\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+\beta_i}\gimel_i(t)-\gamma_i\mathbf{I}^{\sigma_i}\mathfrak{u}_i(\mathrm{z})\Bigg).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathfrak{u}_i(\mathrm{z})&=\sum_{k=0}^{m_i}b_{i,k}t^k+\frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\Bigg(\sum_{k=0}^{n_i}\frac{a_{i,k}k!\mathrm{z}^{m_i+k}}{(m_i+k)!}\\&\quad+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+n_i}\gimel_i(\mathrm{z})+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+\beta_i}\gimel_i(\mathrm{z})-\gamma_i\mathbf{I}^{m_i}\mathfrak{u}_i(\mathrm{z})\Bigg)\\&\quad+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Bigg(\sum_{k=0}^{n_i}\frac{a_{i,k}k!\mathrm{z}^{\sigma_i+k}}{\Gamma(\sigma_i+k+1)}+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+n_i}\gimel_i(\mathrm{z})\\&\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+\beta_i}\gimel_i(\mathrm{z})-\gamma_i\mathbf{I}^{\sigma_i}\mathfrak{u}_i(\mathrm{z})\Bigg).\end{aligned}\tag{3.3}$$

The jth derivative of the general solution implies

$$
\begin{aligned}
\mathfrak{u}_i^{(j)}(\mathrm{z}) = \sum_{k=j}^{m_i} \frac{b_{i,k}k!}{(k-j)!} t^{k-j} + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{a_{i,k}k!\mathrm{z}^{m_i+k-j}}{(m_i+k-j)!} \\
+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i-j} \gimel_i(\mathrm{z}) + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i-j} \gimel_i(\mathrm{z}) - \gamma_i \mathbf{I}^{m_i-j} \mathfrak{u}_i(\mathrm{z}) \Bigg) \\
+ \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{a_{i,k}k!\mathrm{z}^{\sigma_i+k-j}}{\Gamma(\sigma_i+k-j+1)} + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+n_i-j} \gimel_i(\mathrm{z}) \\
+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i-j} \gimel_i(\mathrm{z}) - \gamma_i \mathbf{I}^{\sigma_i-j} \mathfrak{u}_i(\mathrm{z}) \Bigg).
\end{aligned}
$$

Hence, the initial conditions $\mathfrak{u}_i^{(j)}(0) = \mu_{i,j}$ yields that $b_{i,j} = \frac{\mu_{i,j}}{j!}$, $j = 0, 1, 2, \ldots, m_i$. Substituting the values of $a_{i,k}$ and $b_{i,k}$ in (3.3), the integral equation (3.2) is obtained, which completes the proof. □

We describe an operator $\mathbb{Q} : L_q^2 \to L_q^2$ by employing Lemma 3.1, where $\mathbb{Q} := (\mathbb{Q}_1, \mathbb{Q}_2)$; $\mathbb{Q}_i$ is given by

$$
\begin{aligned}
\mathbb{Q}_i(\mathfrak{u}_1, \mathfrak{u}_2)(\mathrm{z}) = \sum_{k=0}^{m_i} \mu_{i,k}\mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\nu_{i,k}\mathrm{z}^{m_i+k}}{(m_i+k)!} \\
+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} \chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) \\
+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} \chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \gamma_i \mathbf{I}^{m_i} \mathfrak{u}_i(\mathrm{z}) \Bigg) \\
+ \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\nu_{i,k}\mathrm{z}^{\sigma_i+k}}{\Gamma(\sigma_i+k+1)} \\
+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+n_i} \chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) \\
+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} \chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \gamma_i \mathbf{I}^{\sigma_i} \mathfrak{u}_i(\mathrm{z}) \Bigg).
\end{aligned}
\tag{3.4}
$$

Observe that the initial value problem (1.2) has a unique solution if the operator equation $\mathbb{Q}(\mathfrak{u}_1, \mathfrak{u}_2) = (\mathfrak{u}_1, \mathfrak{u}_2)$ has a fixed point. In the next result, we shall apply Theorem 2.7 to prove the existence results of the system (1.2).

Theorem 3.2. *Suppose that* $\frac{1}{\sigma} < p \leq \infty$, $p^{-1} + q^{-1} = 1$ *and* $\sigma + \beta$ *belongs to the interval* $(0, 1)$. *Assume that the following assumptions* (H1)–(H3) *are satisfied.*

(H1) $\chi_i(\mathrm{z}, 0, 0) \in L_{1,1-(\sigma+\beta)}$, $i = 1, 2$;

(H2) $\chi_i : [0, \mathrm{z}_\circ] \times \mathbb{R}^d \to \mathbb{R}^d$ *denotes joint Lebesgues measurables functions, there exists* λ_{1i} *and* $\lambda_{2i} \in L^p[0, \mathrm{z}_\circ]$, $i = 1, 2$ *such that*

$$
\|\chi_i(\mathrm{z}, \mathfrak{u}_1, \mathfrak{u}_2) - \chi_i(\mathrm{z}, \bar{\mathfrak{u}}_1, \bar{\mathfrak{u}}_2)\| \leq \lambda_{1i}(\mathrm{z})\|\mathfrak{u}_1 - \bar{\mathfrak{u}}_1\| + \lambda_{2i}(\mathrm{z})\|\mathfrak{u}_2 - \bar{\mathfrak{u}}_2\|,
$$

for each $\mathrm{z} \in [0, \mathrm{z}_\circ]$ *and for all* $\mathfrak{u}_1, \bar{\mathfrak{u}}_1, \mathfrak{u}_2, \bar{\mathfrak{u}}_2 \in \mathbb{R}^n$.

(H3) *The matrix $M_{2\times2}$ defined as*

$$M_{2\times2} = \begin{bmatrix} \overline{C}_1\left\|\lambda_{11}\right\|_p + \overline{\mathcal{D}}_1 & \overline{C}_1\left\|\lambda_{21}\right\|_p \\ \overline{C}_1\left\|\lambda_{12}\right\|_p & \overline{C}_1\left\|\lambda_{22}\right\|_p + \overline{\mathcal{D}}_1 \end{bmatrix}, \tag{3.5}$$

converges to zero, where

$$\begin{aligned} C_i = {} & \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \left\{ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)\Gamma(m_i + n_i)} + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)\Gamma(m_i + \beta_i)} \right\} \\ & + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \left\{ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + n_i)} + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + \beta_i)} \right\}, \end{aligned} \tag{3.6}$$

and

$$\mathcal{D}_i = \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \frac{|\gamma_i|}{\Gamma(\sigma_i)} + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \frac{|\gamma_i|}{\Gamma(m_i)}. \tag{3.7}$$

Then the system (1.2) *has a unique solution in* $L_q[0, 1]$.

Proof. The proof is straightforward when $q = \infty$. Let $\frac{1}{\sigma} < q < \infty$. According to (H3) and Theorem 2.6-(iv), one can find that $I - M_{2\times2}$ is invertible and that its inverse $(I - M_{2\times2})^{-1}$ has nonnegative elements. Then we define

$$\widetilde{U} = \left\{ (\mathrm{u}_1, \mathrm{u}_2) \in L_q^2 \,:\, \|\mathrm{u}_1\|_q \le \widetilde{R}_1, \|\mathrm{u}_2\|_q \le \widetilde{R}_2 \right\},$$

where

$$\begin{bmatrix} \tilde{R}_1 \\ \widetilde{R}_2 \end{bmatrix} \ge (I - M_{2\times2})^{-1} \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix},$$

$\Omega_i = \mathcal{A}_i + \mathcal{B}_i + C_i$, with $\chi_i^{\max} = \max_{\mathrm{z}\in[0,z_\circ]} |\chi_i(\mathrm{z}, 0, 0)|$, $i = 1, 2$,

$$\begin{aligned} \mathcal{A}_i = {} & \frac{(m_i - \sigma_i + 1)(n_i - \beta_i + 1)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(m_i + n_i)} \left| \int_{z_\circ}^{\mathrm{z}} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(\mathrm{z} - \xi)^{1-(m_i+n_i)}} \,\mathrm{d}\xi \right| \\ & + \frac{(m_i - \sigma_i + 1)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(m_i + \beta_i)} \left| \int_{z_\circ}^{\mathrm{z}} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(\mathrm{z} - \xi)^{1-(m_i+\beta_i)}} \,\mathrm{d}\xi \right| \\ & + \frac{(\sigma_i - m_i)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + \beta_i)} \left| \int_{z_\circ}^{\mathrm{z}} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(\mathrm{z} - \xi)^{1-(\sigma_i+\beta_i)}} \,\mathrm{d}\xi \right|, \\ \mathcal{B}_i = {} & \sum_{k=0}^{m_i} \left|\mu_{i,k}\right| \mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{m_i+k}}{(m_i + k)!} + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)}, \end{aligned} \tag{3.8}$$

and

$$\Theta_i = \mathcal{A}_l + \mathcal{B}_l + C_l + \mathcal{D}_l \left\|\mathrm{u}\right\|_q + \mathcal{E}_l \left\|\acute{\mathrm{u}}\right\|_q. \tag{3.9}$$

For each $(\mathfrak{u}_1, \mathfrak{u}_2) \in L_q^2$, we have that

$$
\begin{aligned}
|\mathbb{Q}_i(\mathfrak{u}_1, \mathfrak{u}_2)(\mathrm{z})| \leq & \sum_{k=0}^{m_i} \left|\mu_{i,k}\right| \mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{m_i+k}}{(m_i+k)!} \\
& + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z}))\right| \\
& + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z}))\right| + |\gamma_i| \, \mathbf{I}^{m_i} \left|u_i(\mathrm{z})\right| \Bigg) \\
& + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| t^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)} \\
& + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+n_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z}))\right| \\
& + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z}))\right| + |\gamma_i| \, \mathbf{I}^{\sigma_i} \left|\mathfrak{u}_i(\mathrm{z})\right| \Bigg) \\
\leq & \sum_{k=0}^{m_i} \left|\mu_{i,k}\right| \mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{m_i+k}}{(m_i+k)!} \\
& + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right| \\
& + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right| \\
& + |\gamma_i| \, \mathbf{I}^{m_i} \left|\mathfrak{u}_i(\mathrm{z})\right| \Bigg) + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)} \\
& + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+n_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right| \\
& + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right| \\
& + |\gamma_i| \, \mathbf{I}^{\sigma_i} \left|\mathfrak{u}_i(\mathrm{z})\right| \Bigg) + \frac{(m_i - \sigma_i + 1)(n_i - \beta_i + 1)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} \left|\chi_i(\mathrm{z}, 0, 0)\right| \\
& + \frac{(m_i - \sigma_i + 1)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} \left|\chi_i(\mathrm{z}, 0, 0)\right| \\
& + \frac{(\sigma_i - m_i)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} \left|\chi_i(\mathrm{z}, 0, 0)\right| \\
\leq & \sum_{k=0}^{m_i} \left|\mu_{i,k}\right| \mathrm{z}^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Bigg(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| \mathrm{z}^{m_i+k}}{(m_i+k)!} \\
& + \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right| \\
& + \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} \left|\chi_i(\mathrm{z}, \mathfrak{u}_1(\mathrm{z}), \mathfrak{u}_2(\mathrm{z})) - \chi_i(\mathrm{z}, 0, 0)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ |\gamma_i| \mathbf{I}^{m_i} |\mathfrak{u}_i(z)| \Big) + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Big(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| z^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)} \\
&+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+n_i} |\chi_i(z, \mathfrak{u}_1(z), \mathfrak{u}_2(z)) - \chi_i(z, 0, 0)| \\
&+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} |\chi_i(z, \mathfrak{u}_1(z), \mathfrak{u}_2(z)) - \chi_i(z, 0, 0)| \\
&+ |\gamma_i| \mathbf{I}^{\sigma_i} |\mathfrak{u}_i(z)| \Big) + \frac{(m_i - \sigma_i + 1)(n_i - \beta_i + 1)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+n_i} |\chi_i(z, 0, 0)| \\
&+ \frac{(m_i - \sigma_i + 1)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{m_i+\beta_i} |\chi_i(z, 0, 0)| \\
&+ \frac{(\sigma_i - m_i)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)} \mathbf{I}^{\sigma_i+\beta_i} |\chi_i(z, 0, 0)| \\
\leq &\sum_{k=0}^{m_i} \left|\mu_{i,k}\right| z^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Big(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| z^{m_i+k}}{(m_i + k)!} \\
&+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)\Gamma(m_i + n_i)} \left| \int_{z_\circ}^{z} \frac{\left(\lambda_i(\xi)\left|\mathfrak{u}_1(\xi)\right| + \lambda_{2i}(\xi)\left|\mathfrak{u}_2(\xi)\right|\right)}{(z - \xi)^{1-(m_i+n_i)}} \, d\xi \right| \\
&+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)\Gamma(m_i + \beta_i)} \left| \int_{z_\circ}^{z} \frac{\left(\lambda_{1i}(s)\left|\mathfrak{u}_1(\xi)\right| + \lambda_{2i}(\xi)\left|\mathfrak{u}_2(\xi)\right|\right)}{(z - \xi)^{1-(m_i+\beta_i)}} \, d\xi \right| \\
&+ \frac{|\gamma_i|}{\Gamma(m_i)} \left| \int_{z_\circ}^{z} \frac{\left|\mathfrak{u}_i(\xi)\right|}{(z - \xi)^{1-m_i}} \, d\xi \right| \Big) + \frac{\sigma_i - m_i}{\mathtt{M}(\sigma_i - m_i)} \Big(\sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| z^{\sigma_i+k}}{\Gamma(\sigma_i + k + 1)} \\
&+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + n_i)} \left| \int_{z_\circ}^{z} \frac{\left(\lambda_{1i}(\xi)\left|\mathfrak{u}_1(\xi)\right| + \lambda_{2i}(\xi)\left|\mathfrak{u}_2(\xi)\right|\right)}{(z - \xi)^{1-(\sigma_i+n_i)}} \, d\xi \right| \\
&+ \frac{\beta_i - n_i}{\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + \beta_i)} \left| \int_{z_\circ}^{z} \frac{\left(\lambda_{1i}(\xi)\left|\mathfrak{u}_1(\xi)\right| + \lambda_{2i}(\xi)\left|\mathfrak{u}_2(\xi)\right|\right)}{(z - \xi)^{1-(\sigma_i+\beta_i)}} \, d\xi \right| \\
&+ \frac{|\gamma_i|}{\Gamma(\sigma_i)} \left| \int_{z_\circ}^{z} \frac{\left|\mathfrak{u}_i(\xi)\right|}{(z - \xi)^{1-\sigma_i}} \, d\xi \right| \Big) \\
&+ \frac{(m_i - \sigma_i + 1)(n_i - \beta_i + 1)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(m_i + n_i)} \left| \int_{z_\circ}^{z} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(z - \xi)^{1-(m_i+n_i)}} \, d\xi \right| \\
&+ \frac{(m_i - \sigma_i + 1)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(m_i + \beta_i)} \left| \int_{z_\circ}^{z} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(z - \xi)^{1-(m_i+\beta_i)}} \, d\xi \right| \\
&+ \frac{(\sigma_i - m_i)(\beta_i - n_i)}{\mathtt{M}(\sigma_i - m_i)\mathtt{M}(\beta_i - n_i)\Gamma(\sigma_i + \beta_i)} \left| \int_{z_\circ}^{z} \frac{\left|\chi_i(\xi, 0, 0)\right|}{(z - \xi)^{1-(\sigma_i+\beta_i)}} \, d\xi \right| \\
\leq &\sum_{k=0}^{m_i} \left|\mu_{i,k}\right| z^k + \frac{m_i - \sigma_i + 1}{\mathtt{M}(\sigma_i - m_i)} \Big\{ \sum_{k=0}^{n_i} \frac{\left|\nu_{i,k}\right| z^{m_i+k}}{(m_i + k)!} \\
&+ \frac{n_i - \beta_i + 1}{\mathtt{M}(\beta_i - n_i)\Gamma(m_i + n_i)} \Big[\Big(\int_{z_\circ}^{z} \lambda_{1i}^{p}(\xi) \, d\xi \Big)^{\frac{1}{p}} \Big(\int_{z_\circ}^{z} \frac{\left|\mathfrak{u}_1(\xi)\right|^q}{(z - \xi)^{q-q(m_i+n_i)}} \, d\xi \Big)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\int_{z_\circ}^{z}\lambda_{2i}^{p}\xi\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_2(\xi)\big|^{q}}{(z-\xi)^{q-q(m_i+n_i)}}\,d\xi\Big)^{\frac{1}{q}}\Big]\\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}\Big[\Big(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_1(\xi)\big|^{q}}{(z-\xi)^{q-q(m_i+\beta_i)}}\,d\xi\Big)^{\frac{1}{q}}\\
&+\Big(\int_{z_\circ}^{z}\delta_{i}^{p}\xi\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_2(\xi)\big|^{q}}{(z-\xi)^{q-q(m_i+\beta_i)}}\,d\xi\Big)^{\frac{1}{q}}\Big]\\
&+\frac{|\gamma_{1i}|}{\Gamma(m_i)}\Big(\int_{z_\circ}^{z}\frac{\big|u_i(\xi)\big|^{q}}{(z-\xi)^{q-qm_i}}\,d\xi\Big)^{\frac{1}{q}}\Big\}+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Big\{\sum_{k=0}^{n_i}\frac{\big|\nu_{i,k}\big|\,z^{\sigma_i+k}}{\Gamma(\sigma_i+k+1)}\\
&+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+n_i)}\Big[\Big(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_1(\xi)\big|^{q}}{(z-\xi)^{q-q(\sigma_i+n_i)}}\,d\xi\Big)^{\frac{1}{q}}\\
&+\Big(\int_{z_\circ}^{z}\lambda_{2i}^{p}\xi\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_2(\xi)\big|^{q}}{(z-\xi)^{q-q(\sigma_i+n_i)}}\,d\xi\Big)^{\frac{1}{q}}\Big]\\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\Big[\Big(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_1(\xi)\big|^{q}}{(z-\xi)^{q-q(\sigma_i+\beta_i)}}\,d\xi\Big)^{\frac{1}{q}}\\
&+\Big(\int_{z_\circ}^{z}\lambda_{2i}^{p}\xi\,d\xi\Big)^{\frac{1}{p}}\Big(\int_{z_\circ}^{z}\frac{\big|u_2(\xi)\big|^{q}}{(z-\xi)^{q-q(\sigma_i+\beta_i)}}\,d\xi\Big)^{\frac{1}{q}}\Big]\\
&+\frac{|\gamma_i|}{\Gamma(\sigma_i)}\Big(\int_{z_\circ}^{z}\frac{\big|u_i(\xi)\big|^{q}}{(z-\xi)^{q-q\sigma_i}}\,d\xi\Big)^{\frac{1}{q}}\Big\}\\
&+\frac{(m_i-\sigma_i+1)\,(n_i-\beta_i+1)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)}\Big|\int_{z_\circ}^{z}\frac{\big|\chi_i(\xi,0,0)\big|}{(z-\xi)^{1-(m_i+n_i)}}\,d\xi\Big|\\
&+\frac{(m_i-\sigma_i+1)\,(\beta_i-n_i)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}\Big|\int_{z_\circ}^{z}\frac{\big|\chi_i(\xi,0,0)\big|}{(z-\xi)^{1-(m_i+\beta_i)}}\,d\xi\Big|\\
&+\frac{(\sigma_i-m_i)\,(\beta_i-n_i)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\Big|\int_{z_\circ}^{z}\frac{\big|\chi_i(s,0,0)\big|}{(z-\xi)^{1-(\sigma_i+\beta_i)}}\,d\xi\Big|\\
\le{}&\sum_{k=0}^{m_i}\big|\mu_{i,k}\big|\,z^{k}+\frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\Big\{\sum_{k=0}^{n_i}\frac{\big|\nu_{i,k}\big|\,z^{m_i+k}}{(m_i+k)!}\\
&+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)}\Big[\big\|\lambda_{1i}\big\|_{p}\big\|u_1\big\|_{q,q-q(m_i+n_i)}+\big\|\lambda_{2i}\big\|_{p}\big\|u_2\big\|_{q,q-q(m_i+n_i)}\Big]\\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}\Big[\big\|\lambda_{1i}\big\|_{p}\big\|u_1\big\|_{q,q-q(m_i+\beta_i)}+\big\|\lambda_{2i}\big\|_{p}\big\|u_2\big\|_{q,q-q(m_i+\beta_i)}\Big]\\
&+\frac{|\gamma_i|}{\Gamma(m_i)}\Big[\big\|u_i\big\|_{q,q-qm_i}\Big]\Big\}+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Big\{\sum_{k=0}^{n_i}\frac{\big|\nu_{i,k}\big|\,z^{\sigma_i+k}}{\Gamma(\sigma_i+k+1)}\\
&+\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+n_i)}\Big[\big\|\lambda_{1i}\big\|_{p}\big\|u_1\big\|_{q,q-q(\sigma_i+n_i)}+\big\|\lambda_{2i}\big\|_{p}\big\|u_2\big\|_{q,q-q(\sigma_i+n_i)}\Big]\\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\Big[\big\|\lambda_i\big\|_{p}\big\|u_1\big\|_{q,q-q(\sigma_i+\beta_i)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\lambda_{2i}\right\|_{p}\left\|\mathfrak{u}_{2}\right\|_{q,q-q(\sigma_i+\beta_i)}\Big]+\frac{|\gamma_i|}{\Gamma(\sigma_i)}\Big[\left\|\mathfrak{u}_i\right\|_{q,q-q\sigma_i}\Big]\Big\}\\
&+\frac{(m_i-\sigma_i+1)(n_i-\beta_i+1)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)}\left|\int_{z_\circ}^{z}\frac{\left|\chi_i(\xi,0,0)\right|}{(z-\xi)^{1-(m_i+n_i)}}\,d\xi\right|\\
&+\frac{(m_i-\sigma_i+1)(\beta_i-n_i)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}\left|\int_{z_\circ}^{z}\frac{\left|\chi_i(\xi,0,0)\right|}{(z-\xi)^{1-(m_i+\beta_i)}}\,d\xi\right|\\
&+\frac{(\sigma_i-m_i)(\beta_i-n_i)}{\mathtt{M}(\sigma_i-m_i)\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\left|\int_{z_\circ}^{z}\frac{\left|\chi_i(\xi,0,0)\right|}{(t-s)^{1-(\sigma_i+\beta_i)}}\,d\xi\right|\\
\le{}& \mathcal{A}_i+\mathcal{B}_i+C_i\Big[\left\|\lambda_{1i}\right\|_p\left\|\mathfrak{u}_1\right\|_q+\left\|\lambda_{2i}\right\|_p\left\|\mathfrak{u}_2\right\|_q\Big]+\mathcal{D}_i\Big[\left\|\mathfrak{u}_i\right\|_q\Big].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\mathbb{Q}_1(\mathfrak{u}_1,\mathfrak{u}_2)(z)|^q &\le \Big(\mathcal{A}_1+\mathcal{B}_1+C_1\Big[\left\|\lambda_{11}\right\|_p\left\|\mathfrak{u}_1\right\|_q+\left\|\lambda_{21}\right\|_p\left\|\mathfrak{u}_2\right\|_q\Big]+\mathcal{D}_1\Big[\left\|\mathfrak{u}_1\right\|_q\Big]\Big)^q\\
&\le \Big(\mathcal{A}_1+\mathcal{B}_1+\big(C_1\left\|\lambda_{11}\right\|_p+\mathcal{D}_1\big)\left\|\mathfrak{u}_1\right\|_q+C_1\left\|\lambda_{21}\right\|_p\left\|\mathfrak{u}_2\right\|_q\Big)^q. \qquad (3.10)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Big(\int_{z_\circ}^{z}|\mathbb{Q}_1(\mathfrak{u}_1,\mathfrak{u}_2)(z)|^q\,d\xi\Big)^{\frac{1}{q}}\\
&\quad\le \Big(\mathcal{A}_1+\mathcal{B}_1+\big(C_1\left\|\lambda_{11}\right\|_p+\mathcal{D}_1\big)\left\|\mathfrak{u}_1\right\|_q+C_1\left\|\lambda_{21}\right\|_p\left\|\mathfrak{u}_2\right\|_q\Big)\Big(\int_{z_\circ}^{z}1^q\,d\xi\Big)^{\frac{1}{q}}. \qquad (3.11)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|\mathbb{Q}_1(\mathfrak{u}_1,\mathfrak{u}_2)(z)\right\|_q &\le \overline{\mathcal{A}}_1+\overline{\mathcal{B}}_1+\big(\overline{C}_1\left\|\lambda_{11}\right\|_p+\overline{\mathcal{D}}_1\big)\left\|\mathfrak{u}_1\right\|_q+\overline{C}_1\left\|\lambda_{21}\right\|_p\left\|\mathfrak{u}_2\right\|_q\\
&\le \overline{\mathcal{A}}_1+\overline{\mathcal{B}}_1+\big(\overline{C}_1\left\|\lambda_{11}\right\|_p+\overline{\mathcal{D}}_1\big)\tilde{R}_1+\overline{C}_1\left\|\lambda_{21}\right\|_p\tilde{R}_1\le\tilde{R}_1. \qquad (3.12)
\end{aligned}
$$

Similarly, we can show that $\mathbb{Q}_2$ maps $\widetilde{U}$ into $L_q(J,\mathbb{R})^2$ and

$$
\begin{aligned}
\left\|\mathbb{Q}_1(\mathfrak{u}_1,\mathfrak{u}_2)(z)\right\|_q &\le \overline{\mathcal{A}}_2+\overline{\mathcal{B}}_2+\overline{C}_2\left\|\lambda_{112}\right\|_p\left\|\mathfrak{u}_1\right\|_q+\big(\overline{C}_2\left\|\lambda_{22}\right\|_p+\overline{\mathcal{D}}_2\big)\left\|\mathfrak{u}_2\right\|_q\\
&\le \overline{\mathcal{A}}_2+\overline{\mathcal{B}}_2+\overline{C}_2\left\|\lambda_{12}\right\|_p\tilde{R}_1+\big(\overline{C}_2\left\|\lambda_{22}\right\|_p+\overline{\mathcal{D}}_2\big)\tilde{R}_2\le\tilde{R}_2. \qquad (3.13)
\end{aligned}
$$

Combining (3.10) and (3.13) we obtain

$$
\begin{bmatrix} \|\mathbb{Q}_1(\mathfrak{u}_1,\mathfrak{u}_2)\|_q \\ \|\mathbb{Q}_2(\mathfrak{u}_1,\mathfrak{u}_2)\|_q \end{bmatrix} \le \begin{bmatrix} \widetilde{R}_1 \\ \widetilde{R}_2 \end{bmatrix}.
$$

Hence $\mathbb{Q}(\widetilde{U})\subset\widetilde{U}$. Thus, $\mathbb{Q}$ is a self-map on $\widetilde{U}$. Let

$$
\Theta_i=\mathcal{A}_i+\mathcal{B}_i+C_i\Big[\left\|\lambda_{1i}\right\|_p\left\|\mathfrak{u}_1\right\|_q+\left\|\lambda_{2i}\right\|_p\left\|\mathfrak{u}_2\right\|_q\Big]+\mathcal{D}_i\Big[\left\|u_i\right\|_q\Big].
$$

Thus

$$\left(\int_{z_\circ}^{z}\frac{|\mathbb{Q}_i\mathfrak{u}(z)|^q}{(z-\xi)^{q-q\sigma_i}}\,d\xi\right)^{\frac{1}{q}} \le \left(\int_{z_\circ}^{z}\frac{\Theta_i^q}{(z-\xi)^{q-q\sigma_i}}\,d\xi\right)^{\frac{1}{q}}$$
$$\le \Theta_i\left(\int_{z_\circ}^{z}\frac{1}{(z-\xi)^{q-q\sigma_i}}\,d\xi\right)^{\frac{1}{q}} \le \Theta_i\left(\frac{1}{1-q(1-\sigma_i)}\right)^{\frac{1}{q}} < \infty.$$

This yields $\mathbb{Q} : L_q^2 \to L_q^2$. Now, suppose that $\mathfrak{u}$ and $\acute{\mathfrak{u}}$ are two elements in L_q. So,

$$\begin{aligned}
&\Big|\mathbb{Q}_i\left(\mathfrak{u}_1,\mathfrak{u}_2\right)(z) - \mathbb{Q}_i\left(\acute{\mathfrak{u}}_1,\acute{\mathfrak{u}}_2\right)(z)\Big| \\
&\quad= \frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\Big(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+n_i}\Big|\chi_i(z,\mathfrak{u}_1(z),\mathfrak{u}_2(z)) \\
&\quad\quad-\chi_i(z,\acute{\mathfrak{u}}_1(z),\acute{\mathfrak{u}}_2(z))\Big| + \frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{m_i+\beta_i}\Big|\chi_i(z,\mathfrak{u}_1(z),\mathfrak{u}_2(z)) \\
&\quad\quad-\chi_i(z,\acute{\mathfrak{u}}_1(z),\acute{\mathfrak{u}}_2(z))\Big| - \gamma_i\mathbf{I}^{m_i}\Big|\mathfrak{u}_i(z)-\acute{\mathfrak{u}}_i(z)\Big|\Big) \\
&\quad\quad+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Big(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+n_i}\Big|\chi_i(z,\mathfrak{u}_1(z),\mathfrak{u}_2(z)) \\
&\quad\quad-\chi_i(z,\acute{\mathfrak{u}}_1(z),\acute{\mathfrak{u}}_2(z))\Big| + \frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)}\mathbf{I}^{\sigma_i+\beta_i}\Big|\chi_i(z,\mathfrak{u}_1(z),\mathfrak{u}_2(z)) \\
&\quad\quad-\chi_i(z,\acute{\mathfrak{u}}_1(z),\acute{\mathfrak{u}}_2(z))\Big| - \gamma_i\mathbf{I}^{\sigma_i}\Big|\mathfrak{u}_i(z)-\acute{\mathfrak{u}}_i(z)\Big|\Big) \\
&\quad\le \frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\Big(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)} \\
&\quad\quad\times\Big|\int_{z_\circ}^{z}\frac{\left(\lambda_{1i}(\xi)\Big|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\Big|+\lambda_{2i}(\xi)\Big|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\Big|\right)}{(z-\xi)^{1-(m_i+n_i)}}\,d\xi\Big| \\
&\quad\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)} \\
&\quad\quad\times\Big|\int_{z_\circ}^{z}\frac{\left(\lambda_{1i}(\xi)\Big|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\Big|+\lambda_{2i}(\xi)\Big|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\Big|\right)}{(z-\xi)^{1-(m_i+\beta_i)}}\,d\xi\Big| \\
&\quad\quad-\frac{\gamma_i}{\Gamma(m_i)}\Big|\int_{z_\circ}^{z}\frac{\Big|\mathfrak{u}_i(\xi)-\acute{\mathfrak{u}}_i(\xi)\Big|}{(z-\xi)^{1-m_i}}\,d\xi\Big|\Big) \\
&\quad\quad+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Big(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+n_i)} \\
&\quad\quad\times\Big|\int_{z_\circ}^{z}\frac{\left(\lambda_{1i}(\xi)\Big|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\Big|+\lambda_{1i}(\xi)\Big|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\Big|\right)}{(z-\xi)^{1-(\sigma_i+n_i)}}\,d\xi\Big| \\
&\quad\quad+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)} \\
&\quad\quad\Big|\int_{z_\circ}^{z}\frac{\left(\lambda_{1i}(\xi)\Big|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\Big|+\lambda_{i}(\xi)\Big|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\Big|\right)}{(z-\xi)^{1-(\sigma_i+\beta_i)}}\,d\xi\Big|
\end{aligned}$$

$$- \frac{\gamma_i}{\Gamma(\sigma_i)}\bigg| \int_{z_\circ}^{z} \frac{\left|\mathfrak{u}_i(\xi) - \acute{\mathfrak{u}}_i(\xi)\right|}{(\mathfrak{u}-\xi)^{1-\sigma_i}}\, d\xi\bigg|\bigg).$$

Therefore,

$$\begin{aligned}
&\left|\mathbb{Q}_i\,(\mathfrak{u}_1,\mathfrak{u}_2)\,(z) - \mathbb{Q}_i\,(\acute{\mathfrak{u}}_1,\acute{\mathfrak{u}}_2)\,(z)\right| \\
&\leq \frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\bigg(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)}\bigg(\bigg(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}} \\
&\times\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\right|^{q}}{(z-\xi)^{q-q(m_i+n_i)}}\,d\xi\bigg)^{\frac{1}{q}} \\
&+\bigg(\int_{z_\circ}^{z}\lambda_{2i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\right|^{q}}{(z-\xi)^{q-q(m_i+n_i)}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}\bigg(\bigg(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}} \\
&\times\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\right|^{q}}{(z-\xi)^{q-q(m_i+\beta_i)}}\,d\xi\bigg)^{\frac{1}{q}} \\
&+\bigg(\int_{z_\circ}^{z}\lambda_{2i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\right|^{q}}{(z-\xi)^{q-q(m_i+\beta_i)}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&-\frac{\gamma_i}{\Gamma(m_i)}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_i(\xi)-\acute{\mathfrak{u}}_i(\xi)\right|^{q}}{(z-\xi)^{q-qm_i}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\bigg(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+n_i)}\bigg(\bigg(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}} \\
&\times\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\right|^{q}}{(z-\xi)^{q-q(\sigma_i+n_i)}}\,d\xi\bigg)^{\frac{1}{1/q}} \\
&+\bigg(\int_{z_\circ}^{z}\lambda_{2i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\right|^{q}}{(z-\xi)^{q-q(\sigma_i+n_i)}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\bigg(\bigg(\int_{z_\circ}^{z}\lambda_{1i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}} \\
&\times\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_1(\xi)-\acute{\mathfrak{u}}_1(\xi)\right|^{q}}{(z-\xi)^{q-q(\sigma_i+\beta_i)}}\,d\xi\bigg)^{\frac{1}{q}} \\
&+\bigg(\int_{z_\circ}^{z}\lambda_{2i}^{p}(\xi)\,d\xi\bigg)^{\frac{1}{p}}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_2(\xi)-\acute{\mathfrak{u}}_2(\xi)\right|^{q}}{(z-\xi)^{q-q(\sigma_i+\beta_i)}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&-\frac{\gamma_i}{\Gamma(\sigma_i)}\bigg(\int_{z_\circ}^{z}\frac{\left|\mathfrak{u}_i(\xi)-\acute{\mathfrak{u}}_i(\xi)\right|^{q}}{(z-\xi)^{q-q\sigma_i}}\,d\xi\bigg)^{\frac{1}{q}}\bigg) \\
&\leq \frac{m_i-\sigma_i+1}{\mathtt{M}(\sigma_i-m_i)}\bigg(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+n_i)}\bigg(\left\|\lambda_{1i}\right\|_{p}\left\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\right\|_{q,q-q(m_i+n_i)} \\
&+\left\|\lambda_{2i}\right\|_{p}\left\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\right\|_{q,q-q(m_i+n_i)}\bigg)+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(m_i+\beta_i)}
\end{aligned}$$

$$\times\Big((\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_{q,q-q(m_i+\beta_i)}+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_{q,q-q(m_i+\beta_i)})\Big)$$
$$-\frac{\gamma_i}{\Gamma(m_i)}\big\|\mathfrak{u}_i-\acute{\mathfrak{u}}_i\big\|_{q,q-qm_i}\Big)+\frac{\sigma_i-m_i}{\mathtt{M}(\sigma_i-m_i)}\Big(\frac{n_i-\beta_i+1}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+n_i)}$$
$$\times\Big(\Big(\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_{q,q-q(\sigma_i+n_i)}+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_{q,q-q(\sigma_i+n_i)}\Big)\Big)$$
$$+\frac{\beta_i-n_i}{\mathtt{M}(\beta_i-n_i)\Gamma(\sigma_i+\beta_i)}\Big(\Big(\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_{q,q-q(\sigma_i+\beta_i)}$$
$$+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_{q,q-q(\sigma_i+\beta_i)}\Big)\Big)-\frac{\gamma_i}{\Gamma(\sigma_i)}\big\|\mathfrak{u}_i-\acute{\mathfrak{u}}_i\big\|_{q,q-q\sigma_i}\Big)$$
$$\leq\Big(C_i\Big[\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_q+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_q\Big]+\mathcal{D}_i\Big[\big\|\mathfrak{u}_i-\acute{\mathfrak{u}}_i\big\|_q\Big]\Big),$$

which implies that

$$\Big|\mathbb{Q}_i(\mathfrak{u}_1,\mathfrak{u}_2)(\mathrm{z})-\mathbb{Q}_i(\acute{\mathfrak{u}}_1,\acute{\mathfrak{u}}_2)(\mathrm{z})\Big|^q\leq\Big(C_i\Big[\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_q+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_q\Big]$$
$$+\mathcal{D}_i\Big[\big\|\mathfrak{u}_i-\acute{\mathfrak{u}}_i\big\|_q\Big]\Big)^q.$$

Hence,

$$\Big(\int_{z_\circ}^{\mathrm{z}}\Big|\mathbb{Q}_i(\mathfrak{u}_1,\mathfrak{u}_2)(\mathrm{z})-\mathbb{Q}_i(\acute{\mathfrak{u}}_1,\acute{\mathfrak{u}}_2)(\mathrm{z})\Big|^q\,d\xi\Big)^{\frac{1}{q}}$$
$$\leq\Big(C_i\Big[\big\|\lambda_{1i}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_q+\big\|\lambda_{2i}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_q\Big]$$
$$+\mathcal{D}_i\Big[\big\|\mathfrak{u}_i-\acute{\mathfrak{u}}_i\big\|_q\Big]\Big)\Big(\int_{z_\circ}^{\mathrm{z}}1^q\,d\xi\Big)^{\frac{1}{q}}.$$

Therefore,

$$\Big\|\mathbb{Q}_1(\mathfrak{u}_1,\acute{\mathfrak{u}}_1)(\mathrm{t})-\mathbb{Q}_1(\mathfrak{u}_2,\acute{\mathfrak{u}}_2)(\mathrm{t})\Big\|_q\leq\Big(\overline{C}_1\big\|\lambda_{11}\big\|_p+\overline{\mathcal{D}}_1\Big)\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_q+\overline{C}_1\big\|\lambda_{21}\big\|_p\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_q. \tag{3.14}$$

Similarly, we can obtain

$$\Big\|\mathbb{Q}_2(\mathfrak{u}_1,\acute{\mathfrak{u}}_1)(\mathrm{z})-\mathbb{Q}_2(\mathfrak{u}_2,\acute{\mathfrak{u}}_2)(\mathrm{t})\Big\|_q\leq\overline{C}_1\big\|\lambda_{12}\big\|_p\big\|\mathfrak{u}_1-\acute{\mathfrak{u}}_1\big\|_q+\Big(\overline{C}_1\big\|\lambda_{22}\big\|_p+\overline{\mathcal{D}}_1\Big)\big\|\mathfrak{u}_2-\acute{\mathfrak{u}}_2\big\|_q. \tag{3.15}$$

We can then put (3.14) and (3.15) together and rewrite as

$$\begin{bmatrix}\|\mathbb{Q}_1(\mathfrak{u}_1,\acute{\mathfrak{u}}_1)(\mathrm{z})-\mathbb{Q}_1(\mathfrak{u}_2,\acute{\mathfrak{u}}_2)(\mathrm{z})\|_q\\ \|\mathbb{Q}_2(\mathfrak{u}_1,\acute{\mathfrak{u}}_1)(\mathrm{z})-\mathbb{Q}_2(\mathfrak{u}_2,\acute{\mathfrak{u}}_2)(\mathrm{z})\|_q\end{bmatrix}\leq M_{2\times2}\begin{bmatrix}\|\mathfrak{u}_1-\mathfrak{u}_2\|_q\\ \|\acute{\mathfrak{u}}_1-\acute{\mathfrak{u}}_2\|_q\end{bmatrix}. \tag{3.16}$$

Now, one can apply Theorem 2.7 (Perov's fixed point theorem) to derive the desired result due to (H3). There exists a unique solution $\mathfrak{u}^*\in L^2_{q,q-q\sigma_i}$ $(i=1,2)$ such that $\mathbb{Q}\mathfrak{u}^*=\mathfrak{u}^*$. Since $q-q\sigma_i\in(0,1)$, $i=1,2$, we know from Lemma 2.4 that $L_{q,q-q\sigma}\subseteq L_q$. The proof is completed. □

4. Example and numerical study

In this part of the research note, we prepare an illustrative problem of the given the nonlinear fractional differential equation (1.2) to ensure the correctness of the results obtained above.

Example 4.1. *We consider a nonlinear nonlinear fractional differential equation as follows:*

$$\begin{cases} {}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{13}{4}}\left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{11}{5}}+\frac{\sqrt{11}}{4}\right)u_1(z)=\dfrac{\sqrt{|z|}}{6(1+|z|)}+\dfrac{\sqrt{5}z\sin^2(u_1(z))}{3(1+z^2)(3+\sin^2(u_1(z)))}\\ \quad+\dfrac{\exp(z)\arctan(u_2(z))}{\sqrt{10}(1+\exp(z))(7+\arctan(u_2(z)))},\\ {}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{7}{3}}\left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{11}{4}}+\frac{\sqrt{47}}{5}\right)u_2(z)=\dfrac{z+1.2}{21+|z^3|}+\dfrac{(2.2+z)\tan(u_1(z))}{(z^2+3.5)(4+\tan(u_1(z)))}\\ \quad+\dfrac{u_2(z)}{\sqrt{30}(|z|+1.5)(1+\cos^2(u_2(z)))}, \end{cases} \tag{4.1}$$

for $0<z<1$, *under the following conditions,* $u_1(0)=\mu_{1,0}=15.5$, $u_1^{(1)}(0)=\mu_{1,1}=-13.7$, $u_2(0)=\mu_{2,0}=-6.8$, $u_2^{(1)}(0)=\mu_{2,1}=4.39$, $u_2^{(2)}(0)=\mu_{2,2}=-5.1$, $m_1=2$, $m_2=3$, $n_1=3$ *and* $n_2=2$.

$$\begin{aligned} \left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{11}{5}}+\frac{\sqrt{11}}{4}\right)u_1(0)&=\nu_{1,0}=23.7,\\ \left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{11}{5}}+\frac{\sqrt{11}}{4}\right)^{(1)}u_1(0)&=\nu_{1,1}=-32.5,\\ \left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{11}{5}}+\frac{\sqrt{11}}{4}\right)^{(2)}u_1(0)&=\nu_{1,2}=17.8,\\ \left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{7}{2}}+\frac{\sqrt{47}}{5}\right)u_2(0)&=\nu_{2,0}=14.6,\\ \left({}^{\mathcal{ABC}}\mathbf{D}_{z}^{\frac{7}{2}}+\frac{\sqrt{47}}{5}\right)^{(1)}u_2(0)&=\nu_{2,1}=-9.65. \end{aligned}$$

Clearly, $\beta_1=\frac{13}{4}\in[3,4)$, $\sigma_1=\frac{11}{5}\in(2,3]$, $\gamma_1=\frac{\sqrt{11}}{4}$, $\beta_2=\frac{7}{3}\in[2,3)$, $\sigma_2=\frac{7}{2}\in[3,4)$ *and* $\gamma_2=\frac{\sqrt{47}}{5}$. *We consider*

$$\begin{aligned} \chi_1(z,u_1(z),u_2(z))&=\frac{\sqrt{|z|}}{6(1+|z|)}+\frac{\sqrt{5}z\sin^2(u_1(z))}{3(1+z^2)(3+\sin^2(u_1(z)))}\\ &\quad+\frac{\exp(z)\arctan(u_2(z))}{5\sqrt{2}(1+\exp(z))(7+\arctan(u_2(z)))},\\ \chi_2(z,u_1(z),u_2(z))&=\frac{z+1.2}{21+|z^3|}+\frac{(0.2+z)\tan(u_1(z))}{(z^2+15)(4+\tan(u_1(z)))}\\ &\quad+\frac{u_2(z)}{\sqrt{30}(|z+1.5)(1+\cos^2(u_2(z)))}. \end{aligned}$$

So, we have

$$
\begin{aligned}
&\left|\chi_1(z, u_1(z), u_2(z)) - \chi_1(z, \bar{u}_1(z), \bar{u}_2(z))\right| \\
&\quad = \Bigg|\frac{\sqrt{5}z\sin^2(u_1(z))}{3(1+z^2)(3+\sin^2(u_1(z)))} + \frac{\exp(z)\arctan(u_2(z))}{5\sqrt{2}(1+\exp(z))(7+\arctan(u_2(z)))} \\
&\quad\quad - \left(\frac{\sqrt{5}z\sin^2(\bar{u}_1(z))}{3(1+z^2)(3+\sin^2(\bar{u}_1(z)))} + \frac{\exp(z)\arctan(\bar{u}_2(z))}{5\sqrt{2}(1+\exp(z))(7+\arctan(\bar{u}_2(z)))}\right)\Bigg| \\
&\quad \le \lambda_{11}|u_1(z) - \bar{u}_1(z)| + \lambda_{21}|u_2(z) - \bar{u}_2(z)|,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\chi_2(z, u_1(z), u_2(z)) - \chi_1(z, \bar{u}_1(z), \bar{u}_2(z))\right| \\
&\quad = \Bigg|\frac{(0.2+z)\tan(u_1(z))}{(z^2+15)(4+\tan(u_1(z)))} + \frac{u_2(z)}{\sqrt{30}(|z+1.5)(1+\cos^2(u_2(z)))} \\
&\quad\quad - \left(\frac{(0.2+z)\tan(\bar{u}_1(z))}{(z^2+15)(4+\tan(\bar{u}_1(z)))} + \frac{\bar{u}_2(z)}{\sqrt{30}(|z+1.5)(1+\cos^2(\bar{u}_2(z)))}\right)\Bigg| \\
&\quad \le \lambda_{12}|u_1(z) - \bar{u}_1(z)| + \lambda_{22}|u_2(z) - \bar{u}_2(z)|,
\end{aligned}
$$

where

$$
\lambda_{11} = \frac{\sqrt{5}z}{3(1+z^2)}, \quad \lambda_{21} = \frac{\exp(z)}{5\sqrt{2}(1+\exp(z))}, \quad \lambda_{12} = \frac{0.2+z}{z^2+15}, \quad \lambda_{22} = \frac{1}{\sqrt{30}(|z|+1.5)}.
$$

Table 1 shows the results of λ_{ij} $(i, j = 1, 2)$ *for* $0 < z < 1$. *One can see the 2D-graph of* λ_{ij} $(i, j = 1, 2)$ *for* $0 \le z \le 1$ *in Figure 1.*

Table 1. Numerical values of λ_{11}, λ_{21}, λ_{12}, λ_{22} in Example 4.1 for each $z \in [0, 1]$.

z	λ_{11}	λ_{21}	λ_{12}	λ_{22}
0.00	0.0000	0.000000	0.000000	0.000000
0.10	0.0135	0.022922	0.005304	0.037268
0.20	0.0376	0.033214	0.009098	0.051131
0.30	0.0671	0.041657	0.013129	0.060858
0.40	0.0996	0.049230	0.017463	0.068399
0.50	0.1330	0.056298	0.022092	0.074536
0.60	0.1660	0.063037	0.026991	0.079682
0.70	0.1979	0.069549	0.032132	0.084087
0.80	0.2279	0.075892	0.037488	0.087917
0.90	0.2558	0.082104	0.043029	0.091287
1.00	0.2816	0.088211	0.048729	0.094281

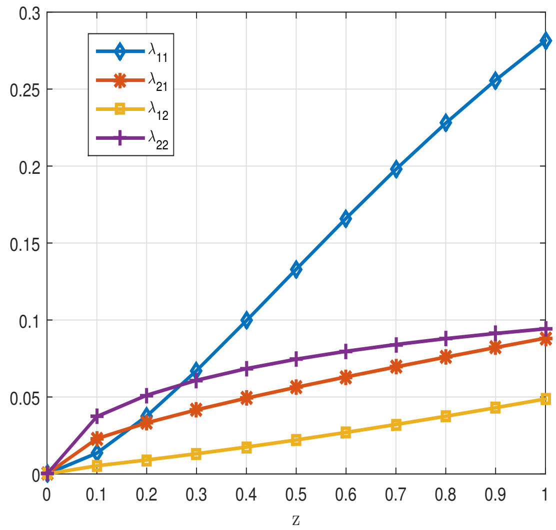


Figure 1. 2D-graph of $\lambda_{ij}\,(i, j = 1, 2)$ for $0 \leq z \leq 1$ in Example 4.1.

Hence, the condition (H3) *in Theorem 3.2 holds. On the other hand, thanks to the normalization function* $\mathtt{M}(\beta) = (1-\beta) + \frac{\beta}{\Gamma(\beta)} > 0$ *and Eqs* (3.6) *and* (3.7)*, we have*

$$
\begin{aligned}
C_1 =& \frac{m_1 - \sigma_1 + 1}{\mathtt{M}(\sigma_1 - m_1)} \left\{ \frac{n_1 - \beta_1 + 1}{\mathtt{M}(\beta_1 - n_1)\Gamma(m_1 + n_1)} + \frac{\beta_1 - n_1}{\mathtt{M}(\beta_1 - n_1)\Gamma(m_1 + \beta_1)} \right\} \\
&+ \frac{\sigma_1 - m_1}{\mathtt{M}(\sigma_1 - m_1)} \left\{ \frac{n_1 - \beta_1 + 1}{\mathtt{M}(\beta_1 - n_1)\Gamma(\sigma_1 + n_1)} + \frac{\beta_1 - n_1}{\mathtt{M}(\beta_1 - n_1)\Gamma(\sigma_1 + \beta_1)} \right\} \\
=& \frac{2 - \frac{11}{5} + 1}{\mathtt{M}\left(\frac{11}{5} - 2\right)} \left\{ \frac{3 - \frac{13}{4} + 1}{\mathtt{M}\left(\frac{13}{4} - 3\right)\Gamma(5)} + \frac{\frac{13}{4} - 3}{\mathtt{M}\left(\frac{13}{4} - 3\right)\Gamma\left(2 + \frac{13}{3}\right)} \right\} \\
&+ \frac{\frac{11}{5} - 2}{\mathtt{M}\left(\frac{11}{5} - 2\right)} \left\{ \frac{3 - \frac{13}{4} + 1}{\mathtt{M}\left(\frac{13}{4} - 3\right)\Gamma\left(\frac{11}{5} + 3\right)} + \frac{\frac{13}{4} - 3}{\mathtt{M}(\frac{13}{4} - 3)\Gamma\left(\frac{11}{5} + \frac{13}{4}\right)} \right\} \simeq 0.052572,
\end{aligned}
$$

$$
\begin{aligned}
C_2 =& \frac{m_2 - \sigma_2 + 1}{\mathtt{M}(\sigma_2 - m_2)} \left\{ \frac{n_2 - \beta_2 + 1}{\mathtt{M}(\beta_2 - n_2)\Gamma(m_2 + n_2)} + \frac{\beta_2 - n_2}{\mathtt{M}(\beta_2 - n_2)\Gamma(m_2 + \beta_2)} \right\} \\
&+ \frac{\sigma_2 - m_2}{\mathtt{M}(\sigma_2 - m_2)} \left\{ \frac{n_2 - \beta_2 + 1}{\mathtt{M}(\beta_2 - n_2)\Gamma(\sigma_2 + n_2)} + \frac{\beta_2 - n_2}{\mathtt{M}(\beta_2 - n_2)\Gamma(\sigma_2 + \beta_2)} \right\} \\
=& \frac{3 - \frac{7}{2} + 1}{\mathtt{M}\left(\frac{7}{2} - 3\right)} \left\{ \frac{2 - \frac{7}{3} + 1}{\mathtt{M}\left(\frac{7}{3} - 2\right)\Gamma(3 + 2)} + \frac{\frac{7}{3} - 2}{\mathtt{M}\left(\frac{7}{3} - 2\right)\Gamma\left(3 + \frac{7}{3}\right)} \right\} \\
&+ \frac{\frac{7}{2} - 3}{\mathtt{M}\left(\frac{7}{2} - 3\right)} \left\{ \frac{2 - \frac{7}{3} + 1}{\mathtt{M}\left(\frac{7}{3} - 2\right)\Gamma\left(\frac{7}{2} + 2\right)} + \frac{\frac{7}{3} - 2}{\mathtt{M}\left(\frac{7}{3} - 2\right)\Gamma\left(\frac{7}{2} + \frac{7}{3}\right)} \right\} \simeq 0.042429,
\end{aligned}
$$

and

$$\mathcal{D}_1 = \frac{\sigma_1 - m_1}{\mathtt{M}(\sigma_1 - m_1)} \frac{|\gamma_1|}{\Gamma(\sigma_1)} + \frac{m_1 - \sigma_1 + 1}{\mathtt{M}(\sigma_1 - m_1)} \frac{|\gamma_1|}{\Gamma(m_1)}$$
$$= \frac{\frac{11}{5} - 2}{\mathtt{M}\left(\frac{11}{5} - 2\right)} \frac{\left|\frac{\sqrt{13}}{4}\right|}{\Gamma\left(\frac{11}{5}\right)} + \frac{2 - \frac{11}{5} + 1}{\mathtt{M}\left(\frac{11}{5} - 2\right)} \frac{\left|\frac{\sqrt{11}}{4}\right|}{\Gamma(2)} \simeq 0.964755,$$
$$\mathcal{D}_2 = \frac{\sigma_2 - m_2}{\mathtt{M}(\sigma_2 - m_2)} \frac{|\gamma_2|}{\Gamma(\sigma_2)} + \frac{m_2 - \sigma_2 + 1}{\mathtt{M}(\sigma_2 - m_2)} \frac{|\gamma_2|}{\Gamma(m_2)}$$
$$= \frac{\frac{7}{2} - 3}{\mathtt{M}\left(\frac{7}{2} - 3\right)} \frac{\left|\frac{\sqrt{47}}{5}\right|}{\Gamma\left(\frac{7}{2}\right)} + \frac{3 - \frac{7}{2} + 1}{\mathtt{M}\left(\frac{7}{2} - 3\right)} \frac{\left|\frac{\sqrt{47}}{5}\right|}{\Gamma(3)} \simeq 0.702050,$$

$\|\lambda_{11}\|_2 = 0.281562$, $\|\lambda_{21}\|_2 = 0.088210$, $\|\lambda_{12}\|_2 = 0.048729$, $\|\lambda_{22}\|_2 = 0.094280$ *and thanks to* (3.5),

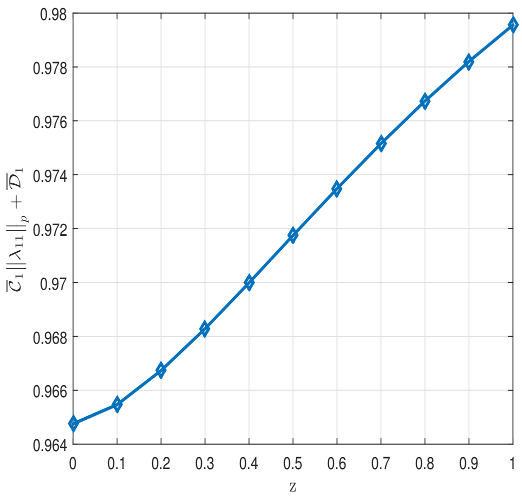
$$M_{2\times 2} = \begin{bmatrix} \overline{C}_1 \left\|\lambda_{11}\right\|_p + \overline{\mathcal{D}}_1 & \overline{C}_1 \left\|\lambda_{21}\right\|_p \\ \overline{C}_1 \left\|\lambda_{12}\right\|_p & \overline{C}_1 \left\|\lambda_{22}\right\|_p + \overline{\mathcal{D}}_1 \end{bmatrix} = \begin{bmatrix} 0.979558 & 0.004637 \\ 0.002561 & 0.969712 \end{bmatrix}, \tag{4.2}$$

Table 2 shows the results of matrix M. Once can see the illustration of Matrix M in Figures 2a, 2b, 3a and 3b. Note that

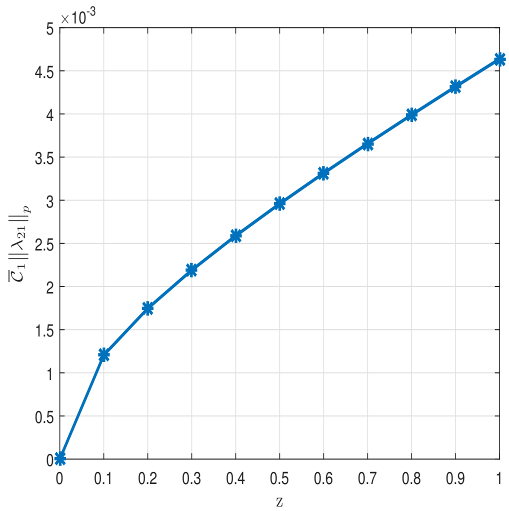
$$\overline{C}_1\|\lambda_{21}\|_p \simeq 0, \quad \overline{C}_1\|\lambda_{12}\|_p \simeq 0, \quad \overline{C}_1\|\lambda_{11}\|_p + \overline{\mathcal{D}}_1 > 0, \quad \overline{C}_1\|\lambda_{22}\|_p + \overline{\mathcal{D}}_1 > 0,$$

and

$$\max\left\{\overline{C}_1\left\|\lambda_{11}\right\|_p + \overline{\mathcal{D}}_1, \overline{C}_1\left\|\lambda_{22}\right\|_p + \overline{\mathcal{D}}_1\right\} < 1.$$

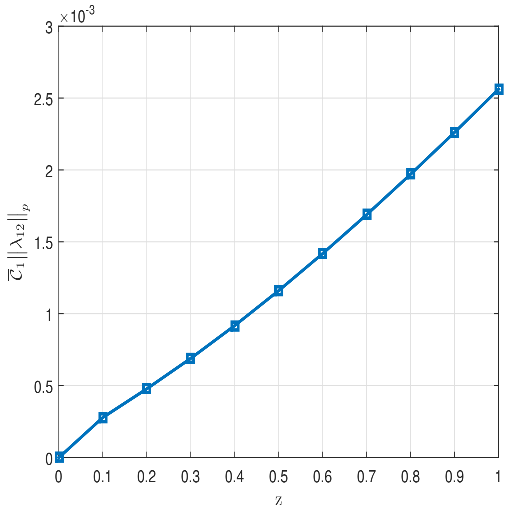


(a) $\overline{C}_1\|\lambda_{11}\|_p + \overline{\mathcal{D}}_1$.

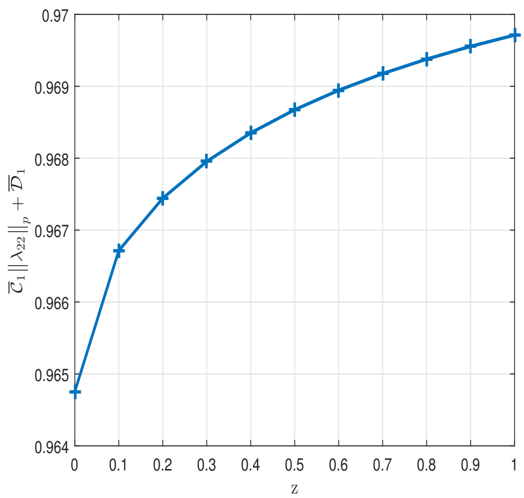


(b) $\overline{C}_1\|\lambda_{21}\|_p$.

Figure 2. Illustration of Matrix M in Example 4.1.



(a) $\overline{C}_1\|\lambda_{12}\|_p$.



(b) $\overline{C}_1\|\lambda_{22}\|_p + \overline{\mathcal{D}}_1$.

Figure 3. Illustration of Matrix M in Example 4.1.

Table 2. Numerical values of $\overline{C}_1\|\lambda_{11}\|_p + \overline{\mathcal{D}}_1$, $\overline{C}_1\|\lambda_{21}\|_p$, $\overline{C}_1\|\lambda_{12}\|_p$, $\overline{C}_1\|\lambda_{22}\|_p + \overline{\mathcal{D}}_1$ in Example 4.1 for all z ∈ [0, 1].

z	$\overline{C}_1\|\lambda_{11}\|_p + \overline{\mathcal{D}}_1$	$\overline{C}_1\|\lambda_{21}\|_p$	$\overline{C}_1\|\lambda_{12}\|_p$	$\overline{C}_1\|\lambda_{22}\|_p + \overline{\mathcal{D}}_1$
0.00	0.9648	0.000000	0.000000	0.964756
0.10	0.9655	0.001205	0.000279	0.966715
0.20	0.9667	0.001746	0.000478	0.967444
0.30	0.9683	0.002190	0.000690	0.967955
0.40	0.9700	0.002588	0.000918	0.968351
0.50	0.9717	0.002960	0.001161	0.968674
0.60	0.9735	0.003314	0.001419	0.968945
0.70	0.9752	0.003656	0.001689	0.969176
0.80	0.9767	0.003990	0.001971	0.969378
0.90	0.9782	0.004316	0.002262	0.969555
1.00	0.9796	0.004637	0.002562	0.969712

Thus $M_{2\times 2}$ converges to zero. Also,

$$\chi_1^{\max} = \max_{z\in[0,1]} |\chi_1(z,0,0)| = \frac{1}{6}, \qquad \chi_2^{\max} = \max_{z\in[0,1]} |\chi_2(z,0,0)| = \frac{2.2}{21},$$

and by using (3.8) *and* (3.9) *we obtain* $\mathcal{A}_1 \simeq 0.00017$, $\mathcal{A}_2 \simeq 0.00040$, $\mathcal{B}_1 \simeq 49.71935$, $\mathcal{B}_2 \simeq 19.02302$, $\Omega_1 = 49.77209$ *and* $\Omega_2 = 19.06585$. *These numerical results are shown in Table 3. Figure 4 shows 2D-graph of* Ω_i $(i = 1, 2)$ *for* $0 \le z \le 1$.

Table 3. Numerical values of $\mathcal{A}_i$, $\mathcal{B}_i$ and Ω_i for $i = 1, 2$ in Example 4.1 $\forall z \in [0, 1]$.

z	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{B}_1$	$\mathcal{B}_2$	Ω_1	Ω_2
0.00	0.00000	0.00000	15.50000	6.80000	15.55257	6.84243
0.10	0.00000	0.00000	17.00285	7.29184	17.05542	7.33427
0.20	0.00000	0.00000	18.80486	7.89781	18.85743	7.94024
0.30	0.00000	0.00000	20.95171	8.63248	21.00428	8.67491
0.40	0.00000	0.00000	23.48934	9.51253	23.54191	9.55496
0.50	0.00000	0.00001	26.46524	10.55666	26.51781	10.59910
0.60	0.00001	0.00003	29.92878	11.78543	29.98136	11.82789
0.70	0.00002	0.00006	33.93138	13.22127	33.98397	13.26376
0.80	0.00005	0.00013	38.52653	14.88844	38.57915	14.93100
0.90	0.00009	0.00023	43.76991	16.81305	43.82257	16.85572
1.00	0.00017	0.00040	49.71935	19.02302	49.77209	19.06585

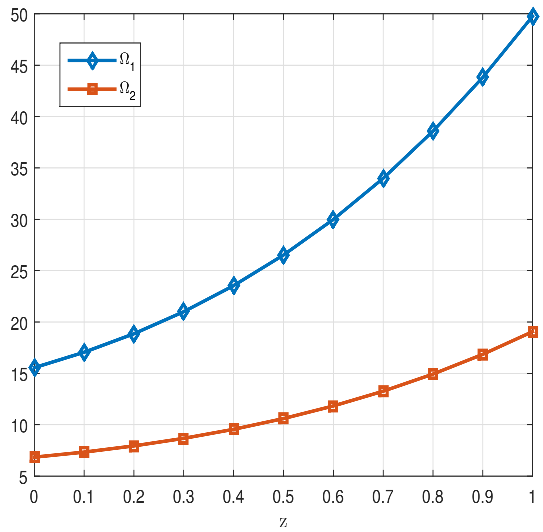


Figure 4. 2D-graph of Ω_i $(i = 1, 2)$ for $0 \leq z \leq 1$ in Example 4.1.

We see that all requirements of Theorem 3.2 are fulfilled. Therefore, this guarantees that the system (4.1) *admits a unique solution in* $L_q[0, 1]$.

5. Conclusions

The authors of this work discussed a the existence of solutions for the coupled Langevin-FDE from perspective of a Perov's fixed point approach. In our presented work, we took into account the used $\mathcal{ABC}$ operator, which has some advantages over the over the classical differential operator. We think that this operator has provided new access points for academics to conduct their research. We used Perov's fixed point theorems and the literature to help us with the existence of the solution. In the the case of Atangana-Baleanu-derivative, we looked into the presence and computational analysis of an example. The graphical results of the numerical simulations comprise the classical solution of the model as a specific example, and we have separate solutions for different orders and dimensions. As the orders approach one other, the graphs depicting various solutions for the various fractional orders get closer. The system of coupled Langvin FDE that makes up the presumptive problem is extremely complex. The readers might reevaluate the posited issue with the aid of various fixed point methodologies to determine whether there are both multiple and singular solutions.

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Conflict of interest

The authors declare no conflict of interest and have equal contributions.

Author contributions

All authors claim equal contribution with no competing interest.

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