Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11709–11726.
DOI: 10.3934/math.2023593
Received: 14 December 2022
Revised: 08 March 2023
Accepted: 09 March 2023
Published: 17 March 2023

Research article

On a mixed nonlinear boundary value problem with the right Caputo fractional derivative and multipoint closed boundary conditions

Bashir Ahmad[1,*], **Manal Alnahdi**[1], **Sotiris K. Ntouyas**[1,2] **and Ahmed Alsaedi**[1]

[1] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

* **Correspondence:** Email: bashirahmad_qau@yahoo.com, bahmad@kau.edu.sa.

Abstract: This paper is concerned with the study of a new class of boundary value problems involving a right Caputo fractional derivative and mixed Riemann-Liouville fractional integral operators, and a nonlocal multipoint version of the closed boundary conditions. The proposed problem contains the usual and mixed Riemann-Liouville integrals type nonlinearities. We obtain the existence and uniqueness results with the aid of the fixed point theorems. Examples are presented for illustrating the abstract results. Our results are not only new in the given configuration but also specialize to some interesting situations.

Keywords: right Caputo fractional derivative; Riemann-Liouville fractional integrals; nonlocal multipoint closed boundary conditions; existence; fixed point
Mathematics Subject Classification: 34A08, 34B10, 34B15

1. Introduction

The topic of boundary value problems is an interesting area of research in view of its applications in applied and technical sciences. In the recent years, the class of nonlocal fractional order boundary value problems involving different fractional derivatives (such as Riemann-Liouville, Caputo, etc.) received an overwhelming interest from many researchers. For the details of a variety of nonlocal single-valued and multivalued boundary value problems involving different types of fractional order derivative operators, we refer the reader to the text [1], articles [2–7] and the references cited therein. There has been shown a great enthusiasm in developing the existence theory for Hilfer, ψ-Hilfer and (k, ψ) Hilfer type fractional differential equations equipped with different types of boundary conditions, for instance, see [8–16].

Nonlocal boundary conditions are found to be more plausible and practical in contrast to the classical boundary conditions in view of their applicability to describe the changes happening within the given domain. Closed boundary conditions are found to be of great help in describing the situation when there is no fluid flow along the boundary or through it. The free slip condition is also a type of the closed boundary conditions which describes the situation when there is a flow along the boundary, but there is no flow perpendicular to it. Such conditions are also useful in the study of sandpile model [17, 18], honeycomb lattice [19], deblurring problems [20], closed-aperture wavefield decomposition in solid media [21], vibration analysis of magneto-electro-elastic cylindrical composite panel [22], etc.

Now we review some works on the boundary value problems with closed boundary conditions. In [23], the authors studied the single-valued and multivalued fractional boundary value problems with open and closed boundary conditions. A three-dimensional Neumann boundary value problem with a generalized boundary condition in a domain with a smooth closed boundary was discussed in [24]. For some interesting results on impulsive fractional differential equations with closed boundary conditions, see the articles [25, 26].

The objective of the present work is to investigate a new class of mixed nonlinear boundary value problems involving a right Caputo fractional derivative, mixed Riemann-Liouville fractional integral operators, and multipoint variant of closed boundary conditions. In precise terms, we consider the following fractional order nonlocal and nonlinear problem:

$$ {}^{C}D^{\alpha}_{T-}y(t) + \lambda I^{\rho}_{T-}I^{\sigma}_{0+}h(t, y(t)) = f(t, y(t)), \;\; t \in J := [0, T], \tag{1.1} $$

$$ y(T) = \sum_{\imath=1}^{m}\Big(p_i y(\xi_i) + Tq_i y'(\xi_i)\Big), \;\; Ty'(T) = \sum_{\imath=1}^{m}\Big(r_i y(\xi_i) + Tv_i y'(\xi_i)\Big), \tag{1.2} $$

where ${}^{C}D^{\alpha}_{T-}$ denote the right Caputo fractional derivative of order $\alpha \in (1, 2]$, I^{ρ}_{T-} and I^{σ}_{0+} represent the right and left Riemann-Liouville fractional integral operators of orders $\rho, \sigma > 0$ respectively, $f, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda, p_i, q_i, r_i, v_i \in \mathbb{R}, i \in \{1, 2, 3, ..., m\}$, and $\xi_i \in (0, T)$. Notice that the integro-differential Eq (1.1) contains the usual and mixed Riemann-Liouville integrals type nonlinearities. The boundary conditions (1.2) can be interpreted as the values of the unknown function and its derivative at the right end-point T of the interval $[0, T]$ are proportional to a linear combination of these values at arbitrary nonlocal positions $\xi_i \in (0, T)$. Physically, the nonlocal multipoint closed boundary conditions provide a flexible mechanism to close the boundary at arbitrary positions in the given domain instead of the left end-point of the domain.

Here we emphasize that much of the literature on fractional differential equations contains the left-sided fractional derivatives and there are a few works dealing with the right-sided fractional derivatives. For instance, the authors in [27, 28] studied the problems involving the right-handed RiemannLiouville fractional derivative operators, while a problem containing the right-handed Caputo fractional derivative was considered in [29]. The problem studied in the present paper is novel in the sense that it solves an integro-differential equation with a right Caputo fractional derivative and mixed nonlinearities complemented with a new concept of nonlocal multipoint closed boundary conditions. The results accomplished for the problems (1.1) and (1.2) will enrich the literature on boundary value problems involving the right-sided fractional derivative operators. The present work is also significant as it produces several new results as special cases as indicated in the last section.

The rest of the paper is arranged as follows. In Section 2, we present an auxiliary lemma which is used to transform the given nonlinear problem into a fixed-point problem. Section 3 contains the main results and illustrative examples. Some interesting observations are presented in the last Section 4.

2. A preliminary result

Let us begin this section with some definitions [30].

Definition 2.1. *The left and right Riemann-Liouville fractional integrals of order* $\beta > 0$ *for* $g \in L_1[a,b]$, *existing almost everywhere on* $[a,b]$, *are respectively defined by*

$$I^{\beta}_{a+}g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds \quad and \quad I^{\beta}_{b-}g(t) = \int_t^b \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} g(s)ds.$$

Definition 2.2. *For* $g \in AC^n[a,b]$, *the right Caputo fractional derivative of order* $\beta \in (n-1,n], n \in \mathbb{N}$, *existing almost everywhere on* $[a,b]$, *is defined by*

$${}^{C}D^{\beta}_{b-}g(t) = (-1)^n \int_t^b \frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)} g^{(n)}(s)ds.$$

In the following lemma, we solve a linear variant of the fractional integro-differential equation (1.1) supplemented with multipoint closed boundary conditions (1.2).

Lemma 2.1. *Let* $H, F \in C[0,T]$ *and* $\Delta \neq 0$. *Then the linear problem*

$$\begin{cases} {}^{C}D^{\alpha}_{T-}y(t) + \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(t) = F(t), \quad t \in J := [0,T], \\ y(T) = \sum_{\imath=1}^{m} \Big(p_i y(\xi_i) + T q_i y'(\xi_i) \Big), \; Ty'(T) = \sum_{\imath=1}^{m} \Big(r_i y(\xi_i) + T v_i y'(\xi_i) \Big), \; 0 < \xi_i < T, \end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$\begin{aligned} y(t) \;=\; & \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} [F(s) - \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(s)]ds \\ & + b_1(t) \Big\{ \sum_{\imath=1}^{m} p_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} [F(s) - \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(s)]ds \\ & - T \sum_{\imath=1}^{m} q_i \int_{\xi_i}^T \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)} [F(s) - \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(s)]ds \Big\} \\ & + b_2(t) \Big\{ \sum_{\imath=1}^{m} r_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} [F(s) - \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(s)]ds \\ & - T \sum_{i=1}^{m} v_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} [F(s) - \lambda I^{\rho}_{T-}I^{\sigma}_{0+}H(s)]ds \Big\}, \end{aligned} \tag{2.2}$$

where

$$b_1(t) \;=\; \frac{1}{\Delta}(tS_6 - S_7 - TS_9 + T), \quad b_2(t) = \frac{1}{\Delta}[(1-S_1)t + S_2 + TS_4 - T],$$

$$
\begin{aligned}
\Delta &= (S_1-1)(S_7+TS_9-T)-S_6(S_2+TS_4-T),\\
S_1 &= \sum_{\imath=1}^{m} p_i,\ \ S_2=\sum_{\imath=1}^{m} p_i\xi_i,\ \ S_3=\sum_{\imath=1}^{m} p_iAi,\ \ S_4=\sum_{\imath=1}^{m} q_i,\ \ S_5=\sum_{\imath=1}^{m} q_iB_i,\\
S_6 &= \sum_{\imath=1}^{m} r_i,\ \ S_7=\sum_{\imath=1}^{m} r_i\xi_i,\ \ S_8=\sum_{\imath=1}^{m} r_iA_i,\ \ S_9=\sum_{\imath=1}^{m} v_i,\ \ S_{10}=\sum_{\imath=1}^{m} v_iB_i,\\
A_i &= I_{T-}^{\alpha}[F(\xi_i)-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}H(\xi_i)],\ \ B_i=-I_{T-}^{\alpha-1}[F(\xi_i)-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}H(\xi_i)]. \qquad (2.3)
\end{aligned}
$$

Proof. Applying the right fractional integral operator I_{T-}^{α} to the integro-differential equation in (2.1), we get

$$y(t)=I_{T-}^{\alpha}F(t)-\lambda I_{T-}^{\alpha+\rho}I_{0+}^{\sigma}H(t)-c_0-c_1t, \qquad (2.4)$$

where c_0 and c_1 are unknown arbitrary constants. Using (2.4) in the nonlocal closed boundary conditions of (2.1), we obtain

$$
\begin{cases}
(S_1-1)c_0+(S_2+TS_4-T)c_1 &= S_3+TS_5,\\
S_6c_0+(S_7+TS_9-T)c_1 &= S_8+TS_{10},
\end{cases} \qquad (2.5)
$$

where $S_i,\ i=1,\dots,10$, are given in (2.3).

Solving the system (2.5) for c_0 and c_1, we find that

$$
\begin{aligned}
c_0 &= \frac{1}{\Delta}[(S_7+TS_9-T)(S_3+TS_5)-(S_2+TS_4-T)(S_8+TS_{10})],\\
c_1 &= \frac{1}{\Delta}[-S_6(S_3+TS_5)+(S_1-1)(S_8+TS_{10})],
\end{aligned}
$$

where Δ is given in (2.3). Substituting the above values of c_0 and c_1 in (2.4) together with the notation (2.3), we obtain the solution (2.2). The converse of this lemma can be obtained by direct computation. This completes the proof.

3. Existence and uniqueness results

This section is devoted to our main results concerning the existence and uniqueness of solutions for the problems (1.1) and (1.2).

In order to convert the problems (1.1) and (1.2) into a fixed point problem, we define an operator $\mathcal{V}:\mathcal{X}\to\mathcal{X}$ by using Lemma 2.1 as follows:

$$
\begin{aligned}
\mathcal{V}y(t) &= \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\\
&\quad +b_1(t)\Bigg\{\sum_{\imath=1}^{m} p_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\\
&\quad -T\sum_{\imath=1}^{m} q_i\int_{\xi_i}^{T}\frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Bigg\}\\
&\quad +b_2(t)\Bigg\{\sum_{\imath=1}^{m} r_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds
\end{aligned}
$$

$$-T\sum_{i=1}^{m}v_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Big\},\ t\in J, \tag{3.1}$$

where $\mathcal{X} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t \in [0,T]\}$. Notice that the fixed point problem $\mathcal{V}y(t) = y(t)$ is equivalent to the boundary value problems (1.1) and (1.2) and the fixed points of the operator $\mathcal{V}$ are its solutions.

In the forthcoming analysis, we use the following estimates:

$$\int_t^T\frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}ds=\int_t^T\frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}\int_0^s\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}du\,ds\le\frac{T^{\sigma}(T-t)^{\alpha+\rho}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)},$$
$$\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}ds=\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}\int_0^s\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}du\,ds\le\frac{T^{\sigma}(T-\xi_i)^{\alpha+\rho}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)},$$

where we have used $u^{\sigma} \le T^{\sigma},\ \rho,\sigma > 0$.

In the sequel, we set

$$\begin{aligned}\Omega_1 &= \frac{1}{\Gamma(\alpha+1)}\Big\{T^{\alpha}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Big]\\ &\quad+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Big]\Big\},\\ \Omega_2 &= \frac{|\lambda|T^{\sigma}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big\{T^{\alpha+\rho}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\\ &\quad+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\},\end{aligned} \tag{3.2}$$

where

$$\overline{b}_1=\max_{t\in[0,T]}|b_1(t)|,\quad \overline{b}_2=\max_{t\in[0,T]}|b_2(t)|.$$

3.1. Existence results

In the following, Krasnosel'skiĭ's fixed point theorem [31] is applied to prove our first existence result for the problems (1.1) and (1.2).

Theorem 3.1. *Assume that:*

(H_1) *There exists* $\mathcal{L} > 0$ *such that* $|f(t,x) - f(t,y)| \le \mathcal{L}|x-y|,\ \forall t \in [0,T],\ x,y \in \mathbb{R}$;
(H_2) *There exists* $\mathcal{K} > 0$ *such that* $|h(t,x) - h(t,y)| \le \mathcal{K}|x-y|,\ \forall t \in [0,T],\ x,y \in \mathbb{R}$;
(H_3) $|f(t,y)| \le \delta(t)$ *and* $|h(t,y)| \le \theta(t)$, *where* $\delta,\theta \in C([0,T],\mathbb{R}^+)$.

Then, the problems (1.1) and (1.2) has at least one solution on $[0,T]$ *if* $\mathcal{L}\gamma_1 + \mathcal{K}\gamma_2 < 1$, *where*

$$\gamma_1=\frac{T^{\alpha}}{\Gamma(\alpha+1)},\ \ \gamma_2=\frac{|\lambda|T^{\alpha+\rho+\sigma}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}. \tag{3.3}$$

Proof. Introduce the ball $B_\eta = \{y \in \mathcal{X} : \|y\| \le \eta\}$, with

$$\eta \ge \|\delta\|\Omega_1 + \|\theta\|\Omega_2. \tag{3.4}$$

Now we verify the hypotheses of Krasnosel'skiĭ's fixed point theorem in three steps by splitting the operator $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ defined by (3.1) on B_η as $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, where

$$\begin{aligned}
\mathcal{V}_1 y(t) &= \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s)) - \lambda I_{T-}^{\rho} I_{0+}^{\sigma} h(s,y(s))]\,ds, \quad t \in J,\\
\mathcal{V}_2 y(t) &= b_1(t)\Big\{\sum_{\imath=1}^{m} p_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s))ds - \lambda I_{T-}^{\rho} I_{0+}^{\sigma} h(s,y(s))]ds\\
&\quad -T\sum_{\imath=1}^{m} q_i \int_{\xi_i}^T \frac{(s-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s)) - \lambda I_{T-}^{\rho} I_{0+}^{\sigma} h(s,y(s))]ds\Big\}\\
&\quad +b_2(t)\Big\{\sum_{\imath=1}^{m} r_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s)) - \lambda I_{T-}^{\rho} I_{0+}^{\sigma} h(s,y(s))]ds\\
&\quad -T\sum_{i=1}^{m} v_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s)) - \lambda I_{T-}^{\rho} I_{0+}^{\sigma} h(s,y(s))]ds\Big\}, \quad t \in J.
\end{aligned}$$

(i) For $y, x \in B_\eta$, we have

$$\begin{aligned}
&\|\mathcal{V}_1 y + \mathcal{V}_2 x\|\\
\le\ & \sup_{t\in[0,T]}\Big\{\int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma} |h(s,y(s))|]ds\\
&+|b_1(t)|\Big\{\sum_{\imath=1}^{m} |p_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big[|f(s,x(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma} |h(s,x(s))|\Big]\,ds\\
&+T\sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[|f(s,x(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma} |h(s,x(s))|]ds\Big\}\\
&+|b_2(t)|\Big\{\sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,x(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma} |h(s,x(s))|]ds\\
&+T\sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[|f(s,x(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma} |h(s,x(s))|]\,ds\Big\}\Big\}\\
\le\ & \|\delta\| \sup_{t\in[0,T]}\Big\{\int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}ds + |b_1(t)|\Big[\sum_{\imath=1}^{m} |p_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+T\sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}ds\Big] + |b_2(t)|\Big[\sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+T\sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}ds\Big]\Big\}\\
&+\|\theta\||\lambda| \sup_{t\in[0,T]}\Big\{\int_t^T \frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} I_{0+}^{\sigma} ds + |b_1(t)|\Big[\sum_{\imath=1}^{m} \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} I_{0+}^{\sigma} ds
\end{aligned}$$

$$
\begin{aligned}
&+T\sum_{\imath=1}^{m}|q_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}ds\Big]+|b_2(t)|\Big[\sum_{\imath=1}^{m}|r_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}ds\\
&+T\sum_{\imath=1}^{m}|v_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}ds\Big]\Big\}\\
\leq\;& \|\delta\|\sup_{t\in[0,T]}\Big\{\frac{(T-t)^{\alpha}}{\Gamma(\alpha+1)}+|b_1(t)|\Big[\sum_{\imath=1}^{m}|p_i|\frac{(T-\xi_i)^{\alpha}}{\Gamma(\alpha+1)}+T\sum_{\imath=1}^{m}|q_i|\frac{(T-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big]\\
&+|b_2(t)|\Big[\sum_{\imath=1}^{m}|r_i|\frac{(T-\xi_i)^{\alpha}}{\Gamma(\alpha+1)}+T\sum_{\imath=1}^{m}|v_i|\frac{(T-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big]\Big\}\\
&+\frac{\|\theta\||\lambda|T^{\sigma}}{\Gamma(\sigma+1)}\sup_{t\in[0,T]}\Big\{\frac{(T-t)^{\alpha+\rho}}{\Gamma(\alpha+\rho+1)}ds+|b_1(t)|\Big[\sum_{\imath=1}^{m}\frac{(T-\xi_i)^{\alpha+\rho}}{\Gamma(\alpha+\rho+1)}+T\sum_{\imath=1}^{m}|q_i|\frac{(T-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}\Big]\\
&+|b_2(t)|\Big[\sum_{\imath=1}^{m}|r_i|\frac{(T-\xi_i)^{\alpha+\rho}}{\Gamma(\alpha+\rho+1)}+T\sum_{\imath=1}^{m}|v_i|\frac{(T-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}\Big]\Big\}\\
\leq\;& \frac{\|\delta\|}{\Gamma(\alpha+1)}\Big\{T^{\alpha}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Big]\\
&+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Big]\Big\}\\
&+\frac{\|\theta\||\lambda|T^{\sigma}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big\{T^{\alpha+\rho}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\\
&+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\}\\
\leq\;& \|\beta\|\Omega_1+\|\theta\|\Omega_2<\eta,
\end{aligned}
$$

where we used (3.4). Thus $\mathcal{V}_1y+\mathcal{V}_2x\in B_\eta$.

(ii) Using (H_1) and (H_2), it is easy to show that

$$
\begin{aligned}
\|\mathcal{V}_1y-\mathcal{V}_1x\| \;\leq\;& \sup_{t\in[0,T]}\Big\{\int_t^T\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,x(s))|ds\\
&+|\lambda|\int_t^T\frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\Big\}\\
\leq\;& (\mathcal{L}\gamma_1+\mathcal{K}\gamma_2)\|y-x\|,
\end{aligned}
$$

which, in view of the condition $\mathcal{L}\gamma_1+\mathcal{K}\gamma_2<1$, implies that the operator $\mathcal{V}_1$ is a contraction.

(iii) Continuity of the functions f,h implies that the operator $\mathcal{V}_2$ is continuous. In addition, $\mathcal{V}_2$ is uniformly bounded on B_η as

$$
\|\mathcal{V}_2y\| \;\leq\; \sup_{t\in[0,T]}\Big\{|b_1(t)|\Big[\sum_{\imath=1}^{m}|p_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|ds
$$

$$
\begin{aligned}
& +|\lambda| \sum_{\imath=1}^{m} |p_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} I_{0+}^{\sigma} |h(s,y(s))| ds \\
& +T \sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| ds \\
& +|\lambda| T \sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)} I_{0+}^{\sigma} |h(s,y(s))| ds \Big] \\
& +|b_2(t)| \Big[\sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))| \\
& +|\lambda| \sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho_1}}{(\alpha+\rho)} I_{0+}^{\sigma} |h(s,y(s))| ds \\
& +T \sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))| ds \\
& +|\lambda| T \sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-2)}}{\Gamma(\alpha+\rho-1)} |h(s,y(s))| ds \Big] \Big\} \\
\leq \; & \|\delta\| \sup_{t\in[0,T]} \Big\{ |b_1(t)| \Big[\sum_{\imath=1}^{m} |p_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} ds + T \sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big] \\
& +|b_2(t)| \Big[\sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} ds + T \sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big] \Big\} \\
& +|\lambda| \|\theta\| \sup_{t\in[0,T]} \Big\{ |b_1(t)| \Big[\sum_{\imath=1}^{m} |p_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} I_{0+}^{\sigma} ds \\
& +T \sum_{\imath=1}^{m} |q_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)} I_{0+}^{\sigma} ds \Big] \\
& +|b_2(t)| \Big[\sum_{\imath=1}^{m} |r_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} I_{0+}^{\sigma} ds + T \sum_{\imath=1}^{m} |v_i| \int_{\xi_i}^{T} \frac{(s-\xi_i)^{\alpha+\rho-2)}}{\Gamma(\alpha+\rho-1)} I_{0+}^{\sigma} ds \Big] \Big\} \\
\leq \; & \|\delta\| \sup_{t\in[0,T]} \Big\{ |b_1(t)| \Big[\sum_{\imath=1}^{m} |p_i| \frac{(T-\xi_i)^{\alpha}}{\Gamma(\alpha+1)} + T \sum_{\imath=1}^{m} |q_i| \frac{(T-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} \Big] \\
& +|b_2(t)| \Big[\sum_{\imath=1}^{m} |r_i| \frac{(T-\xi_i)^{\alpha}}{\Gamma(\alpha+1)} + T \sum_{\imath=1}^{m} |v_i| \frac{(T-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} \Big] \\
& +\frac{|\lambda| \|\theta\| T^{\sigma}}{\Gamma(\sigma+1)} \sup_{t\in[0,T]} \Big\{ |b_1(t)| \Big[\sum_{\imath=1}^{m} |p_i| \frac{(T-\xi_i)^{\alpha+\rho}}{\Gamma(\alpha+\rho+1)} + T \sum_{\imath=1}^{m} |q_i| \frac{(T-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)} \Big] \\
& +|b_2(t)| \Big[\sum_{\imath=1}^{m} |r_i| \frac{(T-\xi_i)^{\alpha+\rho}}{\Gamma(\alpha+\rho+1)} + T \sum_{\imath=1}^{m} |v_i| \frac{(T-\xi_i)^{\alpha+\rho-1)}}{\Gamma(\alpha+\rho)} \Big] \Big\} \\
\leq \; & \|\delta\| (\Omega_1 - \gamma_1) + \|\theta\| (\Omega_2 - \gamma_2),
\end{aligned}
$$

where Ω_i, and γ_i, $i = 1, 2$, are defined in (3.2) and (3.3), respectively. To show the compactness of $\mathcal{V}_2$, we fix $\sup_{(t,y)\in[0,T]\times B_\eta}|f(t,y)| = \overline{f}$, $\sup_{(t,y)\in[0,T]\times B_\eta}|h(t,y)| = \overline{h}$. Then, for $0 < t_1 < t_2 < T$, we have

$$
\begin{aligned}
&|(\mathcal{V}_2 y)(t_2) - (\mathcal{V}_2 y)(t_1)| \\
\leq\ & \Big|b_1(t_2) - b_1(t_1)\Big|\Big\{\sum_{\imath=1}^{m}|p_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big[|f(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|h(s,y(s))|\Big]ds \\
&+T\sum_{\imath=1}^{m}|q_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[|f(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|h(s,y(s))|]ds\Big\} \\
&+\Big|b_2(t_2) - b_2(t_1)\Big|\Big\{\sum_{\imath=1}^{m}|r_i|\int_{\xi}^{T}\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,y(s))| + \lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|h(s,y(s))|ds] \\
&+T\sum_{\imath=1}^{m}|v_i|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[|f(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|h(s,y(s))|]ds\Big\} \\
\leq\ & \frac{|S_6||t_2-t_1|}{|\Delta|}\Big\{\frac{\overline{f}}{\Gamma(\alpha+1)}\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha} + \alpha T\sum_{i=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Big] \\
&+\frac{\overline{h}|\lambda|T^{\sigma}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho} + (\alpha+\rho)T\sum_{i=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\} \\
&+\frac{|S_1-1||t_2-t_1|}{|\Delta|}\Big\{\frac{\overline{f}}{\Gamma(\alpha+1)}\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha} + \alpha T\sum_{i=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Big] \\
&+\frac{\overline{h}|\lambda|T^{\sigma}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho} + (\alpha+\rho)T\sum_{i=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\},
\end{aligned}
$$

which tends to zero, independent of y, as $t_2 \to t_1$. This shows that $\mathcal{V}_2$ is equicontinuous. It is clear from the foregoing arguments that the operator $\mathcal{V}_2$ is relatively compact on B_η. Hence, by the Arzelá-Ascoli theorem, $\mathcal{V}_2$ is compact on B_η.

In view of the foregoing arguments (*i*)–(*iii*), the hypotheses of the Krasnosel'skiı̆'s fixed point theorem [31] are satisfied. Hence, the operator $\mathcal{V}_1 + \mathcal{V}_2 = \mathcal{V}$ has a fixed point, which implies that the problems (1.1) and (1.2) has at least one solution on $[0, T]$. The proof is finished.

Remark 3.1. *Interchanging the roles of the operators $\mathcal{V}_1$ and $\mathcal{V}_2$ in the previous result, the condition $L\gamma_1 + K\gamma_2 < 1$ changes to the following one:*

$$\mathcal{L}(\Omega_1 - \gamma_1) + \mathcal{K}(\Omega_2 - \gamma_2) < 1,$$

where Ω_1, Ω_2 and γ_1, γ_2 are defined in (3.2) and (3.3) respectively.

The following existence result relies on Leray-Schauder nonlinear alternative [32].

Theorem 3.2. *Suppose that the following conditions hold:*

(H_4) *There exist continuous nondecreasing functions $\phi_1, \phi_2 : [0,\infty) \to (0,\infty)$ such that $\forall(t,y) \in [0,1]\times\mathbb{R}$, $|f(t,y)| \leq \omega_1(t)\phi_1(\|y\|)$ and $|h(t,y)| \leq \omega_2(t)\phi_2(\|y\|)$, where $\omega_1, \omega_2 \in C([0,T],\mathbb{R}^+)$;*

(H_5) *There exists a constant $M > 0$ such that*

$$\frac{M}{\|\omega_1\|\phi_1(M)\Omega_1 + \|\omega_2\|\phi_2(M)\Omega_2} > 1.$$

Then, the problems (1.1) and (1.2) has at least one solution on $[0, T]$.

Proof. We firstly show that the operator $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ defined by (3.1) is completely continuous.

(i) $\mathcal{V}$ maps bounded sets into bounded sets in $\mathcal{X}$.

Let $y \in \mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \leq r\}$, where r is a fixed number. Then, using the strategy employed in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
\|\mathcal{V}y\| \leq\ & \frac{\|\omega_1\|\phi_1(r)}{\Gamma(\alpha+1)}\Big\{T^{\alpha} + \overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha} + \alpha T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Big] \\
& + \overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha} + \alpha T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Big]\Big\} \\
& + \frac{|\lambda|T^{\sigma}\|\omega_2\|\phi_2(r)}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big\{T^{\alpha+\rho} + \overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho} \\
& + (\alpha+\rho)T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big] \\
& + \overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho} + (\alpha+\rho)T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\} \\
=\ & \|\omega_1\|\phi_1(r)\Omega_1 + \|\omega_2\|\phi_2(r)\Omega_2 < \infty.
\end{aligned}$$

(ii) $\mathcal{V}$ maps bounded sets into equicontinuous sets.

Let $0 < t_1 < t_2 < T$ and $y \in \mathcal{B}_r$. Then, we obtain

$$\begin{aligned}
& |\mathcal{V}y(t_2) - \mathcal{V}y(t_1)| \\
\leq\ & \Big|\int_{t_2}^{T}\frac{(s-t_2)^{\alpha-1}-(s-t_1)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds + \int_{t_1}^{t_2}\frac{(s-t_1)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds \\
& -\lambda\int_{t_2}^{T}\frac{(s-t_2)^{\alpha+\rho-1}-(s-t_1)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}h(s,y(s))ds - \lambda\int_{t_1}^{t_2}\frac{(s-t_1)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}h(s,y(s))ds\Big| \\
& +\Big|b_1(t_2)-b_1(t_1)\Big|\Big\{\Big|\sum_{\imath=1}^{m}p_i\int_{\xi_i}^{T}\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s)) - \lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Big| \\
& +\Big|T\sum_{\imath=1}^{m}q_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s)) - \lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Big|\Big\} \\
& +\Big|b_2(t_2)-b_2(t_1)\Big|\Big\{\Big|\sum_{\imath=1}^{m}r_i\int_{\xi_i}^{T}\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}[f(s,y(s)) - \lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Big| \\
& +\Big|T\sum_{\imath=1}^{m}v_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}[f(s,y(s)) - \lambda I_{T-}^{\rho}I_{0+}^{\sigma}h(s,y(s))]ds\Big|\Big\} \\
\leq\ & \Big|\int_{t_2}^{T}\frac{(s-t_2)^{\alpha-1}-(s-t_1)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds + \int_{t_1}^{t_2}\frac{(s-t_1)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\Big| \\
& +\Big|\lambda\int_{t_2}^{T}\frac{(s-t_2)^{\alpha+\rho-1}-(s-t_1)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}h(s,y(s))ds + \lambda\int_{t_1}^{t_2}\frac{(s-t_1)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}h(s,y(s))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|S_6||t_2-t_1|}{\Delta}\Bigg|\Bigg\{\Bigg|\sum_{\imath=1}^{m}p_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big[f(s,y(s))-\lambda I^{\rho}_{T-}I^{\sigma}_{0+}h(s,y(s))\Big]ds\Bigg|\\
&+\Bigg|T\sum_{\imath=1}^{m}q_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}\Big[f(s,y(s))-\lambda I^{\rho}_{T-}I^{\sigma}_{0+}h(s,y(s))\Big]ds\Bigg|\Bigg\}\\
&+\frac{|S_1-1||t_2-t_1|}{\Delta}\Bigg\{\Bigg|\sum_{\imath=1}^{m}r_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}\Big[f(s,y(s))-\lambda I^{\rho}_{T-}I^{\sigma}_{0+}h(s,y(s))\Big]ds\Bigg|\\
&+\Bigg|T\sum_{\imath=1}^{m}v_i\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}\Big[f(s,y(s))-\lambda I^{\rho}_{T-}I^{\sigma}_{0+}h(s,y(s))\Big]ds\Bigg|\Bigg\}\\
\le\ &\frac{\omega_1(t)\Phi_1(r)}{\Gamma(\alpha+1)}\Bigg\{\Big|(T-t_2)^{\alpha}-(T-t_1)^{\alpha}\Big|+2\Big|(t_2-t_1)^{\alpha}\Big|\\
&+\frac{|t_2-t_1|}{|\Delta|}\Bigg[|S_6|\Bigg(\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Bigg)\\
&+|S_1-1|\Bigg(\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Bigg)\Bigg]\Bigg\}\\
&+\frac{|\lambda|T^{\sigma}\omega_2(t)\phi_2(r)}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Bigg\{\Big|(T-t_2)^{\alpha+\rho}-(T-t_1)^{\alpha+\rho}+2|t_2-t_1|^{\alpha+\rho}\\
&+\frac{|t_2-t_1|}{|\Delta|}\Bigg[|S_6|\Bigg(\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Bigg)\\
&+|S_1-1|\Bigg(\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Bigg)\Bigg]\Bigg\}.
\end{aligned}
$$

Notice that the right-hand side of the above inequality tends to 0 as $t_2 \to t_1$, independent of $y \in \mathcal{B}_r$. Thus, it follows by the ArzelAscoli theorem that the operator $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ is completely continuous.

The conclusion of the Leray-Schauder nonlinear alternative [32] will be applicable once it is shown that there exists an open set $U \subset C([0,T],\mathbb{R})$ with $y \neq \nu\mathcal{V}y$ for $\nu \in (0,1)$ and $y \in \partial U$. Let $y \in C([0,T],\mathbb{R})$ be such that $y = \nu\mathcal{V}y$ for $\nu \in (0,1)$. As argued in proving that the operator $\mathcal{V}$ is bounded, one can obtain that

$$
|y(t)| = |\nu\mathcal{V}y(t)| \le |\omega_1(t)|\phi(\|y\|)\Omega_1 + |\omega_2(t)|\psi(\|y\|)\Omega_2,
$$

which can be written as

$$
\frac{\|y\|}{\|\omega_1\|\phi(\|y\|)\Omega_1 + \|\omega_2\|\psi(\|y\|)\Omega_2} \le 1.
$$

On the other hand, we can find a positive number M such that $\|y\| \neq M$ by assumption (H_5). Let us set

$$
W = \{y \in \mathcal{X} : \|y\| < M\}.
$$

Clearly, ∂W contains a solution only when $\|y\| = M$. In other words, we cannot find a solution $y \in \partial W$ satisfying $y = \nu\mathcal{V}y$ for some $\nu \in (0,1)$. In consequence, the operator $\mathcal{V}$ has a fixed point $y \in \overline{W}$, which is a solution of the problems (1.1) and (1.2). The proof is finished.

3.2. Uniqueness result

Here we apply Banach contraction mapping principle to establish the uniqueness of solutions for the problems (1.1) and (1.2).

Theorem 3.3. *If the conditions (H_1) and (H_2) hold, then the problems (1.1) and (1.2) has a unique solution on* $[0, T]$ *if*

$$\mathcal{L}\Omega_1 + \mathcal{K}\Omega_2 < 1, \tag{3.5}$$

where Ω_1 *and* Ω_2 *are defined in (3.2).*

Proof. In the first step, we show that $\mathcal{V}\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$, where $\mathcal{B}_\kappa = \{y \in \mathcal{X} : \|y\| \leq \kappa\}$ with

$$\kappa \geq \frac{f_0\Omega_1 + h_0\Omega_2}{1 - (\mathcal{L}\Omega_1 + \mathcal{K}\Omega_2)}, \ f_0 = \sup_{t\in[0,T]} |f(t, 0)|, \ h_0 = \sup_{t\in[0,T]} |h(t, 0)|.$$

For $y \in \mathcal{B}_\kappa$ and using the condition (H_1), we have

$$\begin{aligned} |f(t, y)| &= |f(t, y) - f(t, 0) + f(t, 0)| \leq |f(t, y) - f(t, 0)| + |f(t, 0)| \\ &\leq \mathcal{L}\|y\| + f_0 \leq \mathcal{L}r + f_0. \end{aligned} \tag{3.6}$$

Similarly, using (H_2), we get

$$|h(t, y)| \leq \mathcal{K}r + h_0. \tag{3.7}$$

In view of (3.6) and (3.7), we obtain

$$\begin{aligned} &\|\mathcal{V}y\| \leq \sup_{t\in[0,T]} |\mathcal{V}y(t)| \\ \leq\ & \sup_{t\in[0,T]} \Bigg\{ \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} [|f(s, y(s))| + |\lambda| I^\rho_{T-} I^\sigma_{0+} |h(s, y(s))|] ds \\ &+ |b_1(t)| \Bigg\{ \sum_{\imath=1}^m p_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} [|f(s, y(s))| + |\lambda| I^\rho_{T-} I^\sigma_{0+} |h(s, y(s))|] ds \\ &+ T \sum_{\imath=1}^m |q_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} [|f(s, y(s))| + |\lambda| I^\rho_{T-} I^\sigma_{0+} |h(s, y(s))|] ds \Bigg\} \\ &+ |b_2(t)| \Bigg\{ \sum_{\imath=1}^m r_i \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} [|f(s, y(s))| + |\lambda| I^\rho_{T-} I^\sigma_{0+} |h(s, y(s))|] ds \\ &+ T \sum_{i=1}^m |v_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} [|f(s, y(s))| + |\lambda| I^\rho_{T-} I^\sigma_{0+} |h(s, y(s))|] ds \Bigg\} \Bigg\} \\ \leq\ & (\mathcal{L}r + f_0) \sup_{t\in[0,T]} \Bigg\{ \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\ &+ |b_1(t)| \Big[\sum_{i=1}^m |p_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} ds + T \sum_{\imath=1}^m |q_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big] \\ &+ |b_2(t)| \Big[\sum_{i=1}^m |r_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)} ds + T \sum_{\imath=1}^m |v_i| \int_{\xi_i}^T \frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big] \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&+|\lambda|(\mathcal{K}r+h_0)\sup_{t\in[0,T]}\bigg\{\int_t^T \frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}\,ds\\
&+|b_1(t)|\Big[\sum_{i=1}^m|p_i|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}ds+T\sum_{i=1}^m|q_i|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}ds\Big]\\
&+|b_2(t)|\Big[\sum_{i=1}^m|r_i|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}ds+T\sum_{i=1}^m|v_i|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}ds\Big]\bigg\}\\
\le\ &\frac{(\mathcal{L}r+f_0)}{\Gamma(\alpha+1)}\Big\{T^{\alpha}+\overline{b}_1\Big[\sum_{i=1}^m|p_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{i=1}^m|q_i|(T-\xi_i)^{\alpha-1}\Big]\\
&+\overline{b}_2\Big[\sum_{i=1}^m|r_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{i=1}^m|v_i|(T-\xi_i)^{\alpha-1}\Big]\Big\}\\
&+\frac{T^{\sigma}|\lambda|(\mathcal{K}r+h_0)}{\Gamma(\sigma)\Gamma(\alpha+\rho+1)}\Big\{T^{\alpha+\rho}+\overline{b}_1\Big[\sum_{i=1}^m|p_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{i=1}^m|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\\
&+\overline{b}_2\Big[\sum_{i=1}^m|r_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{i=1}^m|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\}\\
=\ &(\mathcal{L}r+f_0)\Omega_1+(\mathcal{K}r+h_0)\Omega_2\ <\kappa,
\end{aligned}
$$

which implies that $\mathcal{V}y\in\mathcal{B}_\kappa$, for any $y\in\mathcal{B}_\kappa$. Therefore, $\mathcal{V}\mathcal{B}_\kappa\subset\mathcal{B}_\kappa$.

Next, we prove that $\mathcal{V}$ is a contraction. For that, let $x,y\in\mathcal{X}$ and $t\in[0,T]$. Then, by the conditions (H_1) and (H_2), we obtain

$$
\begin{aligned}
&\|\mathcal{V}y-\mathcal{V}x\|=\sup_{t\in[0,T]}\Big|(\mathcal{V}y)(t)-(\mathcal{V}x)(t)\Big|\\
\le\ &\sup_{t\in[0,T]}\bigg\{\int_t^T\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,x(s))|ds\\
&+|\lambda|\int_t^T\frac{(s-t)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\\
&+|b_1(t)|\Big[\sum_{i=1}^m|p_i|\Big(\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,x(s))|ds\\
&+|\lambda|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\Big)\\
&+T\sum_{i=1}^m|q_i|\Big(\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,y(s))-f(s,x(s))|ds\\
&+|\lambda|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\Big)\Big]\\
&+|b_2(t)|\Big[\sum_{i=1}^m|r_i|\Big(\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,x(s))|ds\\
&+|\lambda|\int_{\xi_i}^T\frac{(s-\xi_i)^{\alpha+\rho-1}}{\Gamma(\alpha+\rho)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\Big)
\end{aligned}
$$

$$+T\sum_{\imath=1}^{m}|v_i|\Big(\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,y(s))-f(s,x(s))|ds$$
$$+|\lambda|\int_{\xi_i}^{T}\frac{(s-\xi_i)^{\alpha+\rho-2}}{\Gamma(\alpha+\rho-1)}I_{0+}^{\sigma}|h(s,y(s))-h(s,x(s))|ds\Big)\Big]\Big\}$$
$$\leq \frac{\mathcal{L}}{\Gamma(\alpha+1)}\Big\{T^{\alpha}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha-1}\Big]$$
$$+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha}+\alpha T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha-1}\Big]\Big\}$$
$$+\frac{T^{\sigma}|\lambda|\mathcal{K}}{\Gamma(\sigma+1)\Gamma(\alpha+\rho+1)}\Big\{T^{\alpha+\rho}+\overline{b}_1\Big[\sum_{\imath=1}^{m}|p_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|q_i|(T-\xi_i)^{\alpha+\rho-1}\Big]$$
$$+\overline{b}_2\Big[\sum_{\imath=1}^{m}|r_i|(T-\xi_i)^{\alpha+\rho}+(\alpha+\rho)T\sum_{\imath=1}^{m}|v_i|(T-\xi_i)^{\alpha+\rho-1}\Big]\Big\}\|y-x\|$$
$$= (\mathcal{L}\Omega_1+\mathcal{K}\Omega_2)\|y-x\|,$$

which shows that $\mathcal{V}$ is a contraction in view of the condition (3.5). Therefore, we deduce by Banach contraction mapping principle that there exists a unique fixed point for the operator $\mathcal{V}$, which corresponds to a unique solution for the problems (1.1) and (1.2) on $[0,T]$. The proof is completed.

3.3. Examples

In this subsection, we construct examples for illustrating the abstract results derived in the last two subsections. Let us consider the following problem:

$$\begin{cases} D_{1-}^{9/8}y(t)+3I_{1-}^{7/3}I_{0+}^{3/4}h(t,y(t))=f(t,y(t)),\ t\in J:=[0,1], \\ y(T)=\sum_{\imath=1}^{3}p_iy(\xi_i)+\sum_{\imath=1}^{3}q_iy'(\xi_i),\ y'(T)=\sum_{\imath=1}^{3}r_iy(\xi_i)+\sum_{\imath=1}^{3}v_iy'(\xi_i),\ ,\ 0<\xi_i<1. \end{cases} \tag{3.8}$$

Here $\alpha=9/8,\rho=7/3,\sigma=3/4,\lambda=3,\xi_1=3/7,\xi_2=2/3,\xi_3=4/5,p_1=1/2,p_2=1/3,p_3=1/4,q_1=-2,q_2=-3,q_3=-4,r_1=1,r_2=-1,r_3=3,v_1=-2/7,v_2=-3/7,v_3=-4/7$. Using the given data, it is found that

$$\overline{b}_1=\max_{t\in[0,1]}|b_1(t)|=|b_1(t)|_{t=1}\approx 0.1112461491,\quad \overline{b}_2=\max_{t\in[0,1]}|b_2(t)|=|b_2(t)|_{t=1}\approx 0.3364235041.$$

In consequence, we get $\Omega_1\approx 2.517580993,\Omega_2\approx 0.3543113654$ (Ω_1,Ω_2 are defined in (3.2)).
(i) For illustrating Theorem 3.1, we consider the functions

$$f(t,y)=\frac{m_1}{2t+25}\Big(\frac{y^2}{1+y^2}+\cos 3t+1\Big),\ \ h(t,y)=\frac{m_2}{3\sqrt{t^2+64}}\Big(2\tan^{-1}y+\sin t+e^{-t/2}\Big), \tag{3.9}$$

where m_1 and m_2 are finite positive real numbers. Observe that

$$|f(t,y)|\leq\delta(t)=\frac{m_1(2+\cos 3t)}{2t+25},\qquad |h(t,y)|\leq\theta(t)=\frac{m_2(\pi+\sin t+e^{-t/2})}{3\sqrt{t^2+64}},$$

and $f(t, y)$ and $h(t, y)$ respectively satisfy the conditions (H_1) and (H_2) with $\mathcal{L} = 2m_1/25$ and $\mathcal{K} = m_2/24$. Moreover, $\gamma_1 \approx 0.9438765902$ and $\gamma_2 \approx 0.2972831604$. By the condition $\mathcal{L}\gamma_1 + \mathcal{K}\gamma_2 < 1$, we get

$$0.0755101272m_1 + 0.0123867984m_2 < 1 \tag{3.10}$$

For the values of m_1 and m_2 satisfying the inequality (3.10), the hypothesis of Theorem 3.1 is satisfied. Hence, it follows by the conclusion of Theorem 3.1 that the problem (3.8) with $f(t, y)$ and $h(t, y)$ given in (3.9) has at least one solution on $[0, 1]$. If the values m_1 and m_2 do not satisfy the inequality (3.10), then Theorem 3.1 does not guarantee the existence of at least one solution to the problem (3.8) with $f(t, y)$ and $h(t, y)$ given in (3.9) for such values of m_1 and m_2.

(ii) In order to illustrate Theorem 3.2, we take the following functions (instead of (3.9)) in the problem (3.8):

$$f(t, y) = \frac{e^{-3t}}{t^2 + 3}\big[\sin y + 1/5\big], \qquad h(t, y) = \frac{2}{7\sqrt{t^3 + 1}}\Big(\frac{|y|}{1 + |y|}|y| + \pi/4\Big). \tag{3.11}$$

Observe that the assumption (H_4) is satisfied as $|f(t, y)| \le \omega_1(t)\phi_1(\|y\|)$ and $|h(t, y)| \le \omega_2(t)\phi_2(\|y\|)$, where $\omega_1(t) = e^{-3t}/(t^2 + 3)$, $\phi_1(\|y\|) = (\|y\| + 1/5)$, $\omega_2(t) = 2/(7\sqrt{t^3 + 1})$, $\phi_2(\|y\|) = (\|y\| + \pi/4)$. It is easy to see that $\|\omega_1\| = 1/3$ and $\omega_2\| = 2/7$. By the condition (H_5), we find that $M > 4.151876169$. Thus, all the conditions of Theorem 3.2 are satisfied and hence the problem (3.8) with $f(t, y)$ and $h(t, y)$ given by (3.11) has at least one solution on $[0, 1]$.

(iii) The conditions (H_1) and (H_2) are respectively satisfied by $f(t, y)$ and $h(t, y)$ defined in (3.9) with $\mathcal{L} = 2m_1/25$ and $\mathcal{K} = m_2/24$. By the condition (3.5), we have

$$0.20140647944m_1 + 0.0147629736m_2 < 1. \tag{3.12}$$

Clearly, all the assumptions of Theorem 3.3 hold true with the values of m_1 and m_2 satisfying the inequality (3.12). In consequence, the problem (3.8) with $f(t, y)$ and $h(t, y)$ given in (3.11) has a unique solution on $[0, 1]$. In case, we take $m_1 = m_2 = m$ in (3.9), then the condition (3.12) implies the existence of a unique solution for the problem at hand for $m < 4.62600051$. One can notice that Theorem 3.1 does not guarantee the existence of a unique solution to the problem (3.8) with $f(t, y)$ and $h(t, y)$ given in (3.9) for the values of m_1 and m_2, which do not satisfy the inequality (3.12).

4. Conclusions

In this study, we discussed the existence and uniqueness of solutions under different assumptions for a boundary value problem involving a right Caputo fractional derivative with usual and mixed Riemann-Liouville integrals type nonlinearities, equipped with nonlocal multipoint version of the closed boundary conditions. Our results are not only new in the given configuration, but also yield some new results as special cases. Here are some examples.

- If $\lambda = 0$ in (1.1), then our results correspond to the fractional differential equation ${}^{C}D^{\alpha}_{T-}y(t) = f(t, y(t))$ with the boundary conditions (1.2).
- In case, we take $q_i = 0, r_i = 0, \forall i = 1, \ldots, m$ in the results of this paper, we obtain the ones for the Eq (1.1) supplemented with boundary conditions: $y(T) = \sum_{\imath=1}^{m} p_i y(\xi_i), \;\; y'(T) = \sum_{\imath=1}^{m} v_i y'(\xi_i)$.

- We get the results for the Eq (1.1) complemented with boundary conditions: $y(T) = T\sum_{i=1}^{m} q_i y'(\xi_i),\ \ Ty'(T) = \sum_{i=1}^{m} r_i y(\xi_i)$ by taking $p_i = 0, v_i = 0,\ \forall i = 1, \ldots, m$ in the obtained results.

Acknowledgments

The Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU), Jeddah, Saudi Arabia, has funded this project under grant No. (KEP-PhD: 35-130-1443).

Conflict of interest

The authors declare no conflict of interest.

References

1. B. Ahmad, S. K. Ntouyas, *Nonlocal nonlinear fractional-order boundary value problems*, World Scientific, 2021. https://doi.org/10.1142/12102
2. R. Agarwal, S. Hristova, D. O'Regan, Integral presentations of the solution of a boundary value problem for impulsive fractional integro-differential equations with Riemann-Liouville derivatives, *AIMS Math.*, **7** (2022), 2973–2988. https://doi.org/10.3934/math.2022164
3. J. J. Nieto, Fractional Euler numbers and generalized proportional fractional logistic differential equation, *Fract. Calc. Appl. Anal.*, **25** (2022), 876–886. https://doi.org/10.1007/s13540-022-00044-0
4. L. Peng, Y. Zhou, The existence of mild and classical solutions for time fractional Fokker-Planck equations, *Monatsh. Math.*, **199** (2022), 377–410. https://doi.org/10.1007/s00605-022-01710-4
5. M. Kirane, A. Abdeljabbar, Nonexistence of global solutions of systems of time fractional differential equations posed on the Heisenberg group, *Math. Method. Appl. Sci.*, **45** (2022), 7336–7345. https://doi.org/10.1002/mma.8243
6. A. Alsaedi, M. Alghanmi, B. Ahmad, B. Alharbi, Uniqueness results for a mixed p-Laplacian boundary value problem involving fractional derivatives and integrals with respect to a power function, *Electron. Res. Arch.*, **31** (2023), 367–385. https://doi.org/10.3934/era.2023018
7. A. Samadi, S. K. Ntouyas, J. Tariboon, On a nonlocal coupled system of Hilfer generalized proportional fractional differential equations, *Symmetry*, **14** (2022), 738. https://doi.org/10.3390/sym14040738
8. R. P. Agarwal, A. Assolami, A. Alsaedi, B. Ahmad, Existence results and Ulam-Hyers stability for a fully coupled system of nonlinear sequential Hilfer fractional differential equations and integro-multistrip-multipoint boundary conditions, *Qual. Theory Dyn. Syst.*, **21** (2022), 125. https://doi.org/10.1007/s12346-022-00650-6
9. A. Wongcharoen, S. K. Ntouyas, P. Wongsantisuk, J. Tariboon, Existence results for a nonlocal coupled system of sequential fractional differential equations involving ψ-Hilfer fractional derivatives, *Adv. Math. Phys.*, **2021** (2021), 5554619. https://doi.org/10.1155/2021/5554619

10. K. D. Kucche, A. D. Mali, On the nonlinear (k,ψ)-Hilfer fractional differential equations, *Chaos Soliton. Fract.*, **152** (2021), 111335. https://doi.org/10.1016/j.chaos.2021.111335

11. I. Bouacida, M. Kerboua, S. Segni, Controllability results for Sobolev type ψ-Hilfer fractional backward perturbed integro-differential equations in Hilbert space, *Evol. Equ. Control The.*, **12** (2023), 213–229. https://doi.org/10.3934/eect.2022028

12. A. P. Selvam, V. Govindaraj, Reachability of fractional dynamical systems with multiple delays in control using ψ-Hilfer pseudo-fractional derivative, *J. Math. Phys.*, **63** (2022), 102706. https://doi.org/10.1063/5.0049341

13. Q. Yang, C. Bai, D. Yang, Finite-time stability of nonlinear stochastic ψ-Hilfer fractional systems with time delay, *AIMS Math.*, **7** (2022), 18837–18852. https://doi.org/10.3934/math.20221037

14. A. Salim, M. Benchohra, J. R. Graef, J. E. Lazreg, Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations, *J. Fix. Point Theory A.*, **24** (2022), 7. https://doi.org/10.1007/s11784-021-00920-x

15. S. K. Ntouyas, B. Ahmad, C. Nuchpong, J. Tariboon, On (k,ψ)-Hilfer fractional differential equations and inclusions with mixed (k,ψ)-derivative and integral boundary conditions, *Axioms*, **11** (2022), 403. https://doi.org/10.3390/axioms11080403

16. S. K. Ntouyas, B. Ahmad, J. Tariboon, M. S. Alhodaly, Nonlocal integro-multi-point (k,ψ)-Hilfer type fractional boundary value problems, *Mathematics*, **10** (2022), 2357. https://doi.org/10.3390/math10132357

17. E. V. Ivashkevich, Boundary height correlations in a two-dimensional Abelian sandpile, *J. Phys. A Math. Gen.*, **27** (1994), 3643. https://doi.org/10.1088/0305-4470/27/11/014

18. G. Piroux, P. Ruelle, Boundary height fields in the Abelian sandpile model, *J. Phys. A Math. Gen.*, **38** (2005), 1451. https://doi.org/10.1088/0305-4470/38/7/004

19. N. Azimi-Tafreshi, H. Dashti-Naserabadi, S. Moghimi-Araghi, P. Ruelle, The Abelian sandpile model on the honeycomb lattice, *J. Stat. Mech.*, **2010** (2010), P02004. https://doi.org/10.1088/1742-5468/2010/02/P02004

20. M. Donatelli, S. Serra-Capizzano, Antireflective boundary conditions for deblurring problems, *J. Electr. Comput. Eng.*, **2010** (2010), 241467. https://doi.org/10.1155/2010/241467

21. X. Li, J. Robertsson, A. Curtis, D. van Manen, Internal absorbing boundary conditions for closed-aperture wavefield decomposition in solid media with unknown interiors, *J. Acoust. Soc. Am.*, **152** (2022), 313–329. https://doi.org/10.1121/10.0012578

22. M. Mohammadimehr, S. V. Okhravi, S. M. A. Alavi, Free vibration analysis of magneto-electro-elastic cylindrical composite panel reinforced by various distributions of CNTs with considering open and closed circuits boundary conditions based on FSDT, *J. Vib. Control*, **24** (2018), 1551–1569. https://doi.org/10.1177/1077546316664022

23. B. Ahmad, J. J. Nieto, J. Pimentel, Some boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.*, **62** (2011), 1238–1250. https://doi.org/10.1016/j.camwa.2011.02.035

24. A. V. Setukha, On the three-dimensional Neumann boundary value problem with a generalized boundary condition in a domain with a smooth closed boundary, *Diff. Equat.*, **41** (2005), 1237–1252. https://doi.org/10.1007/s10625-005-0273-4
25. G. Wang, B. Ahmad, L. Zhang, Existence results for nonlinear fractional differential equations with closed boundary conditions and impulses, *Adv. Differ. Equ.*, **2012** (2012), 169. https://doi.org/10.1186/1687-1847-2012-169
26. H. Ergoren, A. Kilicman, Some existence results for impulsive nonlinear fractional differential equations with closed boundary conditions, *Abstr. Appl. Anal.*, **2012** (2012), 387629. https://doi.org/10.1155/2012/387629
27. L. Zhang, B. Ahmad, G. Wang, The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative, *Appl. Math. Lett.*, **31** (2014), 1–6. https://doi.org/10.1016/j.aml.2013.12.014
28. T. Jankowski, Boundary problems for fractional differential equations, *Appl. Math. Lett.*, **28** (2014), 14–19. https://doi.org/10.1016/j.aml.2013.09.004
29. B. Ahmad, M. Alnahdi, S. K. Ntouyas, Existence results for a differential equation involving the right Caputo fractional derivative and mixed nonlinearities with nonlocal closed boundary conditions, *Fractal Fract.*, **7 2023**, 129. https://doi.org/10.3390/fractalfract7020129
30. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 2006. https://doi.org/10.1016/s0304-0208(06)x8001-5
31. M. A. Krasnosel'skiĭ, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk.*, **10** (1995), 123–127.
32. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2005. https://doi.org/10.1007/978-0-387-21593-8