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*Research article*

# Sine hyperbolic fractional orthotriple linear Diophantine fuzzy aggregation operator and its application in decision making

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**Abstract:** The idea of sine hyperbolic fractional orthotriple linear Diophantine fuzzy sets (sinh-FOLDFSs), which allows more uncertainty than fractional orthotriple fuzzy sets (FOFSs) is noteworthy. The regularity and symmetry of the origin are maintained by the widely recognized sine hyperbolic function, which satisfies the experts' expectations for the properties of the multi-time process. Compared to fractional orthotriple linear Diophantine fuzzy sets, sine hyperbolic fractional orthotriple linear Diophantine fuzzy sets (sinh-FOLDFSs) provide a significant idea for enabling more uncertainty. The objective of this research is to provide some reliable sine hyperbolic operational laws for FOLDFSs in order to sustain these properties and the significance of sinh-FOLDFSs. Both the accuracy and score functions for the sinh-FOLDFSs are defined. We define a group of averaging and geometric aggregation operators on the basis of algebraic t-norm and t-conorm operations. The basic characteristics of the defined operators are studied. Using the specified aggregation operators, a group decision-making method for solving real-life decision-making problem is proposed. To verify the validity of the proposed method, we compare our method with other existing methods.

**Keywords:** fractional orthotriple fuzzy sets; sine hyperbolic fractional orthotriple linear Diophantine fuzzy number; sin hyperbolic fractional orthotriple linear Diophantine fuzzy aggregation operators
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## 1. Introduction

Crisp values are being introduced in the actual use of multi-criteria decision-making (MCDM) to broadly depict the information. Due to this, fuzzy sets [14, 20], intuitionistic fuzzy sets [10, 21, 36], and Pythagorean fuzzy sets [7], among others, were used in the fuzzy-rough set [11], and the interval-valued Pythagorean fuzzy set [42]. To deal with uncertainty, Zadeh's fuzzy set theory [44] was widely applied. Atanassov [1] defined the concept of intuitionistic fuzzy set (IFS), in which the total of the membership degree (MD) and non-membership degree (NMG) is less than or equal to one. After that, if the sum of the membership degree and nun-membership degree is greater than one but their sum of squares is less than or equal to one, there may occasionally be erroneous data. Yager [39] suggested a Pythagorean fuzzy set (PFS), a more adaptable and versatile ambiguity-expressing alternative to intuitionistic fuzzy set (IFS). Zhang and Xu [45] created the concept of Pythagorean fuzzy number (PFN). Additionally, they suggested the Pythagorean fuzzy TOPSIS approach, the PFS detailed mathematical form, and an order preference strategy that was similar to the ideal result. This strategy was used within PFNs to address the MCDM problem. A Pythagorean fuzzy maximum and minimum strategy was created by Peng and Yang [24] to approach the multi-criteria group decision-making (MCGDM) problem. They also advocated the procedures of PFN addition and subtraction. Reformat & Yager [30] employed the PFNs to manage the jointly proposed system. The novel generalized Pythagorean fuzzy aggregation operators (AOs), based on Einstein operation, were proposed by Garg in [8]. Albu et al. [5] provided a detailed description of artificial neural networks used in medical applications for control and decision-making.

Yager, [40] developed q-rung orthopair fuzzy sets (q-ROFSs). However, the total of the $q^{th}$ powers of membership degree (MD) and non-membership degree (NMD), i.e., $\mu^{q}_{\acute{E}}(\hat{s}) + \nu^{q}_{\acute{E}}(\hat{s}) \leq 1$, which is less than or equal to 1. As compared to q-ROFS, PyFSs is more IFS-focused. Liu and Wang [17] defined the q-ROF weighted average and geometric AOs. Wei et al. [34] defined a few q-ROF Heronian mean (HM) operators. Ali [2] two novel algorithms were presented to deal with q-ROFSs. Yager et al. [41] explored the notions of probability and certainty as well as plausibility and belief in the q-ROFS data. Yang and Pang [43] defined partitioned BM operators using q-ROF data. Xu et al. [36] defined a few q-rung dual hesitant orthopair fuzzy HM operators. Lei and Xu [16] used the q-ROFRH operator and data with q-rung interval values. Xing et al. [37] presented the concept of point operators for q-ROFSs. Garg and Chen [9] q-ROFS neutrality AOs have been suggested as a solution to group decision-making (DM) issues. Due of its obviously wider base than the IFS, the q-ROFS can convey more muddled information. Qiyas et al. [27] discussed a case study for hospital-based post-acute care cerebrovascular disease using sine hyperbolic q-rung orthopair fuzzy Dombi aggregation operators. The q-ROFS is superior to IFS in managing fuzzy and intuitionistic fuzzy information, because it can handle MCDM problems that intuitionistic fuzzy set cannot, despite the fact that it is evident that IFS is a component of the q-ROFS.

In 2019, Riaz and Hashmi [31] critically examined the restrictions related to membership and non-membership functions in structures of FS, IFS, PyFS, and q-ROFS, and these limitations were pointed out numerically. They introduced the linear Diophantine fuzzy set (LDFS) by adding reference parameters to the structure of IFS to eliminate these restrictions. They stated that the concept of LDFS will eradicate the constraints in the existing methodologies of other sets and enable the free selection of data in practice. The grades of membership, non-membership, and reference parameters in the construction of LDFS are real valued. Mohammad et al. [22] defined some linear Diophantine fuzzy similarity measures and their application in decision-making problem. Qiyas [32] proposed the idea of similarity measures based on q-rung linear Diophantine fuzzy sets and their application in logistics and supply chain management. Hanif et al. [13] developed linear Diophantine fuzzy graphs with a new decision-making approach. Riaz [33] proposed Spherical linear Diophantine fuzzy sets with modeling uncertainties in MCDM. Hashmi et al. [12] defined a Spherical linear Diophantine fuzzy soft rough sets with multi-criteria decision making.

Shahzaib et al. [3] first proposed the concept of the Spherical fuzzy set (SPS) to address this issue that picture fuzzy set (PFS) cannot solve. Then, they found a tonne of spherical fuzzy information-based aggregating techniques. In contrast to PFSs, where all membership degrees must satisfy the requirement $\mu^2_{\acute{E}}(\hat{s}) + \nu^2_{\acute{E}}(\hat{s}) + \eta^2_{\acute{E}}(\hat{s}) \leq 1$, the SFS requires. A more advanced variant of PyFS with a fuzzy set is called a spherical fuzzy set. Additionally, they looked at how the Spherical fuzzy t-norm and conorm are shown in [4].

The membership degree (MD), neutral membership degree (NuMD), and non-membership degree (NMD) are constrained by the PFS and SFS concepts, which have wide applications in a variety of spheres of daily life. To address these difficulties, we created a brand-new extended concept of a fractional orthotriple fuzzy set (FOFS). In the FOFS framework that has been proposed, there are three membership tiers. And for each $\hat{s} \in \acute{S}$ in a finite universe of discourse $\acute{S}$ with $\mu^f_{\acute{E}}(\hat{s}) + \nu^f_{\acute{E}}(\hat{s}) + \eta^f_{\acute{E}}(\hat{s}) \leq 1$, $f \in Q$ (set of positive rational integers). We discover that as rung $f$ increases, the fractional orthotriple fuzzy space broadens, enabling observers to express support for membership over a greater range. It should be mentioned that in order to deal with ambiguity and erroneous information, we were able to generate a fractional orthotriple fuzzy set that produced more precise and accurate rung fuzzy numbers. It is evident that PFS and SFS are the generic forms of FOFS, and by setting $p = q$ for PFS and $p = 2\check{n}, q = \check{n}$ for all $\check{n} \in N$ for SFS, respectively, the corresponding set simplifies to PFS and SFS. As a result of their increased adaptability and improved handling of uncertain information, FOFSs reflect more comprehensive fuzzy data.

The fractional orthotriple fuzzy set was first introduced by Abosuliman et al. [6], in order to generalize the idea of the spherical fuzzy set. Similarity matrices for FOFS were developed by Naeem et al. [23] who also described how they may be applied in emergency situations such as accidents. Utilizing the cosine and cotangent functions, they achieved this. For processing fractional orthotriple fuzzy data, Qiyas et al. [25] developed aggregation methods based on Banzhaf choquet copula. Qiyas et al. [26] created fractional orthotriple fuzzy rough Hamacher AOs and applied them to the wireless network selection. Qiyas et al. [28] developed a decision support system using complex fractional orthotriple fuzzy 2-tuple linguistic AOs. Qiyas et al. [29] defined fractional orthotriple fuzzy Choquet-Frank AOs and their application in the optimal selection for EEG of the depression patients.

According to the study stated above, aggregation operators are crucial in decision-making since they aggregate information from various sources into a single value. FOLDFNs allow experts more flexibility in how it might express a judgment in circumstances needing real-world decision-making. Based on the sine hyperbolic function and FOLDFNs, we developed the concept of sine hyperbolic fractional orthotriple linear Diophantine fuzzy number (sinh-FOLDFN) in this study to address these issues. This provides motivation for the ongoing sinh-FOLDF research work. The sine hyperbolic function is an important function that also benefits from having amplitude and symmetric about the origin and satisfying standards over multiple time scales. The major objective of the entire paper is to establish more sophisticated operational laws for FOLDFNs using algebraic t-norm and t-conorm operations. Based on the stated operating rules, a list of geometric aggregate and averaging operators is provided, together with a thorough explanation of the relevant aspects. To address problems with group decision-making in an emergency, we provide a unique application: alternative selection based on suggested operators. A lot of research is done on how the parameters affect how the alternatives rank.

The remaining sections of the whole study are outlined below: Several key terms relating to FOLDFSs are briefly explained in Sec. 2 of this document. We define sine hyperbolic FOLFFNs in Sec. 3 and go over their characteristics. On the basis of algebraic t-norm and t-conorm, we also deduced sine hyperbolic operational laws for FOFSs. The sine hyperbolic fractional orthotriple linear Diophantine fuzzy weighted averaging (sinh-FOFWA), sine hyperbolic fractional orthotriple linear Diophantine fuzzy ordered weighted averaging (sinh-FOLDFOWA), and sine hyperbolic fractional orthotriple linear Diophantine fuzzy hybrid averaging (sinh-FOLDFHA) were developed in Sec. 4. The terms sine hyperbolic fractional orthotriple linear Diophantine fuzzy weighted geometric (sinh-FOLDFWG), sine hyperbolic fractional orthotriple fuzzy ordered weighted geometric (sinh-FOFOWG), and sine hyperbolic fractional orthotriple linear Diophantine fuzzy hybrid geometric (sinh-FOLDFHG) operators were defined in Sec. 5. We defined an algorithm in accordance with the provided operators. Using described operators in Sec. 6, to solve an example of an alternatives selection problem. To demonstrate the applicability of the suggested methodology, a comparison of numerous approaches is provided in Sec. 7. In Sec. 8 addressed the article's conclusion.

## 2. Preliminaries

Here, we examine a few basic concepts related to our further study.

**Definition 2.1.** [1] An IFS $\acute{E}$ on universal set $\acute{S}$ is defined as;

$$\acute{E} = \left\{\langle \mu_{\acute{E}}(\hat{s}), \nu_{\acute{E}}(\hat{s}) | \hat{s} \in \acute{S} \rangle\right\}, \tag{2.1}$$

where $\mu_{\acute{E}}, \nu_{\acute{E}} : \acute{S} \rightarrow [0,1]$ is the MD and NMD of $\acute{E}$. And $\mu_{\acute{E}}, \nu_{\acute{E}}, \forall \hat{s} \in \acute{S}$, and satisfied the condition, $0 \leq \mu_{\acute{E}} + \nu_{\acute{E}} \leq 1$. Also, $\pi_{\acute{E}}(\hat{s}) = 1 - (\mu_{\acute{E}} + \nu_{\acute{E}})$ is a hesitancy degree of $\hat{s}$ in $\acute{E}$.

**Definition 2.2.** [39] A q-ROFS $\acute{E}$ on universal set $\acute{S}$ is defined as;

$$\acute{E} = \left\{\langle \mu_{\acute{E}}(\hat{s}), \nu_{\acute{E}}(\hat{s}) | \hat{s} \in \acute{S} \rangle\right\}, \tag{2.2}$$

where $\mu_{\acute{E}}, \nu_{\acute{E}} : \acute{S} \rightarrow [0,1]$ is the MD and NMD of $\acute{E}$. And $\mu_{\acute{E}}, \nu_{\acute{E}}$ for all $\hat{s} \in \acute{S}$ and satisfied the condition; $0 \leq \mu_{\acute{E}}^q + \nu_{\acute{E}}^q \leq 1$. Also, $\pi_{\acute{E}}(\hat{s}) = \sqrt{1 - \left(\mu_{\acute{E}}^q + \nu_{\acute{E}}^q\right)}$ is a hesitancy degree of $\hat{s}$ in $\acute{E}$.

**Definition 2.3.** [31] A LDFS $\acute{E}$ on universal set $\acute{S}$ is defined as;

$$\acute{E} = \{(\hat{s}, \langle u_{\acute{E}}(\hat{s}), \nu_{\acute{E}}(\hat{s})\rangle, \langle \alpha_{\acute{E}}, \beta_{\acute{E}}\rangle) | \hat{s} \in \acute{S}\}, \tag{2.3}$$

where $u_{\acute{E}}, \nu_{\acute{E}}, \alpha_{\acute{E}}, \beta_{\acute{E}} \in [0,1]$ are MG and NMG respectively, and satisfies the condition $0 \leq (\alpha_{\acute{E}})\, u_{\acute{E}} + (\beta_{\acute{E}})\, \nu_{\acute{E}} \leq 1, \forall \hat{s} \in \acute{S}$ with $0 \leq \alpha_{\acute{E}} + \beta_{\acute{E}} \leq 1$. The degree of hesitancy as, $\pi_{\acute{E}} = 1 - (\alpha_{\acute{E}})\, u_{\acute{E}} - (\beta_{\acute{E}})\, \nu_{\acute{E}}$.

**Definition 2.4.** [25] A FOFS $\acute{E}$ on universal set $\acute{S}$ is defined as;

$$\acute{E} = \left\{\langle \mu_{\acute{E}}(\hat{s}), \nu_{\acute{E}}(\hat{s}), \eta_{\acute{E}}(\hat{s}) | \hat{s} \in \acute{S} \rangle\right\}, \tag{2.4}$$

where $\mu_{\acute{E}}, \nu_{\acute{E}}, \eta_{\acute{E}} : \acute{S} \to [0,1]$ is the MD, NuMD and NMD of $\acute{E}$. And $\mu_{\acute{E}}, \nu_{\acute{E}}, \eta_{\acute{E}}$, $\forall \hat{s} \in \acute{S}$, and satisfied the condition; $0 \leq \mu_{\acute{E}}^f + \nu_{\acute{E}}^f + \eta_{\acute{E}}^f \leq 1$. Also, $\pi_{\acute{E}}(\hat{s}) = \sqrt{1 - \left(\mu_{\acute{E}}^f + \nu_{\acute{E}}^f + \eta_{\acute{E}}^f\right)}$ is the hesitant degree of $\hat{s}$ in $\acute{E}$.

**Definition 2.5.** A fractional orthotriple linear Diophantine fuzzy set (FOLDFS) on universal set $\acute{S}$ is defined as;

$$D = \{(\hbar, \langle \mu_D(\hat{s}), \nu_D(\hat{s}), \eta_D(\hat{s})\rangle, \langle \alpha_D, \beta_D, \gamma_D\rangle) | \hat{s} \in \acute{S}\}, \tag{2.5}$$

where $\mu_D, \nu_D, \eta_D, \alpha_D, \beta_D, \gamma_D \in [0,1]$ are MG, NuMG, NMG and reference parameters (RPs) respectively. These functions fulfill the restriction; $0 \leq (\alpha_D)^f \mu_D + (\beta_D)^f \nu_D + (\gamma_D)^f \eta_D \leq 1, \forall \hat{s} \in \acute{S}, f \geqslant 1$, with $0 \leq \alpha_D^f + \beta_D^f + \gamma_D^f \leq 1, f \geq 1$. The degree of hesitation is defined as, $\pi_D = \sqrt[q]{1 - ((\alpha_D)^f \mu_D + (\beta_D)^f \nu_D + (\gamma_D)^f \eta_D)}$.

**Definition 2.6.** [25] Suppose $\acute{E}_1 = \left\{\langle \hat{s}, \mu_{\acute{E}_1}(\hat{s}), \nu_{\acute{E}_1}(\hat{s}), \eta_{\acute{E}_1}(\hat{s}) | \hat{s} \in \hat{s}\rangle\right\}$ and $\acute{E}_2 = \left\{\left\langle \hat{s}, \mu_{\acute{E}_2}(\hat{s}), \nu_{\acute{E}_2}(\hat{s}), \eta_{\acute{E}_2}(\hat{s}) | \hat{s} \in \acute{S}\right\rangle\right\}$ are two FOFNs. The fundamental operations are then defined as follows:

(1) $\acute{E}_1 \subseteq \acute{E}_2$, if $\mu_{\acute{E}_1}(\hat{s}) \leq \mu_{\acute{E}_2}(\hat{s}), \nu_{\acute{E}_1}(\hat{s}) \geq \nu_{\acute{E}_2}(\hat{s})$ and $\eta_{\acute{E}_1}(\hat{s}) \geq \eta_{\acute{E}_2}(\hat{s}), \forall \hat{s} \in \acute{S}$;

(2) $\acute{E}_1 = \acute{E}_2$, if $\acute{E}_1 \subseteq \acute{E}_2$ and $\acute{E}_2 \subseteq \acute{E}_1$;

(3) $\acute{E}_1 \cup \acute{E}_2 = \left\{\langle \hat{s}, \max\left\{\mu_{\acute{E}_1}(\hat{s}), \mu_{\acute{E}_2}(\hat{s})\right\}, \min\left\{\nu_{\acute{E}_1}(\hat{s}), \nu_{\acute{E}_2}(\hat{s})\right\}, \min\left\{\eta_{\acute{E}_1}(\hat{s}), \eta_{\acute{E}_2}(\hat{s})\right\}\rangle | \hat{s} \in \acute{S}\right\}$;

(4) $\acute{E}_1 \cap \acute{E}_2 = \left\{\langle \hat{s}, \min\left\{\mu_{\acute{E}_1}(\hat{s}), \mu_{\acute{E}_2}(\hat{s})\right\}, \min\left\{\nu_{\acute{E}_1}(\hat{s}), \nu_{\acute{E}_2}(\hat{s})\right\}, \max\left\{\eta_{\acute{E}_1}(\hat{s}), \eta_{\acute{E}_2}(\hat{s})\right\}\rangle | \hat{s} \in \acute{S}\right\}$;

(5) $\acute{E}_1^c = \left\{\left\langle \eta_{\acute{E}_1}(\hat{s}), \nu_{\acute{E}_1}(\hat{s}), \mu_{\acute{E}_1}(\hat{s})\right\rangle\right\}$.

### *2.1. Sine hyperbolic operational laws based on FOLDFNs*

In this portion, we defined sine hyperbolic fractional orthotriple linear Diophantine fuzzy numbers (sinh-FOLDFNs) and studied some of its key features. Following these, we developed the operation laws for sinh-FOLDFNs using algebraic t-norm and t-conorm operation as well as a few key characteristics.

**Definition 2.7.** Suppose $\acute{E} = \left\{(\hat{s}, \langle \mu_{\acute{E}}(\hat{s}), \nu_{\acute{E}}(\hat{s}), \eta_{\acute{E}}(\hat{s})\rangle, \langle \alpha_{\acute{E}}, \beta_{\acute{E}}, \gamma_{\acute{E}}\rangle) | \hat{s} \in \acute{S}\right\}$ be a FOLDFS. Then,

$$\sinh \acute{E} = \left\{\left(\hat{s}, \left\langle \sinh(\mu_{\acute{E}})^f, \sinh(1 - \nu_{\acute{E}})^f, \sinh(1 - \eta_{\acute{E}})^f\right\rangle, \langle \alpha_{\acute{E}}, \beta_{\acute{E}}, \gamma_{\acute{E}}\rangle\right) | \hat{s} \in \acute{S}\right\}, \tag{2.6}$$

is called sinh-FOLDFS, and the value of $\sinh \acute{E}$ is called sinh-FOLDF number (sinh-FOFN) for each $\hat{s} \in \acute{S}$. Moreover, $\sinh(\mu_{\acute{E}})^f : \acute{S} \to [0,1]$, $\sinh(1 - \nu_{\acute{E}})^f : \acute{S} \to [0,1]$ and $\sinh(1 - \eta_{\acute{E}})^f : \acute{S} \to [0,1]$ is called MD, NuMD, NMD and $\langle \alpha_{\acute{E}}, \beta_{\acute{E}}, \gamma_{\acute{E}}\rangle$ are the reference parameters, respectively, for each $\hat{s} \in \acute{S}$. Also, $\sinh(\mu_{\acute{E}})^f, \sinh(1 - \nu_{\acute{E}})^f, \sinh(1 - \eta_{\acute{E}})^f, \alpha_{\acute{E}}, \beta_{\acute{E}}, \gamma_{\acute{E}}$ should be satisfied the following two conditions;

(1) $\sinh(\mu_{\acute{E}})^f, \sinh(1 - \nu_{\acute{E}})^f, \sinh(1 - \eta_{\acute{E}})^f, \alpha_{\acute{E}}, \beta_{\acute{E}}, \gamma_{\acute{E}} \in [0,1]$.

(2) $0 \leq (\alpha)^f \sinh(\mu_{\acute{E}})^f + (\beta)^f \sinh(1 - \nu_{\acute{E}})^f + (\gamma)^f \sinh(1 - \eta_{\acute{E}})^f \leq 1$ and $0 \leq (\alpha)^f + (\beta)^f + (\gamma)^f \leq 1$.

**Definition 2.8.** Suppose $\sinh \acute{E}_1 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_1}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_1}, \beta_{\acute{E}_1}, \gamma_{\acute{E}_1} \right\rangle\right\}$ and $\sinh \acute{E}_2 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_2}\right)^f, \sinh\left(1-\nu_{\acute{E}_2}\right)^f, \sinh\left(1-\eta_{\acute{E}_2}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_2}, \beta_{\acute{E}_2}, \gamma_{\acute{E}_2} \right\rangle\right\}$ are two sine hyperbolic FOLDFNs and $\lambda > 0$. Then,

(1) $\sinh \acute{E}_1 \subseteq \sinh \acute{E}_2$, if

$$\sinh\left(\mu^f_{\acute{E}_1}\right) \leq \sinh\left(\mu^f_{\acute{E}_2}\right);$$
$$\sinh\left(1-\nu_{\acute{E}_1}\right)^f \geq \sinh\left(1-\nu_{\acute{E}_2}\right)^f;$$
$$\sinh\left(1-\eta_{\acute{E}_1}\right)^f \geq \sinh\left(1-\eta_{\acute{E}_2}\right)^f.$$

And

$$\alpha_{\acute{E}_1} \geq \alpha_{\acute{E}_2}; \beta_{\acute{E}_1} \leq \beta_{\acute{E}_2}; \gamma_{\acute{E}_1} \leq \gamma_{\acute{E}_2}$$

for all $\hat{s} \in \acute{S}$.

(2) $\sinh \acute{E}_1 = \sinh \acute{E}_2$, if $\sinh \acute{E}_1 \subseteq \sinh \acute{E}_2$ and $\sinh \acute{E}_2 \subseteq \sinh \acute{E}_1$.

(3) $\sinh \acute{E}_1 \cup \sinh \acute{E}_2 = \left\{\begin{array}{c} \left\langle \begin{array}{c} \max\left(\sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(\mu_{\acute{E}_2}\right)^f\right), \\ \min\left(\sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_2}\right)^f\right), \\ \min\left(\sinh\left(1-\eta_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_2}\right)^f\right) \end{array} \right\rangle, \\ \left\langle \max\left(\alpha_{\acute{E}_1}, \alpha_{\acute{E}_2}\right), \min\left(\beta_{\acute{E}_1}, \beta_{\acute{E}_2}\right), \min\left(\gamma_{\acute{E}_1}, \gamma_{\acute{E}_2}\right) \right\rangle \end{array}\right\}$.

(4) $\sinh \acute{E}_1 \cap \sinh \acute{E}_2 = \left\{\begin{array}{c} \left\langle \begin{array}{c} \min\left(\sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(\mu_{\acute{E}_2}\right)^f\right), \\ \max\left(\sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f\right) \\ \max\left(\sinh\left(1-\eta_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_1}\right)^f\right) \end{array} \right\rangle, \\ \left\langle \min\left(\alpha_{\acute{E}_1}, \alpha_{\acute{E}_2}\right), \max\left(\beta_{\acute{E}_1}, \beta_{\acute{E}_2}\right), \max\left(\gamma_{\acute{E}_1}, \gamma_{\acute{E}_2}\right) \right\rangle \end{array}\right\}$.

(5) $\sinh \acute{E}^c_1 = \left\{\left\langle \sinh\left(1-\eta_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(\mu_{\acute{E}_1}\right)^f \right\rangle, \left\langle \gamma_{\acute{E}_1}, \beta_{\acute{E}_1}, \alpha_{\acute{E}_1} \right\rangle\right\}$.

**Definition 2.9.** Suppose $\sinh \acute{E}_1 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_1}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_1}, \beta_{\acute{E}_1}, \gamma_{\acute{E}_1} \right\rangle\right\}$ and $\sinh \acute{E}_2 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_2}\right)^f, \sinh\left(1-\nu_{\acute{E}_2}\right)^f, \sinh\left(1-\eta_{\acute{E}_2}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_2}, \beta_{\acute{E}_2}, \gamma_{\acute{E}_2} \right\rangle\right\}$ are two sine hyperbolic FOLDFNs and $\lambda, \lambda_1, \lambda_2 > 0$. Then,

(1) $\sinh \acute{E}_1 \oplus \sinh \acute{E}_2 = \sinh \acute{E}_2 \oplus \sin \acute{E}_1$,

(2) $\sinh \acute{E}_1 \otimes \sinh \acute{E}_2 = \sinh \acute{E}_2 \otimes \sinh \acute{E}_1$,

(3) $\lambda(\sinh \acute{E}_1 \oplus \sinh \acute{E}_2) = \lambda \sinh \acute{E}_1 \oplus \lambda \sinh \acute{E}_2$,

(4) $(\sinh \acute{E}_1 \otimes \sinh \acute{E}_2)^\lambda = \sinh \acute{E}^\lambda_1 \otimes \sinh \acute{E}^\lambda_2$,

(5) $\lambda_1 \sinh \acute{E}_1 \oplus \lambda_2 \sinh \acute{E}_1 = (\lambda_1 + \lambda_2) \sinh \acute{E}_1$,

(6) $\sinh \acute{E}_1^{\lambda_1} \otimes \sin \acute{E}^{\lambda_2} = \sinh \acute{E}_1^{(\lambda_1+\lambda_2)}$,

(7) $\left(\sinh \acute{E}_1^{\lambda_1}\right)^{\lambda_2} = \sinh \acute{E}_1^{\lambda_1\lambda_2}$.

**Definition 2.10.** Suppose $\sinh \acute{E}_1 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_1}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_1}, \beta_{\acute{E}_1}, \gamma_{\acute{E}_1}\right\rangle\right\}$ be a sine hyperbolic FOLDFN. Then, the score function $\Lambda$ and accuracy function $\Gamma$ is defined as follows;

$$\Lambda\left(\sinh \acute{E}_1\right) = \frac{\left(\mu_{\acute{E}_1}^f - \nu_{\acute{E}_1}^f - \eta_{\acute{E}_1}^f\right) + \left(\alpha_{\acute{E}_1} - \beta_{\acute{E}_1} - \gamma_{\acute{E}_1}\right)}{2}, \tag{2.7}$$

$$\Gamma\left(\sinh \acute{E}_1\right) = \frac{\left(\mu_{\acute{E}_1}^f + \nu_{\acute{E}_1}^f + \eta_{\acute{E}_1}^f\right) + \left(\alpha_{\acute{E}_1} + \beta_{\acute{E}_1} + \gamma_{\acute{E}_1}\right)}{2}. \tag{2.8}$$

We define the following comparison rules.

**Definition 2.11.** Let $\sinh \acute{E}_1$ and $\sinh \acute{E}_2$ are any two sinh-FOLDFNs.

(1) If $\Lambda(\sinh \acute{E}_1) < \Lambda(\sinh \acute{E}_2)$, then $\sinh \acute{E}_1 < \sinh \acute{E}_2$,

(2) If $\Lambda(\sinh \acute{E}_1) = \Lambda(\sinh \acute{E}_2)$, then

(a) If $\Gamma(\sinh \acute{E}_1) < \Gamma(\sinh \acute{E}_2)$, then $\sinh \acute{E}_1 < \sinh \acute{E}_2$,

(b) If $\Gamma(\sinh \acute{E}_1) = \Gamma(\sinh \acute{E}_2)$, then $\sinh \acute{E}_1 \sim \sinh \acute{E}_2$.

### 2.2. *Sine hyperbolic fractional orthotriple linear Diophantine fuzzy operation*

In this part, utilizing the algebraic t-norm and t-conorm, we created sine hyperbolic FOLDF operational laws (sinh-FOLDFOLs) for FOLDFNs. The development of some aggregation operators for sinh-FOLDFNs will come after these.

**Definition 2.12.** Suppose $\sinh \acute{E}_1 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_1}\right)^f, \sinh\left(1-\nu_{\acute{E}_1}\right)^f, \sinh\left(1-\eta_{\acute{E}_1}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_1}, \beta_{\acute{E}_1}, \gamma_{\acute{E}_1}\right\rangle\right\}$ and $\sinh \acute{E}_2 = \left\{\left\langle \sinh\left(\mu_{\acute{E}_2}\right)^f, \sinh\left(1-\nu_{\acute{E}_2}\right)^f, \sinh\left(1-\eta_{\acute{E}_2}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_2}, \beta_{\acute{E}_2}, \gamma_{\acute{E}_2}\right\rangle\right\}$ be two sine hyperbolic FOLDFNs, where $f \geq 1$ and $\lambda > 0$. Then, sine hyperbolic FOLDF operational laws (sinh-FOLDFOLs) are defined as;

$$(1)\ \sinh \acute{E}_1 \oplus \sinh \acute{E}_2 = \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{\sinh\left(\mu_{\acute{E}_1}\right)^f + \sinh\left(\mu_{\acute{E}_2}\right)^f - \sinh\left(\mu_{\acute{E}_1}\right)^f . \sinh\left(\mu_{\acute{E}_2}\right)^f}, \\ \sinh\left(1-\nu_{\acute{E}_1}\right)^f . \sinh\left(1-\nu_{\acute{E}_2}\right)^f, \\ \sinh\left(1-\eta_{\acute{E}_1}\right)^f . \sinh\left(1-\eta_{\acute{E}_2}\right)^f \end{array} \right), \\ \left( \begin{array}{c} \sqrt[f]{\left(\alpha_{\acute{E}_1}\right)^f + \left(\alpha_{\acute{E}_2}\right)^f - \left(\alpha_{\acute{E}_1}\right)^f + \left(\alpha_{\acute{E}_2}\right)^f}, \\ \beta_{\acute{E}_1} . \beta_{\acute{E}_2}, \gamma_{\acute{E}_1} . \gamma_{\acute{E}_2} \end{array} \right) \end{array} \right\};$$

$$(2)\ \sinh \acute{E}_1 \otimes \sinh \acute{E}_2 = \left\{ \begin{array}{c} \left( \begin{array}{c} \sinh\left(\mu_{\acute{E}_1}\right)^f . \sinh\left(\mu_{\acute{E}_2}\right)^f, \\ \sqrt[f]{\begin{array}{c} \sinh\left(1-\nu_{\acute{E}_1}\right)^f + \sinh\left(1-\nu_{\acute{E}_2}\right)^f \\ -\sinh\left(1-\nu_{\acute{E}_1}\right)^f . \sinh\left(1-\nu_{\acute{E}_2}\right)^f \end{array}}, \\ \sqrt[f]{\begin{array}{c} \sinh\left(1-\eta_{\acute{E}_1}\right)^f + \sinh\left(1-\eta_{\acute{E}_2}\right)^f - \sinh\left(1-\eta_{\acute{E}_1}\right)^f . \\ \sinh\left(1-\eta_{\acute{E}_2}\right)^f \end{array}} \end{array} \right), \\ \left( \begin{array}{c} \alpha_{\acute{E}_1}.\alpha_{\acute{E}_2}, \sqrt[f]{\left(\beta_{\acute{E}_1}\right)^f + \left(\beta_{\acute{E}_2}\right)^f - \left(\beta_{\acute{E}_1}\right)^f . \left(\beta_{\acute{E}_2}\right)^f}, \\ \sqrt[f]{\left(\gamma_{\acute{E}_1}\right)^f + \left(\gamma_{\acute{E}_2}\right)^f - \left(\gamma_{\acute{E}_1}\right)^f . \left(\gamma_{\acute{E}_2}\right)^f} \end{array} \right) \end{array} \right\};$$

$$(3)\ \lambda \sinh \acute{E}_1 = \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1-\left(1-\sinh\left(\nu_{\acute{E}_1}\right)^f\right)^\lambda}, \left(\sinh\left(1-\nu_{\acute{E}_1}\right)^f\right)^\lambda, \\ \left(\sinh\left(1-\eta_{\acute{E}_1}\right)^f\right)^\lambda \end{array} \right), \\ \left( \sqrt[f]{1-\left(1-\left(\alpha_{\acute{E}_1}\right)^f\right)^\lambda}, \left(\beta_{\acute{E}_1}\right)^\lambda, \left(\gamma_{\acute{E}_1}\right)^\lambda \right) \end{array} \right\};$$

$$(4)\ \left(\sinh \acute{E}_1\right)^\lambda = \left\{ \begin{array}{c} \left( \begin{array}{c} \left(\sinh\left(\nu_{\acute{E}_1}\right)^f\right)^\lambda, \sqrt[f]{1-\left(1-\sinh\left(1-\nu_{\acute{E}_1}\right)^f\right)^\lambda}, \\ \sqrt[f]{1-\left(1-\sinh\left(1-\eta_{\acute{E}_1}\right)^f\right)^\lambda} \end{array} \right), \\ \left( \left(\alpha_{\acute{E}_1}\right)^\lambda, \sqrt[f]{1-\left(1-\left(\beta_{\acute{E}_1}\right)^f\right)^\lambda}, \sqrt[f]{1-\left(1-\left(\gamma_{\acute{E}_1}\right)^f\right)^\lambda} \right) \end{array} \right\}.$$

## 3. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy averagingoperators

The weighted average and geometric aggregation operators are defined in this section using sinh-FOLDFOLs of FOLDFNs as follows.

### 3.1. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy weighted averaging operator

**Definition 3.1.** Suppose $\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}} \right\rangle\right\}$ ($\breve{\imath} = 1, \cdots, \breve{n}$) be a family of sine hyperbolic fractional orthotriple linear Diophantine fuzzy numbers (sinh-FOLDFNs). Then, sinh-FOLDFWA operator is a mapping $\sinh \acute{E}^{\breve{n}} \to \sinh \acute{E}$, such that;

$$\sinh - FOLDFWA(\sinh \acute{E}_1, ..., \sinh \acute{E}_{\breve{n}}) = \bigoplus_{\breve{\imath}=1}^{\breve{n}} (\xi_{\breve{\imath}} \sinh \acute{E}_{\breve{\imath}}), \tag{3.1}$$

where $\xi = (\xi_1, \cdots, \xi_{\breve{n}})^T$ is a weight vector of $\sinh \acute{E}_{\breve{\imath}} (\breve{\imath} = 1, \cdots, \breve{n})$ with $\xi_{\breve{\imath}} > 0$ and $\sum_{\breve{\imath}=1}^{\breve{n}} \xi_{\breve{\imath}} = 1$.

As a result, an auxiliary theorem that explains the observed operations on sinh-FOLDFNs is provided.

**Theorem 3.1.** Suppose $\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}} \right\rangle\right\}$ ($\breve{\imath} = 1, \cdots, \breve{n}$) be a family of (sinh-FOLDFNs). Then, the aggregated value utilizing sinh-FOLDFWA operator is still a sinh-FOLDFN, and

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigoplus_{\check{\imath}=1}^{\check{n}} (\xi_{\check{\imath}} \sinh \acute{E}_{\check{\imath}}) \tag{3.2}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{\check{n}} \left(\sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\check{n}} \left(\sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{\check{n}} \left(\beta_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\check{n}} \left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\},$$

where $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ is the weight vector of $\sinh \acute{E}_{\check{\imath}} (\check{\imath} = 1, \cdots, \check{n})$ with $\xi_{\check{\imath}} > 0$ and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$.

*Proof.* The Eq (3.2) is demonstrated as follows using the mathematical induction principle:

(1) According to sinh-FOLDFNs operations, we have the preceding consequence for $\check{n} = 2$,

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \sinh \acute{E}_2) = \xi_1 \sinh \acute{E}_1 \oplus \xi_2 \sinh \acute{E}_2$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \left(1 - \sinh\left(\mu_{\acute{E}_1}\right)^f\right)^{\xi_1}}, \\ \left(\sinh\left(1 - \nu_{\acute{E}_1}\right)^f\right)^{\xi_1}, \left(\sinh\left(1 - \eta_{\acute{E}_1}\right)^f\right)^{\xi_1} \end{array} \right), \\ \left( \sqrt[f]{1 - \left(1 - \left(\alpha_{\acute{E}_1}\right)^f\right)^{\xi_1}}, \left(\beta_{\acute{E}_1}\right)^{\xi_1}, \left(\gamma_{\acute{E}_1}\right)^{\xi_1} \right) \end{array} \right\} \\ \oplus \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \left(1 - \sinh\left(\mu_{\acute{E}_2}\right)^f\right)^{\xi_2}}, \\ \left(\sinh\left(1 - \nu_{\acute{E}_2}\right)^f\right)^{\xi_2}, \left(\sinh\left(1 - \eta_{\acute{E}_2}\right)^f\right)^{\xi_2} \end{array} \right), \\ \left( \sqrt[f]{1 - \left(1 - \left(\alpha_{\acute{E}_2}\right)^f\right)^{\xi_2}}, \left(\beta_{\acute{E}_2}\right)^{\xi_2}, \left(\gamma_{\acute{E}_2}\right)^{\xi_2} \right) \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\check{\imath}=1}^{2} \left(1 - \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{2} \left(\sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{2} \left(\sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\check{\imath}=1}^{2} \left(1 - \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{2} \left(\left(\beta_{\acute{E}_{\check{\imath}}}\right)\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{2} \left(\left(\gamma_{\acute{E}_{\check{\imath}}}\right)\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\}.$$

Thus, Eq (3.2) is true for $\check{n} = 2$.

(2) Let Eq (3.2) is hold for $\check{n} = \kappa$.

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_\kappa) = \bigoplus_{\check{\imath}=1}^{\kappa} (\xi_{\check{\imath}} \sinh \acute{E}_{\check{\imath}})$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa} \left(1 - \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa} \left(1 - \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\left(\beta_{\acute{E}_{\check{\imath}}}\right)\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\left(\gamma_{\acute{E}_{\check{\imath}}}\right)\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\}.$$

For $\check{n} = \kappa + 1$, then

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_\kappa) = \bigoplus_{\check{\imath}=1}^{\kappa} (\xi_{\check{\imath}} \sinh \acute{E}_{\check{\imath}}) \oplus (\xi_{\kappa+1} \sinh \acute{E}_{\kappa+1})$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa} \left(1 - \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa} \left(1 - \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\beta_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa} \left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\} \\ \oplus \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \left(1 - \sinh\left(\mu_{\acute{E}_{\kappa+1}}\right)^f\right)^{\xi_{\kappa+1}}}, \\ \left(\sinh\left(1 - \nu_{\acute{E}_{\kappa+1}}\right)^f\right)^{\xi_{\kappa+1}}, \left(\sinh\left(1 - \eta_{\acute{E}_{\kappa+1}}\right)^f\right)^{\xi_{\kappa+1}} \end{array} \right), \\ \sqrt[f]{1 - \left(1 - \left(\alpha_{\acute{E}_{\kappa+1}}\right)^f\right)^{\xi_{\kappa+1}}}, \left(\beta_{\acute{E}_{\kappa+1}}\right)^{\xi_{\kappa+1}}, \left(\gamma_{\acute{E}_{\kappa+1}}\right)^{\xi_{\kappa+1}} \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa+1} \left(1 - \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{\kappa+1} \left(\sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa+1} \left(\sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\kappa+1} \left(1 - \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{\kappa+1} \left(\beta_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\kappa+1} \left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\}.$$

Equation (3.2) is consequently correct for $\check{n} = \kappa + 1$. As a result, we show that Eq (3.2) is true for all $\check{n}$. □

The sinh-FOLDFWA operator makes it relatively easy to check the following properties.

**Theorem 3.2. (Idempotency)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{ \left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle \right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOLDFNs, all are identical, i.e., $\sinh \acute{E}_{\breve{\imath}} = \sinh \acute{E}$. Then,

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}}) = \sinh \acute{E}. \tag{3.3}$$

**Theorem 3.3. (Boundedness)**. Suppose

$$\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}}\right\rangle\right\} \; (\breve{\imath} = 1, \cdots, \breve{n})$$

be a family of sinh-FOLDFNs. Suppose $\sinh \acute{E}^- = \min(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}})$ and $\sinh \acute{E}^+ = \max(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}})$. Then,

$$\sinh \acute{E}^- \leq \sinh -FOLDFWA(\sinh \acute{E}, \cdots, \sinh \acute{E}_{\breve{n}}) \leq \sinh \acute{E}^+. \tag{3.4}$$

**Theorem 3.4. (Monotonicity)**. Suppose

$$\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}}\right\rangle\right\} \; (\breve{\imath} = 1, \cdots, \breve{n})$$

be a family of sinh-FOFNs. If $\sinh \acute{E}_{\breve{\imath}} \leq \sinh \acute{E}^{/}_{\breve{\imath}}$. Then,

$$\sinh -FOLDFWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}}) \leq \sinh -FOLDFOWA(\sinh \acute{E}^{/}_1, \cdots, \sinh \acute{E}_{\breve{n}}), \tag{3.5}$$

where, the permutation of $\sinh \acute{E}_{\breve{\imath}} (\breve{\imath} = 1, \cdots, \breve{n})$ is $\sinh \acute{E}^{/}_{\breve{\imath}} (\breve{\imath} = 1, \cdots, \breve{n})$.

### *3.2. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy ordered weighted averaging operator*

**Definition 3.2.** Suppose $\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}}\right\rangle\right\}$ $(\breve{\imath} = 1, \cdots, \breve{n})$ be a family of sinh-FOLDFNs. A sinh-FOLDFOWA operator for $\breve{n}$ dimension is a mapping $\sinh -FOLDFOWA : \sinh \acute{E}^{\breve{n}} \rightarrow \sinh \acute{E}$ with the corresponding weight $\xi = (\xi_1, \cdots, \xi_{\breve{n}})^T$ along with $\xi_{\breve{\imath}} > 0$, and $\sum_{\breve{\imath}=1}^{\breve{n}} \xi_{\breve{\imath}} = 1$, as

$$\sinh -FOLDFOWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}}) = \bigoplus_{\breve{\imath}=1}^{\breve{n}} (\xi_{\breve{\imath}} \sinh \acute{E}_{\sigma(\breve{\imath})}), \tag{3.6}$$

and for $\sinh \acute{E}_{\sigma(\breve{\imath}-1)} \geq \sinh \acute{E}_{\sigma(\breve{\imath})}$ the permutation is $\sigma(1), \cdots, \sigma(\breve{n})$ for all $(\breve{\imath} = 1, \cdots, \breve{n})$.

**Theorem 3.5.** Suppose $\sinh \acute{E}_{\breve{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\breve{\imath}}}, \beta_{\acute{E}_{\breve{\imath}}}, \gamma_{\acute{E}_{\breve{\imath}}}\right\rangle\right\}$ $(\breve{\imath} = 1, \cdots, \breve{n})$ be a family of sinh-FOLDFNs. A sinh-FOLDFOWA operator of $\breve{n}$ dimension is a mapping, $\sinh -FOLDFOWA : \sinh \acute{E}^{\breve{n}} \rightarrow \sinh \acute{E}$ with corresponding weight $\xi = (\xi_1, \cdots, \xi_{\breve{n}})^T$ along with $\xi_{\breve{\imath}} > 0$, and $\sum_{\breve{\imath}=1}^{\breve{n}} \xi_{\breve{\imath}} = 1$. Then,

$$\sinh -FOLDFOWA(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\breve{n}}) = \bigoplus_{\breve{\imath}=1}^{\breve{n}} (\xi_{\breve{\imath}} \sinh \acute{E}_{\sigma(\breve{\imath})})$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1 - \prod_{\breve{\imath}=1}^{\breve{n}} \left(1 - \sinh\left(\mu_{\acute{E}_{\sigma(\breve{\imath})}}\right)^f\right)^{\xi_{\breve{\imath}}}}, \\ \prod_{\breve{\imath}=1}^{\breve{n}} \left(\sinh\left(1 - \nu_{\acute{E}_{\sigma(\breve{\imath})}}\right)^f\right)^{\xi_{\breve{\imath}}}, \prod_{\breve{\imath}=1}^{\breve{n}} \left(\sinh\left(1 - \eta_{\acute{E}_{\sigma(\breve{\imath})}}\right)^f\right)^{\xi_{\breve{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1 - \prod_{\breve{\imath}=1}^{\breve{n}} \left(1 - \left(\alpha_{\acute{E}_{\sigma(\breve{\imath})}}\right)^f\right)^{\xi_{\breve{\imath}}}}, \prod_{\breve{\imath}=1}^{\breve{n}} \left(\beta_{\acute{E}_{\sigma(\breve{\imath})}}\right)^{\xi_{\breve{\imath}}}, \prod_{\breve{\imath}=1}^{\breve{n}} \left(\gamma_{\acute{E}_{\sigma(\breve{\imath})}}\right)^{\xi_{\breve{\imath}}} \right) \end{array} \right\},$$

where $\sigma(1),\cdots,\sigma(\check{n})$ is the permutation of $(\check{\imath}=1,\cdots,\check{n})$, which as $\sinh \acute{E}_{\sigma(\check{\imath}-1)} \geq \sinh \acute{E}_{\sigma(\check{\imath})}$, $\forall(\check{\imath}=1,\cdots,\check{n})$.

Using sinh-FOLDFOWA, the following characteristics are simply illustrated.

**Theorem 3.6. (Idempotency)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\} (\check{\imath}=1,\cdots,\check{n})$$

be a family of sinh-FOLDFNs, all are identical. i.e., $\sinh \acute{E}_{\check{\imath}} = \sinh \acute{E}$. Then,

$$\sinh -FOLDFOWA(\sinh \acute{E}_1,\cdots,\sinh \acute{E}_{\check{n}}) = \sinh \acute{E}. \tag{3.7}$$

**Theorem 3.7. (Boundedness)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\} (\check{\imath}=1,\cdots,\check{n})$$

be a family of sinh-FOLDFNs. Suppose that $\sinh \acute{E}^- = \min(\sinh \acute{E}_1,\cdots,\sinh \acute{E}_{\check{n}})$ and $\sinh \acute{E}^+ = \max(\sinh \acute{E}_1,\cdots,\sinh \acute{E}_{\check{n}})$. Then,

$$\sinh \acute{E}^- \leq \sinh -FOLDFOWA(\sinh \acute{E},\cdots,\sinh \acute{E}_{\check{n}}) \leq \sinh \acute{E}^+. \tag{3.8}$$

**Theorem 3.8. (Monotonicity)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\} (\check{\imath}=1,\cdots,\check{n})$$

be a family of sinh-FOLDFNs. If $\sinh \acute{E}_{\check{\imath}} \leq \sinh \acute{E}^{/}_{\check{\imath}}$. Then,

$$\sinh -FOLDFOWA(\sinh \acute{E}_1,\cdots,\sinh \acute{E}_{\check{n}}) \leq \sinh -FOLDFOWA(\sinh \acute{E}^{/}_1,\cdots,\sinh \acute{E}_{\check{n}}),$$

where $\sinh \acute{E}^{/}_{\check{\imath}}(\check{\imath}=1,\cdots,\check{n})$ is the permutation of $\sinh \acute{E}_{\check{\imath}}(\check{\imath}=1,\cdots,\check{n})$.

We determine that sinh-FOLDFWA operator weights are the precise form of the structured placement of sinh-FOLDF values from Def. (3.1). In the sinh-FOLDFWA and sinh-FOLDFOWA operators, weights denote the number of connected components. Since these components are typically assumed to be the same, we define the sinh-FOLDFHA operator to remove this type of limitation.

### *3.3. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy hybrid averaging operator*

**Definition 3.3.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\}$ $(\check{\imath}=1,\cdots,\check{n})$ be a family of sinh-FOLDFNs. A sinh-FOLDFHA operator of $\check{n}$ dimension is a mapping $\sinh -FOLDFHA : \sinh \acute{E}^{\check{n}} \to \sinh \acute{E}$ with the corresponding weight $\xi = (\xi_1,\cdots,\xi_{\check{n}})^T$ along with $\xi_{\check{\imath}} > 0$, and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$. Then,

$$\sinh -FOLDFHA(\sinh \acute{E}_1,\cdots,\sinh \acute{E}_{\check{n}}) = \bigoplus_{\check{\imath}=1}^{\check{n}} (\xi_{\check{\imath}} \sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})})$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \sqrt[f]{1-\prod_{\check{\imath}=1}^{\check{n}}\left(1-\sinh\left(\widetilde{\mu}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \prod_{\check{\imath}=1}^{\check{n}}\left(\sinh\left(1-\widetilde{\nu}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\check{n}}\left(\sinh\left(1-\widetilde{\eta}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}} \end{array} \right), \\ \left( \sqrt[f]{1-\prod_{\check{\imath}=1}^{\check{n}}\left(1-\left(\widetilde{\alpha}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \prod_{\check{\imath}=1}^{\check{n}}\left(\widetilde{\beta}_{\acute{E}_{\sigma(\check{\imath})}}\right)^{\xi_{\check{\imath}}}, \prod_{\check{\imath}=1}^{\check{n}}\left(\widetilde{\gamma}_{\acute{E}_{\sigma(\check{\imath})}}\right)^{\xi_{\check{\imath}}} \right) \end{array} \right\},$$

where $\sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})}$ is the $\check{\imath}^{th}$ greatest sine hyperbolic fractional orthotriple linear Diophantine fuzzy values $\sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})} = \check{n} w_{\check{\imath}} \sinh \acute{E}_{\check{\imath}} (\check{\imath} = 1, \cdots, \check{n})$, and $w = (w_1, \cdots, w_{\check{n}})^T$ be associated weights of $\sinh \dot{\acute{E}}_{\sigma(\check{\imath})}$ with $w_{\check{\imath}} > 0$ and $\sum_{\check{\imath}=1}^{\check{n}} w_{\check{\imath}} = 1$, where $\check{n}$ is the matching coefficient. In the scenario where $w = (1/\check{n}, \cdots, 1/\check{n})$, sinh-FOLDFWA and sinh-FOLDFOWA operators are regarded as a particular case of the sinh-FOLDFHA operator. So, the sinh-FOLDFHA operator is a generalized version of the sinh-FOFWA and sinh-FOLDFOWA operators, which reflect the structured condition and degree of disagreement statements.

## 4. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy geometric operators

Using sinh-fractional orthotriple linear Diophantine fuzzy operational rules, we defined sine hyperbolic fractional orthotriple linear Diophantine fuzzy geometric operators.

### *4.1. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy weighted geometric operator*

**Definition 4.1.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{ \left\langle \sinh \left( \mu_{\acute{E}_{\check{\imath}}} \right)^f, \sinh \left( 1 - \nu_{\acute{E}_{\check{\imath}}} \right)^f, \sinh \left( 1 - \eta_{\acute{E}_{\check{\imath}}} \right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle \right\}$ $(\check{\imath} = 1, \cdots, \check{n})$ be a family of sine hyperbolic fractional orthotriple linear Diophantine fuzzy numbers (sinh-FOLDFNs). Then, sinh-FOLDFWG operator is a mapping $\sinh \acute{E}^{\check{n}} \to \sinh \acute{E}$, such that;

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigotimes_{\check{\imath}=1}^{\check{n}} (\sinh \acute{E}_{\check{\imath}})^{\xi_{\check{\imath}}}, \tag{4.1}$$

where $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ is the weight of $\sinh \acute{E}_{\check{\imath}} (\check{\imath} = 1, \cdots, \check{n})$ with $\xi_{\check{\imath}} > 0$ and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$.

**Theorem 4.1.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{ \left\langle \sinh \left( \mu_{\acute{E}_{\check{\imath}}} \right)^f, \sinh \left( 1 - \nu_{\acute{E}_{\check{\imath}}} \right)^f, \sinh \left( 1 - \eta_{\acute{E}_{\check{\imath}}} \right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle \right\}$ $(\check{\imath} = 1, \cdots, \check{n})$ be a family of sinh-FOLDFNs. Then, the aggregated value utilizing sinh-FOLDFWG operator is still a sinh-FOLDFNs, and

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigotimes_{\check{\imath}=1}^{\check{n}} (\sinh \acute{E}_{\check{\imath}})^{\xi_{\check{\imath}}} \tag{4.2}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left( \sinh \left( \mu_{\acute{E}_{\check{\imath}}} \right)^f \right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left( 1 - \sinh \left( 1 - \nu_{\acute{E}_{\check{\imath}}} \right)^f \right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left( 1 - \sinh \left( 1 - \eta_{\acute{E}_{\check{\imath}}} \right)^f \right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left( \alpha_{\acute{E}_{\check{\imath}}} \right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left( 1 - \beta_{\acute{E}_{\check{\imath}}} \right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left( 1 - \gamma_{\acute{E}_{\check{\imath}}} \right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\},$$

where $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ is the weight of $\sinh \acute{E}_{\check{\imath}} (\check{\imath} = 1, \cdots, \check{n})$ with $\xi_{\check{\imath}} > 0$ and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$.

*Proof.* The mathematical induction principle is employed to demonstrate this theorem:

(1) When $\breve{n} = 2$, we achieve the following result using sinh-FOLDFNs procedures.

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \sinh \acute{E}_2) = \left(\sinh \acute{E}_1\right)^{\xi_1} \otimes \left(\sinh \acute{E}_2\right)^{\xi_2}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left( \begin{array}{c} \left(\sinh\left(\mu_{\acute{E}_1}\right)^{f}\right)^{\xi_1}, \sqrt[f]{1-\left(1-\sinh\left(1-\nu_{\acute{E}_1}\right)^{f}\right)^{\xi_1}}, \\ \sqrt[f]{1-\left(1-\sinh\left(1-\eta_{\acute{E}_1}\right)^{f}\right)^{\xi_1}} \end{array} \right), \\ \left( \begin{array}{c} \left(\alpha_{\acute{E}_1}\right)^{\xi_1}, \sqrt[f]{1-\left(1-\left(\beta_{\acute{E}_1}\right)^{f}\right)^{\xi_1}}, \\ \sqrt[f]{1-\left(1-\left(\gamma_{\acute{E}_1}\right)^{f}\right)^{\xi_1}} \end{array} \right) \end{array} \right\} \\ \otimes \left\{ \begin{array}{c} \left( \begin{array}{c} \left(\sinh\left(\mu_{\acute{E}_2}\right)^{f}\right)^{\xi_2}, \sqrt[f]{1-\left(1-\sinh\left(1-\nu_{\acute{E}_2}\right)^{f}\right)^{\xi_2}}, \\ \sqrt[f]{1-\left(1-\sinh\left(1-\eta_{\acute{E}_2}\right)^{f}\right)^{\xi_2}} \end{array} \right), \\ \left( \begin{array}{c} \left(\alpha_{\acute{E}_2}\right)^{\xi_2}, \sqrt[f]{1-\left(1-\left(\beta_{\acute{E}_2}\right)^{f}\right)^{\xi_2}}, \\ \sqrt[f]{1-\left(1-\left(\gamma_{\acute{E}_2}\right)^{f}\right)^{\xi_2}} \end{array} \right) \end{array} \right\} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\breve{\imath}=1}^{2}\left(\sinh\left(\mu_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}, \sqrt[f]{1-\prod_{\breve{\imath}=1}^{2}\left(1-\sinh\left(1-\nu_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}}, \\ \sqrt[f]{1-\prod_{\breve{\imath}=1}^{2}\left(1-\sinh\left(1-\eta_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \left(\prod_{\breve{\imath}=1}^{2}\left(\alpha_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}, \sqrt[f]{1-\prod_{\breve{\imath}=1}^{2}\left(1-\left(\beta_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}}, \\ \sqrt[f]{1-\prod_{\breve{\imath}=1}^{2}\left(1-\left(\gamma_{\acute{E}_{\breve{\imath}}}\right)^{f}\right)^{\xi_{\breve{\imath}}}} \end{array} \right) \end{array} \right\}.$$

Thus, Eq (4.2) is true for $\breve{n} = 2$.

(2) Suppose Eq (4.2) is hold for $\breve{n} = \kappa$,

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_\kappa) = \bigotimes_{\breve{\imath}=1}^{\kappa}(\sinh \acute{E}_{\breve{\imath}})^{\xi_{\breve{\imath}}}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa} \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\left(\beta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\}.$$

For $\check{n} = \kappa + 1$, then

$$\sinh -FOFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_\kappa) = \bigotimes_{\check{\imath}=1}^{\kappa} (\sinh \acute{E}_{\check{\imath}})^{\xi_{\check{\imath}}} \otimes (\sinh \acute{E}_{\kappa+1})^{\xi_{\kappa+1}}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa} \left(\sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa} \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\left(\beta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa}\left(1-\left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\}$$

$$\oplus \left\{ \begin{array}{c} \left( \begin{array}{c} \left(\sinh\left(\mu_{\acute{E}_{\kappa+1}}\right)^{f}\right)^{\xi_{\kappa+1}}, \sqrt[f]{1-\left(1-\sinh\left(1-\nu_{\acute{E}_{\kappa+1}}\right)^{f}\right)^{\xi_{\kappa+1}}}, \\ \sqrt[f]{1-\left(1-\sinh\left(1-\eta_{\acute{E}_{\kappa+1}}\right)^{f}\right)^{\xi_{\kappa+1}}} \end{array} \right), \\ \left( \begin{array}{c} \left(\alpha_{\acute{E}_{\kappa+1}}\right)^{\xi_{\kappa+1}}, \sqrt[f]{1-\left(1-\left(\beta_{\acute{E}_{\kappa+1}}\right)^{f}\right)^{\xi_{\kappa+1}}}, \\ \sqrt[f]{1-\left(1-\left(\gamma_{\acute{E}_{\kappa+1}}\right)^{f}\right)^{\xi_{\kappa+1}}} \end{array} \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa+1} \left(\sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa+1}\left(1-\sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa+1}\left(1-\sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\kappa+1} \left(\alpha_{\acute{E}_{\check{\imath}}}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa+1}\left(1-\left(\beta_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1-\prod_{\check{\imath}=1}^{\kappa+1}\left(1-\left(\gamma_{\acute{E}_{\check{\imath}}}\right)^{f}\right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\}.$$

This demonstrates that, for $\check{n} = \kappa + 1$, the theorem is correct. Therefore, (1) and (2) imply that (4.2) holds for any $\check{n}$. □

Based on the sinh-FOLDFWG operator, the following properties can be easily held.

**Theorem 4.2. (Idempotency)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOLDFNs, all are identical, i.e., $\sinh \acute{E}_{\check{\imath}} = \sinh \acute{E}$. Then,

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \sinh \acute{E}. \tag{4.3}$$

**Theorem 4.3. (Boundedness)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOLDFNs. Suppose $\sinh \acute{E}^{-} = \min(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}})$ and $\sinh \acute{E}^{+} = \max(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}})$. Then,

$$\sinh \acute{E}^{-} \leq \sinh -FOLDFDWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) \leq \sinh \acute{E}^{+}. \tag{4.4}$$

**Theorem 4.4. (Monotonicity)**. Suppose $\sinh \acute{E}_{\check{\imath}}$ and $\sinh \acute{E}^{/}_{\check{\imath}} (\check{\imath} = 1, \cdots, \check{n})$ are two family of sinh-FOLDFNs, if $\sinh \acute{E}_{\check{\imath}} \leq \sinh \acute{E}^{/}_{\check{\imath}}$. Then,

$$\sinh -FOLDFWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) \leq \sinh -FOLDFWG(\sinh \acute{E}^{/}_1, \cdots, \sinh \acute{E}^{/}_{\check{n}}).$$

### 4.2. *Sine hyperbolic fractional orthotriple linear Diophantine fuzzy ordered weighted geometric operator*

**Definition 4.2.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\}$ $(\check{\imath} = 1, \cdots, \check{n})$ be a family of sinh-FOLDFNs. A sinh-FOLDFOWG operator for $\check{n}$ dimension is a mapping, $\sinh -FOLDFDOWG : \sinh \acute{E}^{\check{n}} \rightarrow \sinh \acute{E}$ with the corresponding vector $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ along with $\xi_{\check{\imath}} > 0$, and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$, as

$$\sinh -FOLDFOWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigotimes_{\check{\imath}=1}^{m} (\sinh \acute{E}_{\sigma(\check{\imath})})^{\xi_{\check{\imath}}}, \tag{4.5}$$

where $\sigma(1), \cdots, \sigma(\check{n})$ is a permutation of the $(\check{\imath} = 1, \cdots, \check{n})$, for which $\sinh \acute{E}_{\sigma(\check{\imath}-1)} \geq \sinh \acute{E}_{\sigma(\check{\imath})}$

**Theorem 4.5.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^{f}, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^{f}\right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}}\right\rangle\right\}$ $(\check{\imath} = 1, \cdots, \check{n})$ be a family of sinh-FOLDFNs. A sinh-FOLDFOWG operator of dimension $\check{n}$ is a mapping, $\sinh -FOLDFOWG : \sinh \acute{E}^{\check{n}} \rightarrow \sinh \acute{E}$ with corresponding weight vector $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ along with $\xi_{\check{\imath}} > 0$, and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$. Then,

$$\sinh -FOLDFOWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigotimes_{\check{\imath}=1}^{\check{n}} (\sinh \acute{E}_{\sigma(\check{\imath})})^{\xi_{\check{\imath}}}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left(\sinh\left(\mu_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \sinh\left(1 - \nu_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \sinh\left(1 - \eta_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left(\alpha_{\acute{E}_{\sigma(\check{\imath})}}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \left(\beta_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \left(\gamma_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\},$$

where $\sigma(1), \cdots, \sigma(\check{n})$ is the permutation of the $(\check{\imath} = 1, \cdots, \check{n})$, for which $\sinh \acute{E}_{\sigma(\check{\imath}-1)} \geq \sinh \acute{E}_{\sigma(\check{\imath})}$.

The following properties can be easily illustrated by utilizing sinh-FOFOWG.

**Theorem 4.6. (Idempotency)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle\right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOFNs, all are identical, i.e., $\sinh \acute{E}_{\check{\imath}} = \sinh \acute{E}$. Then,

$$\sinh -FOLDFOWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \sinh \acute{E}. \tag{4.6}$$

**Theorem 4.7. (Boundedness)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle\right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOLDFNs. Suppose that $\sinh \acute{E}^- = \min(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}})$ and $\sinh \acute{E}^+ = \max(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}})$. Then,

$$\sinh \acute{E}^- \leq \sinh -FOLDFOWG(\sinh \acute{E}, ..., \sinh \acute{E}_{\check{n}}) \leq \sinh \acute{E}^+. \tag{4.7}$$

**Theorem 4.8. (Monotonicity)**. Suppose

$$\sinh \acute{E}_{\check{\imath}} = \left\{\left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1 - \eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle\right\} \ (\check{\imath} = 1, \cdots, \check{n})$$

be a family of sinh-FOLDFNs, if $\sinh \acute{E}_{\check{\imath}} \leq \sinh \acute{E}^{/}_{\check{\imath}}$. Then,

$$\sinh -FOLDFOWG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) \leq \sinh -FOLDFOWG(\sinh \acute{E}^{/}_1, \cdots, \sinh \acute{E}^{/}_{\check{n}}).$$

where, the permutation of $\sinh \acute{E}_{\check{\imath}}(\check{\imath} = 1, \cdots, \check{n})$ is $\sinh \acute{E}^{/}_{\check{\imath}}(\check{\imath} = 1, \cdots, \check{n})$.

By Def. (4.2), we determine that the sinh-FOLDFWG operator weights are the efficient way to calculate the sinh-FOLDF value. The sinh-FOFOWG operator weight vector is the precise form of the organized position of sinh-FOLDF values. The sinh-FOLDFWG and sinh-FOFOWG operators use weights to express a number of interrelated elements. Since it is typically believed that these aspects will be the same, we create the sinh-FOLDFHG operator to get around this restriction.

*4.3. Sine hyperbolic fractional orthotriple linear Diophantine fuzzy hybrid geometric operator*

**Definition 4.3.** Suppose $\sinh \acute{E}_{\check{\imath}} = \left\{ \left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle \right\}$ ($\check{\imath} = 1, \cdots, \check{n}$) be a family of sinh-FOLDFNs. A sinh-FOLDFHG operator of $\check{n}$ dimension is a mapping, $\sinh -FOLDFHG : \sinh \acute{E}^{\check{n}} \to \sinh \acute{E}$ with corresponding weight $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$ as $\xi_{\check{\imath}} > 0$, and $\sum_{\check{\imath}=1}^{\check{n}} \xi_{\check{\imath}} = 1$. Then,

$$\sinh -FOLDFHG(\sinh \acute{E}_1, \cdots, \sinh \acute{E}_{\check{n}}) = \bigotimes_{\check{\imath}=1}^{\check{n}} \left(\sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})}\right)^{\xi_{\check{\imath}}}$$

$$= \left\{ \begin{array}{c} \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left(\sinh\left(\widetilde{\mu}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \sinh\left(1-\widetilde{\nu}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \sinh\left(1-\widetilde{\eta}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}} \end{array} \right), \\ \left( \begin{array}{c} \prod_{\check{\imath}=1}^{\check{n}} \left(\widetilde{\alpha}_{\acute{E}_{\sigma(\check{\imath})}}\right)^{\xi_{\check{\imath}}}, \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \left(\widetilde{\beta}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}}, \\ \sqrt[f]{1 - \prod_{\check{\imath}=1}^{\check{n}} \left(1 - \left(\widetilde{\gamma}_{\acute{E}_{\sigma(\check{\imath})}}\right)^f\right)^{\xi_{\check{\imath}}}} \end{array} \right) \end{array} \right\},$$

where $\sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})}$ denotes the value of the $\check{\imath}^{th}$ largest weight sine hyperbolic FOLDFNs, and $\sinh \widetilde{\acute{E}}_{\sigma(\check{\imath})} = \left(\sinh \acute{E}_{\check{\imath}}\right)^{\check{n} w_{\check{\imath}}}$ ($\check{\imath} = 1, \cdots, \check{n}$), and $w = (w_1, \cdots, w_{\check{n}})^T$ is the associated weights of $\sinh \dot{\acute{E}}_{\sigma(\check{\imath})}$ with $w_{\check{\imath}} > 0$, and $\sum_{\check{\imath}=1}^{\check{n}} w_{\check{\imath}} = 1$, $\check{n}$ is the balancing coefficient. When the operator is equal to $w = (1/\check{n}, \cdots, 1/\check{n})^T$, sinh-FOLDFWG and sinh-FOLDFOWG are regarded as a special case of sinh-FOLDFHG operator. Since the sinh-FOLDFHG operator denoted the degree and structured condition of the disagreement statements, it is a generalized version of the sinh-FOFWG and sinh-FOLDFOWG operators.

## 5. MCGDM approach based on sinh-fractional orthotriple linear Diophantine fuzzy aggregation operators

Let's assume, we have a DM problem with $m$ possible alternatives $(\mathfrak{R}_1, \cdots, \mathfrak{R}_m)$, $\check{n}$ possible attributes $(\mathfrak{I}_1, \cdots, \mathfrak{I}_{\check{n}})$ and $d$ experts $(E_1, \cdots, E_d)$. Let the expert and attribute weights $\varpi = (\varpi_1, \cdots, \varpi_d)^T$ and $\xi = (\xi_1, \cdots, \xi_{\check{n}})^T$, respectively. Where, $\varpi, \xi \in [0, 1]$ and $\sum_{j=1}^{\check{n}} \varpi_j, \xi_j = 1$ are exist. Each expert $E_\kappa$ evaluates the offered options $\mathfrak{R}_{\check{\imath}}$ in terms of FOLDFNs and rates them according to the attribute $\mathfrak{I}_j$. The steps listed below are used.

**Step 1.** Using the fractional orthotriple fuzzy information developed in the decision matrix.

**Step 2.** If the decision matrix contains cost type data, normalize it according to the following principle, such as

$$\mathfrak{R}_{\check{\imath}j}^{\kappa} = \begin{cases} \left\{ \left\langle \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \alpha_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \gamma_{\acute{E}_{\check{\imath}}} \right\rangle \right\}, & \sinh \acute{E}_{\check{\imath}} \text{ is benefit type;} \\ \left\{ \left\langle \sinh\left(1-\eta_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(1-\nu_{\acute{E}_{\check{\imath}}}\right)^f, \sinh\left(\mu_{\acute{E}_{\check{\imath}}}\right)^f \right\rangle, \left\langle \gamma_{\acute{E}_{\check{\imath}}}, \beta_{\acute{E}_{\check{\imath}}}, \alpha_{\acute{E}_{\check{\imath}}} \right\rangle \right\}, & \sinh \acute{E}_{\check{\imath}} \text{ is cost type.} \end{cases}$$

**Step 3a.** Combine the various preferences of the alternatives $\mathfrak{R}^{\kappa}_{\breve{\imath}j}$ ($\kappa = 1, \cdots, d$) using the sinh-FOLDFWA operator to create $\mathfrak{R}_{\breve{\imath}j}$.

**Step 3b.** Combine the various preferences of the alternatives $\mathfrak{R}^{\kappa}_{\breve{\imath}j}$ ($\kappa = 1, \cdots, d$) using the sinh-FOLDFWG operator to create $\mathfrak{R}_{\breve{\imath}j}$.

**Step 4a.** Get the sum of the values for each $\mathfrak{R}_{\breve{\imath}}$ using the suggested operator (sinh-FOLDFOWA) and the weight vector $\xi = (\xi_1, \cdots, \xi_{\breve{n}})^T$.

**Step 4b.** Get the sum of the values for each $\mathfrak{R}_{\breve{\imath}}$ using the suggested operator (sinh-FOLDFOWG) and the weight vector $\xi = (\xi_1, \cdots, \xi_{\breve{n}})^T$.

**Step 5.** Determine the alternative score values, use Eq (2.7).

**Step 6.** Give each possibility $\mathfrak{R}_{\breve{\imath}}(\breve{\imath} = 1, \cdots, 4)$ a ranking based on the definition of (2.1) and then select the top one.

## 6. Numerical example

Using a numerical example to choose the best industry for investment out of four options, the proposed MCGDM method is illustrated (adapted from [18]).

The company's board of directors decided to make use of idle capital by making an investment in a new sector of the economy. Four industries were chosen as prospective areas for investment after the preliminary assessment. The four alternative $\mathfrak{R}_1$ : Industries are manufacturing industry, $\mathfrak{R}_2$ : Real estate development industry, $\mathfrak{R}_3$ : Education and training industry, and $\mathfrak{R}_4$ : Medical industry. The directors have assembled a group of specialists to decide which investment would be the greatest decision. These professionals were asked to rate the four alternative industries using the following four criteria:

$\mathfrak{I}_1$ : Level of capital gain,

$\mathfrak{I}_2$ : Market potential,

$\mathfrak{I}_3$ : Growth potential,

$\mathfrak{I}_4$ : Political stability,

The three experts, $E_1$, $E_2$, and $E_3$, were permitted to employ sinh-FOLDFNs to provide adequate freedom in their assessments of the values of the key criteria of each alternative industry.

Assume that the weights for the expert and the criteria are, respectively, $\varpi = (0.4, 0.3, 0.3)^T$ and $\xi = (0.3, 0.2, 0.3, 0.2)^T$. The sinh-FOLDFWA and sinh-FOLDFWG operators first remedied the issue. The following matrix includes specific information on expert evaluation.

**Step 1.** Three decision Tables given by three experts are listed below (see Tables 1–3):

**Table 1.** sinh-FOLDF information given by first expert.

| | $\mathfrak{I}_1$ | $\mathfrak{I}_2$ | $\mathfrak{I}_3$ | $\mathfrak{I}_4$ |
|---|---|---|---|---|
| $\mathfrak{R}_1$ | $\begin{pmatrix} \langle 0.6, 0.5, 0.4 \rangle, \\ \langle 0.4, 0.6, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.3 \rangle, \\ \langle 0.7, 0.3, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.5 \rangle, \\ \langle 0.5, 0.4, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.5, 0.2 \rangle, \\ \langle 0.7, 0.3, 0.2 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_2$ | $\begin{pmatrix} \langle 0.7, 0.3, 0.2 \rangle, \\ \langle 0.4, 0.6, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.7, 0.2, 0.5 \rangle, \\ \langle 0.4, 0.5, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.4, 0.3 \rangle, \\ \langle 0.4, 0.5, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.7, 0.5 \rangle, \\ \langle 0.5, 0.4, 0.4 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_3$ | $\begin{pmatrix} \langle 0.5, 0.4, 0.4 \rangle, \\ \langle 0.6, 0.4, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.6, 0.3 \rangle, \\ \langle 0.7, 0.5, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.7, 0.5, 0.2 \rangle, \\ \langle 0.5, 0.4, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.4, 0.5 \rangle, \\ \langle 0.4, 0.5, 0.2 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_4$ | $\begin{pmatrix} \langle 0.7, 0.5, 0.3 \rangle, \\ \langle 0.6, 0.7, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.7, 0.2 \rangle, \\ \langle 0.4, 0.2, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.5, 0.2 \rangle, \\ \langle 0.4, 0.6, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.2, 0.4 \rangle, \\ \langle 0.4, 0.6, 0.3 \rangle \end{pmatrix}$ |

**Table 2.** sinh-FOLDF information given by second expert.

| | $\mathfrak{I}_1$ | $\mathfrak{I}_2$ | $\mathfrak{I}_3$ | $\mathfrak{I}_4$ |
|---|---|---|---|---|
| $\mathfrak{R}_1$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.7 \rangle, \\ \langle 0.5, 0.2, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.5, 0.7 \rangle, \\ \langle 0.3, 0.5, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.6, 0.4 \rangle, \\ \langle 0.6, 0.5, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.6, 0.2 \rangle, \\ \langle 0.5, 0.7, 0.3 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_2$ | $\begin{pmatrix} \langle 0.5, 0.7, 0.3 \rangle, \\ \langle 0.3, 0.6, 0.7 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.6, 0.2 \rangle, \\ \langle 0.5, 0.2, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.2, 0.3 \rangle, \\ \langle 0.3, 0.6, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.4, 0.3 \rangle, \\ \langle 0.2, 0.3, 0.4 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_3$ | $\begin{pmatrix} \langle 0.2, 0.6, 0.3 \rangle, \\ \langle 0.6, 0.5, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.3, 0.4 \rangle, \\ \langle 0.3, 0.7, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.5 \rangle, \\ \langle 0.6, 0.2, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.7, 0.4 \rangle, \\ \langle 0.6, 0.5, 0.2 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_4$ | $\begin{pmatrix} \langle 0.5, 0.4, 0.5 \rangle, \\ \langle 0.5, 0.2, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.5, 0.2 \rangle, \\ \langle 0.4, 0.3, 0.6 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.5, 0.6 \rangle, \\ \langle 0.3, 0.7, 0.4 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.3, 0.6 \rangle, \\ \langle 0.3, 0.6, 0.5 \rangle \end{pmatrix}$ |

**Table 3.** sinh-FOLDF information given by third expert.

| | $\mathfrak{I}_1$ | $\mathfrak{I}_2$ | $\mathfrak{I}_3$ | $\mathfrak{I}_4$ |
|---|---|---|---|---|
| $\mathfrak{R}_1$ | $\begin{pmatrix} \langle 0.6, 0.3, 0.5 \rangle, \\ \langle 0.2, 0.5, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.7, 0.3 \rangle, \\ \langle 0.3, 0.4, 0.6 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.7, 0.5, 0.3 \rangle, \\ \langle 0.3, 0.4, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.3, 0.2 \rangle, \\ \langle 0.5, 0.2, 0.4 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_2$ | $\begin{pmatrix} \langle 0.7, 0.3, 0.5 \rangle, \\ \langle 0.3, 0.4, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.7, 0.5, 0.4 \rangle, \\ \langle 0.6, 0.3, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.3, 0.2 \rangle, \\ \langle 0.5, 0.3, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.4, 0.5 \rangle, \\ \langle 0.7, 0.5, 0.3 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_3$ | $\begin{pmatrix} \langle 0.6, 0.7, 0.2 \rangle, \\ \langle 0.5, 0.3, 0.2 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.3, 0.4, 0.5 \rangle, \\ \langle 0.2, 0.6, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.5 \rangle, \\ \langle 0.3, 0.4, 0.5 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.5, 0.5, 0.4 \rangle, \\ \langle 0.5, 0.3, 0.2 \rangle \end{pmatrix}$ |
| $\mathfrak{R}_4$ | $\begin{pmatrix} \langle 0.3, 0.4, 0.6 \rangle, \\ \langle 0.6, 0.4, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.4, 0.2, 0.4 \rangle, \\ \langle 0.7, 0.5, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.2, 0.5, 0.3 \rangle, \\ \langle 0.6, 0.2, 0.3 \rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.6, 0.4, 0.3 \rangle, \\ \langle 0.2, 0.4, 0.5 \rangle \end{pmatrix}$ |

**Step 2.** Since every criterion represents a benefit, normalization is not necessary.

**Step 3a.** Let the experts' weight, which equals $\varpi = (0.4, 0.3, 0.3)^T$ and $f = 3$, be the deciding factor. Then, using the supplied decision matrix, apply the sinh-FOLDFWA operator to produce a new aggregated Table 4.

**Table 4.** Aggregated values based on sinh-FOLDFWA operator.

| | $\mathfrak{I}_1$ | $\mathfrak{I}_2$ | $\mathfrak{I}_3$ | $\mathfrak{I}_4$ |
|---|---|---|---|---|
| $\mathfrak{R}_1$ | $\begin{pmatrix} \langle 0.35,0.36,0.41\rangle, \\ \langle 0.43,0.13,0.32\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.23,0.35,0.42\rangle, \\ \langle 0.23,0.33,0.27\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.22,0.28,0.36\rangle, \\ \langle 0.25,0.23,0.33\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.36,0.27,0.19\rangle, \\ \langle 0.25,0.25,0.34\rangle \end{pmatrix}$ |
| $\mathfrak{R}_2$ | $\begin{pmatrix} \langle 0.26,0.53,0.29\rangle, \\ \langle 0.61,0.45,0.15\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.46,0.32,0.21\rangle, \\ \langle 0.19,0.24,0.35\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.53,0.38,0.42\rangle, \\ \langle 0.55,0.17,0.25\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.43,0.42,0.23\rangle, \\ \langle 0.53,0.28,0.43\rangle \end{pmatrix}$ |
| $\mathfrak{R}_3$ | $\begin{pmatrix} \langle 0.49,0.14,0.50\rangle, \\ \langle 0.27,0.25,0.36\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.37,0.24,0.32\rangle, \\ \langle 0.41,0.42,0.24\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.31,0.16,0.33\rangle, \\ \langle 0.36,0.23,0.30\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.31,0.34,0.36\rangle, \\ \langle 0.34,0.42,0.15\rangle \end{pmatrix}$ |
| $\mathfrak{R}_4$ | $\begin{pmatrix} \langle 0.56,0.26,0.34\rangle, \\ \langle 0.35,0.47,0.423\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.56,0.13,0.13\rangle, \\ \langle 0.28,0.24,0.42\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.33,0.23,0.24\rangle, \\ \langle 0.46,0.457,0.29\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.25,0.13,0.45\rangle, \\ \langle 0.43,0.236,0.30\rangle \end{pmatrix}$ |

**Step 3b.** Let the experts' weight, which equals $\varpi = (0.4, 0.3, 0.3)^T$ and $f = 3$, be the deciding factor. Then, using the supplied decision matrix, apply the sinh-FOLDFWG operator to produce a new aggregated Table 5.

**Table 5.** Aggregated values based on sinh-FOLDFWG operator.

| | $\mathfrak{I}_1$ | $\mathfrak{I}_2$ | $\mathfrak{I}_3$ | $\mathfrak{I}_4$ |
|---|---|---|---|---|
| $\mathfrak{R}_1$ | $\begin{pmatrix} \langle 0.45,0.49,0.24\rangle, \\ \langle 0.44,0.24,0.52\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.35,0.23,0.52\rangle, \\ \langle 0.43,0.52,0.52\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.43,0.39,0.21\rangle, \\ \langle 0.35,0.23,0.40\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.34,0.52,0.25\rangle, \\ \langle 0.36,0.23,0.23\rangle \end{pmatrix}$ |
| $\mathfrak{R}_2$ | $\begin{pmatrix} \langle 0.53,0.16,0.33\rangle, \\ \langle 0.35,0.43,0.39\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.28,0.51,0.38\rangle, \\ \langle 0.31,0.43,0.42\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.20,0.42,0.36\rangle, \\ \langle 0.51,0.45,0.31\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.31,0.41,0.34\rangle, \\ \langle 0.40,0.31,0.32\rangle \end{pmatrix}$ |
| $\mathfrak{R}_3$ | $\begin{pmatrix} \langle 0.21,0.46,0.27\rangle, \\ \langle 0.23,0.32,0.42\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.46,0.42,0.46\rangle, \\ \langle 0.32,0.25,0.21\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.34,0.25,0.23\rangle, \\ \langle 0.29,0.41,0.59\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.23,0.36,0.47\rangle, \\ \langle 0.24,0.42,0.56\rangle \end{pmatrix}$ |
| $\mathfrak{R}_4$ | $\begin{pmatrix} \langle 0.61,0.31,0.13\rangle, \\ \langle 0.39,0.62,0.35\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.47,0.31,0.20\rangle, \\ \langle 0.54,0.33,0.33\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.21,0.56,0.30\rangle, \\ \langle 0.45,0.32,0.53\rangle \end{pmatrix}$ | $\begin{pmatrix} \langle 0.47,0.26,0.32\rangle, \\ \langle 0.31,0.32,0.36\rangle \end{pmatrix}$ |

**Step 4a.** Using $f = 3$ and the sinh-FOLDFOWA operator, we find the performance values for each alternative $\mathfrak{R}_{\breve{\imath}}(\breve{\imath} = 1, \cdots, 4)$ using the specified weight vector $\xi = (0.3, 0.2, 0.3, 0.2)^T$.

$$\begin{aligned}
\mathfrak{R}_1 &= (\langle 0.672, 0.550, 0.344\rangle, \langle 0.678, 0.614, 0.457\rangle),\\
\mathfrak{R}_2 &= (\langle 0.667, 0.358, 0.455\rangle, \langle 0.567, 0.448, 0.568\rangle),\\
\mathfrak{R}_3 &= (\langle 0.461, 0.543, 0.341\rangle, \langle 0.767, 0.594, 0.367\rangle),\\
\mathfrak{R}_4 &= (\langle 0.558, 0.462, 0.535\rangle, \langle 0.676, 0.348, 0.551\rangle).
\end{aligned}$$

**Step 4b.** Using $f = 3$ and the sinh-FOLDFOWG operator, we find the performance values for each alternative $\mathfrak{R}_{\breve{\imath}}(\breve{\imath} = 1, \cdots, 4)$ using the specified weight vector $\xi = (0.3, 0.2, 0.3, 0.2)^T$.

$$\mathfrak{R}_1 = (\langle 0.750, 0.561, 0.555\rangle, \langle 0.671, 0.588, 0.568\rangle),$$

$$\begin{aligned}
\mathfrak{R}_2 &= (\langle 0.671, 0.535, 0.675\rangle, \langle 0.567, 0.469, 0.547\rangle),\\
\mathfrak{R}_3 &= (\langle 0.567, 0.468, 0.459\rangle, \langle 0.776, 0.517, 0.468\rangle),\\
\mathfrak{R}_4 &= (\langle 0.573, 0.446, 0.365\rangle, \langle 0.539, 0.660, 0.644\rangle).
\end{aligned}$$

**Step 5.** The values of the alternatives $\mathfrak{R}_{\breve{\imath}}(\breve{\imath} = 1, \cdots, 4)$ for all sinh-FOLDFNs are determined using Def. (2.1) as follows;

$$\Lambda(\mathfrak{R}_1) = 0.040, \Lambda(\mathfrak{R}_2) = 0.032, \Lambda(\mathfrak{R}_3) = 0.045, \Lambda(\mathfrak{R}_4) = 0.011.$$

And

$$\Lambda(\mathfrak{R}_1) = -0.005, \Lambda(\mathfrak{R}_2) = -0.121, \Lambda(\mathfrak{R}_3) = 0.010, \Lambda(\mathfrak{R}_4) = -0.173.$$

**Step 6.** The following outcome is obtained by ranking the alternatives $\mathfrak{R}_{\breve{\imath}}(\breve{\imath} = 1, \cdots, 4)$.

$$\mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2 > \mathfrak{R}_4.$$

$\mathfrak{R}_3$ is therefore the most beneficial alternative.

## 7. Comparative analysis

In this section, the comparative study of sinh-FOFWA and sinh-FOFWG operators is established with Spherical fuzzy and fractional orthotriple fuzzy aggregation operators. From our study, we claim that the pre-existing aggregation operators in the environment of SFSs and FOFSs cannot handle the data provided in the form of FOLDFNs.

We compare our defined method with current methods, such as [4, 6, 15, 23, 25, 26], which deal with Spherical fuzzy operators and fractional orthotriple fuzzy operators. Our defined method is based on sinh-FOF information. Operators exhibit advanced reliability in the sphere of actual manipulation. Table 6 presents the results of the comparison. It has been found from the assessment Table 6 that the best alternative produced utilizing the suggested strategy conforms to the results of these earlier experiments. As a result, compared to other existing techniques, our suggested alternative selection methodology using sinh-FOLDFWA and sinh-FOLDFWG operators is more adaptable and efficient. $\mathfrak{R}_3$ is the preferred option according to all procedures, as indicated in Table 6, but different approaches need different computation steps. For instance, the authors of the prior approaches aggregated information using Spherical fuzzy and fractional orthotriple fuzzy operators, however in our suggested method, we use sinh-FOLDF aggregation operators.

**Table 6.** Score values and ranking using different methods.

| Methods | $\mathfrak{R}_1$ | $\mathfrak{R}_2$ | $\mathfrak{R}_3$ | $\mathfrak{R}_4$ | Ranking order |
|---|---|---|---|---|---|
| Kutlu et al. [15] | 0.27 | 0.24 | 0.32 | 0.20 | $\mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2 > \mathfrak{R}_4$ |
| Ashraf et al. [4] | 0.77 | 0.679 | 0.82 | 0.64 | $\mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2 > \mathfrak{R}_4$ |
| Riaz et al. [33] | 0.39 | 0.376 | 0.47 | 0.44 | $\mathfrak{R}_3 > \mathfrak{R}_4 > \mathfrak{R}_1 > \mathfrak{R}_2$ |
| Abosuliman et al. [6] | 0.145 | 0.15 | 0.15 | 0.14 | $\mathfrak{R}_3 > \mathfrak{R}_2 > \mathfrak{R}_1 > \mathfrak{R}_4$ |
| Naeem et al. [23] | 0.46 | 0.41 | 0.49 | 0.43 | $\mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_4 > \mathfrak{R}_2$ |
| Qiyas et al. [25] | 0.81 | 0.86 | 0.93 | 0.88 | $\mathfrak{R}_3 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_1$ |
| Qiyas et al. [26] | 0.72 | 0.72 | 0.75 | 0.68 | $\mathfrak{R}_3 > \mathfrak{R}_1 > \mathfrak{R}_2 > \mathfrak{R}_4$ |

## 8. Conclusions

In this article, we investigated the issue of choosing a better alternative while employing sine hyperbolic fractional orthotriple fuzzy information. We investigated averaging and geometric operations based on algebraic operations in order to create specialized sinh-FOLDF aggregation operators, such as the sinh-FOLDFWA, sinh-FOLDFOWA, sinh-FOLDFHA, sinh-FOLDFWG, sinh-FOLDFOWG, and sinh-FOLDFHG operators. Some properties of the produced operators are discussed. To deal with the ambiguity in the data, we applied operational laws based on sine hyperbolic functions, which will prevent information loss during the study. The suggested strategy takes care of the DM issue as well. The study is based on the representation of the sinh-FOLDFOLs, and the resulting operators are generalizations of the existing operators from the provided instances. The defined AOs are utilized for solving a MCGDM problem. Using a real-world situation, the effectiveness of the suggested operators is evaluated. Finally, a comparison study has been given to demonstrate how the suggested operators are advantageous. As a result, the suggested operators are more broad, consistent, and detailed in order to address DM concerns in the FOLDFS environment.

In the future, we will extend our proposed idea to the Einstein operation, Hamacher operation, Frank operation, and Bonferroni mean operation, Heronian mean operation. Also, in a decision-making problem relate group decision makers, it should be reach a consensus before using aggregation operators to get a collective opinion, such as [19, 35, 38, 46].

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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