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Blow-up phenomena and global existence for nonlinear parabolic problems under nonlinear boundary conditions

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Abstract: In this paper, we consider an initial-boundary value parabolic problem under nonlinear Neumann boundary conditions. By virtue of the modified differential inequality, lower bounds for the blow-up time of the solution are derived in higher dimensional spaces. An upper bound for the blow-up time are specified under appropriate assumptions on the functions a, b, f, g, h and u_0.

Keywords: blow-up; lower bounds; higher dimensional spaces
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1. Introduction

In this paper, we investigate the question of blow-up for nonnegative classical solution $u(x,t)$ of the following initial-boundary value problem defined in higher dimensional spaces

$$(h(u))_t = \sum_{i,j=1}^{N} (a^{ij}(x)u_{x_i})_{x_j} + b(x)f(u), \qquad x \in \Omega, t > 0, \tag{1.1}$$

with the following initial-boundary conditions

$$\begin{aligned} &\sum_{i,j=1}^{N} a^{ij}(x)u_{x_i}\nu_j = g(u), \qquad x \in \partial\Omega, t > 0, \\ &u(x,0) = u_0(x) \geq 0, \qquad x \in \Omega. \end{aligned} \tag{1.2}$$

In (1.1) and (1.2), $\Omega \subseteq \mathbb{R}^N, N \geq 3$ is a bounded star-shaped region with smooth boundary $\partial\Omega$, ν is the outward normal vector to $\partial\Omega$. Moreover, we assume that $h \in C^2(\mathbb{R}^+), 0 < h_m < h'(s) < h_M$ for

$s > 0$, $u_0 \in C^2(\overline{\Omega})$, f, g are nonnegative continuous functions. The $N \times N$ matrix $(a^{ij}(x))$ is a differential positive definite matrix; that is, there exists a constant θ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\eta_i\eta_j \geq \theta|\eta|^2, \ for\ all\ \eta \in \mathbb{R}^N. \tag{1.3}$$

The question of blow-up of solutions to nonlinear parabolic equations and systems has received considerable attention in the literature. We refer to the reader the books of Straughan [1] and Quittner-Souplet [2], the survey paper of Bandle and Brunner [3] and the papers of Vazquez [4] and Weissler [5,6]. Most of the papers concerned with the existence and non-existence of global solutions, blow-up solutions, upper bounds on blow-up time, blow-up rates and asymptotic behavior of solutions. The blow-up phenomena of solutions to nonlinear parabolic equations and systems with nonlinear or linear Neumann boundary conditions was studied by many authors (see [7–10]). Some special cases of (1.1) and (1.2) have been treated already. Imai and Mochizuki [11] and Zhang [12] considered the following problem:

$$(h(u))_t = \Delta u + f(u), \quad x \in \Omega \times (0, T) \tag{1.4}$$

with different boundary conditions, where Ω is a bounded domain of $\mathbb{R}^N$, $(N \geq 2)$ with smooth boundary. Under certain assumptions on the known functions, sufficient conditions were developed for the existence of global solution or blow-up solution. Moreover, an upper bound of the blow-up time was also derived. Gao, Ding and Guo [13] studied the following parabolic equation

$$(h(u))_t = \nabla(a(u)\nabla u) + f(u), \quad x \in \Omega \times (0, T), \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $(N \geq 2)$ with smooth boundary. The authors obtained the conditions therefor the existence of the global solution and blow-up solution. Song and Lv [14, 15] studied the following semilinear parabolic equation with weighted inner source terms

$$(h(u))_t = \Delta u + b(x)f(u), \quad x \in \Omega \times (0, T). \tag{1.6}$$

When the initial-boundary value problem with nonlinear Neumann boundary condition, they obtained the bounds for the blow-up time of the solution in three dimensional space (see [13]). In [14], the initial-boundary value problem with homogeneous Dirichlet or Neumann boundary condition, they derived the bounds for the blow-up rate and the blow-up time in any smooth bounded domain $\Omega \subseteq \mathbb{R}^N, N \geq 3$. Recently, Ma and Fang [16] considered the following equation

$$\begin{cases} u_t = \sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} - b(x)f(u), & x \in \Omega, t > 0, \\ \sum_{i,j=1}^N a^{ij}(x)u_{x_i}\nu_j = g(u), & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.7}$$

Based on the auxiliary function method and the modified differential inequality technique, conditions on weight function and nonlinearities to guarantee the solution exists globally or blows up at finite time were established. Also, the authors derived an upper bound and a lower bound for the blow-up time. For more special cases of (1.1) and (1.2) with inner source term, one can refer to [17–20].

In the present paper, we do not constraint f, g to satisfy the the conditions in [16]. By constructing completely different conditions on the known data and auxiliary functions with those in the paper mentioned above, we use the modified differential inequality technique to seek the conditions which guarantee the solution of (1.1) and (1.2) exists globally or blows up in finite time. Under some certain assumption, we determine a lower bound on blow-up time in a convex bounded domain $\Omega \subset \mathbb{R}^N, N \geq 3$ if blow-up occurs.

This paper is organized as follows: By establishing some appropriate auxiliary functions and using first-order differential inequalities technique, we derive a lower and an upper bounds for the blow-up time in sections 2 and 3, respectively. In section 4, we will establish the conditions to guarantee that the solution to (1.1) and (1.2) exists globally.

2. A lower bound for the blow-up time

We list some Sobolev type inequalities which will be used in this paper.

Lemma 2.1. (*see [12, p976]*) Let Ω be a bounded star-shaped domain in $\mathbb{R}^N$, $N \geq 2$ and $L_0 = \min_{\partial\Omega} x \cdot \mathbf{n}, d = \max_{\overline{\Omega}} |x|$, Then we have

$$\int_{\partial\Omega} w^k dA \leq \frac{N}{L_0} \int_{\Omega} w^k dx + \frac{kd}{L_0} \int_{\Omega} w^{k-1} |\nabla w| dx. \tag{2.1}$$

Lemma 2.2. (*see [13, Corollary IX14, p168]*) Let c_s be a constant depending on Ω and N and $c = \sqrt{2} c_s^{\frac{3}{2}}$, for $N = 3$, and $c = c_s^{\frac{N}{2(N-2)}}$, for $N > 3$. Then we have

$$\Big(\int_{\Omega} w^{\frac{2N}{N-2}} dx \Big)^{\frac{1}{4}} \leq c \Big[\Big(\int_{\Omega} |\nabla w|^2 dx \Big)^{\frac{N}{4(N-2)}} + \Big(\int_{\Omega} w^2 dx \Big)^{\frac{N}{4(N-2)}} \Big]. \tag{2.2}$$

Moreover, we suppose that positive functions a, f, g and h satisfy

$$\begin{aligned}
&(1)\ f(0) = 0,\ f(s) > 0,\ 0 < h'(s) \leq h_M,\ for\ s > 0, \\
&(2)\ \int_s^{\infty} \frac{h'(\eta)}{f(\eta)} d\eta\ is\ bounded\ for\ s > 0, \\
&(3)\ g(s) \leq k_1 f(s) \Big(\int_s^{\infty} \frac{h'(\eta)}{f(\eta)} d\eta \Big)^{-m+2},\ m > 2, \\
&(4)\ \frac{f'(s)}{h'(s)} \int_s^{\infty} \frac{h'(\eta)}{f(\eta)} d\eta \leq (2n+1) - \beta,
\end{aligned} \tag{2.3}$$

where k_1, k_2, n, β, h_M are positive constants. We have the following results:

Theorem 2.1. Assume that the non-negative functions a, f, h and g satisfy the conditions (2.3). If the nonnegative solution $u(x, t)$ of (1.1) and (1.2) becomes unbounded in the measure φ at some finite time t^*, then t^* is bounded from below by

$$t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{\sigma_1 + \sigma_2 \eta + \sigma_3 \eta^{1+\frac{2}{3N-8}} + \sigma_4 \eta^{1+\frac{1}{2(N-2)}}}. \tag{2.4}$$

where $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are positive constants and $\varphi(t)$ will be defined in (2.5).

Proof. Firstly, we define an auxiliary function

$$\varphi(t) = \int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n}dx, \quad n > 2(m-1)(N-2). \tag{2.5}$$

and we compute

$$\begin{aligned}
\varphi'(t) &= 2n \int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}\frac{(h(u))_t}{f(u)}dx \\
&= 2n \int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}\frac{1}{f(u)}\Big(\sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} + b(x)f(u)\Big)dx \\
&\le -2n(2n+1)\int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-2}\frac{h'(u)}{f^2(u)}\sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx \\
&\quad + 2n \int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}\frac{f'(u)}{f^2(u)}\sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx \\
&\quad + 2n \int_{\partial\Omega} \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}\frac{g(u)}{f(u)}dA + 2n\int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}b(x)dx.
\end{aligned} \tag{2.6}$$

Using (1.3) and (2.3), we have

$$\begin{aligned}
\varphi'(t) &\le -\frac{2n\beta\theta}{h_M}\int_\Omega \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-2}\frac{[h'(u)]^2}{f^2(u)}|\nabla u|^2dx \\
&\quad + 2nk_1\int_{\partial\Omega}\Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-m+1}dA + 2n\int_\Omega\Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-2n-1}b(x)dx.
\end{aligned} \tag{2.7}$$

In order to simplify our computations, we let

$$v(u) = \Big[\int_u^\infty \frac{h'(s)}{f(s)}ds\Big]^{-1}, \tag{2.8}$$

and rewrite (2.7) as

$$\begin{aligned}
\varphi'(t) &\le -2n\beta\theta\int_\Omega v^{2n+2}\frac{[h'(u)]^2}{f^2(u)}|\nabla u|^2dx + 2n\int_{\partial\Omega} v^{2n+1}\frac{g(u)}{f(u)}dA + 2n\int_\Omega v^{2n+1}b(x)dx \\
&\le -\frac{2\beta\theta}{n}\int_\Omega |\nabla v^n|^2dx + 2k_1n\int_{\partial\Omega} v^{2n+m-1}dA + 2n\int_\Omega v^{2n+1}b(x)dx,
\end{aligned} \tag{2.9}$$

where we have let $h_M = 1$ for convenience. In (2.9), we have used the fact

$$|\nabla v|^2 = v^4\Big(\frac{h'(u)}{f(u)}\Big)^2|\nabla u|^2. \tag{2.10}$$

By using Lemma 2.1, we have

$$\int_{\partial\Omega} v^{2n+m-1}dA \le \frac{N}{L_0}\int_\Omega v^{2n+m-1}dx + \frac{(2n+m-1)d}{L_0}\int_\Omega v^{2n+m-2}|\nabla v|dx. \tag{2.11}$$

Using Hölder inequality and the Young inequality for (2.11), we have

$$\int_{\partial\Omega} v^{2n+m-1}dA \leq \frac{N}{2L_0}\int_\Omega v^{2n}dx + \Big(\frac{N}{2L_0} + \frac{(2n+m-1)^2d^2}{L_0^2\varepsilon_1}\Big)\int_\Omega v^{2n+2m-2}dx + \frac{\varepsilon_1}{4n^2}\int_\Omega |\nabla v^n|^2dx, \quad (2.12)$$

where ε_1 is a positive constant to be determined later. Substituting (2.12) into (2.9), we obtain

$$\begin{aligned}\varphi'(t) \leq &-(\frac{2\beta\theta}{n} - \frac{k_1\varepsilon_1}{2n})\int_\Omega |\nabla v^n|^2dx + \frac{Nk_1n}{L_0}\int_\Omega v^{2n}dx \\ &+ 2k_1n\Big(\frac{N}{2L_0} + \frac{(2n+m-1)^2d^2}{L_0^2\varepsilon_1}\Big)\int_\Omega v^{2n+2m-2}dx + 2n\int_\Omega v^{2n+1}b(x)dx.\end{aligned} \quad (2.13)$$

Using Hölder inequality and the Young inequality again for (2.12), we have

$$\begin{aligned}\int_\Omega v^{2n+2m-2}dx &\leq \Big(\int_\Omega v^{2n}dx\Big)^{\frac{n-2(N-2)(m-1)}{n}}\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big)^{\frac{2(N-2)(m-1)}{n}} \\ &\leq \frac{n-2(N-2)(m-1)}{n}\Big(\int_\Omega v^{2n}dx\Big) + \frac{2(N-2)(m-1)}{n}\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big),\end{aligned} \quad (2.14)$$

and

$$\begin{aligned}\int_\Omega v^{2n+1}b(x)dx &\leq \Big(\int_\Omega v^{2n}dx\Big)^{\frac{4n-4(N-2)-(2N-3)}{4n}}\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big)^{\frac{6(N-2)}{4n}}\Big(\int_\Omega b^{4n}(x)dx\Big)^{\frac{1}{4n}} \\ &\leq \frac{4n-4(N-2)-(2N-3)}{4n}\int_\Omega v^{2n}dx + \frac{6(N-2)}{4n}\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx + \frac{1}{4n}\int_\Omega b^{4n}(x)dx.\end{aligned} \quad (2.15)$$

Obviously, since $n > 2(N-2)(m-1)$, $4n > 4(N-2)+(2N-3)$. Inserting (2.14) and (2.15) into (2.13), we have

$$\varphi'(t) \leq -(\frac{2\beta\theta}{n} - \frac{k_1\varepsilon_1}{2n})\int_\Omega |\nabla v^n|^2dx + m_1\int_\Omega v^{2n}dx + m_2\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big) + m_3, \quad (2.16)$$

where

$$\begin{aligned}m_1 &= \frac{Nk_1n}{L_0} + 2k_1[n-2(N-2)(m-1)]\Big(\frac{N}{2L_0} + \frac{(2n+m-1)^2d^2}{L_0^2\varepsilon_1}\Big) + \frac{4n-4(N-2)-(2N-3)}{2}, \\ m_2 &= 4k_1(N-2)(m-1)\Big(\frac{N}{2L_0} + \frac{(2n+m-1)^2d^2}{L_0^2\varepsilon_1}\Big) + 3(N-2),\ m_3 = \frac{1}{2}\int_\Omega b^{4n}(x)dx.\end{aligned} \quad (2.17)$$

We use the Schwarz inequality to bound

$$\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx \leq \Big(\int_\Omega v^{2n}dx\Big)^{\frac{3}{4}}\Big(\int_\Omega (v^n)^{\frac{2N}{N-2}}dx\Big)^{\frac{1}{4}}. \quad (2.18)$$

Now, we use Lemma 2.2 with $w = v^n$ for (2.16) to get

$$\begin{aligned}\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx &\leq c\Big(\int_\Omega v^{2n}dx\Big)^{\frac{3}{4}}\Big[\Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{N}{4(N-2)}} + \Big(\int_\Omega v^{2n}dx\Big)^{\frac{N}{4(N-2)}}\Big] \\ &= c\Big(\int_\Omega v^{2n}dx\Big)^{\frac{3}{4}}\cdot\Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{N}{4(N-2)}} + c\Big(\int_\Omega v^{2n}dx\Big)^{\frac{2N-3}{2(N-2)}}.\end{aligned} \quad (2.19)$$

Applying the Young inequality again to obtain

$$\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx \le \frac{c(3N-8)}{4\varepsilon_2^{\frac{N}{3N-8}}(N-2)}\Big(\int_\Omega v^{2n}dx\Big)^{\frac{3(N-2)}{3N-8}} + \frac{Nc\varepsilon_2}{4(N-2)}\int_\Omega |\nabla v^n|^2 dx + c\Big(\int_\Omega v^{2n}dx\Big)^{\frac{2N-3}{2(N-2)}}, \tag{2.20}$$

for $\varepsilon_2 > 0$, inserting (2.19) back into (2.16), we obtain

$$\varphi'(t) \le -\Big[\frac{2\beta\theta}{n} - \frac{k_1\varepsilon_1}{2n} - \frac{m_2Nc\varepsilon_2}{4(N-2)}\Big]\int_\Omega |\nabla v^n|^2 dx + m_1\int_\Omega v^{2n}dx + \frac{cm_2(3N-8)}{4\varepsilon_2^{\frac{N}{3N-8}}(N-2)}\Big(\int_\Omega v^{2n}dx\Big)^{\frac{3(N-2)}{3N-8}} + m_2c\Big(\int_\Omega v^{2n}dx\Big)^{\frac{2N-3}{2(N-2)}} + m_3. \tag{2.21}$$

Now, we choose that

$$\varepsilon_1 = \frac{4\beta\theta}{k_1},\ \varepsilon_2 = \frac{4\beta\theta(N-2)}{m_2Nc}, \tag{2.22}$$

to have

$$\varphi'(t) \le \sigma_1 + \sigma_2\int_\Omega v^{2n}dx + \sigma_3\Big(\int_\Omega v^{2n}dx\Big)^{1+\frac{2}{3N-8}} + \sigma_4\Big(\int_\Omega v^{2n}dx\Big)^{1+\frac{1}{2(N-2)}}, \tag{2.23}$$

where

$$\sigma_1 = m_3,\ \sigma_2 = m_1,\ \sigma_3 = \frac{cm_2(3N-8)}{4\varepsilon_2^{\frac{N}{3N-8}}(N-2)},\ \sigma_4 = m_2c. \tag{2.24}$$

Recalling the definition of $\varphi(t)$ in (2.5), (2.23) can be rewritten as

$$\varphi' \le \sigma_1 + \sigma_2\varphi + \sigma_3\varphi^{1+\frac{2}{3N-8}} + \sigma_4\varphi^{1+\frac{1}{2(N-2)}}. \tag{2.25}$$

Then

$$\frac{\varphi'}{\sigma_1 + \sigma_2\varphi + \sigma_3\varphi^{1+\frac{2}{3N-8}} + \sigma_4\varphi^{1+\frac{1}{2(N-2)}}} \le 1. \tag{2.26}$$

It follows on integrating (2.26) from 0 to t that

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{\sigma_1 + \sigma_2\eta + \sigma_3\eta^{1+\frac{2}{3N-8}} + \sigma_4\eta^{1+\frac{1}{2(N-2)}}} \le t, \tag{2.27}$$

so that letting $t \to t^*$, we conclude that

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{\sigma_1 + \sigma_2\eta + \sigma_3\eta^{1+\frac{2}{3N-8}} + \sigma_4\eta^{1+\frac{1}{2(N-2)}}}. \tag{2.28}$$

Thus, the proof of Theorem 2.1 is complete.

Remark 2.1. The special case $h(u) = u$ and $b(x) = 1$ in (1.1) and (1.2) was considered by [6], and lower bound was derived under some suitable assumption. Obviously, our result is more general.

3. Upper bound of the blow-up time

In this section, we establish some auxiliary functions

$$F(s) = \int_0^s f(\eta)d\eta, \quad G(s) = \int_0^s g(\eta)d\eta, \quad \Phi(t) = \int_\Omega \int_0^u sh'(s)dsdx, \tag{3.1}$$

$$H(t) = 4\int_{\partial\Omega} G(u)dA - 2\int_\Omega \sum_{i,j=1}^N a^{ij}u_{x_i}u_{x_j}dx + 4\int_\Omega b(x)F(u)dx, k_2 > 1. \tag{3.2}$$

We may have the following results:

Theorem 3.1. Let u be a nonnegative solution of (1.1) and (1.2). Assume that f, h and g satisfy the conditions

$$\begin{aligned} &(i)\ h'(s)s^2 \leq k_2 \int_0^s \eta h'(\eta)d\eta,\ s > 0, k_2 > 0,\\ &(ii)\ sf(s) \geq 2k_2(1+q)F(s),\ q > 0,\\ &(iii)\ sg(s) \geq 2k_2(1+p)G(s),\ s > 0,\ p > 0, \end{aligned} \tag{3.3}$$

and $H(0) > 0$. Then the solution $u(x,t)$ of problems (1.1) and (1.2) blows up in L^2-norm at some finite time $t^* \leq T$ with

$$T = \frac{2\Phi(0)}{k_3 pH(0)}. \tag{3.4}$$

We note that $h(s) = s^{\gamma_1},\ k_2 \geq 1+\gamma_1,\ f(s) = s^{\gamma_2},\ 2k_2(1+q) \leq 1+\gamma_2$ and $g(s) = s^{\gamma_3},\ 2k_2(1+p) \leq 1+\gamma_3,\ \gamma_2, \gamma_3 > 3$, $k_3 = min\{k_2(1+p), k_2(1+q)\}$ satisfy these requirements.

Proof. From the definition of $\Phi(t)$ in (3.1), we compute

$$\begin{aligned} \Phi'(t) &= \int_\Omega uh'(u)u_t dx = \int_\Omega u\Big[\sum_{i,j=1}^N (a^{ij}u_{x_i})_{x_j} + b(x)f(u)\Big]dx\\ &= \int_{\partial\Omega} ug(u)dA - \int_\Omega \sum_{i,j=1}^N a^{ij}u_{x_i}u_{x_j}dx + \int_\Omega uf(u)b(x)dx\\ &\geq 2k_2(1+p)\int_{\partial\Omega} G(u)dA - \int_\Omega \sum_{i,j=1}^N a^{ij}u_{x_i}u_{x_j}dx + 2k_2(1+q)\int_\Omega F(u)b(x)dx\\ &\geq \frac{1}{2}k_3 H(t). \end{aligned} \tag{3.5}$$

Differentiating $H(t)$ and using divergence theorem, we can derive

$$\begin{aligned} H'(t) &= 4\int_{\partial\Omega} g(u)u_t dA - 4\int_\Omega \sum_{i,j=1}^N a^{ij}u_{x_i}u_{x_j t}dx + 4\int_\Omega b(x)f(u)u_t dx\\ &= 4\int_{\partial\Omega} g(u)u_t dA - 4\int_{\partial\Omega} \sum_{i,j=1}^N a^{ij}u_{x_i}\nu_j u_t dA + 4\int_\Omega \sum_{i,j=1}^N (a^{ij}u_{x_i})_{x_j}u_t dx + 4\int_\Omega b(x)f(u)u_t dx\\ &= 4\int_\Omega h'(u)u_t^2 dx \geq 0, \end{aligned} \tag{3.6}$$

which implies $H(t) > 0$ for $t > 0$, since $H(0) > 0$.

Using Schwarz inequality and $(3.3)_1$, we have

$$\begin{aligned}[\Phi'(t)]^2 &= \Big(\int_\Omega uh'(u)u_t dx\Big)^2 \le \Big(\int_\Omega h'(u)u^2 dx\Big)\Big(\int_\Omega h'(u)u_t^2 dx\Big)\\ &\le k_2\Big(\int_\Omega\int_0^u sh'(s)dsdx\Big)\Big(\int_\Omega h'(u)u_t^2 dx\Big)\\ &\le \frac{k_2}{4}H'(t)\Phi(t).\end{aligned} \tag{3.7}$$

Combining the above Eq (3.5), we can obtain the inequality

$$H'(t)\Phi(t) \ge (1+p)H(t)\Phi'(t), \tag{3.8}$$

which may be rewritten as

$$\Big[H(t)(\Phi(t))^{-(p+1)}\Big]' \ge 0. \tag{3.9}$$

Integrating (3.9) from 0 to t, we can have

$$H(t)[\Phi(t)]^{-(1+p)} \ge H(0)[\Phi(0)]^{-(1+p)}. \tag{3.10}$$

Substituting (3.10) into (3.5) yields that

$$\Phi'(t) \ge \frac{1}{2}k_3 p + H(0)[\Phi(0)]^{-(1+p)}[\Phi(t)]^{1+p}. \tag{3.11}$$

Integrating now (3.11) from 0 to t, we obtain the inequality

$$[\Phi(t)]^{-p} \le [\Phi(0)]^{-p} - \frac{1}{2}k_3 pH(0)[\Phi(0)]^{-(1+p)}t. \tag{3.12}$$

But this inequality can not hold for

$$t^* \ge T = \frac{2\Phi(0)}{k_3 pH(0)}. \tag{3.13}$$

In conclusion, the solution $u(x,t)$ of problems (1.1) and (1.2) fails to exist by blowing up at some finite time $t^* < T$.

Remark 3.1. If $p = q = 0$, integrating (3.11) from t to ∞, one can see that

$$\Phi(t) \le \Phi(0)exp\{\frac{k_2 H(0)}{2\Phi(0)}t\} \tag{3.14}$$

is valid for all $t > 0$, which implies $t^* = \infty$; that is, $T = \infty$.

In particularly, if the $N \times N$ matrix $(a^{ij}(x))$ is a diagonal matrix,

$$\begin{pmatrix} a(u) & 0 & ... & 0\\ 0 & a(u) & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & a(u)\end{pmatrix}, \tag{3.15}$$

then problems (1.1) and (1.2) can be rewritten as

$$\begin{cases} (h(u))_t = \nabla(a(u)\nabla u) + b(x)f(u), & in\ \Omega\times(0,t^*), \\ \frac{\partial u}{\partial \nu} = g(u), & on \quad \partial\Omega\times(0,t^*), \\ u(x,0) = u_0(x) \geq 0, & in\ \Omega. \end{cases} \tag{3.16}$$

The last term of $(3.16)_1$ means that the system is subjected to external force. Now, we define some auxiliary functions:

$$v(s) = \int_0^s \frac{h'(y)}{a(y)}dy, \quad \psi(t) = \int_\Omega \phi v(u(x,t))dx, \tag{3.17}$$

where ϕ is the associated eigenfunction of the problem defined as

$$\Delta\phi(x) + \lambda_1\phi(x) = 0,\ \phi > 0, \quad x\in\Omega, \quad \frac{\partial\phi}{\partial\nu} = 0, \quad x\in\partial\Omega, \tag{3.18}$$

and λ_1 is the first eigenvalue of the problem (3.18). We have the following theorem:

Theorem 3.2. Let $u(x,t)$ be the nonnegative classical solution of the nonlinear parabolic problem (3.16). Assume that f, g, h satisfy the conditions

$$a'(s) \geq 0,\ s \leq \int_0^s \frac{h'(y)}{a(y)}dy,\ \ f(s) \geq a(s)\Big(\int_0^s \frac{h'(y)}{a(y)}dy\Big)^k,\ s>0,\ k>1, \tag{3.19}$$

and the initial condition satisfies

$$-\lambda_1\psi(0) + \Big(\phi b(x)dx\Big)^{1-k}[\psi(0)]^k > 0. \tag{3.20}$$

Then $\psi(t)$ which was defined in (3.17) must blow up at some finite time t^* and the upper bound for t^* can be given in the form

$$t^* \leq \int_{\psi(0)}^\infty \frac{d\eta}{\Theta(\eta)}. \tag{3.21}$$

Proof. From (3.17) we compute

$$\begin{aligned} \psi'(t) &= \int_\Omega \frac{1}{a}\phi[\nabla(a(u)\nabla u) + b(x)f(u)]dx \\ &= \int_\Omega \frac{a'}{a}\phi|\nabla u|^2 dx - \int_\Omega \nabla\phi\nabla u dx + \int_\Omega \frac{f}{a}b(x)\phi dx \\ &\geq -\lambda_1\int_\Omega \phi u dx + \int_\Omega v^k\phi b(x)dx \\ &\geq -\lambda_1\int_\Omega \phi v dx + \int_\Omega v^k\phi b(x)dx, \end{aligned} \tag{3.22}$$

where we have used Eq (3.16), the divergence theorems (3.17), (3.18) and the assumption (3.19). Making use of Hölder's inequality and (2.3), we have

$$\psi(t) = \int_\Omega \phi v dx \leq \Big(\int_\Omega \phi b(x)v^k dx\Big)^{\frac{1}{k}}\Big(\int_\Omega \phi b^{\frac{1}{1-k}}(x)dx\Big)^{\frac{k-1}{k}}. \tag{3.23}$$

Combining (3.22) and (3.23), we obtain

$$\psi'(t) \geq -\lambda_1\psi(t) + \Big(\int_\Omega \phi b^{\frac{1}{1-k}}(x)dx\Big)^{1-k}[\psi(t)]^k \doteq \Theta(\psi), \tag{3.24}$$

Suppose that the initial data u_0 is large enough to insure $\Theta(\psi(0)) > 0$. Then it can be derived from (3.24) that $\psi(t)$ is increasing for t small. Since $\Theta(\psi)$ is increasing in ψ from its nonnegative minimum, it follows that $\Theta(\psi(t))$ is increasing in t for $t > 0$. This shows that $\psi'(t)$ remains positive, so $\psi(t)$ blows up at some time t^*. From (3.24), we can derive the upper bound for t^*:

$$t^* \leq \int_{\psi(0)}^{\infty} \frac{d\eta}{\Theta(\eta)}. \tag{3.25}$$

4. Global solution

In this section, we derive the conditions on the known functions f, g and h to guarantee existence of global solution. Our main result is the following Theorem 4.1.

Theorem 4.1. Let u be nonnegative classical solution of problems (1.1) and (1.2). We first establish an auxiliary function:

$$\Psi(t) = 2p\int_\Omega\int_0^u s^{2p-1}h'(s)dsdx, \ p > 1. \tag{4.1}$$

Assume that

$$\begin{aligned}&(A1) : f(s(x,t)) \geq \gamma_1(s(x,t))^k,\\ &(A2) : g(s(x,t)) \leq \gamma_2(s(x,t))^\gamma, \ s(x,t) \geq 0,\end{aligned} \tag{4.2}$$

where $\gamma_1 > 0, \gamma_2 > 0, 2\gamma < k+1, \gamma > 1$. Then the solution u of (1.1) and (1.2)can not blow up in finite time. that is to say there must be a global solution.

Proof. We begin with the auxiliary function $\Psi(t)$. Differentiating $\Psi(t)$, using the Eqs (1.1) and (1.2), conditions $(A1), (A2)$, (2.3) and divergence theorem, we have

$$\begin{aligned}\Psi'(t) &= 2p\int_\Omega u^{2p-1}h'(u)u_t dx\\ &= 2p\int_\Omega u^{2p-1}\Big[\sum_{i,j=1}^N (a^{ij}(x)u_{x_i})_{x_j} + b(x)f(u)\Big]dx\\ &= 2p\int_{\partial\Omega} u^{2p-1}g(u)dA - 2p(2p-1)\int_\Omega u^{2p-2}h'(u)\sum_{i,j=1}^N a^{ij}(x)u_{x_i}u_{x_j}dx + 2p\int_\Omega b(x)u^{2p-1}f(u)dx\\ &\leq 2p\gamma_2\int_{\partial\Omega} u^{2p+\gamma-1}dA - 2p(2p-1)\theta\int_\Omega u^{2p-2}h'(u)u_{x_i}u_{x_i}dx + 2p\gamma_1\int_\Omega b(x)u^{2p+k-1}dx\\ &\leq 2p\gamma_2\int_{\partial\Omega} u^{2p+\gamma-1}dA - \frac{2(2p-1)h_m}{p}\theta\int_\Omega |\nabla u^p|^2dx + 2p\gamma_1\int_\Omega b(x)u^{2p+k-1}dx.\end{aligned} \tag{4.3}$$

By using Lemma 2.1, we obtain

$$\begin{aligned}\int_{\partial\Omega} u^{2p+\gamma-1} dA &\le \frac{N}{L_0}\int_\Omega u^{2p+\gamma-1}dx + \frac{(2p+\gamma-1)d}{L_0}\int_\Omega u^{2p+\gamma-2}|\nabla u|dx\\ &= \frac{N}{L_0}\int_\Omega u^{2p+\gamma-1}dx + \frac{(2p+\gamma-1)d}{pL_0}\int_\Omega u^{p+\gamma-1}|\nabla u^p|dx\\ &\le \frac{1}{2}\int_\Omega u^{2p}dx + \frac{1}{2}\epsilon_1\int_\Omega |\nabla u^p|^2dx + \Big[\frac{N^2}{2L_0^2} + \frac{(2p+\gamma-1)^2d^2}{2p^2L_0^2\epsilon_1}\Big]\int_\Omega u^{2p+2\gamma-2}dx,\end{aligned} \tag{4.4}$$

where ϵ_1 is a positive constant to be determined later. Substituting (4.4) into (4.3), we get

$$\begin{aligned}\Psi'(t) &\le p\gamma_2\int_\Omega u^{2p}dx + [\epsilon_1 p\gamma_2 - \frac{2(2p-1)h_m}{p}\theta]\int_\Omega |\nabla u^p|^2dx\\ &\quad + p\gamma_2\Big[\frac{N^2}{L_0^2} + \frac{(2p+\gamma-1)^2d^2}{p^2L_0^2\epsilon_1}\Big]\int_\Omega u^{2p+2\gamma-2}dx + 2p\gamma_1\int_\Omega b(x)u^{2p+k-1}dx.\end{aligned} \tag{4.5}$$

Since $2\gamma < 1 + k$ and $\gamma > 1$, by Hölder's inequality, we have

$$\int_\Omega u^{2p+k-1}dx \ge \Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{2p+k-1}{2p+2\gamma-2}}|\Omega|^{\frac{2\gamma-k-1}{2p+2\gamma-2}}, \tag{4.6}$$

and

$$\Psi(t) \le h_M\int_\Omega u^{2p}dx \le h_M\Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{2p}{2p+2\gamma-2}}|\Omega|^{\frac{\gamma-1}{p+\gamma-1}}. \tag{4.7}$$

Inserting (4.6) and (4.7) into (4.5) and choosing $\epsilon_1 = \frac{2(2p-1)h_m}{p^2\gamma_2}\theta$, we have

$$\begin{aligned}\Psi'(t) &\le a_1\Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{2p}{2p+2\gamma-2}} + a_2\int_\Omega u^{2p+2\gamma-2}dx - a_3\Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{2p+k-1}{2p+2\gamma-2}}\\ &= \Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)\Big[a_1\Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{2-2\gamma}{2p+2\gamma-2}} + a_2 - a_3\Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)^{\frac{k+1-2\gamma}{2p+2\gamma-2}}\Big],\end{aligned} \tag{4.8}$$

where

$$a_1 = p\gamma_2|\Omega|^{\frac{\gamma-1}{p+\gamma-1}},\ a_2 = p\gamma_2\Big[\frac{N^2}{L_0^2} + \frac{(2p+\gamma-1)^2d^2}{p^2L_0^2\epsilon_1}\Big],\ a_3 = 2p\gamma_1|\Omega|^{\frac{2\gamma-k-1}{2p+2\gamma-2}}. \tag{4.9}$$

From (4.7), it follows that

$$\int_\Omega u^{2p+2\gamma-2}dx \ge \Big(\int_\Omega u^{2p}dx\Big)^{\frac{2p+2\gamma-2}{2p}}|\Omega|^{\frac{1-\gamma}{2p}}. \tag{4.10}$$

In light of $2\gamma < 1 + k$ and $\gamma > 1$, we can easily find that $\frac{2-2\gamma}{2p+2\gamma-2} < 0$, $\frac{2p+2\gamma-2}{2p} > 0$. So, combining (4.8) and (4.10), we have

$$\Psi'(t) \le \Big(\int_\Omega u^{2p+2\gamma-2}dx\Big)\Big[a_1(h_M)^{\frac{2\gamma-2}{2p}}|\Omega|^{\frac{1-\gamma}{2p}}\Psi^{\frac{2-2\gamma}{2p}} + a_2 - a_3(h_M)^{\frac{2\gamma-k-1}{2p}}|\Omega|^{\frac{1-\gamma}{2p}}\Psi^{\frac{k+1-2\gamma}{2p}}\Big]. \tag{4.11}$$

From (4.11) we can conclude that $\Psi(t)$ can not blow up in finite time under the conditions (4.2). In fact, if $u(x,t)$ blows up at time t^*, then $\Psi(t)$ becomes unbounded when $t \to t^*$, which leads $\Psi^{\frac{2-2\gamma}{2p}} \to 0$ and $\Psi^{\frac{k+1-2\gamma}{2p}} \to \infty$. So, $\Psi(t)$ decrease in some interval $[t_0, t^*)$ which follows that $\Psi(t) \le \Psi(t_0)$. This is a contradiction.

Now, we consider a particular case of (3.16) with $g(u) = 0$

$$\begin{cases} (h(u))_t = \nabla(a(u)\nabla u) + b(x)f(u), & in\ \Omega \times (0,t^*), \\ \frac{\partial u}{\partial \nu} = 0, & on\ \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \ge 0, & in\ \Omega. \end{cases} \tag{4.12}$$

We still use the auxiliary function $\varphi(t)$ which is defined in (2.5), but with $n > 2(N-2)$. Assuming that f, g, h satisfy the conditions (2.3) and $a(u) \ge \theta$ and referring to (2.8) we write for $g(u) = 0$

$$\varphi'(t) \le -\frac{2\beta\theta}{n}\int_\Omega |\nabla v^n|^2 dx + 2n\int_\Omega v^{2n+1} b(x)dx. \tag{4.13}$$

By Hölder and Schwarz inequalities, we have

$$\begin{aligned} \int_\Omega v^{2n+1}b(x)dx &\le \Big(\int_\Omega v^{2n+2}dx\Big)^{\frac{2n+1}{2n+2}}\Big(\int_\Omega b^{2n+2}(x)dx\Big)^{\frac{1}{2n+2}} \\ &\le \frac{2n+1}{2n+2}\Big(\int_\Omega v^{2n+2}dx\Big) + \frac{1}{2n+2}\Big(\int_\Omega b^{2n+2}(x)dx\Big) \\ &\le \frac{2n+1}{2n+2}\Big(\int_\Omega v^{2n}dx\Big)^{\frac{n-2(N-2)}{n}}\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big)^{\frac{2(N-2)}{n}} + \frac{1}{2n+2}\int_\Omega b^{2n+2}(x)dx \\ &\le \frac{(2n+1)[n-2(N-2)]}{n(2n+2)}\Big(\int_\Omega v^{2n}dx\Big) + \frac{2(2n+1)(N-2)}{n(2n+2)}\Big(\int_\Omega v^{\frac{n(2N-3)}{N-2}}dx\Big) \\ &+ \frac{1}{2n+2}\int_\Omega b^{2n+2}(x)dx. \end{aligned} \tag{4.14}$$

Recalling (2.18) and (2.19), we have from (4.14)

$$\begin{aligned} \int_\Omega v^{2n+1}b(x)dx \le a_1\Big(\int_\Omega v^{2n}dx\Big) + a_2\Big(\int_\Omega v^{2n}dx\Big)^{\frac{3}{4}}\Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{N}{4(N-2)}} \\ + a_2\Big(\int_\Omega v^{2n}dx\Big)^{\frac{2N-3}{2(N-2)}} + a_3, \end{aligned} \tag{4.15}$$

where $a_1 = \frac{(2n+1)[n-2(N-2)]}{n(2n+2)}$, $a_2 = \frac{2(2n+1)(N-2)}{n(2n+2)}c$, $a_3 = \frac{1}{2n+2}\int_\Omega b^{2n+2}(x)dx$. Inserting (4.15) into (4.13), we have

$$\begin{aligned} \varphi'(t) &\le -\frac{2\beta\theta}{n}\int_\Omega |\nabla v^n|^2dx + 2na_1\varphi + 2na_2\varphi^{\frac{3}{4}}\Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{N}{4(N-2)}} \\ &+ 2na_2\varphi^{\frac{2N-3}{2(N-2)}} + 2na_3 \\ &= \Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{N}{4(N-2)}}\Big\{-\frac{2\beta\theta}{n}\Big(\int_\Omega |\nabla v^n|^2dx\Big)^{\frac{3N-8}{4(N-2)}} + 2na_2\varphi^{\frac{3}{4}}\Big\} \\ &+ 2na_1\varphi + 2na_2\varphi^{\frac{2N-3}{2(N-2)}} + 2na_3. \end{aligned} \tag{4.16}$$

Since u has mean value zero it follows that

$$\int_\Omega |\nabla v^n|^2 dx \geq \mu \int_\Omega v^{2n} dx, \tag{4.17}$$

where μ is the first non-zero eigenvalue for

$$\Delta\omega + \mu\omega = 0,\ in\ \Omega; \quad \frac{\partial\omega}{\partial\nu} = 0,\ on\ \partial\Omega. \tag{4.18}$$

Lower bounds for μ can be found, for instance in [21, 22]. From (4.16), we get

$$\begin{aligned}\varphi'(t) \leq & \Big(\int_\Omega |\nabla v^n|^2 dx\Big)^{\frac{N}{4(N-2)}} \varphi^{\frac{3N-8}{4(N-2)}} \Big\{ -\frac{2\beta\theta}{n}\mu^{\frac{3N-8}{4(N-2)}} + 2na_2\varphi^{\frac{1}{2(N-2)}} \Big\} \\ & + 2na_1\varphi + 2na_2\varphi^{\frac{2N-3}{2(N-2)}} + 2na_3 \doteq \pi(\varphi).\end{aligned} \tag{4.19}$$

Now let the initial function u_0 and β satisfy

$$\mu^{\frac{N}{4(N-2)}}\varphi(0)\Big\{ -\frac{2\beta\theta}{n}\mu^{\frac{3N-8}{4(N-2)}} + 2na_2[\varphi(0)]^{\frac{1}{2(N-2)}} \Big\} + 2na_1\varphi(0) + 2na_2[\varphi(0)]^{\frac{2N-3}{2(N-2)}} + 2na_3 < 0. \tag{4.20}$$

It follows then that $\varphi(t)$ is initially decreasing. Since $\pi(\varphi)$ is increasing in φ, it follows that $\pi(\varphi(t))$ is decreasing in t for $t > 0$. This shows that $\varphi'(t)$ remains negative. So $\varphi(t)$ can not blow up at any finite time. In fact

$$\varphi'(t) \leq \varphi\Big\{ -\frac{2\beta\theta}{n}\mu + 2na_2\mu^{\frac{N}{4(N-2)}}\varphi^{\frac{1}{2(N-2)}} + 2na_1 + 2na_2\varphi^{\frac{1}{2(N-2)}} \Big\} + 2na_3. \tag{4.21}$$

If we choose suitbale initial condition and β to satisfy that

$$-\frac{2\beta\theta}{n}\mu + 2na_2\mu^{\frac{N}{4(N-2)}}[\varphi(0)]^{\frac{1}{2(N-2)}} + 2na_1 + 2na_2[\varphi(0)]^{\frac{1}{2(N-2)}} < -\lambda, \tag{4.22}$$

for some positive λ, then

$$\varphi'(t) \leq -\lambda\varphi(t) + 2na_3 \tag{4.23}$$

or

$$\varphi(t) \leq [\varphi(0) - \frac{2na_3}{\lambda}]e^{-\lambda t} + \frac{2na_3}{\lambda}. \tag{4.24}$$

This inequality shows that $\varphi(t)$ decays exponentially in time as $t \to \infty$. This is to say that $\varphi(t)$ remains bounded for $t > 0$. We have established the following theorem.

Theorem 4.2. Assume that f, g, h satisfy the conditions (2.3) and $a(u) \geq \theta$. Then

(1) If the initial condition u_0 and β to satisfy (4.20), then the function $\varphi(t)$ defined by (2.4) remains bounded in L^2.

(2) Furthermore, if the initial condition u_0 and β to satisfy (4.22), $\varphi(t)$ decays exponentially to $\frac{2na_3}{\lambda}$ in time as $t \to \infty$.

5. Conclusions

In this paper, lower bounds for the blow-up time of the solution are derived in higher dimensional spaces by virtue of the modified differential inequality. An upper bound for the blow-up time are specified under appropriate assumptions.

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Conflict of interest

The authors declare that they have no competing interests.

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