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Research article

Homogenization of nonlinear hyperbolic problem with a dynamical boundary condition

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Abstract: In this work, we look at homogenization results for nonlinear hyperbolic problem with a non-local boundary condition. We use the periodic unfolding method to obtain a homogenized nonlinear hyperbolic equation in a fixed domain. Due to the investigation's peculiarity, the unfolding technique must be developed with special attention, creating an unusual two-scale model. We note that the non-local boundary condition caused a damping on the homogenized model.

Keywords: non-local interface condition; nonlinear hyperbolic PDEs; homogenization; the periodic unfolding method
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1. Introduction

The study of integral-type, non-local boundary conditions are fascinating area of rapidly evolving differential equations theory. Non-local boundary conditions are useful in a variety of contexts, including wave equations, electric conduction, petroleum exploitation, heat diffusion, and the elastic behavior of perforated materials, see for instance [1, 15, 24, 26]. Mathematicians, engineers and applied scientist intensively invest gated problems with non-local boundary conditions, from theoretical and computational point of views, see for instance [7, 10, 17–19]. For homogenization for boundary value problems with Dirichlet Nuemann and Robin conditions, we refer to [9, 12, 14]. The first work dealing with homogenization for non-local boundary problems we refer to [10], where the authors investigated the periodic homogenization using the periodic unfolding techniques for the elastic torsion problem of an infinite 3-dimensional rod along with the overall electro-conductivity problem within presence of a significant number of excellent conductors. Their results were further developed to time-dependent problems by Amar et al. in [2, 3], where the authors obtained homogenization for classes of linear parabolic problems with non-local boundaries using the periodic unfolding techniques. The results in [2] are for a linear parabolic problem with special oscillations in the coefficients, whereas the results in [3] are for a linear parabolic problem with time space oscillations in the coefficients. To the best of the author's knowledge, there are no results for homogenization of boundary value problems with non-local boundary conditions in the hyperbolic framework. The goal of this paper is to extend the results of [2] by looking into homogenization results for a nonlinear hyperbolic problem with a non-local boundary condition involving the solution's time derivative. The author's forthcoming studies will concentrate on homogenization of hyperbolic problems with nonlocal boundary conditions when the coefficients oscillate in both the space and time variables at different scales, which is more technical and requires more complicated analysis. As for homogenization of deterministic linear hyperbolic problems with Dirichlet and Neumann boundary conditions, we refer to [13, 16] and the references therein. For homogenization for nonlinear hyperbolic problems with Dirichlet boundary condition, see [25], where the authors obtained homogenization for similar model to the one in this paper, but with Dirichlet condition using multi-scale convergence in fixed domain. We also mention the work on homogenization of hyperbolic SPDEs, see [20–22]. Here, we consider the following nonlinear hyperbolic problem with non-local boundary condition.

$$\begin{cases} \frac{\partial^2 v^\epsilon}{\partial t^2} - \operatorname{div}(A^\epsilon \nabla v^\epsilon) + \beta^\epsilon(v^\epsilon, \nabla v^\epsilon) = f \text{ in } D^\epsilon \times (0,T), \\ \gamma \frac{\partial v^\epsilon}{\partial t} = \frac{1}{\epsilon^n} \int_{\Gamma_\eta^\epsilon} \frac{\partial v^\epsilon}{\partial \nu_{\epsilon A^\epsilon}} d\sigma_x \text{ on } \Gamma_\eta^\epsilon \times (0,T),\ \ \eta \in \mathbb{X}_\epsilon, \\ v^\epsilon = B_\eta(t) \text{ on } \Gamma_\eta^\epsilon \times (0,T),\ \ \eta \in \mathbb{X}_\epsilon, \\ v^\epsilon(0,x) = \frac{\partial v^\epsilon}{\partial t}(0,x) = 0 \text{ in } D, \\ v^\epsilon = 0 \text{ on } \partial D \times (0,T). \end{cases} \tag{1.1}$$

For all $(x, y, t) \in D \times Y_x \times (0, T)$. The domain in which this problem is studied is described further below.

- F is an open subset of $\mathbb{R}^n$ such that $F + z = F$ for all $z \in \mathbb{Z}^n$.
- $Y_x = (0, 1) \times (0, 1) \times \cdots \times (0, 1) \subset \mathbb{R}^n$.
- $F_u = F \cap Y_x$, $F_s = Y_x \backslash \bar{F}$, $\Gamma = \partial F \cap Y_x$ and $\partial F_u \cap \partial Y_x = \phi$ which implies that $\partial F_u = \Gamma$.
- D is an open connected and bounded subset of $\mathbb{R}^n$ and and $D_T = D \times (0, T)$.
- $\mathbb{X}_\epsilon = \{\eta \in \mathbb{Z}^n : \epsilon(\eta + Y_x) \subset D\}$, where ϵ represents a sequence of positive real numbers that tends to zero.
- $F_\eta^\epsilon := \epsilon(F_u + \eta)$ and $\Gamma_\eta^\epsilon = \partial F_\eta^\epsilon$.
- $F^\epsilon = \bigcup_{\eta \in \mathbb{X}_\epsilon} F_\eta^\epsilon$ is disconnected with smooth boundary and $\Gamma^\epsilon = \partial F^\epsilon$.
- $D^\epsilon = D \backslash \bar{F}^\epsilon$ is connected.
- ν is the unit outward normal on Γ and it is extended to $\mathbb{R}^n$ by periodicity.
- $\nu_\epsilon = \nu\left(\frac{x}{\epsilon}\right)$ is the unit outward normal on Γ^ϵ.

Let us mention that $\frac{\partial v^\epsilon}{\partial \nu_{\epsilon A^\epsilon}} = \sum_{i,j=1}^n a_{i,j}\left(\frac{x}{\epsilon}\right) \cdot \nu_j\left(\frac{x}{\epsilon}\right) \frac{\partial v^\epsilon}{\partial x_i}$. For a better understanding of the domain, we add Figure 1.

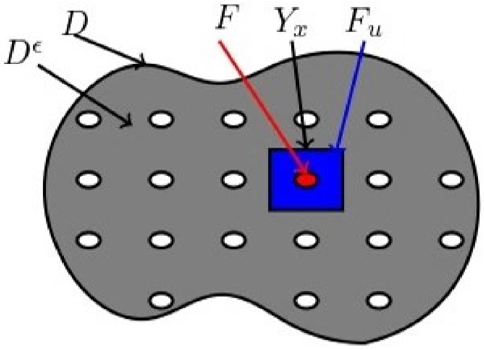

Figure 1. The domain D^ϵ.

Let us state our data assumptions.

(H1) $B_\eta(t)$ is a constant function with respect to the spatial variable x depending on η and t.

(H2) $A = (a_{i,j})_{1\le i,j\le n}$ a symmetric matrix such that $a_{i,j} \in L^\infty(D; L^\infty_{\text{per}}(Y_x))$ where $A^\epsilon(x) = A\left(x, \frac{x}{\epsilon}\right)$ such that

$$\alpha_1|\xi|^2 \le A(x,y)\xi\cdot\xi \le \alpha_2|\xi|^2, \text{ for all } \xi \in \mathbb{R}^n,\ \alpha_1,\alpha_2 > 0. \tag{1.2}$$

(H3) $\beta^\epsilon(t,x,v^\epsilon,\nabla v^\epsilon) = \beta\left(\frac{x}{\epsilon}, v^\epsilon, \nabla v^\epsilon\right)$ is measurable with respect to $(\varphi,\psi)\in\mathbb{R}\times\mathbb{R}^n$ and Y_x-periodic with respect to the first arguments, such that

(a) $|\beta(y,\varphi,\psi)| \le c_0(1+|\varphi|^{r+1}+|\psi|)$.

(b) $\beta(y,\varphi,\psi)\Phi \ge c_1|\varphi|^r\varphi\Phi - c_2(1+|\Phi||\psi|)$.

(c) $\left|\frac{\partial}{\partial\varphi}\beta(y,\varphi,\psi)\right| \le c_0(1+|\varphi|^r)$.

(d) $\left|\nabla_\psi\beta(y,\varphi,\psi)\right| \le c_3$.

(e) For all $(\varphi_1,\psi_1,\Phi_1)$,$(\varphi_2,\psi_3,\Phi_2)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$, we have

$$\begin{aligned}(\beta(y,\varphi_1,\psi_1) - \beta(y,\varphi_2,\psi_2))\,(\Phi_1-\Phi_2)&\\ &\ge -c_4\left(|\varphi_1|^r + |\varphi_2|^r\right)|\varphi_1-\varphi_2||\Phi_1-\Phi_2|\\ &\quad - c_5|\psi_1-\psi_2||\Phi_1-\Phi_2|,\end{aligned} \tag{1.3}$$

where $c_0, c_1, \cdots, c_5$ are positive constants and

$$\begin{cases} r\in[1,\infty), & \text{if } n=1,2,\\ r\in\left[1,\frac{n}{n-2}\right), & \text{if } n\ge 3.\end{cases} \tag{1.4}$$

(H4) $\gamma > 0$.

(H5) $f \in L^2(D_T)$.

We shall refer to both the original function and its extension to the entire of D as v^ϵ for the simplicity's sake. Following [2, 10], we introduce the following spaces

$$\mathcal{H}^\epsilon = \left\{\varphi\in L^2\left(0,T;W_0^\epsilon\right) : \frac{\partial\varphi}{\partial t}\in L^2\left(0,T;L^2_\epsilon\right)\right\}, \tag{1.5}$$

$$\mathcal{H}_1^\epsilon = \left\{\varphi \in L^2\left(0,T;L_\epsilon^2\right) : \frac{\partial \varphi}{\partial t} \in L^2\left(0,T;[W_0^\epsilon]^{'}\right)\right\}, \tag{1.6}$$

where, see [10]

$$L_\epsilon^2 = \left\{\varphi \in L^2(D) : \varphi|_{F_\eta^\epsilon} = C_\eta \text{ for all } \eta \in \mathbb{X}_\epsilon\right\},$$

$$W_0^\epsilon = \left\{\varphi \in W_0^{1,2}(D) : \varphi|_{F_\eta^\epsilon} = C_\eta \text{ for all } \eta \in \mathbb{X}_\epsilon\right\},$$

and $\left[W_0^\epsilon\right]^{'}$ is the dual space of W_0^ϵ. The existence and uniqueness results for system (1.1) for fixed $\epsilon > 0$, are obtained by combining ideas from [1,8]. With this, we can write system (1.1) in the following weak formulation:

$$\begin{aligned}
&\int_0^T \int_{D^\epsilon} \frac{\partial^2 v^\epsilon}{\partial t^2}\varphi dxdt + \int_0^T \int_{D^\epsilon} A^\epsilon \nabla v^\epsilon \nabla \varphi dxdt \\
&\quad + \int_0^T \int_{D^\epsilon} \beta^\epsilon\left(v^\epsilon, \nabla v^\epsilon\right)\varphi dxdt + \sum_{\eta \in \mathbb{X}_\epsilon} \int_0^T \int_{\Gamma_\eta^\epsilon} \frac{\partial v^\epsilon}{\partial \nu_{\epsilon A^\epsilon}} \varphi d\sigma_x dt \\
&\quad = \int_0^T \int_{D^\epsilon} f\varphi dxdt,
\end{aligned} \tag{1.7}$$

for all $\varphi \in \mathcal{D}(D) \times (0,T)$. For a better formulation of our system we introduce the following set of test function

$$\mathcal{U}_\epsilon = \left\{\varphi_\epsilon \in \mathcal{D}(D) : \varphi_\epsilon|_{F_\eta^\epsilon} = C_\eta \text{ for all } \eta \in \mathbb{X}_\epsilon\right\}. \tag{1.8}$$

Now, testing our problem by a function from the set $\mathcal{U}_\epsilon$, we have

$$\begin{aligned}
&\int_0^T \int_{D^\epsilon} \frac{\partial^2 v^\epsilon}{\partial t^2}\varphi_\epsilon dxdt + \int_0^T \int_{D^\epsilon} A^\epsilon \nabla v^\epsilon \nabla \varphi_\epsilon dxdt \\
&\quad + \int_0^T \int_{D^\epsilon} \beta^\epsilon\left(v^\epsilon, \nabla v^\epsilon\right)\varphi_\epsilon dxdt + \frac{\gamma}{|F_u|} \int_0^T \int_{F^\epsilon} \frac{\partial v^\epsilon}{\partial t} \varphi_\epsilon dxdt \\
&\quad = \int_0^T \int_{D^\epsilon} f\varphi_\epsilon dxdt.
\end{aligned} \tag{1.9}$$

2. A priori estimates

Since v^ϵ is somehow taken to be constant in each F_η^ϵ, we may take $\varphi_\epsilon = 2\frac{\partial v^\epsilon}{\partial t}$ in (1.9). We have

$$\begin{aligned}
&2\int_0^t \int_{D^\epsilon} \frac{\partial^2 v^\epsilon}{\partial t^2}\frac{\partial v^\epsilon}{\partial t} dxdt + 2\int_0^t \int_{D^\epsilon} A^\epsilon \nabla v^\epsilon \nabla\left(\frac{\partial v^\epsilon}{\partial t}\right) dxdt \\
&\quad + 2\int_0^t \int_{D^\epsilon} \beta^\epsilon\left(v^\epsilon, \nabla v^\epsilon\right)\frac{\partial v^\epsilon}{\partial t} dxdt + \frac{2\gamma}{|F_u|} \int_0^t \int_{F^\epsilon} \frac{\partial v^\epsilon}{\partial t}\frac{\partial v^\epsilon}{\partial t} dxdt \\
&\quad = 2\int_0^t \int_{D^\epsilon} f\frac{\partial v^\epsilon}{\partial t} dxdt.
\end{aligned} \tag{2.1}$$

Simple calculations on the first and second terms of (2.1) give

$$\begin{aligned}
&\left\|\frac{\partial v^\epsilon}{\partial t}(t)\right\|^2_{L^2_\epsilon} + (A^\epsilon \nabla v^\epsilon(t), \nabla v^\epsilon(t))_{L^2_\epsilon} \\
&+ 2\int_0^t \int_{D^\epsilon} \beta^\epsilon\left(v^\epsilon, \nabla v^\epsilon\right) \frac{\partial v^\epsilon}{\partial t} dxdt + \frac{2\gamma}{|F_u|}\int_0^t \int_{F^\epsilon} \frac{\partial v^\epsilon}{\partial t}\frac{\partial v^\epsilon}{\partial t} dxdt \\
&= 2\int_0^t \int_{D^\epsilon} f \frac{\partial v^\epsilon}{\partial t} dxdt. \qquad (2.2)
\end{aligned}$$

From (H3(b)), we see that

$$\begin{aligned}
&2\int_0^t \int_{D^\epsilon} \beta\left(\frac{x}{\epsilon}, v^\epsilon, \nabla v^\epsilon\right) \frac{\partial v^\epsilon}{\partial t} dxdt \\
&\geq 2c_1 \int_0^t \int_{D^\epsilon} |v^\epsilon|^r v^\epsilon \frac{\partial v^\epsilon}{\partial t} dxdt - 2c_2 \int_0^t \int_{D^\epsilon} \left(1 + \left|\frac{\partial v^\epsilon}{\partial t}\right| |\nabla v^\epsilon|\right) dxdt \\
&= \frac{2c_1}{r+2} \int_{D^\epsilon} \int_0^t \frac{\partial}{\partial t}\left(|v^\epsilon|^{r+2}\right) dtdx - 2c_2 \int_0^t \int_{D^\epsilon} \left(1 + \left|\frac{\partial v^\epsilon}{\partial t}\right| |\nabla v^\epsilon|\right) dxdt \\
&= \frac{2c_1}{r+2} \|v^\epsilon(t)\|^{r+2}_{L^{r+2}_\epsilon} - 2c_2 \int_0^t \int_{D^\epsilon} \left(1 + \left|\frac{\partial v^\epsilon}{\partial t}\right| |\nabla v^\epsilon|\right) dxdt. \qquad (2.3)
\end{aligned}$$

Using (H2), (H4), (2.3) and Young's inequality, we get

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|\frac{\partial v^\epsilon}{\partial t}(t)\right\|^2_{L^2_\epsilon} + \alpha_1 \sup_{t\in[0,T]} \|\nabla v^\epsilon(t)\|^2_{L^2_\epsilon} \\
&\quad + \frac{2c_1}{r+2} \sup_{t\in[0,T]} \|v^\epsilon(t)\|^{r+2}_{L^{r+2}_\epsilon} + \frac{2\alpha_3}{|F_u|} \int_0^T \int_{F^\epsilon} \left|\frac{\partial v^\epsilon}{\partial t}\right|^2 dxdt \\
&\quad \leq L_1 + L_2 \int_0^T \left\{\left\|\frac{\partial v^\epsilon}{\partial t}(t)\right\|^2_{L^2_\epsilon} + \|\nabla v^\epsilon(t)\|^2_{L^2_\epsilon}\right\} dt. \qquad (2.4)
\end{aligned}$$

This inequality and Grownall's inequality gives

$$\sup_{t\in[0,T]} \left\|\frac{\partial v^\epsilon}{\partial t}(t)\right\|^2_{L^2_\epsilon} + \sup_{t\in[0,T]} \|\nabla v^\epsilon(t)\|^2_{L^2_\epsilon} + \sup_{t\in[0,T]} \|v^\epsilon(t)\|^{r+2}_{L^{r+2}_\epsilon} \leq C. \qquad (2.5)$$

From (H3(a)), one easily see that

$$\begin{aligned}
\int_{D^\epsilon} \left|\beta\left(\frac{x}{\epsilon}, v^\epsilon, \nabla v^\epsilon\right)\right|^2 dx &\leq C \int_{D^\epsilon} \left(1 + |v^\epsilon|^{2(r+1)} + |\nabla v^\epsilon|^2\right) dx \\
&\leq C\left(1 + \|v^\epsilon\|^{2(r+1)}_{L^{2(r+1)}_\epsilon} + \|\nabla v^\epsilon\|^2_{L^2_\epsilon}\right). \qquad (2.6)
\end{aligned}$$

Thanks to this and (2.5), we claim that $\beta\left(\frac{x}{\epsilon}, v^\epsilon, \nabla v^\epsilon\right)$ belongs to $L^2(D^\epsilon_T)$, by which and the leading equation in (1.1) we have that

$$\frac{\partial^2 v^\epsilon}{\partial t^2} \in L^2\left(0, T; [W^\epsilon_0]'\right).$$

3. Time-space periodic unfolding operator

In this section, we give some definitions and properties of the time-space periodic unfolding operator, that was essentially given in [2], see also [4, 5]. For this, we set

- $\hat{D}^\epsilon = \text{Int}\left\{\bigcup_{\eta\in\mathbb{X}_\epsilon} \epsilon\left(\eta + \bar{Y}_x\right)\right\}$ and $\Lambda^\epsilon_{x,t} = \hat{D}^\epsilon \times (0, T)$.
- $\mathbb{Y}^\epsilon_x = \epsilon\left([\epsilon^{-1}x]_{Y_x} + Y_x\right)x = \epsilon\left([x\epsilon^{-1}]_{Y_x} + \{x\epsilon^{-1}\}_{Y_x}\right)$, where $[a]$ is the integer part of any real number a.

Definition 3.1. *[2, 4] Let u be a Lebesgue measurable function on the set $D \times (0, T)$, then we define the periodic unfolding operator for this function as*

$$\mathbf{T}_\epsilon(u)(t, x, y) = \begin{cases} u\left(\epsilon\left[x\epsilon^{-1}\right]_{Y_x} + \epsilon y, t\right) & (t, x, y) \in \Lambda^\epsilon_{x,t} \times Y_x, \\ \\ 0 & \textit{otherwise}\ . \end{cases}$$

If u is an integrable function in $D \times (0, T)$, we define the average operator as:

$$\mathbf{A}_\epsilon(u)(t, x) = \begin{cases} (\epsilon^n)^{-1} \int_{\mathbb{Y}^\epsilon_x} u(t, y)dy & (t, x) \in \Lambda^\epsilon_{x,t}, \\ \\ 0 & \textit{otherwise}\ . \end{cases}$$

Note that

$$\mathbf{A}_\epsilon(u)(t, x) = \int_{Y_x} \mathbf{T}_\epsilon(u)(t, x, y, \tau)dyd\tau = \mathcal{M}_{Y_x}\left(\mathbf{T}_\epsilon(u)\right)(t, x),$$

where $\mathcal{M}_V$ stands for the usual integral average on the set V. We also define the oscillation operator as

$$\mathbf{O}_\epsilon(\varphi)(t, x, y) = \mathbf{T}_\epsilon(\varphi)(t, x, y) - \mathbf{A}_\epsilon(\varphi)(t, x).$$

The following proposition gives some of the main properties for the time-space unfolding operator:

Proposition 3.1. *[2, 4, 10] The operator $\mathbf{T}_\epsilon : L^2(D \times (0, T)) \to L^2(D \times (0, T); L^2(\mathcal{E}))$ satisfies the following:*

(1) $\mathbf{T}_\epsilon$ is linear, continuous and

$$\mathbf{T}_\epsilon(\varphi_1\varphi_2) = \mathbf{T}_\epsilon(\varphi_1)\mathbf{T}_\epsilon(\varphi_2),$$

for all $\varphi_1, \varphi_2 \in L^2(D \times (0, T))$.

(2) For every $\varphi \in L^2(D \times (0, T))$, we have

$$||\mathbf{T}_\epsilon(\varphi)||_{L^2(D\times(0,T);L^2(Y_x))} \leq ||\varphi||_{L^2(D\times(0,T))}\,,$$

and

$$\left|\int_0^T \int_D \phi dxdt - \int_0^T \int_D \int_{Y_x} \mathbf{T}_\epsilon(\varphi)dydxdt\right| \leq \int_{(0,T)} \int_{D\setminus\hat{\mathbb{D}}^\epsilon} \phi dxdt. \tag{3.1}$$

(3) For every $\varphi \in W^{1,2}(D \times (0,T))$, the following are true.

(a) $\mathbf{T}_\epsilon(\varphi) \to \varphi$ strongly in $L^2(D \times (0,T); L^2(Y_x))$,

(b) $\mathbf{T}_\epsilon(\nabla\varphi) \to \nabla\varphi$ strongly in $L^2(D \times (0,T); L^2(Y_x))$,

(c) $\mathbf{T}_\epsilon\left(\frac{\partial\varphi}{\partial t}\right) \to \frac{\partial\varphi}{\partial t}$ strongly in $L^2(D \times (0,T); L^2(Y_x))$.

(4) If $\phi^\epsilon \in L^2(Y_x)$ is given by $\varphi^\epsilon(x) = \varphi\left(x\epsilon^{-1}\right)$ for all $x \in \mathbb{R}^n$, then

(a)

$$\mathbf{T}_\epsilon(\varphi^\epsilon)(x,y) = \begin{cases} \varphi(y) & (x,y) \in \hat{D}^\epsilon \times Y_x, \\ \\ 0 & \textit{otherwise} . \end{cases}$$

(b) $\mathbf{T}_\epsilon(\varphi^\epsilon) \to \varphi$ strongly in $L^2(D \times (0,T); L^2(Y_x))$.

(c) Furthermore if $\nabla_y\varphi \in L^2(Y_x)$, then

$$\nabla_y(\mathbf{T}_\epsilon(\varphi\epsilon)) \to \nabla_y\varphi \textit{ strongly in } L^2(D \times Y_x).$$

(5) Let $u^\epsilon \in L^2(D \times (0,T))$ such that $u^\epsilon \to u$ strongly in $L^2(D \times (0,T))$, then

$$\mathbf{T}_\epsilon(u^\epsilon) \to u \textit{ strongly in } L^2(D \times (0,T); L^2(Y_x)). \tag{3.2}$$

(6) Let $\{u^\epsilon\}$ be a bounded sequence in $L^2(D \times (0,T))$, then

$$\mathbf{T}_\epsilon(u^\epsilon) \to u \textit{ weakly in } L^2(D \times (0,T); L^2(Y_x)), \tag{3.3}$$

$$u^\epsilon \to \mathbf{A}_\epsilon(u^\epsilon) \textit{ weakly in } L^2(D \times (0,T)). \tag{3.4}$$

(7) For $u \in L^2(D \times (0,T))$, we have

$$\epsilon^{-1}[\mathbf{O}_\epsilon(u)] \to y^*\nabla_x u \textit{ strongly in } L^2(D \times (0,T); L^2(Y_x)), \tag{3.5}$$

where $y^* = \left(y_1 - \frac{1}{2}, y_2 - \frac{1}{2}, \cdots, y_n - \frac{1}{2}\right)$.

(8) Assume that $\{u^\epsilon\}$ be abounded sequence in $L^2(0,T; W_0^{1,2}(D))$, and that

$$u^\epsilon \rightharpoonup u \textit{ weakly in } L^2(0,T; W_0^{1,2}(D)). \tag{3.6}$$

Then there exists $\hat{u} = \hat{u}(t,x,y)$ in $L^2(D_T; W_\#^{1,2}(Y_x))$, where $\mathcal{M}_{Y_x}(\hat{u}) = 0$ such that up to subsequence

$$\mathbf{T}_\epsilon(\nabla_x u^\epsilon) \rightharpoonup \nabla_x u + \nabla_y \hat{u} \textit{ weakly in } L^2(D_T; L^2(Y_x)), \tag{3.7}$$

$$\epsilon^{-1}[\mathbf{O}_\epsilon(u)] \rightharpoonup y^*\nabla_x u + \hat{u} \textit{ weakly in } L^2(D_T; W_\#^{1,2}(Y_x)). \tag{3.8}$$

4. Homogenization

In this section, we use the unfolding operator to pass to the limit in the weak formulation (1.9). Before that, let us introduce the following spaces, see [2, 11]

$$\mathcal{H} := \left\{\varphi \in L^2\left(0,T; W_0^{1,2}(D)\right) : \frac{\partial\varphi}{\partial t} \in L^2\left(0,T; L^2(D)\right)\right\},$$

$$\mathcal{H}_1 := \left\{\varphi \in L^2\left(0,T;L^2(D)\right) : \frac{\partial\varphi}{\partial t} \in L^2\left(0,T;W^{1,-1}(D)\right)\right\},$$

$$\mathcal{W}^{\Gamma}_{\#}(Y_x) := \left\{\varphi \in W^{1,2}_{\#}(Y_x) : \varphi|_{F_u} = \text{ constant }\right\},$$

$$\mathcal{W}^{\Gamma}(D_T;Y_x) := L^2((0,T)\times Y_x;W^{1,2}_0(D)) \cap W^{1,2}(D_T;L^2_{\#}(Y_x)) \cap L^2(D_T;\mathcal{W}^{\Gamma}_{\#}(Y_x)),$$

$$\mathcal{W}(D_T;Y_x) := \Big\{(v,\hat{v}) : v \in \mathcal{H}, \text{ and } \hat{v} \in L^2(D_T;W^{1,2}_{\#}(Y_x)),$$
$$\mathcal{M}_{Y_x}(\hat{v}) = 0, y^* \cdot \nabla_x v + \hat{v} \text{ is independent of y on } D_T \times F_u\Big\}.$$

Theorem 4.1. *Suppose that (H1)–(H5) hold true and v^ϵ be the solution for system (1.1). There exists a pair $(v,\hat{v}) \in \mathcal{W}(D_T;Y_x)$ such that up to sub-sequence*

$$\mathbf{T}_\epsilon(v^\epsilon) \rightharpoonup v, \text{ weakly in } L^2(D\times(0,T);L^2(Y_x)), \tag{4.1}$$

$$\mathbf{T}_\epsilon\left(\frac{\partial v^\epsilon}{\partial t}\right) \rightharpoonup \frac{\partial v}{\partial t}, \text{ weakly in } L^2(D\times(0,T);L^2(Y_x)), \tag{4.2}$$

$$\mathbf{T}_\epsilon(v^\epsilon) \to v, \text{ strongly in } L^{2(r+1)}(D_T\times Y_x), \tag{4.3}$$

$$\mathbf{T}_\epsilon(v^\epsilon) \to v, \text{ strongly in } L^2(D\times(0,T);L^2(Y_x)), \tag{4.4}$$

$$\mathbf{T}_\epsilon(\nabla_x v^\epsilon) \rightharpoonup \nabla_x v + \nabla_y\hat{v}, \text{ weakly in } L^2(D\times(0,T);L^2(F_s)), \tag{4.5}$$

$$\mathbf{T}_\epsilon(\nabla_x v^\epsilon) \rightharpoonup 0, \text{ weakly in } L^2(D\times(0,T);L^2(F_u)), \tag{4.6}$$

$$\frac{1}{\epsilon}\mathbf{O}_\epsilon(v^\epsilon) \rightharpoonup y^*\nabla_x + \hat{v}, \text{ weakly in } L^2(D\times(0,T);L^2(Y_x)), \tag{4.7}$$

$$y^*\nabla_x + \hat{v}, \text{ is independent of y on } D_T\times F_u\,. \tag{4.8}$$

Proof. Convergences (4.1) and (4.2), easily obtained using Definition 3.1, estimates (2.5) and [13, Theorem 2.19, P. 536]. Regarding convergence (4.3), we not that $v^\epsilon \in L^{2(r+1)}(0,T;W^{1,2}_0(D))$, where $1 < 2(r+1) < \frac{2n}{n-2} = 2^*$. Then, Sobolev embedding theorem [11, Theorem 3.27, P.49], shows that $W^{1,2}_0(D)$ is compactly embedded in $L^{2(r+1)}(D)$, which implies the strong convergence of v^ϵ to v in $L^{2(r+1)}(D\times(0,T)$, this gives (4.3). Similarly, we note that $v^\epsilon \in \mathcal{H}^\epsilon$ and by [11, Theorem 3.59, P. 61], $\mathcal{H}^\epsilon$ is compactly embedded in $L^2(D_T)$, which leads to (4.4). Convergences (4.5)–(4.8) obtained as in [2, Lemma 4.1, P. 1482].

Let us state the main results of this subsection.

Theorem 4.2. *Assume that (H1)–(H5) hold and*

$$\mathbf{T}_\epsilon(A^\epsilon) \to A, \text{ strongly in } L^2(D\times Y_x). \tag{4.9}$$

Then, the pair $(v,\hat{v}) \in \mathcal{W}(D_T;Y_x)$ uniquely satisfies the following system

$$\int_{D_T}\int_{F_s}\left\{\frac{\partial^2 v}{\partial t^2}\phi dydxdt + A\left(\nabla v + \nabla_y\hat{v}\right)\cdot\left(\nabla\phi + \nabla_y\hat{\phi}\right)\right\}dydxdt$$

$$+\int_{D_T}\int_{F_s}\tilde{\beta}(v,\nabla v+\nabla_y\hat{v})\phi dydxdt+\frac{\eta}{|F_u|}\int_{D_T}\int_{F_u}\left(\frac{\partial v}{\partial t}\right)\phi dydxdt$$
$$\int_{D_T}\int_{F_s}f\phi dydxdt, \tag{4.10}$$

where $(\phi,\hat{\phi})\in\mathcal{W}(D_T;Y_x)$.

We first acquire some preliminaries before establishing this result.

Lemma 4.1. *Let* $\bar{\Phi}=(\Phi_1,\Phi_2,\cdots,\Phi_n)$, *where* $\Phi_i\in C_0^\infty(D_T)\otimes C_{per}(Y_x)$ *and* v^ϵ *is the solution of system (1.1). Then*

$$\mathbf{T}_\epsilon(\beta^\epsilon)(t,x,v,\Phi^\epsilon)\rightharpoonup\beta(t,y,v,\Phi),\ \textit{weakly in } L^2\left(D_T;W^{1,2}(Y_x)\right), \tag{4.11}$$

where $\beta^\epsilon(t,x,v^\epsilon,\Phi^\epsilon)=\beta\left(t,\epsilon^{-1}x,v^\epsilon(t,x),\Phi\left(t,x,\epsilon^{-1}x\right)\right)$.

Proof. It is easy to see from estimate (2.6) that $\beta(t,y,v,\Phi)\in L^2(D_T;C_{per}(Y_x)$, we also have

$$\begin{aligned}\mathbf{T}_\epsilon(\beta^\epsilon)(t,x,v,\Phi^\epsilon)\\ =&\beta\left([t,\epsilon^{-1}x]_Y+y,\mathbf{T}_\epsilon(v^\epsilon),\mathbf{T}_\epsilon(\Phi^\epsilon)(t,x,y)\right)\\ &=\beta(t,y,\mathbf{T}_\epsilon(v^\epsilon),\mathbf{T}_\epsilon(\Phi^\epsilon)(t,x,y)).\end{aligned} \tag{4.12}$$

From (4.12) and (1.3), we obtain the following

$$\begin{aligned}&-\int_{D_T}(\mathbf{T}_\epsilon(\beta^\epsilon)(t,x,v,\Phi^\epsilon)-\beta(t,y,v,\Phi))\phi dxdt\\ &\le C_4\int_{D_T}(|\mathbf{T}_\epsilon(v^\epsilon)|^r+|v|^r)|\mathbf{T}_\epsilon(v^\epsilon)-v||\phi|dxdt\\ &+C_5\int_{D_T}|\mathbf{T}_\epsilon(\Phi^\epsilon)-\Phi||\phi|dxdt\\ &\le C_4\left(\|\mathbf{T}_\epsilon(v^\epsilon)\|^r_{L^p(D_T)}+\|v\|^r_{L^p(D_T)}\right)\|\mathbf{T}_\epsilon(v^\epsilon)-v\|_{L^p(D_T)}\|\phi\|_{L^2(D_T)}\\ &+C_5\|\mathbf{T}_\epsilon(\Phi^\epsilon)-\Phi\|_{L^2(D_T)}\|\phi\|_{L^2(D_T)},\end{aligned} \tag{4.13}$$

where we have used the generalized Holder inequality in the first term on the right hand side for $\frac{r}{p}+\frac{1}{p}+\frac{1}{2}=1$ such that $p=2(r+1)$. But,

$$\mathbf{T}_\epsilon(v^\epsilon)\to v \text{ strongly in } L^p(D_T\times Y_x), \tag{4.14}$$
$$\mathbf{T}_\epsilon(\Phi^\epsilon)\to\Phi \text{ strongly in } L^2(D_T;W^{1,2}_{\text{per}}(Y_x)). \tag{4.15}$$

Since ϕ is an arbitrary, we have

$$\lim_{\epsilon\to0}\int_{D_T}(\mathbf{T}_\epsilon(\beta^\epsilon)(t,x,v,\Phi^\epsilon)-\beta(t,y,v,\Phi))\phi dxdt=0. \tag{4.16}$$

Thus,

$$\mathbf{T}_\epsilon(\beta^\epsilon)(t,x,v^\epsilon,\Phi^\epsilon)\rightharpoonup\beta(t,y,v,\Phi)\ \text{ weakly in } L^2(D_T;W^{1,2}_{\text{per}}(\mathcal{E})). \tag{4.17}$$

Remark 4.1. *As in [25], if we let* $\Psi^\epsilon = \varphi_0(t,x) + \varphi_1\left(t, x, \frac{x}{\epsilon}\right)$*, where* $\varphi_0 \in \mathcal{D}(D_T)$ *and* $\varphi_1 \in \mathcal{D}(D_T) \otimes \mathcal{D}_{per}(Y_x)$*, we see that*

$$\mathbf{T}_\epsilon\left(\beta^\epsilon\right)\left(t, x, v^\epsilon, \nabla\Psi^\epsilon\right) \rightharpoonup \beta\left(t, y, v, \nabla\varphi_0 + \nabla_y\varphi_1\right) \text{ weakly in } L^2(D_T; W^{1,2}_{per}(\mathcal{E})). \tag{4.18}$$

Now, we state and proof the following lemma.

Lemma 4.2. *The nonlinear term* $\beta^\epsilon\left(t, x, v^\epsilon, \nabla v^\epsilon\right)$ *satisfies the following convergence*

$$\mathbf{T}_\epsilon\left(\beta^\epsilon\right)\left(t, x, v^\epsilon, \nabla\Psi^\epsilon\right) \rightharpoonup \tilde{\beta}\left(v, \nabla v + \nabla_y\hat{v}\right) \text{ weakly in } L^2(D_T; W^{1,2}_{per}(Y_x)), \tag{4.19}$$

where

$$\tilde{\beta}\left(v, \nabla v + \nabla_y\hat{v}\right) = \int_{Y_x} \beta\left(t, y, v, \nabla v + \nabla_y\hat{v}\right) dy.$$

Proof. First note that the sequence $\mathbf{T}_\epsilon\left(\beta^\epsilon\right)$ is bounded in $L^2(D_T \times Y_x)$ thus, there exists a functions $\beta^* \in L^2(D_T \times Y_x)$ such that up to sub-sequence

$$\mathbf{T}_\epsilon\left(\beta^\epsilon\right)\left(t, x, v^\epsilon, \nabla v^\epsilon\right) \rightharpoonup \beta^* \text{ weakly in } L^2(D_T \times Y_x). \tag{4.20}$$

We use (1.3) to obtain

$$\begin{aligned}
&\int_{D_T}\int_{Y_x} \left[\beta^\epsilon\left(t, y, \mathbf{T}_\epsilon\left(v^\epsilon\right), \mathbf{T}_\epsilon\left(\nabla v^\epsilon\right)\right) - \beta^\epsilon\left(t, y, \mathbf{T}_\epsilon\left(v^\epsilon\right), \mathbf{T}_\epsilon\left(\nabla\Psi^\epsilon\right)\right)\right] \\
&\qquad \cdot \left[\mathbf{T}_\epsilon\left(v^\epsilon\right) - \mathbf{T}_\epsilon\left(\Psi^\epsilon\right)\right] dxdtdy \\
&\qquad + C_5 \int_{D_T}\int_{Y_x} \left|\mathbf{T}_\epsilon\left(\nabla v^\epsilon\right) - \mathbf{T}_\epsilon\left(\nabla\Psi^\epsilon\right)\right| \left|\mathbf{T}_\epsilon\left(v^\epsilon\right) - \mathbf{T}_\epsilon\left(\Psi^\epsilon\right)\right| dxdtdy \\
&\qquad \geq 0.
\end{aligned}$$

Alternatively, to be more specific

$$\begin{aligned}
&\int_{D_T}\int_{Y_x} \left[\beta^\epsilon\left(t, y, \mathbf{T}_\epsilon\left(v^\epsilon\right), \mathbf{T}_\epsilon\left(\nabla v^\epsilon\right)\right) - \beta^\epsilon\left(t, y, \mathbf{T}_\epsilon\left(v^\epsilon\right), \mathbf{T}_\epsilon\left(\nabla\Psi^\epsilon\right)\right)\right] \\
&\cdot \left[\mathbf{T}_\epsilon\left(v^\epsilon\right) - \mathbf{T}_\epsilon\left(\Psi^\epsilon\right)\right] dxdtdy \\
&+ C_5 \left\|\mathbf{T}_\epsilon\left(\nabla v^\epsilon\right) - \mathbf{T}_\epsilon\left(\nabla\Psi^\epsilon\right)\right\|_{L^2(D_T \times Y_x)} \left\|\mathbf{T}_\epsilon\left(v^\epsilon\right) - \mathbf{T}_\epsilon\left(\Psi^\epsilon\right)\right\|_{L^2(D_T \times Y_x)} \\
&\geq 0.
\end{aligned} \tag{4.21}$$

Before passing to the limit in (4.21). We first note that by (4.3) one easily obtain

$$\left\|\mathbf{T}_\epsilon\left(v^\epsilon\right) - \mathbf{T}_\epsilon\left(\Psi^\epsilon\right)\right\|_{L^2(D_T \times Y_x)} \to \left\|v - \varphi_0\right\|_{L^2(D_T \times Y_x)}. \tag{4.22}$$

Using the same steps as in [23, Theorem 2.2], we can demonstrate that

$$\begin{aligned}
&\left\|\mathbf{T}_\epsilon\left(\nabla v^\epsilon\right) - \mathbf{T}_\epsilon\left(\nabla\Psi^\epsilon\right)\right\|_{L^2(D_T \times Y_x)} \\
&\qquad \to \left\|\left(\nabla_x v + \nabla_y\hat{v}\right) - \left(\nabla_x\varphi_0 + \nabla_y\varphi_1\right)\right\|_{L^2(D_T \times Y_x)}.
\end{aligned} \tag{4.23}$$

Now, we use (4.18), (4.20), (4.22) and (4.23) to pass to the limit in (4.21), we get

$$
\begin{aligned}
&\int_{D_T}\int_{Y_x}\left[\beta^* - \beta\left(t,y,v,\nabla\varphi_0+\nabla_y\varphi_1\right)\right][v-\varphi_0]\,dxdtdy\\
&+C_5\,\|v-\varphi_0\|_{L^2(D_T\times Y_x)}\left\|\left(\nabla_x v+\nabla_y\hat{v}\right)-\left(\nabla_x\varphi_0+\nabla_y\varphi_1\right)\right\|_{L^2(D_T\times Y_x)}\\
&\geq 0. \qquad (4.24)
\end{aligned}
$$

Take $\varphi_0 = v - \lambda u$ and $\varphi_1 = \hat{v} - \lambda\hat{u}$ where $(u,\hat{u}) \in \mathcal{W}(D_T; Y_x)$, we are led to

$$
\int_{D_T}\int_{Y_x}\left[\beta^* - \beta\left(t,y,v,(\nabla v-\lambda\nabla u)+(\nabla_y\hat{v}-\lambda\nabla_y\hat{u})\right)\right]\lambda u dxdtdy \qquad (4.25)
$$

$$
+C_5\lambda^2\|u\|_{L^2(D_T\times Y_x)}\left\|\nabla_x u+\nabla_y\hat{u}\right\|_{L^2(D_T\times Y_x)}\geq 0. \qquad (4.26)
$$

Divide both sides of the above inequality by λ and then let $\lambda \to 0$. We get

$$
\int_{D_T}\int_{Y_x}\left[\beta^* - \beta\left(y,\tau,v,\nabla v+\nabla_y\hat{v}\right)\right]udxdtdyd\tau \geq 0. \qquad (4.27)
$$

Because u was arbitrarily chosen, this completes the proof.

Proof of Theorem 4.2. Following [2], we use as test function in (1.7) $\Phi^\epsilon(t,x) = \epsilon\Phi\left(t,x,\frac{x}{\epsilon}\right)$ such that

$$
\Phi(t,x,y) = \mathbf{A}_\epsilon(\psi)(t,x)b(y) + \psi(t,x)c(y), \qquad (4.28)
$$

where $\psi \in C^\infty([0,T];\mathcal{D}(D))$, $b \in \mathcal{D}(Y_x) \cap W^\Gamma_\#(Y_x)$, $c \in C^\infty_\#(Y_x)$ and $c|_{F_v} \equiv 0$. We have

$$
\begin{aligned}
\epsilon\int_0^T\int_{D^\epsilon} & v^\epsilon\left([\mathbf{A}_\epsilon(\psi_{tt})b+\psi_{tt}c]\right)dxdt\\
&+\int_0^T\int_{D^\epsilon}A^\epsilon\nabla v^\epsilon\left(\mathbf{A}_\epsilon(\psi)\nabla_y b+\psi\nabla_y c\right)dxdt\\
&+\epsilon\int_0^T\int_{D^\epsilon}A^\epsilon\nabla v^\epsilon\left(\nabla_x\psi c\right)dxdt\\
&+\epsilon\int_0^T\int_{D^\epsilon}\beta^\epsilon\left(v^\epsilon,\nabla v^\epsilon\right)\left([\mathbf{A}_\epsilon(\psi)b+\psi c]\right)dxdt\\
&-\epsilon\frac{\gamma}{|F_u|}\int_0^T\int_{F^\epsilon}v^\epsilon\left(\mathbf{A}_\epsilon(\psi_t)b+\psi_t c\right)dxdt\\
&=\epsilon\int_0^T\int_{D^\epsilon}f\left(\mathbf{A}_\epsilon(\psi)b+\psi c\right)dxdt. \qquad (4.29)
\end{aligned}
$$

Unfolding (4.29) and using (4.1) to pass to the limit we obtain

$$
\begin{aligned}
&\int_{D_T}\int_{F_s}\mathbf{T}_\epsilon(A^\epsilon)\mathbf{T}_\epsilon(\nabla v^\epsilon)\left(\mathbf{T}_\epsilon(a)\left[\mathbf{T}_\epsilon\left(\mathbf{A}_\epsilon(\psi)\nabla_y b\right)+\mathbf{T}_\epsilon(\psi\nabla_y c)\right]\right)dydxdt\\
&\qquad\to\int_{D_T}\int_{F_s}A\left(\nabla_x v+\nabla_y\hat{v}\right)\cdot\nabla_y(b+c)\psi dydxdt. \qquad (4.30)
\end{aligned}
$$

All of the remaining terms on (4.29) converge to zero. Therefore we have

$$\int_{D_T}\int_{F_s} A\left(\nabla_x v + \nabla_y \hat{v}\right)\cdot\nabla_y(b+c)\psi dydxdt = 0. \tag{4.31}$$

See [2], this can always be written as

$$\int_{D_T}\int_{F_s} A\left(\nabla_x v + \nabla_y \hat{v}\right)\cdot\nabla_y \Psi dydxdt = 0. \tag{4.32}$$

Where $\Psi \in \mathcal{W}^\Gamma(D_T; Y_x)$. Now, we replace our test function by $\Phi^\epsilon(t,x) = \Phi\left(t, x, \frac{x}{\epsilon}\right)$ such that

$$\Phi\,(t,x,y) = \mathbf{A}_\epsilon(\psi)(t,x)b\,(y) + \psi(t,x)\,(1-b\,(y))\,, \tag{4.33}$$

where ψ and b as in (4.28) and $b|_{F_v} \equiv 1$. From this it is clear that

$$\Phi^\epsilon \to \psi \text{ strongly in } L^2(D_T \times Y_x). \tag{4.34}$$

Then we have

$$\begin{aligned}
&\int_0^T\int_{D^\epsilon} v^\epsilon\left(\mathbf{A}_\epsilon(\psi_{tt})b + \psi_{tt}(1-b)\right)dxdt \\
&\quad+\int_0^T\int_{D^\epsilon} A^\epsilon\nabla v^\epsilon\cdot\left(\epsilon^{-1}\left[\mathbf{A}_\epsilon(\psi)-\psi\right]\nabla_y b + \nabla_x\psi(1-b)\right)dxdt \\
&\quad+\int_0^T\int_{D^\epsilon} \beta^\epsilon\left(v^\epsilon,\nabla v^\epsilon\right)\left(\mathbf{A}_\epsilon(\psi)b + \psi(1-b)\right)dxdt \\
&\quad-\frac{\gamma}{|F_u|}\int_0^T\int_{F^\epsilon} v^\epsilon\left(\mathbf{A}_\epsilon(\psi_t)b + \psi_t(1-b)\right)dxdt \\
&\quad=\int_0^T\int_{D^\epsilon} f\left(\mathbf{A}_\epsilon(\psi)b + \psi(1-b)\right)dxdt.
\end{aligned} \tag{4.35}$$

Let us unfold (4.35) and pass to the limit as ϵ goes to zero.

$$\begin{aligned}
&\int_{D_T}\int_{F_s} \mathbf{T}_\epsilon(v^\epsilon)\left(\mathbf{T}_\epsilon(\mathbf{A}_\epsilon(\psi_{tt})b) + \mathbf{T}_\epsilon(\psi_{tt}(1-b))\right)dydxdt \\
&\qquad\qquad\to \int_{D_T}\int_{F_s} v\psi_{tt}dydxdt,
\end{aligned} \tag{4.36}$$

where, we have used (4.1) and (4.34). For the second term on the R.H.S, we have

$$\begin{aligned}
&\int_{D_T}\int_{F_s}\mathbf{T}_\epsilon(A^\epsilon)\mathbf{T}_\epsilon(\nabla v^\epsilon) \\
&\quad\cdot\left(\epsilon^{-1}\left[\mathbf{A}_\epsilon(\psi)-\mathbf{T}_\epsilon(\psi)\right]\mathbf{T}_\epsilon(\nabla_y b) + \mathbf{T}_\epsilon(\nabla_x\psi(1-b))\right)dydxdt \\
&\quad\to \int_{D_T}\int_{F_s} A\left(\nabla_x v + \nabla_y\hat{v}\right)\cdot\left(\nabla_x\psi - \nabla_y\left((y^*\cdot\nabla_x\psi)\,b\right)\right)dydxdt,
\end{aligned} \tag{4.37}$$

where we have used (3.5), (4.5) and (4.9). As for the nonlinear term, we have

$$\begin{aligned}\int_{D_T}\int_{F_s} \mathbf{T}_\epsilon(\beta^\epsilon(v^\epsilon, \nabla v^\epsilon))\,(\mathbf{A}_\epsilon(\psi)\mathbf{T}_\epsilon(b) + \mathbf{T}_\epsilon(\psi(1-b)))\,dydxdt \\ \to \int_{D_T}\int_{F_s} \beta\left(t, y, v, \nabla v + \nabla_y \hat{v}\right)\psi dydxdt,\end{aligned} \tag{4.38}$$

where we have used (4.19) and (4.34). The limit for the boundary term is given by

$$\begin{aligned}\frac{\gamma}{|F_u|}\int_{D_T}\int_{F_u} \mathbf{T}_\epsilon(v^\epsilon)\,(\mathbf{A}_\epsilon\mathbf{T}_\epsilon((\psi_t)b) + \mathbf{T}_\epsilon(\psi_t(1-b)))\,dydxdt \\ \to \frac{\gamma}{|F_u|}\int_{D_T}\int_{F_u} v\psi_t dydxdt.\end{aligned} \tag{4.39}$$

We also have

$$\int_0^T\int_{D^\epsilon} f\,(\mathbf{A}_\epsilon(\psi)b + \psi(1-b))\,dxdt \to \int_{D_T}\int_{F_s} f\psi dydxdt. \tag{4.40}$$

Combining all the above convergences namely, (4.32), (4.36)–(4.40), we obtain the following system

$$\begin{aligned}&\int_{D_T}\int_{F_s} \frac{\partial^2 v}{\partial t^2}\psi dydxdt \\ &+ \int_{D_T}\int_{F_s} A\left(\nabla_x v + \nabla_y \hat{v}\right)\cdot\left(\nabla_x\psi + \nabla_y\left(\Psi - (y^*\cdot\nabla_x\psi)\,b\right)\right)dydxdt \\ &+ \int_{D_T}\int_{F_s} \beta\left(t, y, v, \nabla v + \nabla_y\hat{v}\right)\psi dydxdt + \frac{\gamma}{|F_u|}\int_{D_T}\int_{F_u}\left(\frac{\partial v}{\partial t}\right)\psi dydxdt \\ &= \int_{D_T}\int_{F_s} f\psi dydxdt.\end{aligned} \tag{4.41}$$

This corresponds to (4.10), when putting

$$\hat{\psi}(t, x, y) = \Psi(t, x, y) - (y^*\cdot\nabla_x\psi(t, x))\,b(y) - \int_y \Psi(t, x, y) - (y^*\cdot\nabla_x\psi(t, x))\,b(y)dy.$$

Part (e) of assumption (H3) and estimate (2.2), gives the uniqueness of $(v, \hat{v}) \in \mathcal{W}(D_T; Y_x)$. This gives the convergence of the whole sequence instead of sub-sequence. Following [2, 6, 10], we state that problem (4.32) has the following unique solution

$$\hat{v} = \chi(x, y)\cdot\nabla_x v(t, x), \tag{4.42}$$

such that $\chi(x, y)$, is the corrector of the first-order that uniquely solve the following:

$$\begin{aligned}-\mathrm{div}A(x, y)\nabla_y\left(\chi(x, y) - y\right) &= 0 \text{ in } Y, \\ \mathcal{M}_Y(\chi) &= 0, \\ \chi(x, y) - y \text{ is independent of } y &\text{ on } F_u.\end{aligned} \tag{4.43}$$

By substituting (4.42) into (4.41), we arrive at

$$\begin{aligned}
&\int_{D_T}\int_{F_s}\frac{\partial^2 v}{\partial t^2}\psi dydxdt\\
&\quad+\int_{D_T}\int_{F_s}A_0\nabla_x v\cdot\nabla_x\psi dydxdt\\
&\quad+\int_{D_T}\int_{F_s}\beta\left(v,\nabla v\right)\psi dydxdt+\frac{\gamma}{|F_u|}\int_{D_T}\int_{F_u}\left(\frac{\partial v}{\partial t}\right)\psi dydxdt\\
&\quad=\int_{D_T}\int_{F_s}f\psi dydxdt,
\end{aligned}\tag{4.44}$$

where A_0 is coercive and elliptic matrix given by

$$A_0(x)=\int_Y A(x,y)\nabla_y\left(\chi(x,y)-y\right)\cdot\nabla_y\left(\chi(x,y)-y\right)dy,$$

with this in hand one can easily obtain the strong formulation of (4.44) as the following nonlinear damped wave equation

$$|F_s|\frac{\partial^2 v}{\partial t^2}+\gamma\frac{\partial v}{\partial t}-\operatorname{div}(A_0\nabla v)+\tilde{\beta}\left(v,\nabla v\right)=|F_s|f,\tag{4.45}$$

where

$$\tilde{\beta}\left(v,\nabla v\right)=\int_{F_s}\beta\left(v,\nabla v\right)dy.$$

The initial conditions $v(x,0)=\frac{\partial v}{\partial t}(x,0)=0$ are obtained in the standard way.

Conclusions

Nonlinear hyperbolic problem with special oscillated coefficients and non-local boundary conditions involving the solution's time derivative in perforated domain was considered in this paper. To obtain homogenization results, we used the unfolding periodic operator. The following are the paper's main challenges:

- The non-linearity presented on the function $\beta^\epsilon(v^\epsilon,\nabla v^\epsilon)$, which was introduced in [8] for studying existence and uniqueness for nonlinear hyperbolic problem and in [25] when dealing with homogenization of nonlinear hyperbolic problem with Dirichlet condition in a fixed domain. In this paper we used the concept unfolding operator to pass to the limit in this function in Lemma 4.2.
- In order to address the non-local boundary condition $\gamma\frac{\partial v^\epsilon}{\partial t}=\frac{1}{\epsilon^n}\int_{\Gamma_\eta^\epsilon}\frac{\partial v^\epsilon}{\partial \nu_{\epsilon A^\epsilon}}d\sigma_x$, which prevents the use of unfolding techniques in the conventional sense, we followed the settings in [10] and [2].

With theses, we obtained a damped nonlinear hyperbolic problem in fixed domain.

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Conflict of interest

The author declares that he has no conflict of interest.

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