Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11752–11780.
DOI: 10.3934/math.2023595
Received: 03 January 2023
Revised: 03 March 2023
Accepted: 09 March 2023
Published: 17 March 2023

Research article

Nonlocal impulsive differential equations and inclusions involving Atangana-Baleanu fractional derivative in infinite dimensional spaces

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Abstract: The aim of this paper is to derive conditions under which the solution set of a nonlocal impulsive differential inclusions involving Atangana-Baleanu fractional derivative is a nonempty compact set in an infinite dimensional Banach spaces. Existence results for solutions in the presence of instantaneous or non-instantaneous impulsive effect are given. We considered the case where the right hand side is either a single valued function, or a multifunction. This generalizes recent results to the case when there are impulses, the right hand side is a multifunction, and where the dimension of the space is infinite. Examples are given to illustrate the effectiveness of the established results.

Keywords: Atangana-Baleanu fractional derivative; fractional differential inclusions; instantaneous and noninstantaneous impulses; measure of noncompactness
Mathematics Subject Classification: 34A08, 26A33

1. Introduction

Many real life dynamical systems experience sudden changes or shock. These systems are subject to impulses and can be mathematically modelled using impulsive differential equations. There are two types of such impulses. The first type takes place over a relatively short time compared to the overall duration of the whole process, and is modeled using instantaneous impulsive differential equations. It can be found in many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, and frequency modulated systems [1–3]. In the second type, the changes begin impulsively at some points and remain active over certain time intervals. The mathematical models of these systems use non-instantaneous impulsive differential equations. These equations give rise to new hybrid dynamical systems which contain a continuous-time dynamical system, a discrete-time dynamical system, and an algebraic system [4,5]. Non-instantaneous impulsive differential equations provide an excellent tool to describe the injection of drugs in the bloodstream and their consequent absorption in the body. A generalization of impulsive differential equations are the impulsive differential inclusion, where the right hand side is replaced by multifunction. The studies in [6–10] contain results on the existence of solutions of mild solutions for different kinds of impulsive differential equations and inclusions.

Differential inclusions and differential equations with non-local conditions, in general, arise in many application in engineering and biology [11], including projected dynamical systems, discontinuous and switching dynamical systems, nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion. Many results on nonlocal differential equations and inclusions have been established [12, 13].

Like other differential equations and inclusions, impulsive equations are also generalized to their fractional versions. Fractional calculus has many applications in industry and applied sciences [14–23]. There are many definitions of fractional derivatives, some of which have a singular kernel as those given by Riemann-Liouvill and Caputo [23, 24]. Recently, fractional derivatives with nonsingular kernel were introduced, such as, Caputo and Fabrizio (CF) [25], in which the kernel was based on the exponential function, and Atangana and Baleanu (AB) [26], in which the kernel was based on the Mittag-Leffler function. In [27–29] some application for the AB derivative are given. Several researchers established existence results for solutions of fractional differential equations involving AB derivative in finite dimensional spaces [30–40].

It have been noticed that all existence results in the above cited works, concerning differential equations involving AB derivative, do not contain impulsive effects whether instantaneous or non-instantaneous, and the great majority were concerned exclusively with finite dimensional spaces.

Still, the number of results in the literature about the existence of solutions for differential inclusions (the right side is a multifunction) involving AB derivative is rare. To the extent of the authors' knowledge, the existence of solution for fractional differential equations or inclusions containing AB derivative in the presence of non-local conditions, impulsive effects has not been treated yet. This paper attempts to fill the gap in the literature. The main contributions of this work can be summarized as follows:

(1) A new class of differential equations and differential inclusions containing AB derivative with instantaneous or non-instantaneous impulses and nonlocal conditions in infinite dimensional Banach spaces are formulated.

(2) The existence/uniqueness of solutions for the formulated equations and inclusions were proved.

(3) A generalization of a recent result (Theorem 3.1 in [33]) to infinite dimensional Banach spaces in the presence of both impulses and nonlocal conditions is provided and proved.

(4) The used method helps interested researchers to generalize results to the case where the right hand side is a multifunction, or in the presence of both impulsive effects and nonlocal condition.

Notation 1.1. *Through out this paper, we fix the notation to be as follows:*

- $J = [0, b] \subset \mathbb{R}$, *where* $b > 0$, *and* $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \cdots < s_m < t_{m+1} = b$ *is a partition of* J.
- $J_0 = [0, t_1]$ *and* $J_i = (t_i,\ t_{i+1}], i = 1, 2, ..., m.$

- *E is a real Banach space, and $u_0 \in E$ is a fixed point.*
- *$P_{ck}(E) = \{B \subseteq E : B$ is non-empty, convex, and compact$\}$.*
- *$H^1((a,b),E)$ denotes the Sobolev space $\{u \in L^2((a,b),E) : u' \in L^2((a,b),E)\}$.*
- *$PC(J,E)$ is the Banach space which consists of the functions x: $J \to E$ such that $x_{|J_i} \in C(J_i,E)$, $i = 0,1,2,...,m$, and $x(t_i^-)$, $x(t_i^+)$ exist for each $i = 1,2,...,m$. The norm on $PC(J,E)$ is given by:*

$$\|x\|_{PC(J,E)} := \sup\{\|x(t)\| : t \in J\}.$$

- *For a multifunction F, and $u \in PC(J,E)$, the space $S^1_{F(.,u(.))}$ denotes*

$$\{z \in L^1(J,E) : z(s) \in F(t,u(t)), a.e.\}.$$

In the current study, we consider the following impulsive differential equations.

(1) Differential equation with non-instantaneous impulses:

$$\begin{cases} {}^{ABC}D^{\alpha}_{s_i,t}u(t) = f(t,u(t))\ a.e., & t \in \cup_{i=0}^{m}(s_i,t_{i+1}], \\ u(t) = g_i(t,u(t_i^-)), & t \in \cup_{i=0}^{m}(t_i,s_i], \\ u(0) = u_0 - g(u). \end{cases} \tag{1.1}$$

(2) Differential equation with instantaneous impulses:

$$\begin{cases} {}^{ABC}D^{\alpha}_{0,t}u(t) = f(t,u(t))\ a.e., & t \in [0,b] - \{t_1,t_2,...,t_m\}, \\ u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), & i = 1,2,...,m, \\ u(0) = u_0 - g(u). \end{cases} \tag{1.2}$$

(3) Differential inclusion with non-instantaneous impulses:

$$\begin{cases} {}^{ABC}D^{\alpha}_{s_i,t}u(t) \in \int_{s_i}^{t} F(s,u(s))ds,\ a.e. & t \in \cup_{i=0}^{m}(s_i,\ t_{i+1}], \\ u(t) = g_i(t,u(t_i^-)), & t \in \cup_{i=0}^{m}(t_i,s_i], \\ u(0) = u_0 - g(u), \end{cases} \tag{1.3}$$

where $\alpha \in (0,1)$, and ${}^{ABC}D^{\alpha}_{s_i,t}$ is the Atangana-Baleanu fractional derivative in the Caputo sense of order α with lower limit at s_i. The map f is a function on $[a,b] \times E$ with values in E, and F is a multifunction defined on $J \times E$ whose values are nonempty convex compact subsets of E. The map g: $PC([a,b],E) \to E$ is a continuous map, and the maps g_i: $[t_i,s_i] \times E \longrightarrow E$ and I_i: $E \to E$, $i = 1,2,...,m$, are the impulsive functions.

Remark 1.1. *The initial condition $u(0) = u_0 - g(u)$ is in the general form. It is common to be considered in non-local conditions [11–13]. The advantage of using non-local conditions is that measurements can be combined in more places to obtain better models. For example, for a non-uniform rod, g may be given by*

$$g(x) = \sum_{i=1}^{i=m} \omega_i g(t_i), \tag{1.4}$$

where $\omega_i, i = 1,...,m$ are given constants [41]. In this case, the formula allows the additional measurements at $\omega_i, i = 1,...,m$. A formula similar to (1.4) is also used in [42] to describe the diffusion phenomenon of a small amount of gas in a transparent tube. Also if $g(x) = 0, \forall x \in PC(J,E)$, we obtain the local condition $u(0) = u_0$. If $g(x) = u_0 - u(b)$, we get $u(0) = u(b)$, while $u(0) = -u(b)$, if $g(x) = u_0 + u(b)$.

The paper is organized as follows. In the second section, we recall all needed facts and results. Section 3 presents the existence and uniqueness of solution for problem (1.1). In Section 4, we show the existence of solutions for problem (1.2). Section 5 is devoted to giving the sufficient conditions for the existence results of solutions for problem (1.3). Two examples are given in the last section to illustrate the possible applicability of the provided methods.

2. Preliminaries and background definitions

Definition 2.1. *[26, 30] The Atangana-Baleanu fractional derivative for a function $u \in H^1((a,b),E)$ where $a < b$ in the Caputo sense and in the Riemann-Liouville sense of order α with lower limit at a are defined by*

$$ {}^{ABC}D^{\alpha}_{a,t}u(t) := \frac{M(\alpha)}{(1-\alpha)}\int_a^t u'(x)E_\alpha\left(\frac{-\alpha(t-x)^\alpha}{1-\alpha}\right)dx, \tag{2.1}$$

and

$$ {}^{ABR}D^{\alpha}_{a,t}u(t) := \frac{M(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_a^t u(x)E_\alpha\left(\frac{-\alpha(t-x)^\alpha}{1-\alpha}\right)dx, \tag{2.2}$$

where $M(\alpha) > 0$ is a normalization function satisfying $M(0) = M(1) = 1$, and E_α is the well known Mittag-Leffler function of one variable given by:

$$E_\alpha(\mu) = \sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma(\alpha k+1)}, \mu \in \mathbb{C}. \tag{2.3}$$

Definition 2.2. *The Atangana-Baleanu fractional integral of a function $u \in L^1((a,b),E)$, where $a < b$ and with lower limit at a is given by:*

$$ {}^{AB}I^{\alpha}_{a,t}u(t) = \frac{(1-\alpha)}{M(\alpha)}u(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_a^t u(x)(t-x)^{\alpha-1}dx. \tag{2.4}$$

Lemma 2.1. *[26]. Let $u \in H^1((a,b),E)$.*

(1) ${}^{ABR}D^{\alpha}_{a,t}u(t) = {}^{ABC}D^{\alpha}_{a,t}u(t) + \frac{M(\alpha)}{1-\alpha}u(a)E_\alpha\left(\frac{-\alpha(t-a)^\alpha}{1-\alpha}\right)$.

(2) ${}^{ABR}D^{\alpha}_{0,t}\left({}^{AB}I^{\alpha}_{0,t}u(t)\right) = u(t),\ t \in J$.

(3) ${}^{ABR}D^{\alpha}_{a,t}c = cE_\alpha\left(\frac{-\alpha}{1-\alpha}(t-a)^\alpha\right)$.

The proof of the following lemma can be found in [33].

Lemma 2.2. *Let ϖ: $J \to E$ be continuous with $\varpi(0) = 0$. A function $u \in H^1((0,b),E)$ is a solution for the fractional differential equation*

$$\begin{aligned} {}^{ABC}D^{\alpha}_{0,t}u(t) &= \varpi(t), \quad t \in J, \\ u(0) &= u_0, \end{aligned} \tag{2.5}$$

if

$$u(t) = u_0 + \frac{1-\alpha}{M(\alpha)}\varpi(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varpi(s)ds. \tag{2.6}$$

Using this lemma, the solution for problem (1.1) can be specified as in the following definition:

Definition 2.3. *A function $u \in PC(J,E)$ is a solution for problem (1.1) if*

$$u_{|(s_i,t_{i+1}]} \in H^1((s_i,t_{i+1}),E) \text{ for } i = 0,1,...,m,$$

and

$$u(t) = \begin{cases} u_0 - g(u) + \frac{(1-\alpha)}{M(\alpha)}f(t,u(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds, & t \in [0,t_1], \\ g_i(t,u(t_i^-)), & t \in \cup_{i=1}^{i=m}(t_i,s_i], \\ g_i(s_i,u(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)}f(t-s_i,u(t-s_i)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}f(s,u(s))ds, & \\ t \in \cup_{i=1}^{m}(s_i,t_{i+1}]. & \end{cases} \tag{2.7}$$

Remark 2.1. *The solution function u given by (2.7) is continuous at s_i, hence on $J_i, i = 1,2,...,m$.*

Definition 2.4. *A function $u \in PC(J,E)$ is a solution for problem (1.2) if*

$$u(t) = \begin{cases} u_0 - g(u) + \frac{(1-\alpha)}{M(\alpha)}f(t,u(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds, \quad t \in [0,t_1], \\ u_0 - g(u) + \frac{(1-\alpha)}{M(\alpha)}f(t,u(t)) + \sum_{k=1}^{i} I_k(u(t_k^-)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))ds, \\ t \in J_i, i = 1,2,...,m. \end{cases} \tag{2.8}$$

Remark 2.2. *The solution function given by (2.8) is continuous on J_i and $u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), i = 1,2,...,m$.*

Definition 2.5. *A function $u \in PC(J,E)$ is a solution for problem (1.3) if $u_{|(s_i,t_{i+1}]} \in H^1((s_i,t_{i+1}),E)$ is continuous at s_i; $i = 0,1,..,m$, and*

$$u(t) = \begin{cases} u_0 - g(u) + \frac{(1-\alpha)}{M(\alpha)}f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, & t \in [0,t_1], \\ g_i(t,u(t_i^-)), & t \in \cup_{i=1}^{i=m}(t_i,s_i], \\ g_i(s_i,u(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)}f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}f(s)ds, & t \in \cup_{i=1}^{m}(s_i,t_{i+1}], \end{cases} \tag{2.9}$$

where f: $J \to E$ satisfying $f(t) = \int_{s_i}^t z(s)ds$; $t \in [s_i,t_{i+1}]$; $i = 0,1,..,m$, $z \in S^1_{F(.,u(.))}$.

Remark 2.3. *The solution function u given by (2.9) is continuous at s_i, $i = 1, 2, ..., m$, hence continuous on $J_i = 0, 1, 2, ..., m$.*

In the following lemma, we recall the Schauder fixed point theorem.

Lemma 2.3. *Let X be a Banach space and $S \subseteq X$ be compact, convex, and non-empty. Any continuous operator $T\colon S \to S$ has at least one fixed point.*

3. Existence of a solution for problem (1.1)

Hypothesis 3.1. *We assume the following hypotheses:*

(1) $\underline{\underline{(Hf)}}$ *The function $f\colon J \times E \to E$ is satisfying*

   (a) $f(0, u_0) = 0$,

   (b) *$f(., u)$ continuous for $u \in E$, and for any $t \in J$, the map $u \to f(t, u)$ is uniformly continuous on bounded sets,*

   (c) *there is a continuous function $\varphi\colon J \to \mathbb{R}^+$ satisfying*

$$\|f(t, z)\| \le \varphi(t)(1 + \|z\|), \forall (t, z) \in J \times E, \tag{3.1}$$

   (d) *there is a continuous function $\eta\colon J \to \mathbb{R}^+$ such that for any bounded subset $B \subset E$,*

$$\varkappa(f(t, B)) \le \eta(t)\, \varkappa(B), \text{ for } t \in J, \tag{3.2}$$

   and

$$\sup_{t\in J} |\eta(t)| \left(\frac{4(1-\alpha)}{M(\alpha)} + \frac{4b^\alpha}{M(\alpha)\Gamma(\alpha)} \right) < 1, \tag{3.3}$$

   where $\varkappa$ is the measure of noncompactness on E.

(2) $\underline{\underline{(Hg)}}$ *The function $g\colon PC(J, E) \to E$ is continuous, compact, and there are two positive real numbers a, d such that*

$$\|g(x)\| \le a\|x\| + d, \forall x \in PC(J, E). \tag{3.4}$$

(3) $\underline{\underline{(H)}}$ *For every $i = 1, 2, ..., k$, $g_i\colon [t_i, s_i] \times E \to E$ is defined such that for any $t \in [t_i, s_i]$, the function $x \to g_i(t, x)$ is uniformly continuous and compact on bounded subsets, and there is $\gamma > 0$ with $\|g_i(t, x)\| \le \gamma\|x\|, t \in \cup_{i=1}^m [t_i\,, s_i], x \in E$.*

Theorem 3.1. *If (Hf), (Hg) and (H) are satisfied, then problem (1.1) has a solution provided that*

$$\gamma + a + \varrho \left(\frac{1-\alpha}{M(\alpha)} + \frac{b^\alpha}{M(\alpha)\Gamma(\alpha)} \right) < 1, \tag{3.5}$$

where $\varrho = \sup_{t\in J} |\varphi(t)|$.

Proof. Let $T\colon PC(J,E) \to PC(J,E)$ be defined as

$$T(x)(t) = \begin{cases} u_0 - g(x) + \frac{1-\alpha}{M(\alpha)} f(t,x(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds, & t \in [0,t_1], \\ g_i(t, x(t_i^-)), & t \in \cup_{i=1}^{i=m}(t_i, s_i], \\ g_i(s_i, x(t_i^-)) + \frac{1-\alpha}{M(\alpha)} f(t-s_i, x(t-s_i)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s,x(s))ds, \\ t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}]. \end{cases} \tag{3.6}$$

Our aim is to use Schauder's fixed point theorem to show that T has a fixed point. Set

$$B_{k_0} = \{x \in PC(J,E) : \|x\|_{PC(J,E)} \leq k_0\},$$

$$k_0 = \frac{\|u_0\| + d + \frac{\varrho(1-\alpha)}{M(\alpha)} + \frac{\alpha\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}}{1 - \left[\gamma + a + \frac{\varrho(1-\alpha)}{M(\alpha)} + \frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}\right]}. \tag{3.7}$$

The proof will proceed through the following steps.

- Step 1. In this step, we claim that $T(B_{k_0}) \subseteq B_{k_0}$. Let $x \in B_{k_0}$ and $t \in [0,t_1]$. Using the assumptions (c) in (Hf), and (Hg), we obtain from (3.6)

$$\begin{aligned} \|T(x)(t)\| &\leq \|u_0\| + ak_0 + d + \frac{(1-\alpha)}{M(\alpha)}\varrho(1+k_0) + \frac{\alpha\varrho(1+k_0)}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \|u_0\| + ak_0 + d + \frac{(1-\alpha)}{M(\alpha)}\varrho(1+k_0) + \frac{\varrho(1+k_0)b^\alpha}{M(\alpha)\Gamma(\alpha)}. \end{aligned} \tag{3.8}$$

For $t \in \cup_{i=1}^m (t_i\ s_i]$, we have

$$\|T(x)(t)\| \leq \|g_i(t, x(t_i^-))\| \leq \gamma\, k_0. \tag{3.9}$$

For $t \in (s_i, t_{i+1}]$, by repeating the arguments employed in (3.8), it follows that

$$\|T(x)(t)\| \leq k_0\gamma + \|u_0\| + ak_0 + d + \varrho(1+k_0)\left[\frac{(1-\alpha)}{M(\alpha)} + \frac{b^\alpha}{M(\alpha)\Gamma(\alpha)}\right]. \tag{3.10}$$

From (3.8)–(3.10), we obtain that

$$\begin{aligned} \|T(x)\|_{PC(J,E)} &\leq \|u_0\| + d + \varrho\left[\frac{(1-\alpha)}{M(\alpha)} + \frac{b^\alpha}{M(\alpha)\Gamma(\alpha)}\right] \\ &+ k_0\left[\gamma + a + \frac{\varrho(1-\alpha)}{M(\alpha)} + \frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}\right]. \end{aligned} \tag{3.11}$$

From this relation and (3.7), it follows that $T(x) \in B_{k_0}$.

- Step 2. $T\colon B_{k_0} \to B_{k_0}$ is continuous. Suppose that $x_k \in B_{k_0}$, $x_k \to x$. By definition of T,

$$T(x_k)(t)=\begin{cases} u_0-g(x_k)+\frac{(1-\alpha)}{M(\alpha)}f(t,x_k(t))+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_k(s))ds, & t\in[0,t_1],\\ g_i(t,x_k(t_i^-)), & t\in\cup_{i=1}^{i=m}[t_i,s_i],\\ g_i(s_i,x_k(t_i^-))+\frac{(1-\alpha)}{M(\alpha)}f(t-s_i,x_k(t-s_i))+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_{s_i}^t(t-s)^{\alpha-1}f(s,x_k(s)), & \\ t\in\cup_{i=1}^{i=m}(s_i,t_{i+1}]. & \end{cases} \tag{3.12}$$

By continuity of f, g, and $g_i(t,.)$, and from Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty}T(x_k)=T(x).$$

- Step 3. In this step, we show that the sets $D|_{\overline{J_i}}$ are equicontinuous for any $i=0,1,...,m$, where $D=T(B_{k_0})$ and

$$D|_{\overline{J_i}}=\{u^*\in(\overline{J_i},E):u^*(t)=u(t),t\in(t_i,t_{i+1}],u^*(t_i)=\lim_{t\to t_i^+}u(t),u\in D\}. \tag{3.13}$$

Let $u=T(x),\ x\in B_{k_0}$. We consider the following cases:

Case 1. $i=0$. Let $t, t+\delta$ be two points in $\overline{J_0}=[0,t_1]$. By the uniform continuity of f on bounded sets, we get

$$\begin{aligned}\lim_{\delta\to0}\|u^*(t+\delta)-u^*(t)\| &= \lim_{\delta\to0}\|u(t+\delta)-u(t)\|\\ &\leq \lim_{\delta\to0}\|f(t+\delta,x(t+\delta))-f(t,x(t))\|\\ &+ \lim_{\delta\to0}\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\left\|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}f(s,x(s))ds-\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds\right\|\\ &= \lim_{\delta\to0}\|f(t+\delta,x(t+\delta))-f(t,x(t))\|\\ &+ \frac{\alpha\rho(1+k_0)}{M(\alpha)\Gamma(\alpha)}\lim_{\delta\to0}\left\|\int_0^{t+\delta}(t+\delta-s)^{\alpha-1}ds-\int_0^{t+\delta}(t-s)^{\alpha-1}ds\right\|\\ &+ \frac{\varrho\alpha}{M(\alpha)\Gamma(\alpha)}\lim_{\delta\to0}\left\|\int_0^{t+\delta}(t-s)^{\alpha-1}ds-\int_0^t(t-s)^{\alpha-1}ds\right\|\\ &\leq \lim_{\delta\to0}\|f(t+\delta,x(t+\delta))-f(t,x(t))\|\\ &+ \frac{\alpha\gamma(1+k_0)}{M(\alpha)\Gamma(\alpha)}\lim_{\delta\to0}\int_0^{t+\delta}|(t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1}|ds\\ &+ \frac{\alpha\gamma(1+k_0)}{M(\alpha)\Gamma(\alpha)}\lim_{\delta\to0}\int_t^{t+\delta}(t-s)^{\alpha-1}ds.\end{aligned} \tag{3.14}$$

By condition (b) in (Hf), we get

$$\begin{aligned}&\lim_{\delta\to0}\|f(t+\delta,x(t+\delta))-f(t,x(t))\|\\ \leq\ &\lim_{\delta\to0}\|f(t+\delta,x(t))-f(t,x(t))\|+\lim_{\delta\to0}\|f(t+\delta,x(t+\delta))-f(t+\delta,x(t))\|.\end{aligned} \tag{3.15}$$

Since $f\ (.,x(t))$ is continuous, and $f(t+\delta,.)$ is uniformly continuous, it follows that

$$\lim_{\delta\to 0}\|f(t+\delta,x(t+\delta))-f(t,x(t))\|=0, \tag{3.16}$$

independently of x. Therefore,

$$\lim_{\delta\to 0}\|u^*(t+\delta)-u^*(t)\|=0, \tag{3.17}$$

independently of x.

Case 2. $i\geq 1$. let $t, t+\delta$ be two points in

$$J_i=(t_i,t_{i+1}]=(t_i,s_i)\cup[s_i,t_{i+1}].$$

If $t, t+\delta$ are in (t_i,s_i), then by the uniform continuous of $g_i(t,.)$ on bounded sets of E , one has

$$\begin{aligned}\lim_{\delta\to 0}\|u^*(t+\delta)-u^*(t)\| &= \lim_{\delta\to 0}\|u(t+\delta)-u(t)\|\\ &\leq \lim_{\delta\to 0}\|g_i(t+\delta,x(t_i^-))-g_i(t,x(t_i^-))\|=0,\end{aligned} \tag{3.18}$$

independently of x. If $t, t+\delta$ are in (s_i,t_{i+1}), then using the same arguments as in case 1, we obtain that $\lim_{\delta\to 0}\|u(t+\delta)-u(t)\|=0$, independently of x. If $t=s_i$, then

$$\begin{aligned}\lim_{\delta\to 0}\|u^*(t+\delta)-u^*(t)\| &= \lim_{\delta\to 0}\ \|u(s_i+\delta)-u(s_i)\|\\ &\leq \lim_{\delta\to 0}\|g_i(s_i,x(t_i^-))\|+\frac{(1-\alpha)}{M(\alpha)}\|f(\delta,x(\delta))\|\\ &+ \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\left\|\int_{s_i}^{s_i+\delta}(s_i+\delta-s)^{\alpha-1}f(s,x(s))ds-g_i(s_i,x(t_i^-))\right\|\\ &= \lim_{\delta\to 0}\left[\frac{(1-\alpha)}{M(\alpha)}\|f(\delta,x(\delta))\|+\frac{\alpha\gamma(1+k_0)}{M(\alpha)\Gamma(\alpha)}\int_{s_i}^{s_i+\delta}(s_i+\delta-s)^{\alpha-1}ds\right]\\ &= 0.\end{aligned} \tag{3.19}$$

When $t=t_i$,

$$\begin{aligned}&\lim_{\delta\to 0}\|u^*(t_i+\delta)-u^*(t_i)\|\\ &\lim_{\delta\to 0}\lim_{\lambda\to t_1^+}\|u(t_i+\delta)-u(\lambda)\|\\ \leq\ &\lim_{\delta\to 0}\lim_{\lambda\to t_i^+}\|f(t_i+\delta,x(t_i+\delta))-f(\lambda,x(\lambda))\|\\ +\ &\lim_{\delta\to 0}\lim_{\lambda\to t_i^+}\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\left\|\int_0^{t_i+\delta}(t_i+\delta-s)^{\alpha-1}f(s,x(s))ds-\int_0^{\lambda}(\lambda-s)^{\alpha-1}f(s.x(s))ds\right\|\\ =\ &0.\end{aligned} \tag{3.20}$$

Thus, our claim for this step is proved.

- Step 4. Set $B_n=T(B_{n-1})$, $n\geq 1$ and $B=\cap_{n=1}^{\infty}B_n$.

The sequence (B_n) is a non-decreasing, non-empty convex, bounded and closed sets. Our goal in this step is to show that B is relatively compact, hence compact. According to the generalized Cantor's intersection property [43], it is enough to show that

$$\lim_{n\to\infty} \varkappa_{PC}(B_n) = 0, \tag{3.21}$$

where $\varkappa_{PC}$ is Haudorff measure of noncompactness in $PC(J, E)$, which is defined by

$$\varkappa_{PC}(B_n) = \max_{0\le i\le m}\{\chi_i(B_{n|\overline{J_i}})\},$$

where χ_i is Hausdorff measure of noncompacness on $C(\overline{J_i}, E)$ [43]. Let $\epsilon > 0$, and $n \ge 1$ be a fixed, then there is a sequence (u_k) in B_n such that

$$\varkappa_{PC}(B_n) \;\le\; 2\varkappa_{PC}\{u_k : k \ge 1\} + \epsilon = 2\max_{0\le i\le m}\{\chi_i(D_{|\overline{J_i}}\}, \tag{3.22}$$

where $D = \{u_k : k \ge 1\}$ [44]. The set $D_{|\overline{J_i}}$ is defined in (3.13). From Step 2, $D_{|\overline{J_i}}$ is equicontinuous, hence relation (3.22) becomes

$$\varkappa_{PC}(B_n) \le 2\max_{t\in J}\chi\{u_k(t) : k \ge 1\} + \epsilon. \tag{3.23}$$

Since $u_k \in B_n = T(B_{n-1})$, there is $x_k \in B_{n-1}$with $u_k = T(x_k)$. One has by the definition of T that

$$u_k(t) = \begin{cases} u_0 - g(x_k) + \frac{(1-\alpha)}{M(\alpha)} f(t, x_k(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_k(s))ds, & t \in [0, t_1], \\ g_i(t, x_k(t_i^-)), & t \in \cup_{i=1}^{i=m}(t_i, s_i], \\ g_i(s_i, x_k(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)} f(t-s_i), x_k(t-s_i)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s, x_k(s))ds, & \\ t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}]. & \end{cases} \tag{3.24}$$

By compactness of both g and $g_i(t, .)$, $t \in \cup_{i=1}^{i=m}[t_i, s_i]$, it follows that

$$\chi\{g(x_k) : k \ge 1\} = 0 \;\text{ and }\; \chi\{g_i(t, x_k(t_i^-)) : k \ge 1\} = 0,$$

for any $t \in \cup_{i=1}^{i=m}[t_i, s_i]$. Moreover, in view of (d) in (Hf), for $t \in J$,

$$\chi\{f_k(t, x_k(t)) : k \ge 1\} \le \eta(t)\,\chi\{x_k(t) : k \ge 1\}. \tag{3.25}$$

By the properties of χ, it follows that for $t \in J$, we have

$$\begin{aligned} \chi\left(\int_0^t (t-s)^{\alpha-1} f(s, x_k(s))ds : k \ge 1\right) \;&\le\; 2\int_0^t (t-s)^{\alpha-1}\chi\,(f_k(s) : k \ge 1)\,ds \\ &=\; 2\chi_{PC}(B_{n-1})\int_0^t (t-s)^{\alpha-1}\eta(s)ds \\ &=\; 2\chi_{PC}(B_{n-1})\sup_{t\in J}|\eta(t)|\,\frac{b^\alpha}{\alpha}. \end{aligned} \tag{3.26}$$

Thus, by (3.22)–(3.26), it follows that

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_{n-1})\left(\frac{4(1-\alpha)}{M(\alpha)}\sup_{t\in J}|\eta(t)| + \frac{4b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\sup_{t\in J}|\eta(t)|\right) + \epsilon. \tag{3.27}$$

Since ϵ is arbitrary, we get that for all $t \in J$

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_{n-1})\sup_{t\in J}|\eta(t)|\left(\frac{4(1-\alpha)}{M(\alpha)} + \frac{4b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right). \tag{3.28}$$

Since this relation holds for every n, we get

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_1)\sup_{t\in J}|\eta(t)|\left(\frac{4(1-\alpha)}{M(\alpha)} + \frac{4b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right)^{n-1}, \tag{3.29}$$

which with (3.3) establishes (3.21).

- Step 5. Applying Schauder's fixed point theorem, the map $T\colon B \to B$ has a fixed point which is a solution for problem (1.1).

Remark 3.1. *Conditions (3.5) is necessary to show that there exists $k_0 > 0$ such that $T(B_{k_0}) \subseteq B_{k_0}$. Moreover, if*

$$g(x) = 0, \forall x \in PC(J, E),$$

and

$$g_i(t, x) = 0, \forall (t, x) \in [s_i, t_i];\ i = 1, 2, ..., m,$$

then this condition becomes

$$\varrho\left[\frac{(1-\alpha)}{M(\alpha)} + \frac{b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right] < 1. \tag{3.30}$$

which appears often in the literature, see for example, Theorem 3.1 in [33].

Theorem 3.2. *If we replace (Hf) in the statement of Theorem 3.1 by the following condition: $\underline{\underline{(Hf)^*}}$*

(1) $f(0, u_0) = 0$.

(2) There exists $L > 0$ such that

$$\|f(t_1, z_1) - f(t_2, z_2)\| \leq L\left(|t_2 - t_1| + \|z_2 - z_1\|\right), \forall (t_1, z_1), (t_2, z_2) \in J \times E, \tag{3.31}$$

then problem (1.1) has a solution provided that

$$\gamma + a + \frac{L(1-\alpha)}{M(\alpha)} + \frac{Lb^{\alpha}}{M(\alpha)\Gamma(\alpha)} < 1, \tag{3.32}$$

and

$$8L\left(\frac{1-\alpha}{M(\alpha)} + \frac{b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right) < 1. \tag{3.33}$$

If in addition, $g(x) = 0, \forall x \in PC(J, E)$, then this solution is unique.

Proof. By (3.31), f is uniformly continuous on bounded sets, and for any $(t, z) \in J \times E$,

$$\|f(t, z)\| \le \|f(t, z) - f(0, 0)\| + \|f(0, 0)\| \le bL + L\|z\| + \|f(0, 0)\|. \tag{3.34}$$

Moreover, if B is a bounded set, then for any (t, z_1), $(t, z_2) \in J \times B$, one has

$$\|f(t, z_1) - f(t, z_2)\| \le L\|z_1 - z_2\|. \tag{3.35}$$

Thus,

$$\varkappa(f(t, B)) \le L\,\mu(B) \le 2L\,\varkappa(B), \text{ for } t \in J, \tag{3.36}$$

where μ is Kuratwaski measure of noncompactness. Therefore, relation (3.2) holds with $\eta(t) = 2L$.

Next, using the arguments in the proof of Theorem 3.1, it follows that $T(B_{k_0^*}) \subseteq B_{k_0^*}$, where

$$k_0^* = \frac{\|u_0\| + \ d + (Lb + \|f(0, 0)\|)\left(\frac{(1-\alpha)}{M(\alpha)} + \frac{b^\alpha}{M(\alpha)\Gamma(\alpha)}\right)}{1 - \left[\gamma + a + \frac{L(1-\alpha)}{M(\alpha)} + \frac{Lb^\alpha}{M(\alpha)\Gamma(\alpha)}\right]}. \tag{3.37}$$

By applying Theorem (3.1), problem (1.1) has a solution if (3.32) and (3.33) are satisfied.

Now, assume that $g(x) = 0, \forall x \in E$, and let u, v be two solutions for problem (1.1). If $t \in [0, t_1]$, then

$$\begin{aligned}\|u(t) - v(t)\| &\le \frac{(1-\alpha)}{M(\alpha)}\|f(t, u(t)) - f(t, v(t))\| + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, u(s)) - f(s, v(s))\|ds \\ &\le \frac{L(1-\alpha)}{M(\alpha)}\|u(t)) - v(t)\| + \frac{L\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)) - v(s)\|ds. \end{aligned} \tag{3.38}$$

Thus,

$$\|u(t) - v(t)\| \le \left(1 - \frac{L(1-\alpha)}{M(\alpha)}\right)^{-1} \frac{L\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)) - v(s)\|ds. \tag{3.39}$$

Note that relation (3.33) implies that $\frac{L(1-\alpha)}{M(\alpha)} < 1$. Applying the generalized Gronwall inequality [Corollary 2, [45]], we obtain

$$u(t) = v(t), \forall t \in [0, t_1].$$

Thus,

$$g_i(t, u(t_1^-)) = g_i(t, v(t_1^-));\ t \in (t_1, s_1], \text{ and } u(t) = v(t), \forall t \in (t_1, s_1].$$

Next, let $t \in (s_1, t_2]$. If $t - s_1 \le s_1$, then

$$u(t - s_1) = v(t - s_1).$$

So,

$$\|u(t - s_1) - v(t - s_1)\| \le \sup_{\theta \in (s_1, t_2]} \|u(\theta) - v(\theta)\|. \tag{3.40}$$

Therefore,

$$\|u(t) - v(t)\| \le \frac{L(1-\alpha)}{M(\alpha)} \sup_{\theta \in (s_1, t_2]} \|u(\theta) - v(\theta)\| + \frac{L\alpha}{M(\alpha)\Gamma(\alpha)} \sup_{\theta \in (s_1, t_2]} \|u(\theta) - v(\theta)\| \int_{s_1}^t (t-s)^{\alpha-1}ds, \tag{3.41}$$

from which it follows

$$\begin{aligned}\sup_{\theta\in(s_1,t_2]}\|u(\theta)-v(\theta)\| &\leq \frac{L(1-\alpha)}{M(\alpha)}\sup_{\theta\in(s_1,t_2]}\|u(\theta)-v(\theta)\|+\frac{Lb^\alpha}{M(\alpha)\Gamma(\alpha)}\sup_{\theta\in(s_1,t_2]}\|u(\theta)-v(\theta)\| \\ &\leq \left[\frac{L(1-\alpha)}{M(\alpha)}+\frac{Lb^\alpha}{M(\alpha)\Gamma(\alpha)}\right]\sup_{\theta\in(s_1,t_2]}\|u(\theta)-v(\theta)\|. \qquad (3.42)\end{aligned}$$

This relation together with (3.33) implies that

$$\sup_{\theta\in(s_1,t_2]}\|u(\theta)-v(\theta)\|=0,$$

hence

$$u(t)=v(t)\forall t\in(s_1,t_2].$$

By repeating the same argument, we conclude that $u(t)=v(t), \forall t\in J$.

Remark 3.2. *Assumption (Hf) enables us to apply Schauder's fixed-point theorem to prove that the operator T has a fixed point, which is a solution to problem (1.1), but it does not yield any information about the uniqueness of the solution. The condition $(Hf)^*$ enables us to show that there is a solution to problem (1.1), as well as the uniqueness by using the generalized Gronwall inequality.*

4. Existence of solutions for problem (1.2)

Theorem 4.1. *If in addition to the assumptions (Hf) and (Hg),we assume the following condition.*
(HI): For any $i=1,2,...,m$, I_i: $E\to E$ is continuous and compact and there is $\xi>0$ with

$$\sup_{i=1,2,...,m}\|I_i(x)\|\leq\xi\|x\|,\ \forall x\in E,$$

then problem (1.2) has a solution provided that (3.2) and

$$a+\frac{\varrho(1-\alpha)}{M(\alpha)}+\xi+\frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}<1 \qquad (4.1)$$

are satisfied.

Proof. We follow similar arguments to those used in the proof of Theorem (3.1). Therefore, we focus only on the differences. Set

$$\eta=\frac{\|u_0\|+\ d+\frac{\varrho(1-\alpha)}{M(\alpha)}+\frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}}{1-\left[a+\frac{\varrho(1-\alpha)}{M(\alpha)}+\xi+\frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}\right]}. \qquad (4.2)$$

Define a function T^*: $PC(J,E)\to PC(J,E)$ as

$$T^*(x)(t)=\begin{cases} u_0-g(x)+\frac{(1-\alpha)}{M(\alpha)}f(t,x(t))+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds, & t\in[0,t_1], \\ u_0-g(x)+\frac{(1-\alpha)}{M(\alpha)}f(t,x(t))+\sum_{k=1}^i I_k(x(t_k^-))+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds, \\ t\in J_i, i=1,2,...,m. \end{cases} \qquad (4.3)$$

If $t \in J_i,\ i = 1, 2, ..., m$ and $x \in B_\eta = \{z \in P(J, E) : \|z\| \leq \eta\}$, then

$$\begin{aligned} \|T^*(x)(t)\| &\leq \|u_0\| + a\eta + d + \frac{(1-\alpha)}{M(\alpha)}\varrho(1+\eta) + \xi\eta + \frac{\alpha\varrho(1+\eta)}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\ &\leq \|u_0\| + d + \frac{\varrho(1-\alpha)}{M(\alpha)} + \frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)} + \eta\left[a + \frac{\varrho(1-\alpha)}{M(\alpha)} + \xi + \frac{\varrho b^\alpha}{M(\alpha)\Gamma(\alpha)}\right], \end{aligned} \tag{4.4}$$

By this and using (4.1), we obtain $T(B_\eta) \subseteq B_\eta$.

By continuity and compactness of $I_i, i = 1, 2, ..., m$, and using same arguments as in the proof of Theorem (3.1), we can show that T^* has a fixed point, which is a solution for problem (1.2).

Theorem 4.2. *If $(Hf)^*$, (HI) and $g(x) = 0, \forall x \in PC(J, E)$ are satisfied, then problem (1.2) has a unique solution provided that (3.33) holds.*

Proof. Let T^*: $PC(J, E) \to PC(J, E)$ be defined as in the proof of Theorem 4.1. Following the arguments used in proving Theorem (4.1), one can show the existence of solutions. Let u and v be two solutions for problem (1.2). Similar to the proof of Theorem (3.2), we show that

$$u(t) = v(t), \forall t \in [0, t_1].$$

Let $t \in (t_1, t_2]$. Since $u(t_1^-) = v(t_1^-)$, then $I_1(u(t_1^-)) = I_1(v(t_1^-))$, and

$$\|u(t) - v(t)\| \leq \frac{L(1-\alpha)}{M(\alpha)} \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\| + \frac{L\alpha}{M(\alpha)\Gamma(\alpha)} \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\| \int_0^t (t-s)^{\alpha-1} ds, \tag{4.5}$$

from which it follows

$$\begin{aligned} \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\| &\leq \frac{L(1-\alpha)}{M(\alpha)} \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\| + \frac{Lb^\alpha}{M(\alpha)\Gamma(\alpha)} \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\| \\ &\leq \left[\frac{L(1-\alpha)}{M(\alpha)} + \frac{Lb^\alpha}{M(\alpha)\Gamma(\alpha)}\right] \sup_{\theta\in(t_1,t_2]} \|u(\theta) - v(\theta)\|. \end{aligned} \tag{4.6}$$

In view of (3.33), it yields from this equation, $u(\theta) = v(\theta), \forall \theta \in [t_1, t_2]$.

In the following theorem, we provide a version for the existence and uniqueness of the solution for problem (1.2) without assuming that $g(x) = 0, \forall x \in PC(J, E)$.

Theorem 4.3. *If the following conditions hold:*

(1) $\underline{\underline{(Hf)^*}}$ *The function f: $J \times E \to E$ satisfies*

   (a) $f(0, u_0) = 0$.

   (b) There is $L > 0$ such that

$$\|f(t, z_1) - f(t, z_2)\| \leq L\|z_2 - z_1\|, \forall t \in J \text{ , and } \forall z_1, z_2 \in E. \tag{4.7}$$

(2) $\underline{\underline{(Hg)^*}}$ *There is $\nu > 0$ such that*

$$\|g(x) - g(y)\| \leq \nu\|x - y\|_{PC(J,E)}, \forall x, y \in PC(J, E). \tag{4.8}$$

(3) $\underline{\underline{(HI)^*}}$ *For any* $i \in \{1, 2, ..., m\}$*, there is* $\sigma_i > 0$ *such that*

$$\|I_i(u) - I_i(v)\| \le \sigma_i \, \|u - v\|_{PC(J,E)}, \forall u, v \in E. \tag{4.9}$$

Then problem (1.2) has a unique solution provided that

$$\nu + \frac{L(1-\alpha)}{M(\alpha)} + \frac{L\alpha b^\alpha}{M(\alpha)\Gamma(\alpha)} + m\sigma < 1, \tag{4.10}$$

where $\sigma = \sum_{i=1}^{i=m} \sigma_i$.

Proof. Let T^*: $PC(J,E) \to PC(J,E)$ be defined as in (4.3). We show that T^* is a contraction. Let $x, y \in PC(J,E)$, and $t \in [0, t_1]$. Using $(Hf)^{**}$ and $(Hg)^*$ to get

$$\begin{aligned}
\|T^*(x)(t) - T^*(y)(t)\| &\le \|g(x) - g(y)\| + \frac{(1-\alpha)}{M(\alpha)}\|f(t, x(t)) - f(t, x(t))\| \\
&+ \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s)) - f(s, y(s))\| ds \\
&\le \|x - y\|_{PC(J,E)} \left[\nu + \frac{L(1-\alpha)}{M(\alpha)} + \frac{L\alpha b^\alpha}{M(\alpha)\Gamma(\alpha)}\right].
\end{aligned} \tag{4.11}$$

Similarly, if $t \in J_i, i = 1, 2, .., m$, then by using $(Hf)^{**}$, $(Hg)^*$ and $(HI)^*$ we obtain

$$\|T^*(x)(t) - T^*(y)(t)\| \le \|x - y\|_{PC(J,E)} \left[\nu + \frac{L(1-\alpha)}{M(\alpha)} + \frac{L\alpha b^\alpha}{M(\alpha)\Gamma(\alpha)} + m\sigma\right]. \tag{4.12}$$

Then,

$$\|T^*(x) - T^*(y)\|_{PC(J,E)} \le \|x - y\|_{PC(J,E)} \left[\nu + \frac{L(1-\alpha)}{M(\alpha)} + \frac{L\alpha b^\alpha}{M(\alpha)\Gamma(\alpha)} + m\sigma\right]. \tag{4.13}$$

Since,

$$\nu + \frac{L(1-\alpha)}{M(\alpha)} + \frac{L\alpha b^\alpha}{M(\alpha)\Gamma(\alpha)} + m\sigma < 1,$$

T^* is a contraction, and hence, by Banach's fixed point theorem, it has a unique fixed point, which is the unique solution for problem (1.2).

Remark 4.1. *Theorem 4.3 generalizes Theorem 3.1 in [33] to infinite dimensional Banach spaces and in the presence of non-local conditions and instantaneous impulses. If there are not impulses effect and*

$$g(x) = 0, \forall x \in PC(J,E),$$

the inequality (4.10) becomes the same condition assumed in Theorem 3.1 in [33] .

Remark 4.2. *If the function f satisfies Lipschitz condition, then applying Banach's fixed point theorem shows that the solution operator has a unique fixed point, which is a unique solution for our problems. Alternatively, we applied Schauder's fixed point theorem to show the existence of fixed point of the solution operator, then by using the generalized Gronwall inequality, we showed the uniqueness.*

- *In Theorem 3.1, the function f does not satisfies the Lipschitz condition. Therefore, we were only able to show the existence of a solution to problem (1.1). In Theorem 3.2, the funtion f satisfies $(Hf)^*$, which implies Lipschitz condition. Consequently, we obtained both existence and uniqueness of a solution to problem (1.1). Note that since the impulsive functions g_i, and $I_i, i = 1, 2, ..., m$, do not satisfy Lipschitz's condition, we can not apply Banach's fixed point theorem.*

- *For problem (1.2), in Theorem 4.1, with the assumption of (Hf), we only have proved the existence of a solution, in Theorem 4.2, we investigated the existence and uniqueness with the assumption that f satisfies $(Hf)^*$ and $g(x) = 0, \forall x \in PC(J, E)$. In Theorem 4.3, we obtained the existence and uniqueness when all functions f, g and L satisfy Lipschitz's condition. Note that Theorem 4.2 is not a special case of Theorem 4.3.*

5. Existence of solutions for problem (1.3)

In this section, we derive sufficient conditions under which the solution set of problem (1.3) will be not empty and compact. We need to the following fixed points theorems for multifunctions. For more information about multifunctions, we refer the reader to [46].

Lemma 5.1. *[[47], Corollary 3.3.1]. Let W be a closed convex subset of E and R: $E \to P_{ck}(W)$ be a closed multifunction which is $\vartheta-$condensing on every bounded subset of W, where ϑ is a nonsingular measure of noncompactness defined on subsets of W, then the set of fixed points for R is not empty.*

Lemma 5.2. *[[47], Proposition 3.5.1]. Let W be a closed subset of E and R: $W \to P_{ck}(E)$ be a closed multifunction which is $\vartheta-$condensing on every bounded subset of W, where ϑ is a monotone measure of noncompactness defined on W. If the set of fixed points for R is a bounded subset of E, then it is compact.*

Hypothesis 5.1. *We employ the following hypothesis: $\underline{\underline{(Hf)}}$ The function F: $J \times E \to P_{ck}(E)$ is defined such that:*

(a) For every $x \in E$, $t \to F(t, x)$ is measurable.

(b) For almost $t \in J$, $x \to F(t, x)$ is upper semi-continuous.

(c) There is a function $\psi \in L^1(J, \mathbb{R}^+)$ with

$$\sup_{y \in F(t,x)} \|y\| \le \psi(t)(1 + \|x\|), \text{ for a.e., } t \in J. \tag{5.1}$$

(d) There is a function $\eta \in L^1(J, \mathbb{R}^+)$ such that for any bounded subset $B \subset E$,

$$\varkappa(F(t, B)) \le \eta(t)\, \varkappa(B), \text{ for } t \in J, \tag{5.2}$$

and

$$\|\eta\|_{L^1(J,\mathbb{R}^+)} \left(\frac{2(1-\alpha)}{M(\alpha)} + \frac{4b^\alpha}{M(\alpha)\Gamma(\alpha)} \right) < 1, \tag{5.3}$$

where $\varkappa$ is the measure of noncompactness on E.

Theorem 5.1. *If* (Hf), (Hg) *and* (H) *are satisfied, then the solution set for problem (1.3) is non-empty and compact provided that*

$$\gamma + a + \frac{\rho(1-\alpha)}{M(\alpha)} + \frac{\alpha\rho b^{\alpha}}{M(\alpha)\Gamma(\alpha+1)} < 1, \tag{5.4}$$

where $\rho = \|\psi\|_{L^1(J,\mathbb{R}^+)}$.

Proof. Due to (a) in (Hf), for any $x \in PC(J,E)$, $S^1_{F(.,x(.))}$ is not empty, and so, multifunction R: $PC(J,E) \to 2^{PC(J,E)}$ can be defined such that $u \in R(x)$ if and only if

$$u(t) = \begin{cases} u_0 - g(x) + \frac{(1-\alpha)}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, & t \in [0,t_1], \\ g_i(t, x(t_i^-)), & t \in \cup_{i=1}^{i=m}[t_i, s_i], \\ g_i(s_i, x(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s)ds, & t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}], \end{cases} \tag{5.5}$$

where for $t \in \cup_{i=0}^{i=m}[s_i, t_{i+1}]$,

$$f(t) = \int_{s_i}^t z(s)ds, \ z \in S^1_{F(.,u(.))}.$$

Our aim is using Lemma 5.1, to show that R has a fixed point. Set

$$B_\zeta = \{x \in PC(J,E) : \|x\|_{PC(J,E)} \le \zeta\},$$

$$\zeta = \frac{\|u_0\| + d + \frac{\rho(1-\alpha)}{M(\alpha)} + \frac{\rho b^{\alpha}}{M(\alpha)\Gamma(\alpha)}}{1 - \left[\gamma + a + \frac{\rho(1-\alpha)}{M(\alpha)} + \frac{\alpha\rho b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right]}. \tag{5.6}$$

The proof will proceed in the following steps.

- Step 1. In this step, we claim that $R(B_\zeta) \subseteq B_\zeta$. Let $x \in B_\zeta$ and $u \in R(x)$.

Then, there is $z \in S^1_{F(.,u(.))}$ such that u satisfies (5.5), where

$$f(t) = \int_{s_i}^t z(s)ds, t \in [s_i, t_{i+1}], i = 0, 1, ..., m.$$

Using (c) of (Hf), we get

$$\|f(t)\| \le (1+\zeta)\rho, \quad \forall t \in \cup_{i=0}^{i=m}[s_i, t_{i+1}]. \tag{5.7}$$

Let $t \in [0, t_1]$. Using (5.7) and (Hg), we obtain

$$\begin{aligned} \|u(t)\| &\le \|u_0\| + a\zeta + d + \frac{(1-\alpha)}{M(\alpha)}\rho(1+\zeta) + \frac{\alpha\rho(1+\zeta)}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\le \|u_0\| + a\zeta + d + \frac{(1-\alpha)}{M(\alpha)}\rho(1+\zeta) + \frac{\alpha\rho(1+\zeta)b^{\alpha}}{M(\alpha)\Gamma(\alpha+1)}. \end{aligned} \tag{5.8}$$

Let $t \in \cup_{i=1}^m (t_i\ s_i]$. Then, by (H)

$$\|u(t)\| \le \|g_i(t, x(t_i^-))\| \le \gamma\ \zeta. \tag{5.9}$$

Let $t \in (s_i, t_{i+1}]$. By repeating the arguments employed in (5.8), it follows

$$\|u(t)\| \leq \zeta\gamma + \|u_0\| + a\zeta + d + \frac{(1-\alpha)}{M(\alpha)}\varrho(1+\zeta) + \frac{\alpha\rho(1+\zeta)b^\alpha}{M(\alpha)\Gamma(\alpha+1)}. \quad (5.10)$$

From (5.8)–(5.10), we get

$$\|u\|_{PC(J,E)} \leq \|u_0\| + d + \frac{\rho(1-\alpha)}{M(\alpha)} + \frac{\alpha\rho b^\alpha}{M(\alpha)\Gamma(\alpha+1)} + \zeta\left[\gamma + a + \frac{\rho(1-\alpha)}{M(\alpha)} + \frac{\alpha\rho b^\alpha}{M(\alpha)\Gamma(\alpha+1)}\right]. \quad (5.11)$$

It yields from this relation and (5.4), that $u \in B_\zeta$.

- Step 2. In this step, we show that, if $x_k \in B_\zeta$, $u_k \in R(x_k)$, $x_k \to x$ and $u_k \to u$, then $u \in R(x)$. By definition of R, there is

$$f_k = \int_{s_i}^{t} z_k(s)ds; t \in [s_i, t_{i+1}],\ (i = 0, 1, 2, ..m), z_k \in S^1_{F(.,x_k(.))},$$

such that

$$u_k(t) = \begin{cases} u_0 - g(x_k) + \frac{1-\alpha}{M(\alpha)} f_k(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_k(s)ds, & t \in [0, t_1], \\ g_i(t, x_k(t_i^-)), & t \in \cup_{i=1}^{i=m}[t_i, s_i], \\ g_i(s_i, x_k(t_i^-)) + \frac{1-\alpha}{M(\alpha)} f_k(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f_k(s)ds, & t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}]. \end{cases} \quad (5.12)$$

From (d) in (Hf), it follows,

$$\varkappa\{z_k(t) : k \geq 1\} \leq \varkappa F(t, \{x_k(t) : k \geq 1\}) \leq \eta(t)\varkappa\{x_k(t) : k \geq 1\} = 0. \quad (5.13)$$

Moreover, by (c) in (Hf) and $\|z_k(t)\| \leq \psi(t)(1+\zeta), a.e.$, then $\{z_k\colon k \geq 1\}$ is semicompact in $L^1(J, E)$, and hence, it is weakly compact in $L^1(J, E)$ [47]. By Mazure Lemma, without loss of generality, there exists a subsequence $(z_k^*), k \geq 1$ of convex combinations of (z_k) and converging almost everywhere to a function z. Note that $z_k^*(t) \in F(t, (x_k(t)), a.e., \forall k \geq 1$. Due to the upper semicontinuity of $F(t, .), a, e.$, it follows that $z \in S^1_{F(,.x(.))}$. Set

$$\overline{u}_k(t) = \begin{cases} u_0 - g(z_k) + \frac{(1-\alpha)}{M(\alpha)} f_k^*(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_k^*(s)ds, & t \in [0, t_1], \\ g_i(t, x_k(t_i^-)), & t \in \cup_{i=1}^{i=m}[t_i, s_i], \\ g_i(s_i, x_k(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)} f_k^*(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f_k^*(s)ds, & t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}], \end{cases} \quad (5.14)$$

where $f_k^*(t) = \int_{s_i}^t z_k(s)ds; t \in [s_i, t_{i+1}], (i = 0, 1, 2, ..., m)$. Clearly $(\overline{u}_k)$ is a subsequence of (u_k).

Moreover, by continuity of both g and $g_i(t,.)$, and by taking the limit $k \to \infty$, in (5.14), $\overline{u}_k(_k)$ converges to $\overline{u}$, where

$$\overline{u}(t) = \begin{cases} u_0 - g(z) + \frac{(1-\alpha)}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, & t \in [0, t_1], \\ g_i(t, x(t_i^-)), & t \in \cup_{i=1}^{i=m} [t_i, s_i], \\ g_i(s_i, x(t_i^-)) + \frac{(1-\alpha)}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s)ds, & t \in \cup_{i=1}^{i=m} [s_i, t_{i+1}], \end{cases} \tag{5.15}$$

and

$$f(t) = \int_{s_i}^t z(s)ds; \; t \in [s_i, t_{i+1}], \quad (i = 0, 1, 2, ..., m).$$

By the uniqueness of the limit, $u = \overline{u} \in R(x)$.

- Step 3. $R(x), \; x \in B_\zeta$ is compact.

Suppose that (u_k) is a sequence in $R(x)$: $x \in B_\zeta$. By arguing as in Step 2, there is a subsequence of (u_k) converging to $\overline{u} \in R(x)$.

- Step 4. Our goal in this step is to show that the sets $D|_{\overline{J_i}}$ are equicontinuous for any $i = 0, 1, ..., m$, where $D = R(B_\zeta)$ and

$$D|_{\overline{J_i}} = \{u^* \in (\overline{J_i}, E) : u^*(t) = u(t), t \in (t_i, t_{i+1}], u^*(t_i) = \lim_{t \to t_i^+} u(t), u \in D\}. \tag{5.16}$$

Let $u \in R(x), \; x \in B_\zeta$. We consider the following cases:

Case 1. $i = 0$. Let $t, t + \delta$ be two points in $\overline{J_0} = [0, t_1]$. Then,

$$u(t) = u_0 - g(x) + \frac{(1-\alpha)}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{5.17}$$

where $f(t) = \int_0^t z(s)ds, z \in S^1_{F(.,x(.))}$. By using (c) in (Hf), one has,

$$\|f(t)\| \le (1+\zeta) \int_0^t \psi(s)ds \le (1+\zeta) \|\psi\|_{L^1(J,\mathbb{R}^+)}. \tag{5.18}$$

Therefore,

$$\begin{aligned} \lim_{\delta \to 0} \|u^*(t+\delta) - u^*(t)\| &= \lim_{\delta \to 0} \|u(t+\delta) - u(t)\| \\ &\le \lim_{\delta \to 0} \|f(t+\delta) - f(t)\| \\ &+ \lim_{\delta \to 0} \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \| \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} f(s)ds - \int_0^t (t-s)^{\alpha-1} f(s)ds \| \\ &\le (1+\zeta) \lim_{\delta \to 0} \int_t^{t+\delta} \psi(s)ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \lim_{\delta\to 0} \| \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} f(s)ds - \int_0^{t+\delta} (t-s)^{\alpha-1} f(s)ds \| \\
&+ \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \lim_{\delta\to 0} \| \int_0^{t+\delta} (t-s)^{\alpha-1} f(s)ds - \int_0^{t} (t-s)^{\alpha-1} f(s)ds \| \\
&\leq (1+\zeta) \lim_{\delta\to 0} \int_t^{t+\delta} \psi(s)ds \\
&+ \frac{\alpha(1+\zeta)\|\psi\|_{L^1(J,\mathbb{R}^+)}}{M(\alpha)\Gamma(\alpha)} \lim_{\delta\to 0} \int_0^{t+\delta} |(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}| ds \\
&+ \frac{\alpha\gamma(1+\zeta)}{M(\alpha)\Gamma(\alpha)} \|\psi\|_{L^1(J,\mathbb{R}^+)} \lim_{\delta\to 0} \int_t^{t+\delta} (t-s)^{\alpha-1} ds \\
&= 0.
\end{aligned} \tag{5.19}
$$

Case 2. $i \geq 1$. Let t, $t+\delta$ be two points in

$$
J_i = (t_i, t_{i+1}] = (t_i, s_i) \cup [s_i, t_{i+1}].
$$

If t, $t+\delta$ are in (t_i, s_i), then

$$
\begin{aligned}
\lim_{\delta\to 0} \|u^*(t+\delta) - u^*(t)\| &= \lim_{\delta\to 0} \|u(t+\delta) - u(t)\| \\
&\leq \lim_{\delta\to 0} \|g_i(t+\delta, x(t_i^-)) - g_i(t, x(t_i^-))\| = 0.
\end{aligned} \tag{5.20}
$$

If $t, t+\delta$ are in (s_i, t_{i+1}), then by using the same arguments as in Case 1, we can arrive to

$$
\lim_{\delta\to 0} \|u(t+\delta) - u(t)\| = 0.
$$

When $t = t_i^+$,

$$
\begin{aligned}
& \lim_{\delta\to 0} \|u^*(t_i+\delta) - u^*(t_i)\| \\
& \lim_{\delta\to 0} \lim_{\lambda\to t_1^+} \|u(t_i+\delta) - u(\lambda)\| \\
\leq \; & \lim_{\delta\to 0} \lim_{\lambda\to t_i^+} \|f(t_i+\delta) - f(\lambda)\| \\
+ \; & \lim_{\delta\to 0} \lim_{\lambda\to t_i^+} \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \| \int_0^{t_i+\delta} (t_i+\delta-s)^{\alpha-1} f(s)ds - \int_0^{\lambda} (\lambda-s)^{\alpha-1} f(s)ds \| \\
= \; & 0.
\end{aligned} \tag{5.21}
$$

Thus, the claim is proved in this step.

- Step 5. Set $B_n = R(B_{n-1})$, $n \geq 1$ and $B = \cap_{n=1}^{\infty} B_n$.

Then, (B_n) is a non-decreasing sequence of non-empty, convex, bounded and closed sets. Our goal in this step is to show that B is relatively compact, and hence it is compact. According to the generalized Cantor's intersection property [43], it is enough to show that

$$
\lim_{n\to\infty} \varkappa_{PC}(B_n) = 0, \tag{5.22}
$$

where $\varkappa_{PC}$ is the measure of noncompactness in $PC(J,E)$. Let $\epsilon > 0$, and $n \geq 1$ be a fixed. There is (see [44]) a sequence (u_k) in B_n such that

$$\begin{aligned} \varkappa_{PC}(B_n) &\leq 2\varkappa_{PC}\{u_k : k \geq 1\} + \epsilon \\ &= 2\max_{0\leq i\leq m}\{\chi_i(Z_{|\overline{J_i}})\}, \end{aligned} \tag{5.23}$$

where

$$Z = \{u_k : k \geq 1\},$$

$Z_{|\overline{J_i}}$ is defined similar to (5.16) and χ_i is the Hausdorff measure of noncompactness in $C(\overline{J_i}\ , E)$ [43]. By Step 2, $Z_{|\overline{J_i}}$ is equicontinuous, and hence relation (5.23) becomes

$$\varkappa_{PC}(B_n) \leq 2\max_{t\in J}\chi\{u_k(t) : k \geq 1\} + \epsilon. \tag{5.24}$$

Since

$$u_k \in B_n = R(B_{n-1}),$$

there is $x_k \in B_{n-1}$ with $u_k \in R(x)$. According to the definition of R, there is $z_k \in S^1_{F(.,x_k(.))}$ such that

$$u_k(t) = \begin{cases} u_0 - g(x_k) + \frac{1-\alpha}{M(\alpha)}f_k(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_k(s)ds, & t \in [0,t_1], \\ g_i(t, x_k(t_i^-)), & t \in \cup_{i=1}^{i=m}[t_i, s_i], \\ g_i(s_i, x_k(t_i^-)) + \frac{1-\alpha}{M(\alpha)}f_k(t-s_i) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}f_k(s)ds, & t \in \cup_{i=1}^{i=m}(s_i, t_{i+1}], \end{cases} \tag{5.25}$$

where

$$f_k(t) = \int_{s_i}^t z_k(s)ds; t \in [s_i, t_{i+1}], i = 0, 1, ..., m.$$

By the compactness of both g and $g_i(t,.)$, $t \in \cup_{i=1}^{i=m}[t_i, s_i]$, we obtain

$$\chi\{g(x_k) : k \geq 1\} = 0,$$

and

$$\chi\{g_i(t, x_k(t_i^-)) : k \geq 1\} = 0,$$

for any $t \in \cup_{i=1}^{i=m}[t_i, s_i]$. Moreover, in view of (4.10), for $t \in J$,

$$\begin{aligned} \chi\{z_k(t) : k \geq 1\} &\leq \chi F(t, \{x_k(t) : k \geq 1\}) \\ &\leq \eta(t)\chi\{x_k(t) : k \geq 1\} \\ &\leq \eta(t)\chi_{PC}(B_{n-1}). \end{aligned} \tag{5.26}$$

Then, for $t \in [s_i, t_{i+1}]$, $i = 0, 1, ..., m$

$$\begin{aligned} \chi\{f_k(t) : k \geq 1\} &\leq \chi\{\int_{s_i}^t z_k(s)ds : k \geq 1\} \\ &\leq 2\int_{s_i}^t \chi\{z_k(s) : k \geq 1\}ds \end{aligned}$$

$$\leq 2\chi_{PC}(B_{n-1})\int_{s_i}^{t}\eta(s)ds$$
$$\leq 2\chi_{PC}(B_{n-1})\|\eta\|_{L^1(J,\mathbb{R}^+)}. \tag{5.27}$$

Due to the properties of χ, it yields from (5.27) for $t \in J$,

$$\begin{aligned}\chi\{\int_0^t (t-s)^{\alpha-1} f_k(s)ds : k \geq 1\} &\leq 2\int_0^t (t-s)^{\alpha-1}\chi\{f_k(s) : k \geq 1\}ds \\ &= 4\chi_{PC}(B_{n-1})\|\eta\|_{L^1(J,\mathbb{R}^+)}\int_0^t (t-s)^{\alpha-1}ds \\ &= 4\chi_{PC}(B_{n-1})\|\eta\|_{L^1(J,\mathbb{R}^+)}\frac{b^\alpha}{\alpha}.\end{aligned} \tag{5.28}$$

Thus, by (5.24), (5.25) and (5.28),

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_{n-1})\|\eta\|_{L^1(J,\mathbb{R}^+)}\left(\frac{2(1-\alpha)}{M(\alpha)} + \frac{4b^\alpha}{M(\alpha)\Gamma(\alpha)}\right) + \epsilon. \tag{5.29}$$

Since ϵ is arbitrary, we get for all $t \in J$

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_{n-1})\|\eta\|_{L^1(J,\mathbb{R}^+)}\left(\frac{2(1-\alpha)}{M(\alpha)} + \frac{4b^\alpha}{M(\alpha)\Gamma(\alpha)}\right). \tag{5.30}$$

Since this relation holds for every n, we get

$$\varkappa_{PC}(B_n) \leq \chi_{PC}(B_1)\left[\|\eta\|_{L^1(J,\mathbb{R}^+)}\left(\frac{2(1-\alpha)}{M(\alpha)} + \frac{4b^\alpha}{M(\alpha)\Gamma(\alpha)}\right)\right]^{n-1}, \tag{5.31}$$

which with (5.3) gives that (5.22) is satisfied.

- Step 6. Applying Corollary 3.3.1 in [47], the multifunction R: $B \to P_{ck}(B)$ has a fixed point which is a solution for problem (1.3). Furthermore, by arguing as in step 1, one can show that the set of fixed points of R is bounded, and hence by Lemma 5.2, the solution set for problem (1.3) is compact.

6. Examples

Example 6.1. *Let $\alpha \in (0, 1)$, E be a Hilbert space, $J = [0, 1]$, and*

$$s_0 = 0, s_i = \frac{2i}{9}, t_i = \frac{2i-1}{9}, i = 1, 2, 3, 4, t_5 = 1.$$

Let F: $J \times E \to P_{ck}(E)$ be a multifunction defined by

$$F(t,u) = \frac{\rho_1\|u\|\sin t}{\sigma\,(1+\|u\|)}K, \quad (t,u) \in J \times E, \tag{6.1}$$

where K is a convex and compact subset of E with $0 \in K$, $\rho_1 > 0$, σ is a constant such that

$$\sup\{\ \|z\|\ : z \in K\} = \sigma.$$

For any $u \in PC(J, E)$, the function

$$z(t) = \frac{\varrho\|u\| \ \sin t}{\sigma\ (1 + \|u\|)} z_0,\ z_0 \in Z \tag{6.2}$$

is an element of $S^1_{F(.,u(.))}$, and

$$z(t) \in F(t, u(t)),\ t \in J.$$

Hence, $S^1_{F(.,u(.))}$ is not empty. Moreover, for any $u, v \in E$ and $t \in J$, we have

$$\sup_{y \in F(t,x)} \|y\| \leq \varrho_1 \sin(t)(1 + \|x\|), \tag{6.3}$$

and

$$\begin{aligned} H(F(t,u), F(t,v)) &\leq \varrho_1 |\sin t| \left| \frac{\|u\|}{(1 + \|u\|)} - \frac{\|v\|}{(1 + \|v\|)} \right| \\ &\leq \varrho_1 |\sin t|\ \|u - v\|, \end{aligned} \tag{6.4}$$

where H is the Hausdorff distance. Thus, (Hf) is satisfied, where

$$\zeta(t) = \varrho |\sin t|,\ \eta(t) = 2\varrho_1 |\sin t|,\ t \in J.$$

Let $g\colon PC(J, E) \to E$ be defined by

$$g(x) = a\ proj_K x, \tag{6.5}$$

where $a > 0$. Note that, $g(0) = 0$, g is continuous and compact. Moreover,

$$\|g(x) - g(0)\| = a\ \|proj_K x - proj_K 0\| \leq a\ \|x\|, \tag{6.6}$$

which yields, $\|g(x)\| \leq a\ \|x\|$, and therefore, (Hg) is satisfied.

Next, for any $i \in \mathbb{N}$, let $g_i\colon [t_i, s_i] \times E \to E$, be defined as:

$$g_i(t, x) := it\ \varrho_2\ proj\ _K x,\ \ (t, x) \in [t_i, s_i] \times E, i = 1, 2, 3, 4, \tag{6.7}$$

where ϱ_2 is a positive real number. For any

$$t \in [t_i, s_i],\ i = 1, 2, 3, 4,$$

the maps $g_i(t, .)$ is continuous and compact, where

$$\|g_i(t, x)\| \leq 4\varrho_2\ \|x\|.$$

Hence, (H) holds with $\gamma = 4\varrho_2$. By applying Theorem 5.1, with $u_0 = 0$, the solution set of the following problem:

$$\begin{cases} {}^{ABC}D^{\alpha}_{s_i,t} u(t) \in -K \frac{\varrho_1 \|u\| \cot s}{\sigma\ (1+\|u\|)}, a.e. & t \in \cup_{i=0}^{4} (s_i, t_{i+1}], \\ u(t) = it\ \varrho_2\ proj\ _K u(t_i^-), & t \in \cup_{i=0}^{4} (t_i\ s_i], \\ u(0) = -a\ proj_K u, \end{cases} \tag{6.8}$$

is not empty and compact, provided that

$$\left(\frac{4(1-\alpha)}{M(\alpha)}+\frac{8b^{\alpha}}{M(\alpha)\Gamma(\alpha)}\right)\varrho_1<1, \tag{6.9}$$

and

$$4b\varrho_2+a+\frac{2(1-\alpha)}{M(\alpha)}+\varrho_1\frac{\varrho b^{\alpha}}{M(\alpha)\Gamma(\alpha)}<1, \tag{6.10}$$

were ϱ_1, a and ϱ_2 appear in (6.1), (6.5), and (6.7). By choosing ϱ_1, a and ϱ_2 sufficiently small, we can obtain (6.9) and (6.10).

Example 6.2. *Let $E = L^2[0,\pi]$, α, J, s_0, s_i, t_i $(i = 1,2,3,4)$, t_5 as be in Example 6.1. Consider the function $f\colon J\times E\to E$, defined by*

$$f(t,x)(\theta)=r(\sin x(\theta)+\sin t),\ t\in J,\ x\in E,\ \theta\in[0,\pi], \tag{6.11}$$

where $r>0$. Let $u_0\in L^2[0,\pi]$ be the zero function. Note that

$$f(0,u_0)(\theta)=0,\ \forall\theta\in[0,\pi].$$

We show that f satisfies $(Hf)^*$. Let

$$z_1,z_2\in E=L^2[0,\pi]$$

and $t_1,t_2\in J$. One has

$$\begin{aligned}\|f(t_1,z_1)-f(t_t,z_2)\|_{L^2[0,\pi]}&=\left(\int_0^{\pi}|(r\sin z_2(\theta)-r\sin z_1(\theta))+r(\sin t_2-\sin t_1)|^2ds\right)^{\frac{1}{2}}\\&\le r\left(\int_0^{\pi}|\sin z_2(\theta)-\sin z_1(\theta)|^2d\theta\right)^{\frac{1}{2}}+k\left(\int_0^{\pi}|\sin t_2-\sin t_1|^2d\theta\right)^{\frac{1}{2}}\\&=r\|z_2-z_1\|_{L^2[0,\pi]}+r\sqrt{\pi}|t_2-t_1|.\end{aligned} \tag{6.12}$$

Thus, $(Hf)^*$is satisfied, with

$$L=r(1+\sqrt{\pi}).$$

Next, let

$$g:PC(J,E)\to E,\ g_i:[t_i,s_i]\times E\to E,(i=1,2,3,4)$$

be defined as in (6.5) and (6.7), with K is a convex and compact subset of $L^2[0,\pi]$ and $0\in K$. By applying Theorem (3.2) with u_0=0, there is a solution for the following problem:

$$\begin{cases}\left({}^{ABC}D^{\alpha}_{s_i,t}u(t)\right)(s)=r(\sin u(s)+\sin t),\ a.e., & t\in\cup_{i=0}^{m}(s_i,t_{i+1}],\ s\in[0,\pi],\\ u(t)=it\,\varrho_2\,proj_Ku(t_i^-), & t\in\cup_{i=0}^{4}(t_i,s_i],\\ u(0)=-a\,proj_Ku,\end{cases} \tag{6.13}$$

provided that

$$4\varrho_2+a+r\left(1+\sqrt{\pi}\right)\left(\frac{1-\alpha}{M(\alpha)}+\frac{1}{M(\alpha)\Gamma(\alpha)}\right)<1, \tag{6.14}$$

and

$$8r\left(1+\sqrt{\pi}\right)\left(\frac{1-\alpha}{M(\alpha)}+\frac{1}{M(\alpha)\Gamma(\alpha)}\right)<1. \tag{6.15}$$

By choosing ϱ_2, a and r sufficiently small, we can obtain (6.14) and (6.15). If

$$g(x)=0,\ \forall x\in PC(J,E),$$

then the problem

$$\begin{cases} \left({}^{ABC}D^{\alpha}_{s_i,t}u(t)\right)(s)=r(\sin u(s)+\sin t),\ a.e., & t\in\cup_{i=0}^{m}(s_i,t_{i+1}],\ s\in[0,\pi],\\ u(t)=it\,\varrho_2\ proj_{K}u(t_i^-), & t\in\cup_{i=0}^{4}(t_i,s_i],\\ u(0)=0, \end{cases} \tag{6.16}$$

has a unique solution provided that (6.14) and (6.15) are satisfied with $a=0$.

7. Discussion and conclusions

Recently, some existence results of solutions for differential equations, involving the Atangana-Baleanu fractional derivative were done in finite dimensional spaces. In this work, we investigate the existence in infinite dimensional spaces, for differential equations and differential inclusions containing the Atangana-Baleanu fractional derivative. A new class of differential equations and differential inclusions containing AB derivative with instantaneous or non-instantaneous impulses and nonlocal conditions in infinite dimensional Banach spaces are formulated. The existence and uniqueness of solutions for problems (1.1) and (1.2) were proved. For problem (1.3), we were able to show the existence of a solution. The used method is based on properties of both multifunctions and the Hausdorff measure of noncompactness. It is noteworthy that there is no uniqueness of solutions for differential inclusions. Theorem 4.3 in this paper, generalizes a recent result (Theorem 3.1 in [33]) to infinite dimensional Banach spaces. The used method might help researcher who aim to generalize any of the above mentioned results to the case where the right hand side is a multifunction, or in the presence of impulsive effects. For directions of future work, the authors suggest:

(1) Extending the results of this paper to the case where the interval $[0, b]$ is replaced by $[0,\infty)$.

(2) Extending the results in [30–40], to the infinite dimensional case and/or in the presence of impulses and/or to the case where the function is a multi-valued.

(3) Proving results analogue to those in [48] when the Caputo derivative is replaced by the Atangana-Baleanu fractional derivative.

Acknowledgments

The authors would like to thank King Faisal university in Saudi Arabia for thier continual support.

Conflict of interest

The authors declare no conflicts of interest.

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