Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11837–11850.
DOI: 10.3934/math.2023599
Received: 28 October 2022
Revised: 05 March 2023
Accepted: 08 March 2023
Published: 20 March 2023

Research article

The new general solution for a class of fractional-order impulsive differential equations involving the Riemann-Liouville type Hadamard fractional derivative

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Abstract: In this paper, the new general solution for a class of higher-order impulsive fractional differential equations (IFDEs) involving the Riemann-Liouville (R-L) type Hadamard fractional derivative (FD) is presented. Specifically, the necessary and sufficient conditions of the solution are obtained by converting boundary value problems (BVPs) into integral equations and applying analytical techniques. The results in the paper provide a new method for converting BVPs or initial value problems (IVPs) for IFDEs to integral equations. Finally, some examples are devoted to explaining the application of the theorem.

Keywords: higher-order fractional differential equation; general solution; Riemann-Liouville type Hadamard fractional derivative
Mathematics Subject Classification: 34A08, 34A37

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1. Introduction

Fractional calculus is used as a powerful tool to describe the memory and hereditary properties of various materials and processes [1–4]. Compared with the classical integer-order differential equation models, the fractional-order differential equation models become more practical and realistic by the effect of these characteristics of the FDs. As a matter of fact, in many engineering and scientific disciplines, fractional differential equations (FDEs) all exist, among which include physics, chemistry, biology, control theory, economics, signal and image processing, blood flow phenomena, aerodynamics, biophysics, fitting of experimental data, and so on [1,3,4]. For further developments about this topic, please refer to the references [5–10] and see the related references therein.

It is important to study the IFDE. The existence of solutions about various IVPs for IFDEs included with Caputo FDs or the R-L FDs was studied by authors in papers [11–17] recently. The solvability of various BVPs of impulsive DEs involved with the R-L FDs was researched in the references [18–25] in the past few years.

However, we should notice that most of the works on this subject in the past were based on R-L FDEs and Caputo FDEs. In 1892, another kind of FD was introduced by Hadamard, called the Hadamard type FDE, which differs from the previous FDE in that the kernel about the integral and derivative contains a logarithmic function of the arbitrary exponent. One similar property for both the Hadamard and R-L FDs is based on the fact that the derivative of a constant is not equal to zero. That's because their definitions include the general derivatives beyond the integral.

Recently, some results have been achieved in the study of BVPs or IVPs for IFDEs, which involve IFDEs with Hadamard derivative with a single starting point see the references [26–31]. For example, the existence of solutions for a class of IVPs for impulsive Hadamard FDEs was established by Wang etc in [29].

$$\begin{aligned} &{}^{H}D_{1^{+}}^{\alpha}u(t) = f(t, u(t)), t \in (1, e] \setminus \{t_1, \cdots, t_m\}, \\ &{}^{H}I_{1^{+}}^{1-\alpha}u\left(t_i^{+}\right) - {}^{H}I_{1^{+}}^{1-\alpha}u\left(t_i^{-}\right) = p_i \in \mathbb{R}, i = 1, 2, \cdots, m, \\ &{}^{H}I_{1^{+}}^{1-\alpha}u(1) = u_0 \in \mathbb{R}, \end{aligned} \tag{1}$$

where $\alpha \in (0, 1)$, ${}^{H}D_{1+}^{\alpha}$ is the left-side Hadamard FD of order α with the single starting point 1 and ${}^{H}I_{1+}^{1-\alpha}$ denotes left-side Hadamard fractional integral (FI) of order $1 - \alpha$. The existence of the solution was proved by using the Banach contraction principle and Schauder's fixed point theorem on the weight spaces of piecewise continuous functions.

The following Cauchy problem with R-L Hadamard FD and the impulsive effect was studied in [31]:

$$\begin{aligned} &{}^{H}D_{a^{+}}^{\alpha}u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ &\Delta^{H}D_{a^{+}}^{\alpha-1}u(t_i) = \phi_i(u(t_i)), i = 1, 2, \cdots, m, \\ &\Delta^{H}I_{a^{+}}^{2-\alpha}u(t_i) = \psi_i(u(t_i)), i = 1, 2, \cdots, m, \\ &{}^{H}D_{a^{+}}^{\alpha-1}u(a) = u_1 \in \mathbb{R}, {}^{H}I_{a^{+}}^{2-\alpha}u(a) = u_2 \in \mathbb{R}, \end{aligned} \tag{2}$$

where $\alpha \in (1, 2]$, ${}^{H}D_{a^{+}}^{*}$ is the R-L Hadamard FD of order $*$, ${}^{H}I_{a^{+}}^{*}$ is the Hadamard FI of order $*, a = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$ are the impulse points, $f : (a, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\phi_i, \psi_i \in C(\mathbb{R}, \mathbb{R})$. The jump conditions are defined by $\Delta u(t_i) = u\left(t_i^{+}\right) - u(t_i), i = 1, 2, \cdots, m.$

We find that some of the lemmas about the BVPs for IFDEs involving Hadamard FDs in [31] are wrong (see Remarks 3.1 and 3.2 and the counterexample in Section 3), despite ideas in these documents are very good. For this reason, the article gives the following new general solution for the FDE

$${}^{H}D_{1^{+}}^{\alpha}x(t) = g(t), \text{ a.e. }, t \in (t_i, t_{i+1}], i \in \mathbb{Z}_0^m, \tag{3}$$

where

(i) $\alpha \in (n-1, n)$ which n is a positive integer, $\mathbb{Z}_b^c = [b, b+1, b+2, \cdots, c]$ with b, c being nonnegative integers,

(ii) $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$ and $t_i (i = 1, 2, \cdots, m)$ are constants (i.e., impulse points), $q \in \mathbb{Z}_1^{n-1}$,

(iii) $g : (1, e] \to \mathbb{R}$, $g \in C^0(1, e]$, $|g(t)| \le (\ln t)^k (1 - \ln t)^l$, a.e., $t \in (1, e], k > -1, l \le 0, 1 + k + l > 0$,

(iv) ${}^H D_{1^+}^{*}$ is the standard R-L Hadamard FD of order $*$ with the starting point $t = 1$.

A function $x : (1, e] \to \mathbb{R}$ is called a piecewise continuous solution of (3) if $x|_{(t_i, t_{i+1}]} \left(i \in \mathbb{Z}_0^m\right)$ is continuous, $\lim_{t \to t_i^+} (\ln t - \ln t_i)^{n-\alpha} x(t)$ is finite, ${}^H D_{1^+}^{\alpha} x \in L^1(1, e]$ and satisfies (3) for almost all $t \in (1, e]$.

In Section 2, some definitions are given, and we establish the new expression of the general solution of (3) in Theorem 2.1. In Section 3, we point some incorrect lemmas out in recently published papers by remarks and counterexamples.

2. Main results

Let us recall some basic definitions of fractional calculus [1,25]. Then the main result will be proved. Let the Gamma function and the Beta function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, B(p, q) = \int_0^1 (1 - x)^{p-1} x^{q-1} dx,$$

respectively for $\alpha > 0, p, q > 0$. We denote $\log_e t$ by $\ln t$ for $t > 0$.

Definition 2.1. *[25]* Let $\alpha > 0$ and $h : (a, b) \to \mathbb{R}$ be a sufficiently good function. The left side Hadamard FI of order α about h is given by

$${}^H I_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln s)^{\alpha-1} h(s) \frac{ds}{s}, t > a,$$

provided that the right-hand side makes sense.

The left side Hadamard FD of order $\alpha \in (n - 1, n)$ which n is a positive integer about h is given by

$${}^H D_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^n \left[\int_a^t (\ln t - \ln s)^{n-\alpha-1} h(s) \frac{ds}{s}\right], t > a,$$

provided that the right-hand side makes sense.

Let $\alpha \in (n - 1, n)$ with n being a positive integer, $q \in \mathbb{Z}_1^{n-1}$. The left side mixed Hadamard FD of order α of h is given by

$${}^q D_{a^+}^{\alpha} h(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^q \left[\int_a^t (\ln t - \ln s)^{n-\alpha-1} h(s) \frac{ds}{s}\right], t > a,$$

provided that the right-hand side makes sense.

Remark 2.1. Let $\alpha \in (n-1, n)$ which n is a positive integer. It is easy to know that ${}^n D_{a^+}^{\alpha} h(t) = {}^H D_{a^+}^{\alpha} h(t)$.

Remark 2.2. Let $\alpha > 0$ and $h : (a, b) \to \mathbb{R}$ be a piecewise continuous function, i.e., h is continuous on each subinterval $(t_i, t_{i+1}]$ $(i \in \mathbb{Z}_a^m = \{0, 1, 2, \cdots, m\}, a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e)$. The

Hadamard FI of order $\alpha > 0$ of h at the point $t \in (t_i, t_{i+1}]$ is given by

$$ {}^{H}I_{a^+}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\ln t - \ln s)^{\alpha-1}h(s)\frac{ds}{s} $$

$$ = \sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s} + \int_{t_i}^{t}\frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], $$

provided that each term of the right-hand side makes sense (it means $\int_{t_i}^{t_{i+1}}(\ln t - \ln s)^{\alpha-1}h(s)\frac{ds}{s}(i \in \mathbb{Z}_0^m)$ and $\int_{t_i}^{t}(\ln t - \ln s)^{\alpha-1}h(s)\frac{ds}{s}$ exist).

Let $\alpha > 0$ and $h : (a, b) \to \mathbb{R}$ be a piecewise continuous function, i.e., h is continuous on each subinterval $(t_i, t_{i+1}]\,(i \in \mathbb{Z}_a^m = \{0, 1, 2, \cdots, m\}, a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e)$. The R-L Hadamard FD of order α about the function $h : (a, b) \to \mathbb{R}$ at the point $t \in (t_i, t_{i+1}]$ is given by

$$ {}^{H}D_{a^+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \frac{h(s)}{(\ln t - \ln s)^{\alpha-n+1}}\frac{ds}{s} $$

$$ = \sum_{i=0}^{m}\left(t\frac{d}{dt}\right)^n\int_{t_i}^{t_{i+1}}\frac{(\ln t - \ln s)^{n-\alpha-1}}{\Gamma(n-\alpha)}h(s)\frac{ds}{s} + \left(t\frac{d}{dt}\right)^n\int_{t_i}^{t}\frac{(\ln t - \ln s)^{n-\alpha-1}}{\Gamma(n-\alpha)}h(s)\frac{ds}{s}, $$

where $n - 1 < \alpha < n$, provided that each term of the right-hand side exists (it means both $\left(t\frac{d}{dt}\right)^n\int_{t_i}^{t_{i+1}}(\ln t - \ln s)^{n-\alpha-1}h(s)\frac{ds}{s}\left(i \in \mathbb{Z}_0^m\right)$ and $\left(t\frac{d}{dt}\right)^n\int_{t_i}^{t}(\ln t - \ln s)^{n-\alpha-1}h(s)\frac{ds}{s}$ exist).

Now, we establish the new expression of the general solution of (3).

Theorem 2.1 Suppose that (i)–(iv) hold. Then x is a solution of (3) if and only if there exist constants $c_{j,v} \in \mathbb{R}\ (v = 1, 2, \cdots, n,\ \ j = 0, 1, 2, \cdots, m)$ such that

$$ x(t) = \sum_{j=0}^{i}\sum_{v=1}^{n}c_{j,v}\left(\ln t - \ln t_j\right)^{\alpha-v} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)}g(s)\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{Z}_0^m, \tag{4} $$

Proof. **Step I.** Suppose that x is a solution of (4). We prove that x is a piecewise continuous solution of (3).

Since $|g(t)| \le (\ln t)^k(1 - \ln t)^l$ for all $t \in (1, e)$ and g is a continuous function on $(1, e]$, we obtain for $t \in (1, e]$ that

$$ \left(\ln t - \ln t_j\right)^{n-\alpha}\left|\int_1^t \frac{(\ln t - \ln s)^{\alpha-i-1}}{\Gamma(\alpha-i)}g(s)\frac{ds}{s}\right| \le \left(\ln t - \ln t_j\right)^{n-\alpha}\int_1^t \frac{(\ln t - \ln s)^{\alpha-i-1}}{\Gamma(\alpha-i)}|g(s)|\frac{ds}{s} $$

$$ \le \left(\ln t - \ln t_j\right)^{n-\alpha}\int_1^t \frac{(\ln t - \ln s)^{\alpha-i-1}}{\Gamma(\alpha-i)}(\ln s)^k(1 - \ln s)^l\frac{ds}{s} $$

$$ \le \left(\ln t - \ln t_j\right)^{n-\alpha}\int_1^t \frac{(\ln t - \ln s)^{\alpha+l-i-1}}{\Gamma(\alpha-i)}(\ln s)^k\frac{ds}{s}(\text{ by } \frac{\ln s}{\ln t} = w) $$

$$ = \left(\ln t - \ln t_j\right)^{n-\alpha}(\ln t)^{\alpha-i+k+l}\int_0^1 \frac{(1-w)^{\alpha+l-i-1}}{\Gamma(\alpha-i)}w^k d. $$

Then $x|_{(t_j,t_{j+1}]} \in C\left(t_j, t_{j+1}\right]$ and $\lim_{t\to t_j^+}\left(\ln t - \ln t_j\right)^{n-\alpha}x(t)$ is finite by $1 + k + l > 0$.

From (4), for $t \in (t_i, t_{i+1}]$ and Remark 2.1, we have

$$
\begin{aligned}
{}^H D_{1^+}^{\alpha} x(t) &= \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln s)^{n-\alpha-1} x(s) \frac{ds}{s} \\
&= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1} x(s)\frac{ds}{s} + \left(t\frac{d}{dt}\right)^n \int_{t_i}^{t} (\ln t - \ln s)^{n-\alpha-1} x(s)\frac{ds}{s}}{\Gamma(n-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1}\left(\sum_{\kappa=0}^{j}\sum_{v=1}^{n} c_{\kappa,v}(\ln s - \ln t_\kappa)^{\alpha-v} + \int_1^s \frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u)\frac{du}{u}\right)\frac{ds}{s}}{\Gamma(n-\alpha)} \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \int_{t_i}^{t} (\ln t - \ln s)^{n-\alpha-1}\left(\sum_{\kappa=0}^{i}\sum_{v=1}^{n} c_{\kappa,v}(\ln s - \ln t_\kappa)^{\alpha-v} + \int_1^s \frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u)\frac{du}{u}\right)\frac{ds}{s}}{\Gamma(n-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}\sum_{v=1}^{n} c_{\kappa,v} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1}(\ln s - \ln t_\kappa)^{\alpha-v}\frac{ds}{s}}{\Gamma(n-\alpha)} \qquad \left(\text{by } \frac{\ln s - \ln t_\kappa}{\ln t - \ln t_\kappa} = w\right) \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i}\sum_{v=1}^{n} c_{\kappa,v} \int_{t_i}^{t} (\ln t - \ln s)^{n-\alpha-1}(\ln s - \ln t_\kappa)^{\alpha-v}\frac{ds}{s}}{\Gamma(n-\alpha)} \qquad \left(\text{by } \frac{\ln s - \ln t_\kappa}{\ln t - \ln t_\kappa} = w\right) \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \int_1^t \left(\int_u^t (\ln t - \ln s)^{n-\alpha-1} \frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right) g(u)\frac{du}{u}}{\Gamma(n-\alpha)} \qquad \left(\text{by } \frac{\ln s - \ln u}{\ln t - \ln u} = w\right) \\
&= \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i-1}\sum_{j=\kappa}^{i-1}\sum_{v=1}^{n} c_{\kappa,v}(\ln t - \ln t_\kappa)^{n-v} \int_{\frac{\ln_j - \ln t_\kappa}{\ln t - \ln t_\kappa}}^{\frac{\ln t_{j+1} - \ln t_\kappa}{\ln t - \ln t_\kappa}} (1-w)^{n-\alpha-1} w^v dw}{\Gamma(n-\alpha)} \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i}\sum_{v=1}^{n} c_{\kappa,v}(\ln t - \ln t_\kappa)^{n-v} \int_{\frac{\ln t_i - \ln t_\kappa}{\ln t - \ln t_\kappa}}^{1} (1-w)^{n-\alpha-1} w^v dw}{\Gamma(n-\alpha)} \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln u)^{n-1}\left(\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha}-1}{\Gamma(\alpha)}dw\right) g(u)\frac{du}{u}}{\Gamma(n-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i}\sum_{v=1}^{n} c_{\kappa,v}(\ln t - \ln t_\kappa)^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^v dw}{\Gamma(n-\alpha)} \\
&\quad + \frac{\left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln u)^{n-1}\left(\int_0^1 (1-w)^{n-\alpha-1}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dw\right) g(u)\frac{du}{u}}{\Gamma(n-\alpha)} \\
&= g(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m.
\end{aligned}
$$

It follows that x is a piecewise continuous solution of (3).

Step II. We prove that x satisfies (4) if x is a piecewise continuous solution of (3). For $t \in (t_0, t_1]$, we have from (3) that

$$
\begin{aligned}
\int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\frac{ds}{s} &= {}^H I_{1^+}^{\alpha} g(t) = {}^H I_{1^+}^{\alpha}\, {}^H D_{1^+}^{\alpha} x(t) \\
&= \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} \left(s\frac{d}{ds}\right)^n \left[\int_1^s \frac{(\ln s - \ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\frac{ds}{s}
\end{aligned}
$$

$$= \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} d\left(s\frac{d}{ds}\right)^{n-1}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]$$

$$= -\left(s\frac{d}{ds}\right)^{n-1}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}$$

$$+ \int_1^t \frac{(\ln t-\ln s)^{\alpha-2}}{\Gamma(\alpha-1)}\left(s\frac{d}{ds}\right)^{n-1}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\frac{ds}{s}$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)}$$

$$+ \int_1^t \frac{(\ln t-\ln s)^{\alpha-n}}{\Gamma(\alpha-n+1)}\left(s\frac{d}{ds}\right)\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\frac{ds}{s}$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)}$$

$$+t\left[\int_1^t \frac{(\ln t-\ln s)^{\alpha-n+1}}{(\alpha-n+1)\Gamma(\alpha-n+1)}\left(s\frac{d}{ds}\right)\left(\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right)\frac{ds}{s}\right]_t'$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)}$$

$$+t\left[\int_1^t \frac{(\ln t-\ln s)^{\alpha-n}}{\Gamma(\alpha-n+1)}\left(\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right)\frac{ds}{s}\right]_t'$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)}$$

$$+t\left[\int_1^t \int_u^t \frac{(\ln t-\ln s)^{\alpha-n}}{\Gamma(\alpha-n+1)}\frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)}\frac{ds}{s} x(u)\frac{du}{u}\right]_t'$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)}$$

$$+t\left[\int_1^t \int_0^1 \frac{(1-w)^{\alpha-n}}{\Gamma(\alpha-n+1)}\frac{w^{n-\alpha-1}}{\Gamma(n-\alpha)} dw\, x(u)\frac{du}{u}\right]_t'$$

$$= -\sum_{v=1}^n \left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right]\Bigg|_{t=1} \frac{(\ln t)^{\alpha-v}}{\Gamma(\alpha-v+1)} + x(t),\ t\in(t_0,t_1]\,.$$

It follows that there exist constants $c_{0,v} \in \mathbb{R}\ (v = 1, \cdots, n)$ such that

$$x(t) = \sum_{v=1}^n c_{0,v}(\ln t)^{\alpha-v} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\frac{ds}{s},\ t\in(t_0,t_1]$$

$$c_{0,v} = -\left(s\frac{d}{ds}\right)^{n-v}\left[\int_1^s \frac{(\ln s-\ln u)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(u)\frac{du}{u}\right].$$

We know (4) holds for $i = 0$. Supposing that (4) holds for $0, 1, 2, \cdots, i$, we will prove that (4) holds for $i + 1$. Then by mathematical induction method, we obtain that (4) holds for all $i \in \mathbb{Z}_0^m$.

In fact, we suppose that

$$x(t) = \Phi(t) + \sum_{j=0}^{i}\sum_{v=1}^{n} c_{j,v}\left(\ln t - \ln t_j\right)^{\alpha-v} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\frac{ds}{s},\ t\in(t_{i+1},t_{i+2}]\,. \tag{5}$$

Then for $t \in (t_{i+1}, t_{i+2}]$, we have

$$g(t) = {}^H D^{\alpha}_{1^+} x(t) = \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1} x(s) \frac{ds}{s} + \left(t\frac{d}{dt}\right)^n \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} x(s) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1} \left(\sum_{\kappa=0}^{j} \sum_{v=1}^{n} c_{\kappa,v} (\ln s - \ln t_\kappa)^{\alpha-v} + \int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u) \frac{du}{u}\right) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} \left(\Phi(s) + \sum_{\kappa=0}^{i+1} \sum_{v=1}^{n} c_{\kappa,v} (\ln s - \ln t_\kappa)^{\alpha-v} + \int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u) \frac{du}{u}\right) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1} \left(\sum_{\kappa=0}^{j} \sum_{v=1}^{n} c_{\kappa,v} (\ln s - \ln t_\kappa)^{\alpha-v}\right) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} \left(\sum_{\kappa=0}^{i+1} \sum_{v=1}^{n} c_{\kappa,v} (\ln s - \ln t_\kappa)^{\alpha-v}\right) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln s)^{n-\alpha-1} \left(\int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u) \frac{du}{u}\right) \frac{ds}{s}}{\Gamma(n-\alpha)} + \frac{\left(t\frac{d}{dt}\right)^n \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} \Phi(s) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln s)^{n-\alpha-1} \left(\int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} g(u) \frac{du}{u}\right) \frac{ds}{s}}{\Gamma(n-\alpha)} + \frac{\left(t\frac{d}{dt}\right)^n \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} \Phi(s) \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^n \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{v=1}^{n} c_{\kappa,v} \int_{t_j}^{t_{j+1}} (\ln t - \ln s)^{n-\alpha-1} (\ln s - \ln t_\kappa)^{\alpha-v} \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i+1} \sum_{v=1}^{n} c_{\kappa,v} \int_{t_{i+1}}^{t} (\ln t - \ln s)^{n-\alpha-1} (\ln s - \ln t_\kappa)^{\alpha-v} \frac{ds}{s}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_1^t \int_u^t (\ln t - \ln s)^{n-\alpha-1} \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s} g(u) \frac{du}{u}}{\Gamma(n-\alpha)} + {}^H D^{\alpha}_{t^+_{i+1}} \Phi(t)$$

$$= \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i} \sum_{j=\kappa}^{i} \sum_{v=1}^{n} c_{\kappa,v} (\ln t - \ln t_\kappa)^{n-v} \int_{\frac{\ln t_j - \ln_\kappa}{\ln t - \ln t_\kappa}}^{\frac{\ln t_{j+1} - \ln t_\kappa}{\ln t - \ln t_\kappa}} (1-w)^{n-\alpha-1} w^{\alpha-v} dw}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \sum_{\kappa=0}^{i+1} \sum_{v=1}^{n} c_{\kappa,v} (\ln t - \ln t_\kappa)^{n-v} \int_{\frac{\ln t_{i+1} - \ln t_\kappa}{\ln t - \ln t_\kappa}}^{1} (1-w)^{n-\alpha-1} w^{\alpha-v} dw}{\Gamma(n-\alpha)}$$

$$+ \frac{\left(t\frac{d}{dt}\right)^n \int_1^t (\ln t - \ln s)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw g(u) \frac{du}{u}}{\Gamma(n-\alpha)} + {}^H D^{\alpha}_{t^+_{i+1}} \Phi(t)$$

$$= \frac{\left(t\frac{d}{dt}\right)^n \sum_{k=0}^{i+1} \sum_{v=1}^{n} c_{\kappa,v} (\ln t - \ln t_\kappa)^{n-v} \int_0^1 (1-w)^{n-\alpha-1} w^{\alpha-v} dw}{\Gamma(n-\alpha)} + g(t) + {}^H D^{\alpha}_{t^+_{i+1}} \Phi(t).$$

We get

$$g(t) = {}^H D^{\alpha}_{1^+} x(t) = g(t) + {}^H D^{\alpha}_{t^+_{i+1}} \Phi(t).$$

So ${}^H D^{\alpha}_{t^+_{i+1}} \Phi(t) = 0$ on $(t_{i+1}, t_{i+2}]$. Then there exist constants $c_{i+1,v} \in \mathbb{R}\left(v \in \mathbb{Z}_1^n\right)$ such that $\Phi(t) = \sum_{v=1}^{n} c_{i+1,v} \frac{(\ln t - \ln t_{i+1})^{\alpha-v}}{\Gamma(v+1)}$ on $(t_{i+1}, t_{i+2}]$. Substituting Φ into (5), we get that (4) holds for $i+1$. By mathematical induction method, we obtain that (4) holds for all $i \in \mathbb{Z}_0^m$. So x satisfies (4) if x is a piecewise continuous solution of (3). The proof is completed.

3. Remarks and counterexamples

In this section, we present some remarks and counterexamples to illustrate the application of the main theorem given in Section 2. At the same time, we correct the incorrect lemmas in recently published papers by remarks and counterexamples.

Remark 3.1. Let $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$ and $\alpha \in (1, 2)$. From Theorem 2.1, we know that x is a solution of

$$\begin{aligned} &{}^H D_{1^+}^{\alpha} u(t) = f(t, u(t)), t \in (t_i, t_{i+1}], i = 0, 1, 2, \cdots, m, \\ &\Delta^H D_{1^+}^{\alpha-1} u(t_i) = \phi_i(u(t_i)), i = 1, 2, \cdots, m, \\ &\Delta^H I_{1^+}^{2-\alpha} u(t_i) = \psi_i(u(t_i)), i = 1, 2, \cdots, m, \\ &{}^H D_{1^+}^{\alpha-1} u(1) = u_1 \in \mathbb{R}, {}^H I_{1^+}^{2-\alpha} u(1) = u_2 \in \mathbb{R}, \end{aligned} \tag{6}$$

if and only if

$$\begin{aligned} x(t) = &\tfrac{u_1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \tfrac{u_2}{\Gamma(\alpha-1)}(\ln t)^{\alpha-2} \\ &+ \textstyle\sum_{j=1}^{i} \tfrac{\phi_j(x(t_j))}{\Gamma(\alpha)} \left(\ln t - \ln t_j\right)^{\alpha-1} + \sum_{j=1}^{i} \tfrac{\psi_j(x(t_j))}{\Gamma(\alpha-1)} \left(\ln t - \ln t_j\right)^{\alpha-2} \\ &+ \textstyle\int_1^t \tfrac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \tfrac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbb{Z}_0^m. \end{aligned}$$

We note that IVP(6) (that is similar to (6)) was studied in [31]. Corollary 3.5 in [31] is as follows:

Result 3.1. Suppose $\alpha \in (1, 2)$, $\lambda, h \in \mathbb{R}$. Then IVP(6) is equivalent to the following integral equation:

$$u(t) = \begin{cases} \frac{u_1}{\Gamma(\alpha)} \left(\ln \frac{t}{a}\right)^{\alpha-1} + \frac{u_2}{\Gamma(\alpha-1)} \left(\ln \frac{t}{a}\right)^{\alpha-2} + \int_a^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \frac{ds}{s}, t \in (a, t_1], \\ \frac{u_1}{\Gamma(\alpha)} \left(\ln \frac{t}{a}\right)^{\alpha-1} + \frac{u_2}{\Gamma(\alpha-1)} \left(\ln \frac{t}{a}\right)^{\alpha-2} + \int_a^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \frac{ds}{s} \\ + \sum_{i=1}^{k} \left[\frac{\psi_i(u(t_i))}{\Gamma(\alpha-1)} \left(\ln \frac{t}{t_i}\right)^{\alpha-2} + \frac{\phi_i(u(t_i))}{\Gamma(\alpha)} \left(\ln \frac{t}{t_i}\right)^{\alpha-1} \right] \\ \quad - \sum_{i=1}^{k} \left[\lambda \psi_i(u(t_i)) + h \phi_i(u(t_i))\right] \left[\frac{u_1}{\Gamma(\alpha)} \left(\ln \frac{t}{a}\right)^{\alpha-1} + \frac{u_2}{\Gamma(\alpha-1)} \left(\ln \frac{t}{a}\right)^{\alpha-2} \right. \\ + \int_a^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \frac{ds}{s} \\ - \frac{u_1 + \int_a^{t_i} f(s, u(s)) \frac{ds}{s}}{\Gamma(\alpha)} \left(\ln \frac{t}{t_i}\right)^{\alpha-1} - \frac{u_1 \ln \frac{t_i}{a} + u_2 + \int_a^{t_i} (\ln t_i - \ln s) f(s, u(s)) \frac{ds}{s}}{\Gamma(\alpha-1)} \left(\ln \frac{t}{a}\right)^{\alpha-2} \\ \left. - \int_{t_i}^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) \frac{ds}{s} \right], t \in (t_i, t_{i+1}], i \in \mathbb{Z}_1^m. \end{cases}$$

Example 3.1. [31, Example 1] Consider the following problem:

$$
\begin{aligned}
&{}^{H}D_{1^+}^{\frac{3}{2}}u(t) = \ln t, t \in (1,3], t \neq 2,\\
&\Delta^{H}D_{1^+}^{\frac{1}{2}}u(2) = \bar{\delta}, \Delta^{H}I_{1^+}^{\frac{1}{2}}u(2) = \delta, \quad {}^{H}D_{1^+}^{\frac{1}{2}}u(1) = u_1 \in \mathbb{R}, {}^{H}I_{1^+}^{\frac{1}{2}}u(1) = u_2 \in \mathbb{R}.
\end{aligned}
$$

By Result 3.1, Zhang in [31] get that

$$
u(t) = \begin{cases}
\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t - \ln s)^{\frac{1}{2}}}{\Gamma(3/2)} \ln s \frac{ds}{s}, t \in (1,2],\\
\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t - \ln s)^{\frac{1}{2}}}{\Gamma(3/2)} \ln s \frac{ds}{s}\\
+\left[\frac{\delta}{\Gamma(1/2)}\left(\ln \frac{t}{2}\right)^{-\frac{1}{2}} + \frac{\bar{\delta}}{\Gamma(3/2)}\left(\ln \frac{t}{2}\right)^{\frac{1}{2}}\right]\\
-[\lambda\delta + h\bar{\delta}]\left[\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t - \ln s)^{\frac{1}{2}}}{\Gamma(3/2)} \ln s \frac{ds}{s}\right.\\
-\frac{u_1 + \int_1^2 \ln s \frac{ds}{s}}{\Gamma(3/2)}\left(\ln \frac{t}{2}\right)^{\frac{1}{2}} - \frac{u_1 \ln 2 + u_2 + \int_1^2 (\ln 2 - \ln s) \ln s \frac{ds}{s}}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}\\
\left.- \int_2^t \frac{(\ln t - \ln s)^{\frac{1}{2}}}{\Gamma(3/2)} \ln s \frac{ds}{s}\right], t \in (2,3]
\end{cases}
$$

$$
= \begin{cases}
\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)}, t \in (1,2],\\
\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)} + \frac{\delta}{\Gamma(1/2)}\left(\ln \frac{t}{2}\right)^{-\frac{1}{2}} + \frac{\bar{\delta}}{\Gamma(3/2)}\left(\ln \frac{t}{2}\right)^{\frac{1}{2}}\\
-[\lambda\delta + h\bar{\delta}]\left[\frac{u_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)}\right.\\
-\frac{u_1 + \frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}\left(\ln \frac{t}{2}\right)^{\frac{1}{2}} - \frac{u_1 \ln 2 + u_2 + \frac{1}{6}(\ln 2)^3}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}\\
\left.-\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(3/2)} \int_{\ln 2}^1 (1-w)^{\frac{1}{2}} w dw\right], t \in (2,3].
\end{cases}
$$

We find that for $t \in (2,3]$

$$
\begin{aligned}
{}^{H}D_{1^+}^{\frac{3}{2}}u(t) &= \frac{\left(t\frac{d}{dt}\right)^2 \int_1^t (\ln t - \ln s)^{-\frac{1}{2}} u(s) \frac{ds}{s}}{\Gamma(1/2)}\\
&= \frac{\left(t\frac{d}{dt}\right)^2 \int_1^2 (\ln t - \ln s)^{-\frac{1}{2}} u(s) \frac{ds}{s} + \left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t - \ln s)^{-\frac{1}{2}} u(s) \frac{ds}{s}}{\Gamma(1/2)}\\
&= \frac{\left(t\frac{d}{dt}\right)^2 \int_1^2 (\ln t - \ln s)^{-\frac{1}{2}} \left[\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} + \frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right] \frac{ds}{s}}{\Gamma(1/2)}\\
&+\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t - \ln s)^{-\frac{1}{2}} \left[\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}\right.\\
&+\frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} + \frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)} + \frac{\delta}{\Gamma(1/2)}\left(\ln \frac{s}{2}\right)^{-\frac{1}{2}} + \frac{\bar{\delta}}{\Gamma(3/2)}\left(\ln \frac{s}{2}\right)^{\frac{1}{2}}\\
&-[\lambda\delta + h\bar{\delta}]\left(\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}} + \frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} + \frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right.\\
&\left.\left.-\frac{u_1 + \frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}\left(\ln \frac{s}{2}\right)^{\frac{1}{2}} - \frac{u_1 \ln 2 + u_2 + \frac{1}{6}(\ln 2)^3}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} - \frac{(\ln s)^{\frac{5}{2}}}{\Gamma(3/2)} \int_{\ln 2}^1 \ln s (1-w)^{\frac{1}{2}} w dw\right)\right] \frac{ds}{s}
\end{aligned}
$$

$$= \frac{\left(t\frac{d}{dt}\right)^2 \int_1^t (\ln t-\ln s)^{-\frac{1}{2}}\left[\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right]^{\frac{ds}{s}}}{\Gamma(1/2)}$$

$$+\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t-\ln s)^{-\frac{1}{2}}\left[\frac{\delta}{\Gamma(1/2)}\left(\ln\frac{s}{2}\right)^{-\frac{1}{2}}+\frac{\bar{\delta}}{\Gamma(3/2)}\left(\ln\frac{s}{2}\right)^{\frac{1}{2}}\right]\frac{ds}{s}$$

$$-[\lambda\delta+h\bar{\delta}]\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right.$$

$$\left.-\frac{u_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}\left(\ln\frac{s}{2}\right)^{\frac{1}{2}}-\frac{u_1\ln 2+u_2+\frac{1}{6}(\ln 2)^3}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}-\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(3/2)}\int_{\ln 2}^1(1-w)^{\frac{1}{2}}wdw\right)\frac{ds}{s}$$

$$+\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2\int_0^1(1-w)^{-\frac{1}{2}}\frac{\delta}{\Gamma(1/2)}w^{-\frac{1}{2}}dw+\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2\left(\ln\frac{t}{2}\right)\int_0^1(1-w)^{-\frac{1}{2}}\frac{\bar{\delta}}{\Gamma(3/2)}w^{\frac{1}{2}}dw$$

$$-[\lambda\delta+h\bar{\delta}]\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right.$$

$$\left.-\frac{u_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}\left(\ln\frac{s}{2}\right)^{\frac{1}{2}}-\frac{u_1\ln 2+u_2+\frac{1}{6}(\ln 2)^3}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}-\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(3/2)}\int_{\ln 2}^1(1-w)^{\frac{1}{2}}wdw\right)\frac{ds}{s}$$

$$=\ln t-[\lambda\delta+h\bar{\delta}]\frac{1}{\Gamma(1/2)}\left(t\frac{d}{dt}\right)^2 \int_2^t (\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{u_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{u_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right.$$

$$\left.-\frac{u_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}\left(\ln\frac{s}{2}\right)^{\frac{1}{2}}-\frac{u_1\ln 2+u_2+\frac{1}{6}(\ln 2)^3}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}-\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(3/2)}\int_{\ln 2}^1(1-w)^{\frac{1}{2}}wdw\right)\frac{ds}{s}$$

$$\neq \ln t,\ \text{if } [\lambda\delta+h\bar{\delta}]\neq 0.$$

It is easy to see that Theorem 3.4 in [31] is incorrect.

Remark 3.2. Let $\alpha\in(0,1)$. Wang and Zhang (Lemma 2.9 in [29])proved that u is a solution of

$$\begin{cases} {}^HD_{1^+}^{\alpha}u(t)=f(t,u(t)), t\in(1,e]\setminus\{t_1,\cdots,t_m\}, \\ {}^HI_{1^+}^{1-\alpha}u\left(t_i^+\right)-{}^HI_{1^+}^{1-\alpha}u\left(t_i^-\right)=p_i\in\mathbb{R}, i=1,2,\cdots,m, \\ {}^HI_{1^+}^{1-\alpha}u(1)=u_0\in\mathbb{R}, \end{cases} \tag{7}$$

if and only if x satisfies

$$x(t)=\begin{cases} \frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, t\in(1,t_1], \\ \frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\sum_{j=1}^i\frac{p_j}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{Z}_1^m. \end{cases} \tag{8}$$

Let $A_0=\frac{u_0}{\Gamma(\alpha)}, A_i=\frac{p_i}{\Gamma(\alpha)}$ for $i\in\mathbb{Z}_1^m$. We can rewrite this expression by

$$x(t)=\sum_{j=0}^i A_j(\ln t)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{Z}_1^m.$$

Then for $t\in(t_i,t_{i+1}]$, we obtain

$$
\begin{aligned}
{}^{H}D_{1^+}^{\alpha}x(t) &= \frac{\left(t\frac{d}{dt}\right)\int_1^t(\ln t-\ln s)^{-\alpha}x(s)\frac{dd}{s}}{\Gamma(1-\alpha)}\\
&= \frac{\left(t\frac{d}{dt}\right)\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{-\alpha}x(s)\frac{ds}{s}+\left(t\frac{d}{dt}\right)\int_{t_i}^{t}(\ln t-\ln s)^{-\alpha}x(s)\frac{ds}{s}}{\Gamma(1-\alpha)}\\
&= \frac{\left(t\frac{d}{dt}\right)\sum_{\tau=0}^{i-1}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{-\alpha}\left(\sum_{j=0}^{\tau}A_j(\ln s)^{\alpha-1}+\int_1^s\frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)}f(u,x(u))\frac{du}{u}\right)\frac{ds}{s}}{\Gamma(1-\alpha)}\\
&= \frac{\left(t\frac{d}{dt}\right)\int_{t_i}^{t}(\ln t-\ln s)^{-\alpha}\left(\sum_{j=0}^{i}A_j(\ln s)^{\alpha-1}+\int_1^s\frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)}f(u,x(u))\frac{du}{u}\right)\frac{ds}{s}}{\Gamma(1-\alpha)}\\
&\quad+\frac{\left(t\frac{d}{dt}\right)\sum_{\tau=0}^{i-1}\sum_{j=0}^{\tau}A_j\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{-\alpha}(\ln s)^{\alpha-1}\frac{ds}{s}}{\Gamma(1-\alpha)}\\
&= \frac{\left(t\frac{d}{dt}\right)\sum_{j=0}^{i-1}\sum_{\tau=j}^{i-1}A_j\int_{\frac{\ln t_\tau}{\ln t}}^{t}(1-w)^{-\alpha}w^{\alpha-1}dw}{\Gamma(1-\alpha)}\\
&\quad+\frac{\left(t\frac{d}{dt}\right)\sum_{j=0}^{i}A_j\int_{\frac{\ln t_j}{\ln t}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw+\left(t\frac{d}{dt}\right)\int_1^t\int_u^t(\ln t-\ln s)^{-\alpha}\frac{(\ln s-\ln u)^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}f(u,x(u))\frac{du}{u}}{\Gamma(1-\alpha)}\\
&= \frac{\left(t\frac{d}{dt}\right)\sum_{j=0}^{i-1}A_j\int_{\frac{\ln t_j}{\ln t}}^{\ln t_i}(1-w)^{-\alpha}w^{\alpha-1}dw}{\Gamma(1-\alpha)}\\
&\quad+\frac{\left(t\frac{d}{dt}\right)\sum_{j=0}^{i}A_j\int_{\frac{\ln t_j}{\ln t}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw+\left(t\frac{d}{dt}\right)\int_1^t\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwf(u,x(u))\frac{du}{u}}{\Gamma(1-\alpha)}\\
&= f(t,x(t))+\frac{\left(t\frac{d}{dt}\right)\sum_{j=0}^{i}A_j\int_{\frac{\ln t_j}{\ln t}}^{1}(1-w)^{-\alpha}w^{\alpha-1}dw}{\Gamma(1-\alpha)}\neq f(t,x(t)),\ \text{if } A_i\neq 0.
\end{aligned}
$$

It shows that Lemma 2.9 in [31] is wrong.

By Theorem 2.1 ($n=1$), we know that x is a solution of ${}^{H}D_{1^+}^{\alpha}u(t)=f(t,u(t)), t\in(1,e]\setminus\{t_1,\cdots,t_m\}$ if and only if there exist constants $c_i\in\mathbb{R}$ such that

$$x(t)=\sum_{j=0}^{i}c_i\left(\ln t-\ln t_j\right)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{Z}_0^m.$$

Then

$${}^{H}I_{1^+}^{1-\alpha}x(t)=\Gamma(\alpha)\sum_{j=0}^{i}c_i+\int_1^t f(s,x(s))\frac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{Z}_0^m.$$

Hence ${}^{H}I_{1^+}^{1-\alpha}u\left(t_i^+\right)-{}^{H}I_{1^+}^{1-\alpha}u\left(t_i^-\right)=p_i, i\in\mathbb{Z}_1^m$ and ${}^{H}I_{1^+}^{1-\alpha}u(1)=u_0$ imply that $c_i=\frac{p_i}{\Gamma(\alpha)}, i\in\mathbb{Z}_1^m$ and $c_0=\frac{u_0}{\Gamma(\alpha)}$.

It follows that $x(t)=\frac{u_0(\ln t)^{\alpha-1}}{\Gamma(\alpha)}+\sum_{j=1}^{i}\frac{p_j}{\Gamma(\alpha)}\left(\ln t-\ln t_j\right)^{\alpha-1}+\int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}, t\in(t_i,t_{i+1}], i\in\mathbb{Z}_0^m$.

This is a corrected form of the solution to the above-mentioned problem.

4. Conclusions

The authors believe that the article will be appreciated by the researchers working in the field of impulsive fractional calculus and be helpful for study on the boundary value problems for impulsive fractional differential equations involving Hadamard fractional derivatives and in the nonlinear area and the numerical simulation, especially for study in the the solvability of boundary value problems, initial value problems or numerical solutions of boundary value problems for impulsive fractional differential equation involving the Hadamard fractional derivatives

Acknowledgments

The work described in this paper is financially supported by Basic and Applied Basic Projects of Guangzhou, China (No. 202002030228).

Conflict of interest

The authors declare no conflicts of interest.

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