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Research article

η-Ricci-Bourguignon solitons with a semi-symmetric metric and semi-symmetric non-metric connection

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Abstract: We consider a generalization of a Ricci soliton as η-Ricci-Bourguignon solitons on a Riemannian manifold endowed with a semi-symmetric metric and semi-symmetric non-metric connection. We find some properties of η-Ricci-Bourguignon soliton on Riemannian manifolds equipped with a semi-symmetric metric and semi-symmetric non-metric connection when the potential vector field is torse-forming with respect to a semi-symmetric metric and semi-symmetric non-metric connection.

Keywords: η-Ricci-Bourguignon soliton; semi-symmetric metric connection; semi-symmetric non-metric connection; torse-forming vector field; quasi-Einstein manifold; hyper-generalized quasi-Einstein manifold
Mathematics Subject Classification: 35Q51, 53C07, 53C25

1. Introduction

A semi-symmetric connection is a linear connection on a Riemannian manifold (M, g) whose torsion tensor T is of the form

$$T(\zeta_1, \zeta_2) = \phi(\zeta_2)\zeta_1 - \phi(\zeta_1)\zeta_2,$$

where ϕ is a 1-form defined by $\phi(\zeta_1) = g(\zeta_1, U)$ and U is a vector field on M [11].

If ∇ is the Levi-Civita connection of a Riemannian manifold (M, g), then the semi-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{\nabla}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + \phi(\zeta_2)\zeta_1 - g(\zeta_1, \zeta_2)U, \tag{1.1}$$

where ζ_1, ζ_2, U are vector fields on M [20]. Let $\widetilde{R}$ and R denote Riemannian curvature tensor fields of $\widetilde{\nabla}$ and ∇, respectively. Then from (1.1), it is easy to see that

$$\widetilde{R}(\zeta_1, \zeta_2)\zeta_3 = R(\zeta_1, \zeta_2)\zeta_3 - \alpha(\zeta_2, \zeta_3)\zeta_1 + \alpha(\zeta_1, \zeta_3)\zeta_2 \tag{1.2}$$

$$-g(\zeta_2,\zeta_3)B\zeta_1 + g(\zeta_1,\zeta_3)B\zeta_2,$$

where

$$\alpha(\zeta_1,\zeta_2) = g(B\zeta_1,\zeta_2) = (\nabla_{\zeta_1}\phi)\zeta_2 - \phi(\zeta_1)\phi(\zeta_2) + \frac{1}{2}g(\zeta_1,\zeta_2). \tag{1.3}$$

Denote by $\widetilde{Ric}$ and Ric the Ricci tensor fields of the connections $\widetilde{\nabla}$ and ∇, respectively. Then from (1.2), it is easy to see that

$$\widetilde{Ric} = Ric - (n-2)\alpha - (tr\alpha)g, \text{(see [20])}. \tag{1.4}$$

The semi-symmetric non-metric connection $\overset{\circ}{\widetilde{\nabla}}$ is defined by

$$\overset{\circ}{\widetilde{\nabla}}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + \phi(\zeta_2)\zeta_1, \tag{1.5}$$

where ζ_1, ζ_2 are vector fields on M and ∇ is the Levi-Civita connection of a Riemannian manifold (M,g) [1]. Let $\overset{\circ}{\widetilde{R}}$ and R denote the Riemannian curvature tensor fields of $\overset{\circ}{\widetilde{\nabla}}$ and ∇, respectively. Then from (1.5), it is easy to see that

$$\overset{\circ}{\widetilde{R}}(\zeta_1,\zeta_2)\zeta_3 = R(\zeta_1,\zeta_2)\zeta_3 - \sigma(\zeta_2,\zeta_3)\zeta_1 + \sigma(\zeta_1,\zeta_3)\zeta_2, \tag{1.6}$$

where

$$\sigma(\zeta_1,\zeta_2) = g(B\zeta_1,\zeta_2) = (\nabla_{\zeta_1}\phi)\zeta_2 - \phi(\zeta_1)\phi(\zeta_2). \tag{1.7}$$

Denote by $\overset{\circ}{\widetilde{Ric}}$ and Ric the Ricci tensor fields of the connections $\overset{\circ}{\widetilde{\nabla}}$ and ∇, respectively. Then from (1.6), we have

$$\overset{\circ}{\widetilde{Ric}} = Ric - (n-1)\sigma, \text{(see [1])}. \tag{1.8}$$

Let (M,g) be a Riemannian manifold. R. S. Hamilton [12] presented the Ricci flow for the first time as

$$\tfrac{\partial}{\partial t}g(t) = -2Ric(g(t)).$$

The Ricci flow is an evolution equation for Riemannian metrics. Ricci solitons correspond to self-similar solutions of Ricci flow. In the recent years, the geometry of Ricci solitons has been studied by many geometers. See, for example, [3, 8, 15, 17].

Another generalization of Ricci soliton is η-Ricci-Bourguignon soliton. An η-Ricci-Bourguignon soliton (see [18]) is defined by

$$\frac{1}{2}\pounds_\lambda g + Ric = (\alpha^* + \beta\tau)g + \gamma\eta\otimes\eta, \tag{1.9}$$

where λ is the potential vector field, η is a 1-form on M, $\pounds_\lambda g$ denotes the Lie derivative of g in the direction of λ, Ric is the Ricci curvature, τ is scalar curvature and α^*, β, γ are real numbers. η-Ricci-Bourguignon solitons on submanifolds were studied in [5].

In the present study, we consider some properties of η-Ricci-Bourguignon soliton on Riemannian manifolds equipped with a semi-symmetric metric connection and semi-symmetric non-metric connection when the potential vector field is torse-forming with respect to a semi-symmetric metric connection and semi-symmetric non-metric connection. As recent studies on torse-forming vector fields see [4, 14, 15].

The paper is organized as follows: In Section 2, η-Ricci-Bourguignon solitons on Riemannian manifolds with a semi-symmetric metric connection are studied. In Section 3, η-Ricci-Bourguignon solitons on Riemannian manifolds endowed with a semi-symmetric non-metric connection is considered.

2. η-Ricci-Bourguignon solitons on Riemannian manifolds equipped with a semi-symmetric metric connection

In this section, we consider Ricci solitons on Riemannian manifolds endowed with a semi-symmetric metric connection.

The Euclidean 3-space, hyperbolic 3-space and Minkowski motion group are included in the following 3-parameter family of Riemannian homogeneous spaces $(\mathbb{R}^3, g\,[\mu_1,\mu_2,\mu_3])$ with left invariant metric

$$g\,[\mu_1,\mu_2,\mu_3] = e^{-2\mu_1 t}dx^2 + e^{-2\mu_2 t}dy^2 + \mu_3^2 dt^2.$$

Here μ_1,μ_2 are real constants and μ_3 is a positive constant.

The Lie group $G(\mu_1,\mu_2,\mu_3)$ can be realised as a closed subgroup of affine transformation group $GL_3\mathbb{R} \ltimes \mathbb{R}^3$ of $\mathbb{R}^3$.

The Levi-Civita connection ∇ of $G(\mu_1,\mu_2,\mu_3)$ is given by the following formula:

$$\begin{gathered}\nabla_{E_1}E_1 = \frac{\mu_1}{\mu_3}E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -\frac{\mu_1}{\mu_3}E_1,\\ \nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = \frac{\mu_2}{\mu_3}E_3,\ \nabla_{E_2}E_3 = -\frac{\mu_2}{\mu_3}E_2,\\ \nabla_{E_3}E_1 = \nabla_{E_3}E_2 = \nabla_{E_3}E_3 = 0.\end{gathered} \tag{2.1}$$

The Ricci tensor field Ric of G is given by

$$R_{11} = -\frac{\mu_1(\mu_1+\mu_2)}{\mu_3^2},\ R_{22} = -\frac{\mu_2(\mu_1+\mu_2)}{\mu_3^2}, R_{33} = -\frac{\mu_1^2+\mu_2^2}{\mu_3^2}$$

and the scalar curvature τ of G is given by

$$\tau = -\frac{2}{\mu_3^2}\left(\mu_1^2+\mu_2^2+\mu_1\mu_2\right).\text{ (see [13]).}$$

Using (2.1), the Levi-Civita connection ∇ of $G(-1,1,1)$ is given by the following formula:

$$\begin{gathered}\nabla_{E_1}E_1 = -E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = E_1,\\ \nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = E_3,\ \nabla_{E_2}E_3 = -E_2,\\ \nabla_{E_3}E_1 = \nabla_{E_3}E_2 = \nabla_{E_3}E_3 = 0.\end{gathered}$$

Then we can state the following example:

Example 1. *Assume that $\lambda = 2\sqrt{2}E_2+4E_3$ is the potential vector field. If η is the 1-form corresponding to the vector field $P = \sqrt{2}E_2 + 2E_3$, then $G(-1, 1, 1)$ is an η-Ricci-Bourguignon soliton with respect to a semi-symmetric metric connection.*

Using the Eq (1.1), we get the Lie derivative as follows

$$(\pounds_\lambda g)(\zeta_1, \zeta_2) = g(\widetilde{\nabla}_{\zeta_1}\lambda, \zeta_2) + g(\zeta_1, \widetilde{\nabla}_{\zeta_2}\lambda) - 2\phi(\lambda)g(\zeta_1, \zeta_2) + g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1). \tag{2.2}$$

Therefore, using Eq (2.2), the soliton Eq (1.9) with respect to a semi-symmetric metric connection can be written as

$$\frac{1}{2}\left(g(\widetilde{\nabla}_{\zeta_1}\lambda, \zeta_2) + g(\zeta_1, \widetilde{\nabla}_{\zeta_2}\lambda)\right) - \phi(\lambda)g(\zeta_1, \zeta_2) + \frac{1}{2}\left(g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1)\right) + Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau)g(\zeta_1, \zeta_2) + \gamma\eta(\zeta_1)\eta(\zeta_2). \tag{2.3}$$

A vector field λ on a Riemannian manifold (M, g) is called torse-forming [19], if

$$\nabla_{\zeta_1}\lambda = c\zeta_1 + \varpi(\zeta_1)\lambda,$$

where c is a smooth function, ϖ is a 1-form and ∇ is the Levi-Civita connection of g.

Specifically, if $\varpi = 0$, then λ is called a concircular vector field [10] and if $c = 0$, then λ is called a recurrent vector field [17].

A non-flat Riemannian manifold (M, g) $(n \geq 3)$ is called a hyper-generalized quasi-Einstein manifold [16], if its Ricci tensor field is not likewise zero and provides

$$Ric = b_1 g + b_2\omega_1 \otimes \omega_1 + b_3(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + b_4(\omega_1 \otimes \omega_3 + \omega_3 \otimes \omega_1),$$

where b_1, b_2, b_3 and b_4 are functions and ω_1, ω_2 and ω_3 are non-zero 1-forms. If $b_4 = 0$, then M is called a generalized quasi-Einstein manifold in the sense of Chaki [7]. If $b_3 = b_4 = 0$, then M is called a quasi-Einstein manifold [6]. Suppose that $b_2 = b_3 = b_4 = 0$, then (M, g) is an Einstein manifold [2]. The functions b_1, b_2, b_3 and b_4 are called associated functions.

A non-flat Riemannian manifold (M, g) $(n \geq 3)$ is called a generalized quasi-Einstein manifold in the sense of De and Ghosh [9], if its Ricci tensor field is not identically zero and satisfies

$$Ric = b_1 g + b_2\omega_1 \otimes \omega_1 + b_3\omega_2 \otimes \omega_2,$$

where b_1, b_2 and b_3 are functions. The functions b_1, b_2 and b_3 are called associated functions.

Now let (M, g) be a Riemannian manifold equipped with a semi-symmetric metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric metric connection on M. Then $\widetilde{\nabla}_{\zeta_1}\lambda = c\zeta_1 + \varpi(\zeta_1)\lambda$. So by (2.3), we can write

$$Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2) - \frac{1}{2}\{g(\zeta_2, \lambda)\varpi(\zeta_1) + g(\zeta_1, \lambda)\varpi(\zeta_2)\} - \frac{1}{2}\{g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1)\} + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

Thus, the following theorem can be stated:

Theorem 1. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric metric connection on M. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if*

$$Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2) \tag{2.4}$$

$$-\frac{1}{2}\{g(\zeta_2, \lambda)\varpi(\zeta_1) + g(\zeta_1, \lambda)\varpi(\zeta_2)\}$$
$$-\frac{1}{2}\{g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1)\} + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

If λ is a concircular potential vector field with respect to a semi-symmetric metric connection, then the following corollaries can be stated:

Corollary 1. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection and λ a concircular potential vector field with respect to a semi-symmetric metric connection on M. If (M, g) is an η-Ricci-Bourguignon soliton and ϕ is the g dual of λ, then M is a generalized quasi Einstein manifold in the sense of De and Ghosh with associated functions $\left(\alpha^* + \beta\tau - c + \|\lambda\|^2\right), -1$ and γ.*

Corollary 2. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection and λ a concircular potential vector field with respect to a semi-symmetric metric connection on M. If (M, g) is an η-Ricci-Bourguignon soliton and η is the g dual of λ, then M is a generalized quasi Einstein manifold in the sense of Chaki with associated functions $(\alpha^* + \beta\tau - c + \phi(\lambda)), \gamma$ and $-\frac{1}{2}$.*

Now assume that λ is a torse-forming potential vector field and the 1-form η is the g-dual of λ. Then from (2.4), we have

$$Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2)$$
$$-\frac{1}{2}\{\eta(\zeta_1)\varpi(\zeta_2) + \eta(\zeta_2)\varpi(\zeta_1)\}$$
$$-\frac{1}{2}\{\eta(\zeta_1)\phi(\zeta_2) + \eta(\zeta_2)\phi(\zeta_1)\} + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

Then we obtain:

Theorem 2. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric metric connection on M. Suppose that the 1-form η is the g dual of λ. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if M is a hyper-generalized quasi-Einstein manifold with associated functions $(\alpha^* + \beta\tau - c + \phi(\lambda)), \gamma, -\frac{1}{2}$ and $-\frac{1}{2}$.*

Using (1.4), the Eq (2.4) can be written as

$$\widetilde{Ric}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda) - tr\alpha)\, g(\zeta_1, \zeta_2)$$
$$-\frac{1}{2}\{g(\zeta_1, \lambda)\varpi(\zeta_2) + g(\zeta_2, \lambda)\varpi(\zeta_1)\} - \frac{1}{2}\{g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1)\}$$
$$+ \gamma\eta(\zeta_1)\eta(\zeta_2) - (n-2)\alpha(\zeta_1, \zeta_2). \tag{2.5}$$

Thus, the following corollary can be expressed:

Corollary 3. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric metric connection on M. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if the Ricci tensor field of a semi-symmetric metric connection is of the form (2.5).*

Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection, i.e., $\nabla U = 0$ and $\|U\| = 1$. Then

$$(\nabla_{\zeta_1}\phi)\zeta_2 = \nabla_{\zeta_1}\phi(\zeta_2) - \phi(\nabla_{\zeta_1}\zeta_2) = 0.$$

So from (1.3), $\alpha(\zeta_1, \zeta_2) = -\phi(\zeta_1)\phi(\zeta_2) + \frac{1}{2}g(\zeta_1, \zeta_2)$ and $tr\alpha = \frac{n}{2} - 1$. Thus by (2.5), we have

$$\widetilde{Ric}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda) - n + 2)g(\zeta_1, \zeta_2)$$

$$-\frac{1}{2}\{g(\zeta_1, \lambda)\varpi(\zeta_2) + g(\zeta_2, \lambda)\varpi(\zeta_1)\} - \frac{1}{2}\{g(\zeta_1, \lambda)\phi(\zeta_2) + g(\zeta_2, \lambda)\phi(\zeta_1)\}$$

$$+\gamma\eta(\zeta_1)\eta(\zeta_2) + (n - 2)\,\phi(\zeta_1)\phi(\zeta_2).$$

If λ is a concircular potential vector field and ϕ is the g dual of λ, then

$$\widetilde{Ric}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \|\lambda\|^2 - n + 2)g(\zeta_1, \zeta_2)$$

$$+\gamma\eta(\zeta_1)\eta(\zeta_2) + (n - 3)\,\phi(\zeta_1)\phi(\zeta_2).$$

Hence we have:

Theorem 3. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric metric connection, U a parallel unit vector field with respect to the Levi-Civita connection ∇ and λ a concircular potential vector field with respect to a semi-symmetric metric connection on M. Suppose that the 1-form ϕ is the g-dual of λ. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if M is a generalized quasi-Einstein manifold in the sense of De and Ghosh with respect to a semi-symmetric metric connection with associated functions $\left(\alpha^* + \beta\tau - c + \|\lambda\|^2 - n + 2\right), \gamma, (n - 3)$.*

3. η-Ricci-Bourguignon on Riemannian manifolds equipped with a semi-symmetric non-metric connection

In this section, we consider Ricci solitons on Riemannian manifolds endowed with a semi-symmetric non-metric connection.

Using (2.1), the Levi-Civita connection ∇ of $G(-1, -1, 1)$ is given by the following formula:

$$\nabla_{E_1}E_1 = -E_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = E_1,$$

$$\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = -E_3,\ \nabla_{E_2}E_3 = E_2,$$

$$\nabla_{E_3}E_1 = \nabla_{E_3}E_2 = \nabla_{E_3}E_3 = 0.$$

Then we can state the following example:

Example 2. *Assume that $\lambda = 2E_1 - 3E_2 + E_3$ is the potential vector field. If η is the 1-form corresponding to the vector field $P = 2E_3$, then $G(-1, -1, 1)$ is an η-Ricci-Bourguignon soliton with respect to a semi-symmetric non-metric connection.*

Using the Eq (1.5), we get the Lie derivative as follows

$$(\pounds_\lambda g)(\zeta_1, \zeta_2) = g(\overset{\circ}{\tilde{\nabla}}_{\zeta_1}\lambda, \zeta_2) + g(\zeta_1, \overset{\circ}{\tilde{\nabla}}_{\zeta_2}\lambda) - 2\phi(\lambda)g(\zeta_1, \zeta_2). \tag{3.1}$$

Therefore, using Eq (3.1), the soliton Eq (1.9) with respect to a semi-symmetric non-metric connection can be written as

$$\frac{1}{2}\left(g(\overset{\circ}{\tilde{\nabla}}_{\zeta_1}\lambda, \zeta_2) + g(\zeta_1, \overset{\circ}{\tilde{\nabla}}_{\zeta_2}\lambda)\right) \tag{3.2}$$
$$+Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau + \phi(\lambda))g(\zeta_1, \zeta_2) + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

Let us suppose that U is a parallel unit vector field with respect to the Levi-Civita connection ∇. Using (1.5), we get

$$\overset{\circ}{\tilde{\nabla}}_{\zeta_1}U = \zeta_1.$$

Thus we have:

Proposition 1. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection. If U is a parallel unit vector field with respect to the Levi-Civita connection ∇, then U is a torse-forming potential vector field with respect to a semi-symmetric non-metric connection of the form $\overset{\circ}{\tilde{\nabla}}_{\zeta_1}U = \zeta_1$.*

Now let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric non-metric connection on M. Then $\overset{\circ}{\tilde{\nabla}}_{\zeta_1}\lambda = c\zeta_1 + \varpi(\zeta_1)\lambda$. So by (3.2), we can write

$$Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2) \tag{3.3}$$
$$-\frac{1}{2}\{g(\zeta_2, \lambda)\varpi(\zeta_1) + g(\zeta_1, \lambda)\varpi(\zeta_2)\} + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

Thus, the following theorem can be expressed:

Theorem 4. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection and λ a torse-forming potential vector field with respect a semi-symmetric non-metric connection on M. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if the Ricci tensor field of the Levi-Civita connection is of the form (3.3).*

If λ is a concircular potential vector field with respect a semi-symmetric non-metric connection, then the following corollary can be stated:

Corollary 4. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection and λ a concircular potential vector field with respect to a semi-symmetric non-metric connection on M. If (M, g) is an η-Ricci-Bourguignon soliton, then M is a generalized quasi-Einstein manifold in the sense of De and Ghosh with associated functions $(\alpha^* + \beta\tau - c + \phi(\lambda))$ and γ.*

Now let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric non-metric connection on M. Then (M, g) is an η-Ricci-Bourguignon soliton and η is the g dual of λ if and only if

$$Ric(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2) \tag{3.4}$$

$$-\frac{1}{2}\{\eta(\zeta_2)\varpi(\zeta_1) + \eta(\zeta_1)\varpi(\zeta_2)\} + \gamma\eta(\zeta_1)\eta(\zeta_2).$$

Thus, the following theorem can be stated:

Theorem 5. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection and λ a torse-forming potential vector field with respect to a semi-symmetric non-metric connection on M. Suppose that the 1-form η is the g dual of λ. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if M is a hyper-generalized quasi-Einstein manifold with associated functions $(\alpha^* + \beta\tau - c + \phi(\lambda))\,, \gamma, -\frac{1}{2}$ and 0.*

Using (1.8), the Eq (3.3) can be written as

$$\overset{\circ}{\widetilde{Ric}}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))\, g(\zeta_1, \zeta_2)$$

$$-\frac{1}{2}\{g(\zeta_1, \lambda)\varpi(\zeta_2) + g(\zeta_2, \lambda)\varpi(\zeta_1)\} + \gamma\eta(\zeta_1)\eta(\zeta_2) - (n-1)\sigma(\zeta_1, \zeta_2). \tag{3.5}$$

Now assume that U is a parallel unit vector field with respect to the Levi-Civita connection, i.e., $\nabla U = 0$ and $||U|| = 1$. Then

$$(\nabla_{\zeta_1}\phi)\zeta_2 = \nabla_{\zeta_1}\phi(\zeta_2) - \phi(\nabla_{\zeta_1}\zeta_2) = 0.$$

So from (1.7), $\sigma(\zeta_1, \zeta_2) = -\phi(\zeta_1)\phi(\zeta_2)$. Thus by (3.5), we have

$$\overset{\circ}{\widetilde{Ric}}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))g(\zeta_1, \zeta_2)$$

$$-\frac{1}{2}\{g(\zeta_1, \lambda)\varpi(\zeta_2) + g(\zeta_2, \lambda)\varpi(\zeta_1)\} + \gamma\eta(\zeta_1)\eta(\zeta_2) + (n-1)\,\phi(\zeta_1)\phi(\zeta_2).$$

If λ is a concircular potential vector field , then

$$\overset{\circ}{\widetilde{Ric}}(\zeta_1, \zeta_2) = (\alpha^* + \beta\tau - c + \phi(\lambda))g(\zeta_1, \zeta_2)$$

$$+\gamma\eta(\zeta_1)\eta(\zeta_2) + (n-1)\,\phi(\zeta_1)\phi(\zeta_2).$$

Hence we get:

Theorem 6. *Let (M, g) be a Riemannian manifold endowed with a semi-symmetric non-metric connection, U a parallel unit vector field with respect to the Levi-Civita connection ∇ and λ a concircular potential vector field with respect a semi-symmetric non-metric connection on M. Then (M, g) is an η-Ricci-Bourguignon soliton if and only if M is a generalized quasi-Einstein manifold in the sense of De and Ghosh with respect to a semi-symmetric non-metric connection with associated functions $(\alpha^* + \beta\tau - c + \phi(\lambda))\,, \gamma$ and $(n-1)$.*

Conflict of interest

The author declares no conflict of interest.

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