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Research article

Different characterization of soft substructures in quantale modules dependent on soft relations and their approximations

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Abstract: The quantale module introduced by Abramsky and Vickers, engaged a large number of researchers. This research article focuses the combined behavior of rough set, soft set and an algebraic structure quantale module with the left action. In fact, the paper reflects the generalization of rough soft sets. This combined effect is totally dependent on soft binary relation including aftersets and foresets. Different soft substructures in quantale modules are defined. The characterizations of soft substructures in quantale modules based on soft binary relation are presented. Further, in quantale modules, we define soft compatible and soft complete relations in terms of aftersets and foresets. Furthermore, we use soft compatible and soft complete relations to approximate soft substructures of quantale modules and these approximations are interpreted by aftersets and foresets. This concept generalizes the concept of rough soft quantale modules. Additionally, we describe the algebraic relationships between the upper (lower) approximations of soft substructures of quantale modules and the upper (lower) approximations of their homomorphic images using the concept of soft quantale module homomorphism.

Keywords: quantale module and its substructures; rough set; soft set; aftersets and foresets; compatible relations
Mathematics Subject Classification: 06D72, 60Lxx, 60L70

List of Acronyms∗

Acronyms	Representation
C_{ltc}	Complete lattice
Q_S	Quantale submodule
Q_I	Quantale submodule ideal
LO_{ap}	Lower approximation
UP_{ap}	Upper approximation
URQ_S	Upper rough quantale submodule
LRQ_S	Lower rough quantale submodule
URQ_I	Upper rough quantale submodule ideal
LRQ_I	Lower rough quantale submodule ideal
GU_rS	Generalized upper soft
GL_rS	Generalized lower soft
SBIR	Soft binary relation
SCMR	Soft compatible relation
SCMPR	Soft complete relation
S.P	Set of Parameters
SWMH	Soft weak quantale module isomorphism

1. Introduction

Molodtsov [6] proposed the soft set (S-set) theory which has many applications to find solutions of problems in economics, medicine, engineering and social sciences. A S-set over a set U under consideration is a pair $(F, A) = \{F(\alpha) \subseteq U: \forall\ \alpha \in A\}$ where F is a function from A (set of parameter) to $P(U)$. The S-sets are generalization of conventional sets. Molodstov also discussed how this approach could be used to elaborate a variety of problems. The specification of a parameter is not required in S-set theory. This makes S-set theory a natural mathematical framework for approximate logic. Numerous theories were proposed to deal with uncertainty and imprecision following the development of S-set theory. Some of those are extensions of S-sets, while others strive to deal with uncertainty in another suitable way. To get better and precise result, rough set (R-set) presented by Pawlak [27] is combined with S-set named as rough soft set. In R-set theory an important part is played by an equivalence relation. There is always a question left whether an equivalence relation is simple to obtain. Thus, the combined R-set with S-set is termed as rough soft set defined by Feng et al., [7] is as follows: Let $G = (f, A)$ be a soft set over U. Then the pair $P = (U, G)$ is called soft approximation space. Based on P, following are defined as $\underline{apr_P}(X) = \{u \in U: \exists\, a \in A[u \in f(a) \subseteq X]\}$ and $\overline{apr}_P(X) = \{u \in U: \exists\, a \in A[u \in f(a) \cap X \neq \varphi]\}$ where $X \subseteq U$, $\underline{apr_P}(X)$ and $\overline{apr}_P(X)$ are called lower and upper soft rough sets. In this way, further approximation of S-sets in a different way was proposed by Shabir et al., [14] to give a proper illustration of the information and allow a greater degree of freedom and flexibility in representing uncertainty which is as follows: Let (F, V) be a soft binary relation from K_1 to K_2 (where K_1 to K_2 are universal sets under consideration). Thus, $(\underline{F^M}, V)$ and $(\overline{F}^M, V)$ are the lower and upper approximation of S-set (M, V) over K_2 with respect to aftersets, are essentially two soft sets over K_1 defined as $\underline{F^M}(v) = \{k \in K_1: \varphi \neq kF(v) \subseteq M(v)\}$ and $(\overline{F}^M(v) = \{k \in K_1: kF(v) \cap M(v) \neq \varphi\}$. From above discussion it is clear to understand that approximation of soft sets with respect to either aftersets or foresets by soft relations are simple and more suitable to handle different situations in different field of sciences.

1.1. Background and importance of quantale module

The quantale module, introduced by Abramsky and Vickers [20], engaged a large number of researchers. By replacing rings by quantale and abelian group with complete lattices in the module over ring, the concept of the quantale module was developed. Abramsky and Vickers used the concept of a quantale module for the unified treatment of process semantics. The concept of modules over a commutative unital quantaleas by Rosenthal [11] provided a family of models of full linear logic. Russo [4] introduced an approach to data compression algorithms using quantale module homomorphism as an application. A quantale-theoretic approach to propositional deductive [5] systems has been developed in recent years, based on the notion that any propositional deductive system may be represented as a quantale module. However, despite of their multiple applications, the first systematic studies on the categories of quantale modules are rather recent [2–4,22]. On the other hand, the results presented in [15] and [21] clearly suggest that the algebraic categories of quantales, unital quantales, and quantale modules are worth to be further investigated.

1.2. Literature review

The categories of modules over unital quantales were introduced by Russo. The main categorical properties were established and a special class of operators, called Q-module transforms, was defined [2]. Some applications of quantale modules with applications to logic and image processing were introduced Russo [4] including Free modules, hom-sets, products and coproducts, Q-module transforms, projective and injective Q-modules. Further decomposition and projectivity of quantale modules were discussed by Slesinger [18] and showed that every quantale module join-generated by its sub-set of join-irreducible elements can be uniquely decomposed into a collection of further indecomposable submodules. Further, he characterized regular projective essential modules that admitted this product decomposition as products of such cyclic quantale modules. The concept of Q-P quantale modules was defined by Liang and discussed categorical properties of Q-P quantale modules [23]. Algebraic properties of the category of Q-P quantale modules were defined by [23] and the structure of the free Q-P quantale modules generated by a set were obtained. Modules on involutive quantales were defined by Heymans and Stubbe [9]. They defined Canonical Hilbert structure, an application of sheaf theory. In 2018, rough set was applied to substructures of quantale modules and defined rough Q-submodule. Further the concepts of set-valued homomorphism and strong set-valued homomorphism of Q-modules were introduced, and related properties are investigated [25]. With the help of soft relations, substructures of quantale modules were approximated by Qurashi et al., [24]. An application of rough set theory based on Multi source information fusion was presented by Zhang et al., [16]. Zhang et al., introduces further heterogeneous feature selection dependent on neighborhood combination entropy [17].

1.3. Research gap in the current literature and motivation of the study

The literature overview above highlights some developments in both classical and R-set theory. Furthermore, even though several results about rough submodule and rough submodule ideals of quantale module and generalized lower and upper approximations operators based on set-valued homomorphism of quantale modules have been demonstrated but there are still some open questions remain to be answered.

(1) In classical quantale module theory, there is a lot of contributions but there is no attention on its generalization, for example soft quantale module, different characterization of fuzzy substructures in quantale modules like $(\epsilon, \epsilon \vee q)$-fuzzy substructures, $(\epsilon, \epsilon \vee q_k)$-fuzzy substructures of quantale modules and $(\epsilon_\gamma, \epsilon_\gamma \vee q_\delta)$-fuzzy substructures of Quantale modules. Rough neutrosophic soft substructures in quantale modules, fuzzy bipolar soft substructures in quantale modules.

(2) Study of different substructures of quantale modules by soft relations is present in existing literature. Further, approximation of soft ideals by soft relations in semigroups was proposed by Kanwal and Shabir [19]. Since soft substructures in quantale modules are generalization of its substructures so therefore, it is important to understand the characterization of soft substructures in quantale modules dependent on soft relations.

(3) Roughness of substructures with the help of congruence relations and set-valued homomorphism is in the lecturer [25]. A natural question comes into mind, what will be the behavior of roughness of soft substructures by soft relations is a logical question to ask.

(4) Some fundamental and important theorems of quantale module homomorphism are discussed in [24]. Therefore, it is necessary to discuss these remarkable theorems in the context of soft quantale module homomorphism.

(5) Numerous algebraic aspects of substructures of quantale module with and without by soft relations have been studied in the literature. In the context of soft substructures of quantale modules, these studies have yet to be examined from a broader perspective.

The ultimate goal of this research is to address the aforementioned open problems and fulfil the knowledge gap in the existing literature.

1.4. Comparative study and limitations of the current research

The results proved in this paper are valid for substructures in quantale modules. Moreover, every fuzzy set is an IFS, so the present study can also be applied to fuzzy substructures and intuitionistic fuzzy substructures in quantale modules by soft relations. Further, approximations of Pythagorean fuzzy sets by soft binary relations were presented by Bilal and Shabir [13]. So, we can define approximation of Pythagorean fuzzy substructures in quantale modules by soft relations. However, we cannot apply these results directly to q-rung orthopair fuzzy ideals, picture fuzzy ideals and fuzzy soft hyper substructures in quantale modules. Therefore, separate studies are recommended for these generalized structures. This is the main limitation of our research.

The detail of paper is as follows. In Section 2, some necessary definitions related to substructures of quantale module are presented. Further, rough set, soft sets, soft substructures in quantale module and soft binary relations are discussed. In Section 3, some characterization of subsets of quantale modules are described. Moreover, different rough soft substructures with respect to aftersets and foresets will be expressed in Section 4. In the last section, soft quantale module homomorphism with its relation to upper (lower) approximations and homomorphic images are described.

2. Preliminaries

In this section, we define soft substructures of quantale modules and present some basic notions of soft sets, rough sets and substructures of quantale modules, which are the main tools in our study.

Definition 2.1. [1] Let Ϙ be a C_{ltc}. Define an associative binary operation $\otimes$ on Ϙ satisfying,

1) $\mathfrak{r} \otimes (\vee_{l\in L} z_l) = \vee_{l\in L}(\mathfrak{r} \otimes z_l)$;
2) $(\vee_{l\in L}\mathfrak{r}_l) \otimes z = \vee_{l\in L}(\mathfrak{r}_l \otimes z)$.

$\forall\, \mathfrak{r}, z \in$ Ϙ and $\{z_l\}, \{\mathfrak{r}_l\} \subseteq$ Ϙ $(l \in L)$. Then (Ϙ,$\otimes$) is a quantale.

Let $\mathcal{R}_l, \mathcal{R}_1, \mathcal{R}_2 \subseteq$ Ϙ. Then the following are defined

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = \{\mathfrak{r}_1 \otimes \mathfrak{r}_2 : \mathfrak{r}_1 \in \mathcal{R}_1, \mathfrak{r}_2 \in \mathcal{R}_2\};$$

$$\mathcal{R}_1 \vee \mathcal{R}_2 = \{\mathfrak{r}_1 \vee \mathfrak{r}_2 : \mathfrak{r}_1 \in \mathcal{R}_1, \mathfrak{r}_2 \in \mathcal{R}_2\};$$

$$\vee_{l\in L}\mathcal{R}_l = \{\vee_{l\in L}\mathfrak{r}_l : \mathfrak{r}_l \in \mathcal{R}_l\}.$$

Definition 2.2. [20] Let Ϙ be a quantale and Q be a sup-lattice equipped with a left action $\star$: Ϙ $\times\, Q \longrightarrow Q$. Then Q is called left Ϙ-module over the quantale Ϙ, if it satisfies the following criteria,

1) $(\vee_{l\in L}\, a_l) \star x = \vee_{l\in L}\, (a_l \star x)$;
2) $a \star (\vee_{i\in I}\, x_i) = \vee_{i\in I}\, (a \star x_i)$;
3) $(a \otimes b) \star x = a \star (b \star x)$.

for any $a, b \in$ Ϙ, $\{a_l\} \subseteq$ Ϙ $(l \in L), x \in Q$, and $\{x_i\} \subseteq Q\ (i \in I)$ way. We write Q for left Ϙ-module over the quantale throughout in this thesis. For a Ϙ-module Q, $A \subseteq$ Ϙ $and\ m \in Q$ we have,

$$\mathrm{A} \star m = \{a \star m \mid a \in \mathrm{A}\};$$

$$\mathrm{A} \star \mathrm{B} = \{a \star b \mid a \in \mathrm{A}, b \in \mathrm{B}\}.$$

where $B \subseteq Q. For\ A, B, A_l \subseteq Q\ (l \in L)$, We write

$$A \vee B = \{a \vee b \mid a \in A, b \in B\};$$

$$\vee_{l\in L}\, A_l = \{\vee_{l\in L}\, a_l \mid a_l \in A_l\}.$$

Example 2.3. Let Ϙ $= \{\mathcal{L}, y, z, \text{Ŧ}\}$ be the C_{ltc} shown in Figure 1 and operation $\otimes$ on Ϙ is shown in Table 1. Then(Ϙ,$\otimes$) is a quantale. Let $Q = \{\mathcal{L}, x, \text{Ŧ}\}$ be a sup lattice. The order relation of Q is given in Figure 2. And the left action on $\mathcal{Q}$ i.e., $\star : Ϙ \times \mathcal{Q} \to \mathcal{Q}$ is shown in Table 2. Then it is easy to verify that $\mathcal{Q}$ is Ϙ-module.

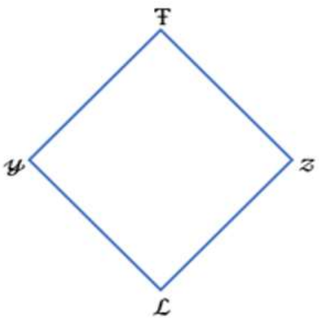

Figure 1. Description of Ϙ.

Figure 2. Description of $\mathcal{Q}$.

Table 1. Binary operation subject to $\otimes$.

$\otimes$	ℒ	𝓎	𝓏	Ŧ
ℒ	ℒ	ℒ	ℒ	ℒ
𝓎	ℒ	𝓎	ℒ	𝓎
𝓏	ℒ	ℒ	𝓏	𝓏
Ŧ	ℒ	𝓎	𝓏	Ŧ

Table 2. Left action subject to $\star$.

$\star$	ℒ	𝓍	Ŧ
ℒ	ℒ	ℒ	ℒ
𝓎	ℒ	ℒ	ℒ
𝓏	ℒ	𝓍	Ŧ
Ŧ	ℒ	𝓍	Ŧ

Example 2.4. Every quantale Ϙ is a certainly a Ϙ-module over Ϙ.

Definition 2.5. [20] Let $\mathcal{Q}$ be a Ϙ-module. If a subset $\mathcal{Q}_1 \subseteq \mathcal{Q}$ satisfies the following axioms for any $m \in \mathcal{Q}_1, \{m_i\} \subseteq \mathcal{Q}_1$ *and* $\gamma \in Ϙ$, we have

1) $\vee_{i\in I}\, m_i \in \mathcal{Q}_1\ \forall\, m_i \in \mathcal{Q}_1$;
2) $\gamma \star m \in \mathcal{Q}_1\ \forall\, m \in \mathcal{Q}_1, \gamma \in Ϙ$.

Then $\mathcal{Q}_1$ is called Ϙ-submodule ($Ϙ_S$) of $\mathcal{Q}$.

Definition 2.6. [20] Let $\mathcal{J} \neq \emptyset$ be a subset of Ϙ-module $\mathcal{Q}$. Then $\mathcal{J}$ is called Ϙ-sub module ideal ($Ϙ_I$) of $\mathcal{Q}$ if following holds;

1) $A \subseteq \mathcal{J}$ implies $\vee A \subseteq \mathcal{J}$;
2) $x \in \mathcal{J}$ and $b \leq x$ implies $b \in \mathcal{J}$;
3) $x \in \mathcal{J}$ implies $\gamma \star x \in \mathcal{J}, \forall\, \gamma \in Ϙ$.

Example 2.7. Consider the quantale module given in Example 2.3. Then $\{ℒ\}$, $\{ℒ, x\}$, $\{ℒ, x, Ŧ\}$ are $Ϙ_I$ of $\mathcal{Q}$.

Definition 2.8. [5] If Л is a mapping given by $Л : V \to P(\mathcal{Q})$ where $V \subseteq$ E (S.P), then the pair $(Л, V)$ is called a soft set over $\mathcal{Q}$.

Definition 2.9. [8] Assume $(\mathcal{M}, V_1)$ and $(\mathcal{N}, V_2)$ be two soft sets over Q. Then we called $(\mathcal{M}, V_1)$ soft subset $(\mathcal{N}, V_2)$ if the conditions listed below are satisfied,

- $V_1 \subseteq V_2$;
- $\mathcal{M}(v) \subseteq \mathcal{N}(v)\ \forall\ v \in V_1$.

We will represent soft subsets defined in above manner by $(\mathcal{M}, V_1) \subseteq (\mathcal{N}, V_2)$.

Definition 2.10. [8] Let $(Л, V)$ be a soft set over $Q \times Q$, i.e, $Л : V \to P(Q \times Q)$. Then $(Л, V)$ is called a soft binary relation (SBIR) over Q. A SBIR from $Q_1$to Q_2 is a soft set $(Л, V)$ from $Q_1$to Q_2. That is $Л : V \to P(Q_1 \times Q_2)$.

Definition 2.11. Let $(Л, V)$ be a soft set over quantale module Q. Then the soft substructures of quantale modules are defined as,

1) $(Л, V)$ is called soft quantale submodule (Q_S) over Q iff $Л(v)$ is a Q_S of Q, $\forall\ v \in V$.
2) $(Л, V)$ is called soft quantale submodule ideal (Q_I) over Q iff $Л(v)$ is a Q_I of Q,$\forall\ v \in V$.

Definition 2.12. [27] Let $\mathbb{Q}$ be an equivalence relation on a non-empty finite set Q. Then $(Q, \mathbb{Q})$ is called an approximation space. Let C be a subset of Q. Then C may or may not be written as union of the equivalence classes of Q. We say that C is definable, if C can be written as union of some equivalence classes of Q. Otherwise, it is called not definable. In case, if C is not definable, then C can be approximated by two definable subsets called the lower and upper approximations of C. These approximations are defined as follows,

$$\underline{\mu}(C) = \{q \in Q : [q]_\mu \subseteq C\} \text{ and } \overline{\mu}(C) = \{q \in Q : [q]_\mu \cap C \neq \emptyset\}.$$

A rough set is a pair $(\underline{\mu}(C), \overline{\mu}(C))$ if $\underline{\mu}(C) \neq \overline{\mu}(C)$.

3. Approximation of soft subsets of quantale module by soft relation

In this section, we approximate the subsets of quantale module by using soft relations.

Definition 3.1. [14] Assume V is the subset of E (S.P) and $(Л, V)$ be a SBIR from Q_1 *to* Q_2 i.e., $Л: V \to P(Q_1 \times Q_2)$. Thus, the LO_{ap} $(\underline{Л}^{\mathcal{M}}, V)$ and UP_{ap} $(\overline{Л}^{\mathcal{M}}, V)$ w.r.t the afterset of soft set $(\mathcal{M}, V)$ over Q_2 are essentially two soft sets over Q_1 defined as

$$\underline{Л}^{\mathcal{M}}(v) = \{\gamma_1 \in Q_1 : \emptyset \neq \gamma_1 Л(v) \subseteq \mathcal{M}(v)\}$$

and

$$\overline{Л}^{\mathcal{M}}(v) = \{\gamma_1 \in Q_1 : \gamma_1 Л(v) \cap \mathcal{M}(v) \neq \emptyset\}.$$

$$\forall\ v \in V.$$

The LO_{ap} $(^{\mathcal{N}}\underline{Л}, V)$ and UP_{ap} $(^{\mathcal{N}}\overline{Л}, V)$ of w.r.t the foreset of a soft set $(\mathcal{N}, V)$ over Q_1 are two soft sets over Q_2 defined as,

$$^{\mathcal{N}}\underline{Л}(v) = \{\gamma_2 \in Q_2 : \emptyset \neq Л(v)\gamma_2 \subseteq \mathcal{N}(v)\}.$$

and

$$^{\mathcal{N}}\overline{\text{Л}}(v) = \{\gamma_2 \in Q_2 : \text{Л}(v)\gamma_2 \cap \mathcal{N}(v) \neq \emptyset\} \; \forall\, v \in V,$$

where $\gamma_1\text{Л}(v) = \{\gamma_2 \in Q_2 : (\gamma_1, \gamma_2) \in \text{Л}(v)\}$ is called the afterset of γ_1 and $\text{Л}(v)\gamma_2 = \{\gamma_1 \in Q_1 : (\gamma_1, \gamma_2) \in \text{Л}(v)\}$ is called the foreset of γ_2.

Remark 3.2. (1) For each soft set $(\mathcal{M}, V)$ over Q_2, $\underline{\text{Л}}^{\mathcal{M}} : V \to P(Q_1)$ and $\overline{\text{Л}}^{\mathcal{M}} : V \to P(Q_1)$.

(2) For each soft set $(\mathcal{N}, V)$ over Q_1, $^{\mathcal{N}}\underline{\text{Л}} : V \to P(Q_2)$ and $^{\mathcal{N}}\overline{\text{Л}} : V \to P(Q_2)$.

Theorem 3.3. [14] Let $(\text{Л}, V)$ and $(\text{Ꝙ}, V)$ be two SBIR from a non-empty set Q_1 to a non-empty set Q_2 and consider $(\mathcal{M}_1, V)$ and $(\mathcal{M}_2, V)$ be two soft set over Q_2. Then

(1) $(\mathcal{M}_1, V) \subseteq (\mathcal{M}_2, V) \Rightarrow (\underline{\text{Л}}^{\mathcal{M}_1}, V) \subseteq (\underline{\text{Л}}^{\mathcal{M}_2}, V)$,

(2) $(\mathcal{M}_1, V) \subseteq (\mathcal{M}_2, V) \Rightarrow \left(\overline{\text{Л}}^{\mathcal{M}_1}, V\right) \subseteq \left(\overline{\text{Л}}^{\mathcal{M}_2}, V\right)$,

(3) $\left(\underline{\text{Л}}^{\mathcal{M}_1}, V\right) \cap \left(\underline{\text{Л}}^{\mathcal{M}_2}, V\right) = \left(\underline{\text{Л}}^{\mathcal{M}_1 \cap \mathcal{M}_2}, V\right)$,

(4) $\left(\overline{\text{Л}}^{\mathcal{M}_1}, V\right) \cap \left(\overline{\text{Л}}^{\mathcal{M}_2}, V\right) \supseteq \left(\overline{\text{Л}}^{\mathcal{M}_1 \cap \mathcal{M}_2}, V\right)$,

(5) $\left(\underline{\text{Л}}^{\mathcal{M}_1}, V\right) \cup \left(\underline{\text{Л}}^{\mathcal{M}_2}, V\right) \subseteq \left(\underline{\text{Л}}^{\mathcal{M}_1 \cup \mathcal{M}_2}, V\right)$,

(6) $\left(\overline{\text{Л}}^{\mathcal{M}_1}, V\right) \cup \left(\overline{\text{Л}}^{\mathcal{M}_2}, V\right) = \left(\overline{\text{Л}}^{\mathcal{M}_1 \cup \mathcal{M}_2}, V\right)$,

(7) $(\text{Л}, V) \subseteq (\text{Ꝙ}, V) \Rightarrow \left(\underline{\text{Ꝙ}}^{\mathcal{M}_1}, V\right) \subseteq \left(\underline{\text{Л}}^{\mathcal{M}_1}, V\right)$,

(8) $(\text{Л}, V) \subseteq (\text{Ꝙ}, V) \Rightarrow \left(\overline{\text{Ꝙ}}^{\mathcal{M}_1}, V\right) \supseteq \left(\overline{\text{Л}}^{\mathcal{M}_1}, V\right)$.

Theorem 3.4. Let $(\text{Л}, V)$ and $(\text{Ꝙ}, V)$ are two SBIR from $Q_1 \neq \emptyset$ to $Q_2 \neq \emptyset$. Then for any soft set $(\mathcal{M}, V)$ over Q_2, we have

(1) $\left(\overline{\text{Л} \cap \text{Ꝙ}}^{\mathcal{M}}, V\right) \subseteq \left(\overline{\text{Л}}^{\mathcal{M}}, V\right) \cap \left(\overline{\text{Ꝙ}}^{\mathcal{M}}, V\right)$.

(2) $\left(\underline{\text{Л} \cap \text{Ꝙ}}^{\mathcal{M}}, V\right) \supseteq \left(\underline{\text{Л}}^{\mathcal{M}}, V\right) \cup \left(\underline{\text{Ꝙ}}^{\mathcal{M}}, V\right)$.

Proof. The proof is obvious and will be immediately concluded from part (7) and (8) of Theorem 3.3.

In general converse of above Theorem is not true we will present an example to justify this as follows.

Example 3.5. Assume $Q_1 = \{\mathcal{L}, \mathfrak{r}, \text{Ŧ}\}$ and $Q_2 = \{\mathcal{L}', w, x, y, z, \text{Ŧ}'\}$ be two C_{ltc} as shown in Figures 3 and 4 respectively. The associative binary operation $\otimes_1$ and $\otimes_2$ on Q_1 and Q_2 is defined as,

(1) $a \otimes_1 b = a \wedge b$

(2) $a \otimes_2 b = \mathcal{L}'$

Then Q_1 and Q_2 are quantales by (1) and (2), and Q_1 and Q_2 are quantale modules by Tables 3 and 4.

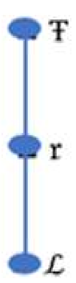

Figure 3. Description of Q_1.

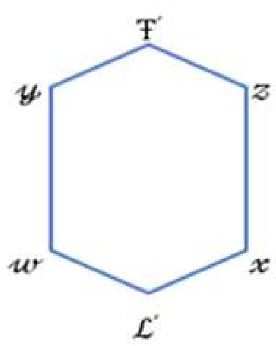

Figure 4. Description of Q_2.

Table 3. Left action subject to $\star_1$.

$\star_1$	$\mathcal{L}$	$\mathfrak{r}$	$Ŧ$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathfrak{r}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$Ŧ$	$\mathcal{L}$	$\mathfrak{r}$	$Ŧ$

Table 4. Left action subject to $\star_2$.

$\star_2$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$\mathcal{L}'$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$\mathcal{w}$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$\mathcal{x}$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$\mathcal{y}$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$\mathcal{z}$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$
$Ŧ'$	$\mathcal{L}'$	$\mathcal{w}$	$\mathcal{x}$	$\mathcal{y}$	$\mathcal{z}$	$Ŧ'$

Consider $V = \{v_1, v_2\}$ and define $Л: V \longrightarrow P(Q_1 \times Q_2)$ and $Ꝏ : V \longrightarrow P(Q_1 \times Q_2)$ by,

$$Л(v_1) = \left\{ \begin{matrix} (\mathcal{L}, \mathcal{L}'), (\mathfrak{r}, \mathcal{w}), (Ŧ, \mathcal{x}), (Ŧ, Ŧ'), (\mathcal{L}, Ŧ'), (\mathfrak{r}, \mathcal{y}) \\ (Ŧ, \mathcal{z}), (\mathfrak{r}, \mathcal{x}) \end{matrix} \right\},$$

$$Л(v_2) = \{(\mathcal{L}, \mathcal{L}'), (\mathfrak{r}, \mathcal{x}), (\mathcal{L}, \mathcal{w}), (Ŧ, \mathcal{z}), (Ŧ, Ŧ')\}$$

and

$$Ꝏ(v_1) = \{(\mathcal{L}, \mathcal{L}'), (Ŧ, \mathcal{x}), (Ŧ, \mathcal{z}), (\mathfrak{r}, \mathcal{L}'), (\mathcal{L}, Ŧ')\},$$

$$Ꝏ(v_2) = \{(\mathcal{L}, \mathcal{L}'), (Ŧ, Ŧ'), (\mathcal{L}, \mathcal{w}), (\mathfrak{r}, \mathcal{z})\}.$$

$$(Л \cap Ꝏ)(v_1) = \{(\mathcal{L}, \mathcal{L}'), (Ŧ, \mathcal{x}), (\mathcal{L}, Ŧ'), (Ŧ, \mathcal{z})\}$$

and

$$(Л \cap Ꝏ)(v_2) = \{(\mathcal{L}, \mathcal{L}'), (\mathcal{L}, \mathcal{w}), (Ŧ, Ŧ')\}.$$

Following are the aftersets corresponding to $Л(v_1)$ and $Ꝏ(v_1)$,
$\mathcal{L}\, Л(v_1) = \{\mathcal{L}', Ŧ'\}, \mathfrak{r}\, Л(v_1) = \{\mathcal{x}, \mathcal{w}, \mathcal{y}\}$ and $Ŧ\, Л(v_1) = \{\mathcal{x}, \mathcal{z}, Ŧ'\}$,
$\mathcal{L}\, Ꝏ(v_1) = \{\mathcal{L}', Ŧ'\}, \mathfrak{r}\, Ꝏ(v_1) = \{\mathcal{L}'\}$ and $Ŧ\, Ꝏ(v_1) = \{\mathcal{x}, \mathcal{z}\}$.
Also, $\mathcal{L}\,(Л \cap Ꝏ)(v_1) = \{\mathcal{L}', Ŧ'\}, \mathfrak{r}\,(Л \cap Ꝏ)(v_1) = \emptyset$ and $Ŧ\,(Л \cap Ꝏ)(v_1) = \{\mathcal{x}, \mathcal{z}\}$. Now, we define soft set $(\mathcal{M}_1, V)$ over Q_2 by, $\mathcal{M}_1(v_1) = \{\mathcal{L}', \mathcal{x}\}$ and $\mathcal{M}_1(v_2) = \{\mathcal{L}', \mathcal{x}, \mathcal{z}, Ŧ'\}$. Thus, $\overline{Л}^{\mathcal{M}_1}(v_1) = \{\mathcal{L}, \mathfrak{r}, Ŧ\}$, $\overline{Ꝏ}^{\mathcal{M}_1}(v_1) = \{\mathcal{L}, \mathfrak{r}, Ŧ\}$ and $\overline{(Л \cap Ꝏ)}^{\mathcal{M}_1}(v_1) = \{\mathcal{L}, Ŧ\}$ $\Rightarrow \overline{Л}^{\mathcal{M}_1}(v_1) \cap \overline{Ꝏ}^{\mathcal{M}_1}(v_1) = \{\mathcal{L}, \mathfrak{r}, Ŧ\}$. This shows that $\overline{Л}^{\mathcal{M}_1}(v_1) \cap \overline{Ꝏ}^{\mathcal{M}_1}(v_1) \nsubseteq \overline{(Л \cap Ꝏ)}^{\mathcal{M}_1}(v_1)$.
Now, consider $V = (v_1, v_2)$ and define $Л : V \longrightarrow P(Q_1 \times Q_2)$ and $Ꝏ : V \longrightarrow P(Q_1 \times Q_2)$ by,

$$Л(v_1) = \left\{ \begin{matrix} (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, \mathcal{w}), (Ŧ, \mathcal{L}'), (\mathfrak{r}, \mathcal{y}) \\ (Ŧ, \mathcal{z}), (\mathfrak{r}, Ŧ') \end{matrix} \right\},$$

$$Л(v_2) = \{(\mathcal{L}, \mathcal{L}'), (\mathfrak{r}, \mathcal{x}), (\mathcal{L}, \mathcal{w}), (Ŧ, \mathcal{z}), (Ŧ, Ŧ')\}$$

and

$$Ꞷ(v_1) = \{(\mathcal{L},\mathcal{L}'),(\mathcal{L},x),(Ŧ,Ŧ'),(Ŧ,\mathcal{L}'),(ɍ,Ŧ'),(ɍ,z)\},$$

$$Ꞷ(v_2) = \{(\mathcal{L},\mathcal{L}'),(Ŧ,Ŧ'),(\mathcal{L},\mathscr{w}),(ɍ,z)\}.$$

So, $(Л \cap Ꞷ)(v_1) = \{(\mathcal{L},\mathcal{L}'),(Ŧ,\mathcal{L}'),(ɍ,Ŧ')\}$ and $(Л \cap Ꞷ)(v_2) = \{(\mathcal{L},\mathcal{L}'),(\mathcal{L},\mathscr{w}),(Ŧ,Ŧ')\}$. Following are the aftersets corresponding to $Л(v_1)$ and $Ꞷ(v_1)$,

$$\mathcal{L}\,Л(v_1) = \{\mathcal{L}',\mathscr{w}\}, ɍ\,Л(v_1) = \{Ŧ',\mathscr{y}\} \text{ and } Ŧ\,Л(v_1) = \{\mathcal{L}',z\}$$

$$\mathcal{L}\,Ꞷ(v_1) = \{\mathcal{L}',x\}, ɍ\,Ꞷ(v_1) = \{Ŧ',z\} \text{ and } Ŧ\,Ꞷ(v_1) = \{\mathcal{L}',Ŧ'\}.$$

Also, $\mathcal{L}(Л \cap Ꞷ)(v_1) = \{\mathcal{L}'\}, ɍ(Л \cap Ꞷ)(v_1) = \{Ŧ'\}$ and $Ŧ(Л \cap Ꞷ)(v_1) = \{\mathcal{L}'\}$.
Now, we define soft set $(\mathcal{M}_2,V)$ over Q_2 by, $\mathcal{M}_2(v_1) = \{\mathcal{L}',Ŧ'\}$ and $\mathcal{M}_2(v_2) = \{\mathcal{L}'\}$. Thus, $\underline{Л}^{\mathcal{M}_2}(v_1) = \emptyset, \underline{Ꞷ}^{\mathcal{M}_2}(v_1) = \{Ŧ\}$ and $\underline{(Л \cap Ꞷ)}^{\mathcal{M}_2}(v_1) = \{\mathcal{L},ɍ,Ŧ\}$. Thereby, $\underline{Л}^{\mathcal{M}_2}(v_1) \cup \underline{Ꞷ}^{\mathcal{M}_2}(v_1) = \{Ŧ\}$. This shows that, $\underline{Л}^{\mathcal{M}_2}(v_1) \cup \underline{Ꞷ}^{\mathcal{M}_2}(v_1) \not\supseteq \underline{(Л \cap Ꞷ)}^{\mathcal{M}_2}(v_1)$.

Definition 3.6. For SBIR $(Л,V)$ from Q_1 to Q_2 i.e., $Л : V \longrightarrow P(Q_1 \times Q_2)$ the soft compatible relation is defined as, for all $\in Q_1, w \in Q_2, f_l \subseteq Q_1$ and $g_l \subseteq Q_2$ for $(l \in L)$ we have

(1) $(f_l, g_l) \in Л(v) \Rightarrow (\vee_{l\in L} f_l, \vee_{l\in L} g_l) \in Л(v)$,

(2) $(u,w) \in Л(v) \Rightarrow (\gamma_1 \star_1 u, \gamma_2 \star_2 w) \in Л(v)\ \forall \gamma_1 \in \mathrm{Q}_1, \gamma_2 \in \mathrm{Q}_2$.

Example 3.7. Let $\mathrm{Q}_1 = \{\mathcal{L},\mathscr{j},\mathscr{k},Ŧ\}$ and $\mathrm{Q}_2 = \{\mathcal{L}',x',y',z',Ŧ'\}$ be two C_{lts} described in Figures 5 and 6 respectively. The associative binary operation $\otimes_1$ and $\otimes_2$ on Q_1 and Q_2 is defined as,

(1) $a \otimes_1 b = a \wedge b$

(2) $a \otimes_2 b = \mathcal{L}'$

We define $\star_1$ and $\star_2$ the left action on Q_1 and Q_2, respectively as shown in Tables 5 and 6. Then, Q_1 and Q_2 are quantale modules. Let $V = \{v_1,v_2\}$ and the SBIR $(Л,V)$ from Q_1 to Q_2 be defined by,

$$Л(v_1) = \left\{\begin{matrix}(\mathcal{L},\mathcal{L}'),(\mathscr{j},x'),(\mathcal{L},x'),(\mathscr{k},y'),(\mathcal{L},y'),(\mathcal{L},Ŧ'),\\ (\mathscr{j},Ŧ'),(\mathscr{k},Ŧ'),(Ŧ,Ŧ')\end{matrix}\right\},$$

$$Л(v_2) = \left\{\begin{matrix}(\mathscr{j},x'),(\mathscr{j},y'),(\mathscr{j},z'),(\mathscr{j},Ŧ'),(\mathscr{k},\mathcal{L}'),(\mathscr{k},x'),(\mathcal{L},\mathcal{L}'),(Ŧ,y'),(\mathcal{L},y'),(Ŧ,Ŧ')\\ (\mathcal{L},x'),(Ŧ,z'),(Ŧ,x'),(\mathscr{k},z'),(Ŧ,\mathcal{L}'),(\mathscr{k},y'),(\mathcal{L},Ŧ'),(\mathscr{k},Ŧ'),(\mathscr{j},\mathcal{L}'),(\mathcal{L},z')\end{matrix}\right\}.$$

Then $(Л,V)$ is an SCMR.

❖ Throughout in this paper, we consider that $(Л,V)$ be a SBIR from Q_1 *to* Q_2.

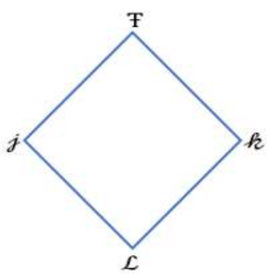

Figure 5. Description of Q_1.

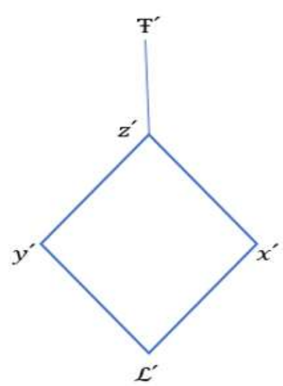

Figure 6. Description of Q_2.

Table 5. Left action subject to $\star_1$.

$\star_1$	$\mathcal{L}$	j	k	Ŧ
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
j	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
k	$\mathcal{L}$	j	k	Ŧ
Ŧ	$\mathcal{L}$	j	k	Ŧ

Table 6. Left action subject to $\star_2$.

$\star_2$	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$
$\mathcal{L}'$	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$
x'	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$
y'	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$
z'	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$
Ŧ$'$	$\mathcal{L}'$	x'	y'	z'	Ŧ$'$

Lemma 3.8. If $(Л, V)$ is a SCMR form quantale module $\mathcal{Q}_1$ to $\mathcal{Q}_2$, then for $\gamma_1 \in \mathrm{Q}_1$ and $g, h \in \mathcal{Q}_1$ we have

(1) $\gamma_1 Л(v) \star_2 gЛ(v) \subseteq (\gamma_1 \star_1 g)Л(v)$.

(2) $gЛ(v) \vee hЛ(v) \subseteq (g \vee h)Л(v)$.

Proof. (1) Let $f \in \gamma_1 Л(v) \star_2 gЛ(v)$. Then for some $m \in \gamma_1 Л(v)$ and $n \in gЛ(v)$, we have $f = m \star_2 n$. Thereby, $(\gamma_1, m) \in Л(v)$ and $(g, n) \in Л(v)$. By SCMR , we have $(\gamma_1 \star_1 g, m \star_2 n) \in Л(v)$. Thus, $(m \star_2 n) \in (\gamma_1 \star_1 g)\, Л(v)$. Consequently, $\gamma_1\, Л(v) \star_2 g\, Л(v) \subseteq (\gamma_1 \star_1 g)Л(v)$.

(2) Let $f \in gЛ(v) \vee hЛ(v)$. Then for some $m \in gЛ(v)$ and $n \in hЛ(v)$, we have $f = m \vee n$. Thereby, $(g, m) \in Л(v)$ and $(h, n) \in Л(v)$. By SCMR, we have $(g \vee h, m \vee n) \in Л(v)$. Thus, $(m \vee n) \in (g \vee h)\, Л(v)$. Consequently, $g\, Л(v) \vee h\, Л(v) \subseteq (g \vee h)\, Л(v)$.

Lemma 3.9. If $(Л, V)$ is a SCMR from a quantale module $\mathcal{Q}_1$ to $\mathcal{Q}_2$, then for $\gamma_2 \in \mathrm{Q}_2$ and $x, y \in \mathcal{Q}_2$ we have

(1) $Л(v)\gamma_2 \star_1 Л(v)y \subseteq Л(v)(\gamma_2 \star_2 y)$.

(2) $Л(v)x \vee Л(v)y \subseteq Л(v)(x \vee y)$.

Proof. (1) Assume $p \in Л(v)\gamma_2 \star_1 Л(v)y$. Then for some $u \in Л(v)\gamma_2$ and $w \in Л(v)y$, we have $p = u \star_1 w$. Thereby, $(u, \gamma_2) \in Л(v)$ and $(w, y) \in Л(v)$. By SCMR, we have $(u \star_1 w, \gamma_2 \star_2 y) \in Л(v)$. Thus, $(u \star_1 w) \in Л(v)(\gamma_2 \star_2 y)$. Consequently, $Л(v)\gamma_2 \star_1 Л(v)y \subseteq Л(v)(\gamma_2 \star_2 y)$

(2) Assume $p \in Л(v)x \vee Л(v)y$. Then for some $u \in Л(v)x$ and $w \in Л(v)y$, we have $p = u \vee w$. Thereby, $(u, x) \in Л(v)$ and $(w, y) \in Л(v)$. By SCMR, we have $(u \vee w, x \vee y) \in Л(v)$. Thus, $(u \vee w) \in Л(v)(x \vee y)$. Consequently, $Л(v)x \vee Л(v)y \subseteq Л(v)(x \vee y)$.

Definition 3.10. A SCMR $(Л, V)$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ w.r.t aftersets is called soft complete relation (SCMPR) if $\forall\, u, w \in \mathcal{Q}_1, \gamma_1 \in \mathrm{Q}_1$ we have

(1) $uЛ(v) \vee wЛ(v) = (u \vee w)Л(v)$.

(2) $\gamma_1 Л(v) \star_2 uЛ(v) = (\gamma_1 \star_1 u)Л(v)\ \forall\, v \in V$.

If a SCMR $(Л, V)$ w.r.t the aftersets satisfies condition (1) only, then it is called $\vee$-complete. If a SCMR $(Л, V)$ w.r.t the aftersets satisfies condition (2) only, then it is called $\star$-complete.

A SCMR $(\mathbb{Q}, V)$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ w.r.t foresets is called soft complete relation (SCMPR) if $\forall$

$\mathscr{s}, \mathscr{t} \in \mathcal{Q}_2, \gamma_2 \in \mathcal{Q}_2$ we have,

(1) $\mathbb{Q}(v)\mathscr{s} \vee \mathbb{Q}(v)\mathscr{t} = \mathbb{Q}(v)(\mathscr{s} \vee \mathscr{t})$.

(2) $Л(v)\gamma_2 \star_1 Л(v)\mathscr{t} = Л(v)(\gamma_2 \star_2 \mathscr{t}) \ \forall\, v \in V$.

If a SCMR $(Л, V)$ w.r.t the foresets satisfies condition (1) only, then it is called $\vee$-complete. If a SCMR $(Л, V)$ w.r.t the foresets satisfies condition (2) only, then it is called $\star$-complete.

Example 3.11. Consider $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be quantale modules described in Example 3.5. Let $V = (v_1, v_2)$ and the SBIR $(Л, V)$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ be defined by,

$$Л(v_1) = \begin{Bmatrix} (\mathfrak{r}, z), (\mathfrak{r}, \mathcal{L}'), (\mathcal{L}, z), (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, x) \\ (Ŧ, \mathcal{L}'), (Ŧ, x), (Ŧ, z), (\mathfrak{r}, x) \end{Bmatrix},$$

$$Л(v_2) = \begin{Bmatrix} (\mathfrak{r}, w), (\mathfrak{r}, \mathcal{L}'), (\mathcal{L}, w), (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, y), (Ŧ, Ŧ') \\ (Ŧ, \mathcal{L}'), (Ŧ, w), (Ŧ, y), (\mathfrak{r}, y), (\mathcal{L}, Ŧ'), (\mathfrak{r}, Ŧ') \end{Bmatrix}.$$

Then $(Л, V)$ is SCMR. Following are the aftersets corresponding to $Л(v_1)$ and $Л(v_2)$.

$\mathcal{L}Л(v_1) = \{\mathcal{L}', x, z\}$, $\mathfrak{r}\,Л(v_1) = \{\mathcal{L}', x, z\}$, $ŦЛ(v_1) = \{\mathcal{L}', x, z\}$.

$\mathcal{L}Л(v_2) = \{\mathcal{L}', y, w, Ŧ'\}$ $\mathfrak{r}\,Л(v_2) = \{\mathcal{L}', y, w, Ŧ'\}$, $ŦЛ(v_2) = \{\mathcal{L}', y, w, Ŧ'\}$.

Then it is very easy to verify that $(Л, V)$ w.r.t the aftersets is SCMPR from $\mathcal{Q}_1$ to $\mathcal{Q}_2$.

Now, consider $V = (v_1, v_2)$ and the SBIR $(Л, V)$ from $\mathcal{Q}_1$to $\mathcal{Q}_2$ is defined by,

$$Л(v_1) = \begin{Bmatrix} (\mathfrak{r}, z), (\mathfrak{r}, \mathcal{L}'), (\mathcal{L}, z), (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, x), (\mathfrak{r}, Ŧ') \\ (\mathcal{L}, Ŧ'), (\mathfrak{r}, x), (\mathcal{L}, y), (\mathfrak{r}, y), (\mathcal{L}, w), (\mathfrak{r}, w) \end{Bmatrix},$$

$$Л(v_2) = \begin{Bmatrix} (\mathfrak{r}, w), (\mathfrak{r}, \mathcal{L}'), (\mathfrak{r}, x), (\mathfrak{r}, y), (\mathfrak{r}, z), (Ŧ, Ŧ') \\ (Ŧ, \mathcal{L}'), (Ŧ, w), (Ŧ, y), (Ŧ, x), (Ŧ, z), (\mathfrak{r}, Ŧ') \end{Bmatrix}.$$

Then $(Л, V)$ is SCMR. Following are foresets corresponding to $Л(v_1)$ and $Л(v_2)$,

$Л(v_1)\mathcal{L}' = \{\mathcal{L}, \mathfrak{r}\}$, $Л(v_1)x = \{\mathcal{L}, \mathfrak{r}\}$, $Л(v_1)y = \{\mathcal{L}, \mathfrak{r}\}$, $Л(v_1)z = \{\mathcal{L}, \mathfrak{r}\}$, $Л(v_1)w = \{\mathcal{L}, \mathfrak{r}\}$, $Л(v_1)Ŧ' = \{\mathcal{L}, \mathfrak{r}\}$.

$Л(v_2)\mathcal{L}' = \{\mathfrak{r}, Ŧ\}$, $Л(v_2)x = \{\mathfrak{r}, Ŧ\}$, $Л(v_2)y = \{\mathfrak{r}, Ŧ\}$, $Л(v_2)z = \{\mathfrak{r}, Ŧ\}$, $Л(v_2)w = \{\mathfrak{r}, Ŧ\}$, $Л(v_2)Ŧ' = \{\mathfrak{r}, Ŧ\}$.

Then it is very easy to verify that $(Л, V)$ w.r.t the foresets is SCMPR from $\mathcal{Q}_1$to $\mathcal{Q}_2$.

Remark 3.12. Generally, neither SCMPR w.r.t aftersets implies SCMPR w.r.t foresets nor SCMPR w.r.t foresets implies SCMPR w.r.t aftersets.

Theorem 3.13. Let $(Л, V)$ be a SCMR from $\mathcal{Q}_1$ to $\mathcal{Q}_2$. Then for any soft set $(\mathcal{M}_2, V)$ over $\mathcal{Q}_2$ and $(\mathbb{Q}_2, V)$ over $\mathcal{Q}_2$, we have

(1) $\left(\overline{Л}^{\mathbb{Q}_2}, V\right) \star_1 \left(\overline{Л}^{\mathcal{M}_2}, V\right) \subseteq \left(\overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}, V\right)$.

(2) $\bigvee_{l \in L} \left(\overline{Л}^{\mathcal{M}_l}, V\right) \subseteq \left(\overline{Л}^{\vee_{l \in L} \mathcal{M}_l}, V\right)$.

Proof. (1) For arbitrary $v \in V$, let $f \in \overline{Л}^{\mathbb{Q}_2}(v) \star_1 \overline{Л}^{\mathcal{M}_2}(v)$. Then, for some $e \in \overline{Л}^{\mathbb{Q}_2}(v)$ and $g \in \overline{Л}^{\mathcal{M}_2}(v)$ we have $f = e \star_1 g$. Thereby, $eЛ(v) \cap \mathbb{Q}_2(v) \neq \emptyset$ and $gЛ(v) \cap \mathcal{M}_2(v) \neq \emptyset$. So, for $j \in \mathcal{Q}_2$ and $k \in \mathcal{Q}_2$, we have $j \in eЛ(v) \cap \mathbb{Q}_2(v)$ and $k \in gЛ(v) \cap \mathcal{M}_2(v) \Rightarrow j \in eЛ(v)$, $k \in gЛ(v)$, $j \in \mathbb{Q}_2(v)$ and $k \in \mathcal{M}_2(v)$. Thereby, $(e, j) \in Л(v)$, $(g, k) \in Л(v)$. Since $Л(v)$ is SCMR thus, $(e \star_1 g, j \star_2 k) \in Л(v)$ i.e., $(j \star_2 k) \in (e \star_1 g)Л(v)$ and $(j \star_2 k) \in \mathbb{Q}_2(v) \star_2 \mathcal{M}_2(v)$. Thus, $(j \star_2 k) \in (e \star_1 g)Л(v) \cap \mathbb{Q}_2(v) \star_2 \mathcal{M}_2(v) \Rightarrow (e \star_1 g)Л(v) \cap \mathbb{Q}_2(v) \star_2 \mathcal{M}_2(v) \neq \emptyset$. Hence, $f = (e \star_1 g) \in \overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v)$.

(2) Now for arbitrary $v \in V$, assume $f \in \vee_{l\in L} \overline{Л}^{\mathcal{M}_l}(v)$. Then $f = \vee_{l\in L} g_l$ for some $g_l \in \overline{Л}^{\mathcal{M}_l}(v)$. Thereby, $g_l\, Л(v) \cap \mathcal{M}_l(v) \neq \emptyset$. So there exists $e_l \in g_l\, Л(v)$ i.e., $(g_l, e_l) \in Л(v)$ *and* $e_l \in \mathcal{M}_l(v)$. Since $Л(v)$ is SCMR thus, $(\vee_{l\in L} g_l, \vee_{l\in L} e_l) \in Л(v) \Rightarrow \vee_{l\in L} e_l \in \vee_{l\in L} g_l Л(v)$ and $\vee_{l\in L} e_l \in \vee_{l\in L} \mathcal{M}_l(v)$. Thus, $\vee_{l\in L} e_l \in \vee_{l\in L} g_l\, Л(v) \cap \vee_{l\in L} \mathcal{M}_l(v)$. So, $\vee_{l\in L} g_l\, Л(v) \cap \vee_{l\in L} \mathcal{M}_l(v) \neq \emptyset$. Hence, $f = \vee_{l\in L} g_l \in \overline{Л}^{\vee_{l\in L}\mathcal{M}_l}(v)$.

Example 3.14. Consider Q_1 and Q_2 be two quantale modules described in Example 3.7. Let $V = \{v_1, v_2\}$ and the SBIR $(Л, V)$ from Q_1 to Q_2 be defined by,

$$Л(v_1) = \begin{Bmatrix} (\mathcal{L}, \mathcal{L}'), (j, y'), (\mathcal{L}, z'), (k, y'), (\mathcal{L}, y'), (Ŧ, \mathcal{L}'), \\ (j, z'), (k, Ŧ'), (Ŧ, Ŧ'), (j, \mathcal{L}'), (\mathcal{L}, Ŧ') \end{Bmatrix},$$

$$Л(v_2) = \begin{Bmatrix} (j, x'), (\mathcal{L}, y'), (k, y'), (\mathcal{L}, z'), (k, z'), \\ (\mathcal{L}, \mathcal{L}'), (Ŧ, z'), (\mathcal{L}, x'), (j, z') \end{Bmatrix}.$$

Then $(Л, V)$ is SCMR. Following are the aftersets corresponding to $Л(v_1)$ and $Л(v_2)$.

$\mathcal{L}Л(v_1) = \{\mathcal{L}', y', z', Ŧ'\}, jЛ(v_1) = \{\mathcal{L}', y', z'\}, kЛ(v_1) = \{y', Ŧ'\}, ŦЛ(v_1) = \{\mathcal{L}', Ŧ'\}$.

$\mathcal{L}Л(v_2) = \{\mathcal{L}', x', y', z'\}, jЛ(v_2) = \{x', z'\}, kЛ(v_2) = \{y', z'\}, ŦЛ(v_2) = \{z'\}$.

Now let $V = \{v_1, v_2\}$ and define soft set $(\mathbb{Q}_2, V)$ over Q_2 and $(\mathcal{M}_2, V)$ over Q_2.

$\mathbb{Q}_2(v_1) = \{z'\}, \mathbb{Q}_2(v_2) = \{x'\}$ and $\mathcal{M}_2(v_1) = \{x', y'\}$, $\mathcal{M}_2(v_2) = \{y'\}$. Then $\overline{Л}^{\mathbb{Q}_2}(v_1) = \{\mathcal{L}, j\}$ and $\overline{Л}^{\mathcal{M}_2}(v_1) = \{\mathcal{L}, j, k\}$. So, $\overline{Л}^{\mathbb{Q}_2}(v_1) \star_1 \overline{Л}^{\mathcal{M}_2}(v_1) = \{\mathcal{L}, j\} \star_1 \{\mathcal{L}, j, k\} = \{\mathcal{L}\}$ and $\mathbb{Q}_2(v_1) \star_2 \mathcal{M}_2(v_1) = \{z'\} \star_2 \{x', y'\} = \{x', y'\}$. UPap of $\mathbb{Q}_2(v_1) \star_2 \mathcal{M}_2(v_1)$ is $\overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v_1) = \{\mathcal{L}, j, k\}$. Therefore $\overline{Л}^{\mathbb{Q}_2}(v_1) \star_1 \overline{Л}^{\mathcal{M}_2}(v_1) = \{\mathcal{L}, j\} \star_1 \{\mathcal{L}, j, k\} = \{\mathcal{L}\} \subseteq \{\mathcal{L}, j, k\} = \overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v_1)$.

Also $\overline{Л}^{\mathbb{Q}_2}(v_2) = \{\mathcal{L}, j\}$ and $\overline{Л}^{\mathcal{M}_2}(v_2) = \{\mathcal{L}, k\}$. Thus $\overline{Л}^{\mathbb{Q}_2}(v_2) \star_1 \overline{Л}^{\mathcal{M}_2}(v_2) = \{\mathcal{L}, j\} \star_1 \{\mathcal{L}, k\} = \{\mathcal{L}\}$ and $\mathbb{Q}_2(v_2) \star_2 \mathcal{M}_2(v_2) = \{x'\} \star_2 \{y'\} = \{y'\}$. UPap of $\mathbb{Q}_2(v_2) \star_2 \mathcal{M}_2(v_2)$ is $\overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v_2) = \{\mathcal{L}, k\}$. Therefore $\overline{Л}^{\mathbb{Q}_2}(v_2) \star_1 \overline{Л}^{\mathcal{M}_2}(v_2) = \{\mathcal{L}, j\} \star_1 \{\mathcal{L}, k\} = \{\mathcal{L}\} \subset \{\mathcal{L}, k\} = \overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v_2)$. Hence, $\left(\overline{Л}^{\mathbb{Q}_2}, V\right) \star_1 \left(\overline{Л}^{\mathcal{M}_2}, V\right) \subseteq \left(\overline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}, V\right)$. Similarly, we can find an example that supports the argument for Theorem 3.13 (2).

Theorem 3.15. Let $(Л, V)$ be a SCMR w.r.t foresets from Q_1 to Q_2. Then for any soft set $(\mathcal{N}_1, V)$ over Q_1 and $(\mathbb{Q}_1, V)$ over Q_1, we have

(1) $\left({}^{\mathbb{Q}_1}\overline{Л}, V\right) \star_2 \left({}^{\mathcal{N}_1}\overline{Л}, V\right) \subseteq \left({}^{\mathbb{Q}_1 \star_1 \mathcal{N}_1}\overline{Л}, V\right)$.

(2) $\vee_{l\in L} \left({}^{\mathcal{N}_l}\overline{Л}, V\right) \subseteq \left({}^{\vee_{l\in L}\mathcal{N}_l}\overline{Л}, V\right)$.

Proof. Similar to proof of Theorem 3.13.

Theorem 3.16. Let $(Л, V)$ be a SCMPR w.r.t afterset from Q_1 to Q_2. Then for any soft set $(\mathcal{M}_2, V)$ over Q_2 and $(\mathbb{Q}_2, V)$ over Q_2, we have

(1) $\left(\underline{Л}^{\mathbb{Q}_2}, V\right) \star_1 \left(\underline{Л}^{\mathcal{M}_2}, V\right) \subseteq \left(\underline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}, V\right)$.

(2) $\vee_{l\in L}\left(\underline{Л}^{\mathcal{M}_l}, V\right) \subseteq \left(\underline{Л}^{\vee_{l\in L}\mathcal{M}_l}, V\right)$.

Proof. (1) For arbitrary $v \in V$, let $f \in \underline{Л}^{\mathbb{Q}_2}(v) \star_1 \underline{Л}^{\mathcal{M}_2}(v)$. Then for some $e \in \underline{Л}^{\mathbb{Q}_2}(v)$ and $g \in \underline{Л}^{\mathcal{M}_2}(v)$, we have $f = e \star_1 g$. Thereby, $\emptyset \neq eЛ(v) \subseteq \mathbb{Q}_2(v)$ and $\emptyset \neq gЛ(v) \subseteq \mathcal{M}_2(v)$. Then by SCMPR, $(e \star_1 g)Л(v) = eЛ(v) \star_2 gЛ(v) \subseteq \mathbb{Q}_2(v) \star_2 \mathcal{M}_2(v) \Rightarrow (e \star_1 g)Л(v) \subseteq \mathbb{Q}_2(v) \star_2 \mathcal{M}_2(v)$. Thus, $f = (e \star_1 g) \in \underline{Л}^{\mathbb{Q}_2 \star_2 \mathcal{M}_2}(v)$.

(2) Now for arbitrary $v \in V$, assume $f \in \vee_{l\in L} \underline{Л}^{\mathcal{M}_l}(v)$. Then, $f = \vee_{l\in L} g_l$ for some $g_l \in \underline{Л}^{\mathcal{M}_l}(v)$. Thereby, $\emptyset \neq g_l\, Л(v) \subseteq \mathcal{M}_l(v)$. By SCMPR, we have $\vee_{l\in L}(g_l Л(v)) = (\vee_{l\in L} g_l)Л(v) \subseteq \vee_{l\in L} \mathcal{M}_l(v)$. So, $(\vee_{l\in L} g_l)Л(v) \subseteq \vee_{l\in L} \mathcal{M}_l(v)$. Hence, $f = \vee_{l\in L} g_l \in \underline{Л}^{\vee_{l\in L}\mathcal{M}_l}(v)$.

Theorem 3.17. Let $(Л, V)$ be a SCMPR w.r.t foresets from $Q_1$to Q_2. Then for any soft set $(\mathcal{N}_1, V)$ over Q_1 and $(\mathbb{Q}_1, V)$ over $Ϙ_1$, we have

(1) $\left({}^{\mathbb{Q}_1}\underline{Л}, V\right) \star_2 \left({}^{\mathcal{N}_1}\underline{Л}, V\right) \subseteq \left({}^{\mathbb{Q}_1 \star_1 \mathcal{N}_1}\underline{Л}, V\right)$.

(2) $\vee_{l\in L}\left({}^{\mathcal{N}_l}\underline{Л}, V\right) \subseteq \left({}^{\vee_{l\in L}\mathcal{N}_l}\underline{Л}, V\right)$.

Proof. Similar to proof of Theorem 3.16.

4. Approximation of soft substructures in quantale modules

In this section, we use SBIR from Q_1 to Q_2 to approximate different soft substructures of quantale modules Q_1 and Q_2. We will show that by using SCMR, UP_{ap} of a soft substructure of quantale module is again a soft substructure of quantale module, and we will provide counter examples to show that the converse is not true. We'll also show that by using SCMPR, LO_{ap} of a soft substructure of quantale modules is again a soft substructure of quantale module and provide a counter-example to show that the converse is not true.

Throughout in this section, we consider $(Л, V)$ be the SBIR from Q_1 to Q_2 and $bЛ(v) \neq \emptyset$ for all $\in Q_1$, $v \in V$ and $Л(v)c \neq \emptyset\ \forall\ c \in Q_2$, $v \in V$ unless otherwise specified. Consider two soft sets $(\mathcal{N}, V)$ and $(\mathcal{M}, V)$ over Q_1 and Q_2 respectively, by the rule,

$\mathcal{N}(v) = Q_1$ and $\mathcal{M}(v) = Q_2\ \forall\ v \in V$. Then,

${}^{\mathcal{N}}\underline{Л}(v) = \{\gamma \in Q_2 : Л(v)\gamma \subseteq Q_1\} \subseteq Q_2,$

${}^{\mathcal{N}}\overline{Л}(v) = \{\gamma \in Q_2 : Л(v)\gamma \cap Q_1 \neq \emptyset\} \subseteq Q_2,$

$\underline{Л}^{\mathcal{M}}(v) = \{\rho \in Q_1 : Л(v)\rho \subseteq Q_2\} \subseteq Q_1.$

$\overline{Л}^{\mathcal{M}}(v) = \{\rho \in Q_1 : Л(v)\rho \cap Q_2 \neq \emptyset\} \subseteq Q_1,$

Definition 4.1. Let $(Л, V)$ be the SBIR from Q_1 to Q_2. A soft set $(\mathcal{M}, V)$ over Q_2 is called GU_rS (generalized upper soft) $Ϙ_S$ of Q_1 w.r.t aftersets if $UP_{ap}\left(\overline{Л}^{\mathcal{M}}, V\right)$ is soft $Ϙ_s$ of Q_1.

Definition 4.2. Let $(Л, V)$ be the SBIR from $Q_1$to Q_2. A soft set $(\mathcal{M}, V)$ over Q_2 is called GU_rS left (right) $Ϙ_I$ of Q_1w.r.t aftersets if $UP_{ap}\left(\overline{Л}^{\mathcal{M}}, V\right)$ is soft left (right) $Ϙ_I$ of Q_1.

Definition 4.3. Let $(Л, V)$ be the SBIR from $Q_1$to Q_2. A soft set $(\mathcal{N}, V)$ over Q_1 is called GU_rS quantale sub-module of Q_2 w.r.t foresets if $UP_{ap}\left({}^{\mathcal{N}}\overline{Л}, V\right)$ is soft $Ϙ_s$ of Q_2.

Definition 4.4. Let $(Л, V)$ be the SBIR from $Q_1$to Q_2. A soft set $(\mathcal{N}, V)$ over Q_1 is called GU_rS left (right) $Ϙ_I$ of Q_2 w.r.t foresets if $UP_{ap}\left({}^{\mathcal{N}}\overline{Л}, V\right)$ is soft left (right) $Ϙ_I$ of Q_2.

Theorem 4.5. Let $(Л, V)$ be a SCMR from $Q_1$to Q_2. If $(\mathcal{M}, V)$ is a soft $Ϙ_S$ of Q_2, then $(\mathcal{M}, V)$ is a GU_rS $Ϙ_s$ of Q_1w.r.t the aftersets.

Proof. (1) Assume that $(\mathcal{M}, V)$ is soft $Ϙ_S$ of Q_2, then $\emptyset \neq \overline{Л}^{\mathcal{M}}(v)$ for any $v \in V$. Let $\mu_l \in \overline{Л}^{\mathcal{M}}(v)\ for\ l \in L$. Thereby, $\mu_l\, Л(v) \cap \mathcal{M}(v) \neq \emptyset$. So there exist, $w_l \in \mu_l\, Л(v)\ i.e., (\mu_l, w_l) \in Л(v)\ and\ w_l \in \mathcal{M}(v)$. By SCMR, $(\vee_{l\in L}\mu_l, \vee_{l\in L} w_l) \in Л(v) \Rightarrow \vee_{l\in L} w_l \in \vee_{l\in L}\mu_l\, Л(v)$. As $(\mathcal{M}, V)$ is soft $Ϙ_S$ of Q_2 so, we have $\vee_{l\in L} w_l \in \mathcal{M}(v)$. Thus, $\vee_{l\in L} w_l \in \vee_{l\in L}\mu_l\, Л(v) \cap \mathcal{M}(v)$. So, $\vee_{l\in L}\mu_l\, Л(v) \cap \mathcal{M}(v) \neq \emptyset$. Hence, $\vee_{l\in L}\mu_l \in \overline{Л}^{\mathcal{M}}(v)$.

(2) Let $\gamma \in \overline{Л}^{\mathbb{Q}}(v) \subseteq Ϙ_1$ and $\mu \in \overline{Л}^{\mathcal{M}}(v)$, where $(\mathbb{Q}, V)\ is\ a\ subset\ of\ (Ϙ_2, V)$. Thereby, $\gamma Л(v) \cap \mathbb{Q}(v) \neq \emptyset$ and $\mu Л(v) \cap \mathcal{M}(v) \neq \emptyset$. So, for $\hbar \in Ϙ_2$ and $g \in Q_2$, we have $\hbar \in \gamma Л(v) \cap \mathbb{Q}(v)$ and $g \in \mu Л(v) \cap \mathcal{M}(v) \Rightarrow \hbar \in \gamma Л(v)$, $g \in \mu Л(v)$, $\hbar \in \mathbb{Q}(v), g \in \mathcal{M}(v)$. Thereby, $(\gamma, \hbar) \in Л(v)$, $(\mu, g) \in Л(v)$. Since $Л(v)$ is SCMR thus, $(\gamma \star_1 \mu, \hbar \star_2 g) \in Л(v)$, i.e., $(\hbar \star_2 g) \in (\gamma \star_1 \mu)Л(v)$ and $(\hbar \star_2 g) \in \mathbb{Q}(v) \star_2 \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is soft $Ϙ_S$ of Q_2 so, $(\hbar \star_2 g) \in$

$\mathbb{Q}(v) \star_2 \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thus, $(\hbar \star_2 g) \in (\gamma \star_1 \mu)Л(v) \cap \mathcal{M}(v) \Rightarrow (\gamma \star_1 \mu)Л(v) \cap \mathcal{M}(v) \neq \emptyset$. Hence, $(\gamma \star_1 \mu) \in \overline{Л}^{\mathcal{M}}(v)$.

Theorem 4.6. Let $(Л, V)$ be a SCMR from $Q_1$to Q_2. If $(\mathcal{N}, V)$ is a soft Q_S of Q_1, then $(\mathcal{N}, V)$ is a GU$_r$S Q_S of Q_2 w.r.t the foresets.

Proof. Similar to Proof of Theorem 4.5.

Theorem 4.7. Let $(Л, V)$ be a soft $\vee$ −complete relation from Q_1 to Q_2 w.r.t aftersets. If $(\mathcal{M}, V)$ is a soft left (right) Q_I of Q_2, then $(\mathcal{M}, V)$ is a GU$_r$S left (right) Q_I of Q_1 w.r.t the aftersets.

Proof. (1) Assume that $(\mathcal{M}, V)$ is soft Q_I of Q_2, then $\emptyset \neq \overline{Л}^{\mathcal{M}}(v)$ for any $v \in V$. Let $\rho_1 \in \overline{Л}^{\mathcal{M}}(v)$ *and* $\rho_2 \in \overline{Л}^{\mathcal{M}}(v)$. Thereby, $\rho_1 Л(v) \cap \mathcal{M}(v) \neq \emptyset$ *and* $\rho_2 Л(v) \cap \mathcal{M}(v) \neq \emptyset$. So there exist $w_1 \in \rho_1 Л(v)$ *i.e.*, $(\rho_1, w_1) \in Л(v)$ *and* $w_1 \in \mathcal{M}(v)$ and $w_2 \in \rho_2 Л(v)$ *i.e.*, $(\rho_2, w_2) \in Л(v)$ *and* $w_2 \in \mathcal{M}(v)$. By SCMR, $(\rho_1 \vee \rho_2, w_1 \vee w_2) \in Л(v) \Rightarrow w_1 \vee w_2 \in (\rho_1 \vee \rho_2) Л(v)$. As $(\mathcal{M}, V)$ is soft Q_I of Q_2, so we have $w_1 \vee w_2 \in \mathcal{M}(v)$. Thus, $w_1 \vee w_2 \in (\rho_1 \vee \rho_2) Л(v) \cap \mathcal{M}(v)$. So, $(\rho_1 \vee \rho_2) Л(v) \cap \mathcal{M}(v) \neq \emptyset$. Hence, $(\rho_1 \vee \rho_2) \in \overline{Л}^{\mathcal{M}}(v)$.

(2) Let $\rho_1, \rho_2 \in Q_1$, $\rho_1 \leq \rho_2$ and $\rho_2 \in \overline{Л}^{\mathcal{M}}(v)$. So, $\rho_1 \vee \rho_2 = \rho_2 \in \overline{Л}^{\mathcal{M}}(v)$. Since $\rho_2 \in \overline{Л}^{\mathcal{M}}(v) \Rightarrow \rho_2 Л(v) \cap \mathcal{M}(v) \neq \emptyset$. So there exist $w_2 \in \rho_2 Л(v)$ *i.e.*, $(\rho_2, w_2) \in Л(v)$ *and* $w_2 \in \mathcal{M}(v)$. As $(Л, V)$ is a soft $\vee$ −complete relation, thereby, $w_2 \in \rho_2 Л(v) = (\rho_1 \vee \rho_2)Л(v) = \rho_1 Л(v) \vee \rho_2 Л(v) \Rightarrow w_2 = u \vee v$, for some $u \in \rho_1 Л(v)$ *and* $v \in \rho_2 Л(v)$. Thus, $u \leq w_2$ *and* $w_2 \in \mathcal{M}(v)$. As $M(v)$ is Q_I so, $u \in \mathcal{M}(v)$. Thus, $u \in \rho_1 Л(v) \cap \mathcal{M}(v)$. Consequently, $\rho_1 \in \overline{Л}^{\mathcal{M}}(v)$.

(3) Let $\beta \in \overline{Л}^{\mathbb{Q}}(v) \subseteq Q_1$ and $\rho \in \overline{Л}^{\mathcal{M}}(v)$, where $(\mathbb{Q}, V)$ *is a subset of* (Q_2, V). Thereby, $\beta Л(v) \cap \mathbb{Q}(v) \neq \emptyset$ and $\rho Л(v) \cap \mathcal{M}(v) \neq \emptyset$. So, for $u \in Q_2$ and $v \in Q_2$, we have $u \in \beta Л(v) \cap \mathbb{Q}(v)$ and $v \in \rho Л(v) \cap \mathcal{M}(v)$. $\Rightarrow u \in \beta Л(v)$, $v \in \rho Л(v)$, $u \in \mathbb{Q}(v)$ and $v \in \mathcal{M}(v)$. Thereby, $(\beta, u) \in Л(v)$, $(\rho, v) \in Л(v)$. Since $(Л, V)$ is SCMR thus, $(\beta \star_1 \rho, u \star_2 v) \in Л(v)$, i.e., $(u \star_2 v) \in (\beta \star_1 \rho)Л(v)$ and $(u \star_2 v) \in \mathbb{Q}(v) \star_2 \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is soft Q_I of Q_2 so, $(u \star_2 v) \in \mathbb{Q}(v) \star_2 \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thus $(u \star_2 v) \in (\beta \star_1 \rho)Л(v) \cap \mathcal{M}(v) \Rightarrow (\beta \star_1 \rho)Л(v) \cap \mathcal{M}(v) \neq \emptyset$. Hence, $(\beta \star_1 \rho) \in \overline{Л}^{\mathcal{M}}(v)$.

Theorem 4.8. Let $(Л, V)$ be a soft $\vee$ −complete relation from Q_1 to Q_2 w.r.t foresets. If $(\mathcal{N}, V)$ is a soft left (right) Q_I of Q_1, then $(\mathcal{N}, V)$ is a GU$_r$S left (right) Q_I of Q_2 w.r.t the foresets.

Proof. Similar to proof of Theorem 4.7.

Definition 4.9. Let $(Л, V)$ be the SBIR from Q_1 to Q_2. A soft set $(\mathcal{M}, V)$ over Q_2 is called GL$_r$S (generalized lower soft) Q_S of Q_1w.r.t aftersets, if $LO_{ap}\left(\underline{Л}^{\mathcal{M}}, V\right)$ is soft Q_S of Q_1.

Definition 4.10. Let $(Л, V)$ be the SBIR from Q_1 to Q_2. A soft set $(\mathcal{M}, V)$ over Q_2 is called GL$_r$S left (right) Q_I of Q_1 w.r.t aftersets, if $LO_{ap}\left(\underline{Л}^{\mathcal{M}}, V\right)$ is soft left (right) Q_I of Q_1.

Definition 4.11. Let $(Л, V)$ be the SBIR from Q_1 to Q_2. A soft set $(\mathcal{N}, V)$ over Q_1 is called GL$_r$S quantale sub-module of Q_2 w.r.t foresets, if $LO_{ap}\left({}^{\mathcal{N}}\underline{Л}, V\right)$ is soft Q_S of Q_2.

Definition 4.12. Let $(Л, V)$ be the SBIR from Q_1 to Q_2. A soft set $(\mathcal{N}, V)$ over Q_1 is called GL$_r$S left (right) Q_I of Q_2 w.r.t foresets, if $LO_{ap}\left({}^{\mathcal{N}}\underline{Л}, V\right)$is soft left (right) Q_I of Q_2.

Theorem 4.13. Let $(Л, V)$ be a SCMPR from Q_1 to Q_2. If $(\mathcal{M}, V)$ is a soft Q_S of Q_2, then $(\mathcal{M}, V)$ is a GL$_r$S Q_S of Q_1 w.r.t the aftersets.

Proof. (1) Assume that $(\mathcal{M}, V)$ is soft Q_S of Q_2, then $\emptyset \neq \underline{Л}^{\mathcal{M}}(v)$ for any $v \in V$. Let $x_l \in \underline{Л}^{\mathcal{M}}(v)$ *for* $(l \in L)$. Thereby, $x_l Л(v) \subseteq \mathcal{M}(v)$. Since $(Л, V)$ is a SCMPR from Q_1 to Q_2 therefore, $\vee_{l \in L}\left(x_l Л(v)\right) = (\vee_{l \in L} x_l) Л(v) \subseteq \vee_{l \in L} \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is a soft Q_S of Q_2 So, $\vee_{l \in L} \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thereby, $(\vee_{l \in L} x_l) Л(v) \subseteq \mathcal{M}(v)$. Consequently, $\vee_{l \in L} x_l \in \underline{Л}^{\mathcal{M}}(v)$.

(2) Let $\gamma \in \underline{Л}^{\mathbb{Q}}(v) \subseteq Q_1$ and $x \in \underline{Л}^{\mathcal{M}}(v)$, where $(\mathbb{Q}, V)$ *is a subset of* (Q_2, V). Thereby, $\gamma Л(v) \subseteq \mathbb{Q}(v)$ and $x Л(v) \subseteq \mathcal{M}(v)$. Since $(Л, V)$ is a SCMPR from Q_1 to Q_2. Therefore, $(\gamma \star_1 x)Л(v) = \gamma Л(v) \star_2 x Л(v) \subseteq \mathbb{Q}(v) \star_2 \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is soft Q_S of Q_2 so, $(\gamma \star_1 x)Л(v) \subseteq \mathbb{Q}(v) \star_2 \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thus, $(\gamma \star_1 x)Л(v) \subseteq \mathcal{M}(v)$. Hence $(\gamma \star_1 x) \in \underline{Л}^{\mathcal{M}}(v)$.

Theorem 4.14. Let $(Л, V)$ be a SCMPR from $Q_1$to Q_2. If $(\mathcal{N}, V)$ is a soft Q_S of Q_1, then $(\mathcal{N}, V)$ is a GL$_r$S Q_S of Q_2 w.r.t the foresets.

Proof. Similar to proof of Theorem 4.13.

The converse of above Theorems is not true in general. To substantiate our claim, we will provide the following example:

Example 4.15. Let $Q_1 = \{\mathcal{L}, a, b, c, d, Ŧ\}$ and $Q_2 = \{\mathcal{L}', e, f, g, h, Ŧ'\}$ be two C_{lts} described in Figures 7 and 8 respectively. The associative binary operation $\otimes_1$ and $\otimes_2$ on Q_1 and Q_2 is defined as,

(1) $a \otimes_1 b = \mathcal{L}$

(2) $a \otimes_2 b = \mathcal{L}'$

We define $\star_1$ and $\star_2$ the left action on Q_1 and Q_2, respectively as shown in Tables 7 and 8.

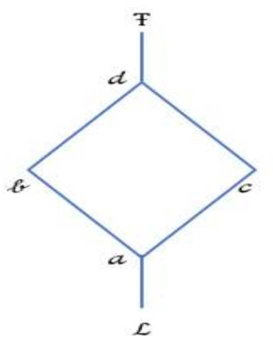

Figure 7. Description of Q_1.

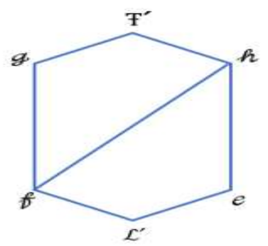

Figure 8. Description of Q_2.

Table 7. Left action subject to $\star_1$.

$\star_1$	$\mathcal{L}$	a	b	c	d	Ŧ
$\mathcal{L}$	$\mathcal{L}$	a	b	c	d	Ŧ
a	$\mathcal{L}$	a	b	c	d	Ŧ
b	$\mathcal{L}$	a	b	c	d	Ŧ
c	$\mathcal{L}$	a	b	c	d	Ŧ
d	$\mathcal{L}$	a	b	c	d	Ŧ
Ŧ	$\mathcal{L}$	a	b	c	d	Ŧ

Table 8. Left action subject to $\star_2$.

$\star_2$	$\mathcal{L}'$	e	f	g	h	Ŧ′
$\mathcal{L}'$	$\mathcal{L}'$	e	f	g	h	Ŧ′
e	$\mathcal{L}'$	e	f	g	h	Ŧ′
f	$\mathcal{L}'$	e	f	g	h	Ŧ′
g	$\mathcal{L}'$	e	f	g	h	Ŧ′
h	$\mathcal{L}'$	e	f	g	h	Ŧ′

Ŧ′	$\mathcal{L}'$	e	$\mathscr{f}$	$\mathscr{g}$	$\mathscr{h}$	Ŧ′

Then, Q_1 and Q_2 are quantale modules. Let $V = \{v_1, v_2\}$ and the SBIR $(Л, V)$ from Q_1 to Q_2 be defined by,

$$Л(v_1) = \begin{Bmatrix} (Ŧ, \mathscr{h}), (Ŧ, e), (c, \mathscr{h}), (\mathscr{b}, \mathscr{h}), (d, e), (a, e), \\ (\mathcal{L}, \mathscr{h}), (a, \mathscr{h}), (c, e), (\mathscr{b}, e), (d, \mathscr{h}), (\mathcal{L}, e) \end{Bmatrix},$$

$$Л(v_2) = \begin{Bmatrix} (\mathcal{L}, \mathcal{L}'), (c, \mathscr{f}), (a, \mathscr{f}), (Ŧ, \mathscr{f}), (d, \mathcal{L}'), (Ŧ, \mathcal{L}'), \\ (\mathcal{L}, \mathscr{f}), (\mathscr{b}, \mathcal{L}), (\mathscr{b}, \mathscr{f}), (a, \mathcal{L}'), (d, \mathscr{f}), (c, \mathcal{L}') \end{Bmatrix}.$$

Then $(Л, V)$ is SCMR. Following are the aftersets corresponding to $Л(v_1)$ and $Л(v_2)$,
$\mathcal{L}Л(v_1) = \{e, \mathscr{h}\}$, $aЛ(v_1) = \{e, \mathscr{h}\}$, $\mathscr{b}Л(v_1) = \{e, \mathscr{h}\}$, $cЛ(v_1) = \{e, \mathscr{h}\}$, $dЛ(v_1) = \{e, \mathscr{h}\}$, $ŦЛ(v_1) = \{e, \mathscr{h}\}$,
$\mathcal{L}Л(v_2) = \{\mathscr{f}, \mathcal{L}'\}$, $aЛ(v_2) = \{\mathscr{f}, \mathcal{L}'\}$, $\mathscr{b}Л(v_2) = \{\mathscr{f}, \mathcal{L}'\}$, $cЛ(v_2) = \{\mathscr{f}, \mathcal{L}'\}$, $dЛ(v_2) = \{\mathscr{f}, \mathcal{L}'\}$, $ŦЛ(v_2) = \{\mathscr{f}, \mathcal{L}'\}$.
Then $(Л, V)$ is SCMPR from Q_1 to Q_2 w.r.t aftersets. Define soft set $(\mathcal{M}, V)$ over Q_2 by the rule, $\mathcal{M}(v_1) = \{e, \mathscr{h}, \mathscr{g}\}$ and $\mathcal{M}(v_2) = \{\mathcal{L}', \mathscr{f}, \mathscr{g}, \mathscr{h}\}$. Then $(\mathcal{M}, V)$ is not a soft $Ϙ_S$ of Q_2. But $\underline{Л}^{\mathcal{M}}(v_1) = \{\mathcal{L}, a, \mathscr{b}, c, d, Ŧ\}$ and $\underline{Л}^{\mathcal{M}}(v_2) = \{\mathcal{L}, a, \mathscr{b}, c, d, Ŧ\}$ are $Ϙ_S$ of Q_1. So, $(\mathcal{M}, V)$ is a GL$_r$S $Ϙ_S$ of Q_1 w.r.t aftersets.

Theorem 4.16. Let $(Л, V)$ be a SCMPR from Q_1 to Q_2. If $(\mathcal{M}, V)$ is a soft $Ϙ_I$ of Q_2, then $(\mathcal{M}, V)$ is a GL$_r$S $Ϙ_I$ of Q_1 w.r.t the aftersets.

Proof. (1) Assume that $(\mathcal{M}, V)$ is soft $Ϙ_I$ of Q_2, then $\emptyset \neq \underline{Л}^{\mathcal{M}}(v)$ for any $v \in V$. Let $x_1, x_2 \in \underline{Л}^{\mathcal{M}}(v)$. Thereby, $x_1\, Л(v) \subseteq \mathcal{M}(v)$ and $x_2\, Л(v) \subseteq \mathcal{M}(v)$. Since $(Л, V)$ is a SCMPR from Q_1 to Q_2 therefore $x_1\, Л(v) \vee x_2\, Л(v) = (x_1 \vee x_2)\, Л(v) \subseteq \mathcal{M}(v) \vee \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is a soft $Ϙ_I$ of Q_2. So, $\mathcal{M}(v) \vee \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thereby, $(x_1 \vee x_2)\, Л(v) \subseteq \mathcal{M}(v)$. Consequently, $x_1 \vee x_2 \in \underline{Л}^{\mathcal{M}}(v)$.
(2) Let $x_1, x_2 \in Q_1$, $x_1 \leq x_2$ and $x_2 \in \underline{Л}^{\mathcal{M}}(v)$. So, $x_1 \vee x_2 = x_2 \in \underline{Л}^{\mathcal{M}}(v)$. Since $x_2 \in \underline{Л}^{\mathcal{M}}(v) \Rightarrow x_2\, Л(v) \subseteq \mathcal{M}(v)$. Suppose $\mathscr{y}_1 \in x_1\, Л(v)$ and $\mathscr{y}_2 \in x_2\, Л(v)$. As $(Л, V)$ is a SCMPR thereby, $\mathscr{y}_1 \vee \mathscr{y}_2 \in x_1\, Л(v) \vee x_2\, Л(v) = (x_1 \vee x_2)Л(v)$. i.e., $\mathscr{y}_1 \vee \mathscr{y}_2 \in x_2\, Л(v) \subseteq \mathcal{M}(v)$. As $M(v)\ is\ Ϙ_I$, so $\mathscr{y}_1 \leq \mathscr{y}_1 \vee \mathscr{y}_2 \in \mathcal{M}(v)$. Thus, $\mathscr{y}_1 \in \mathcal{M}(v)$. Hence, $x_1\, Л(v) \subseteq \mathcal{M}(v)$. Consequently, $x_1 \in \overline{Л}^{\mathcal{M}}(v)$.
(3) Let $\beta \in \underline{Л}^{\mathbb{Q}}(v) \subseteq Ϙ_1$ and $x \in \underline{Л}^{\mathcal{M}}(v)$, where $(\mathbb{Q}, V)\ is\ a\ subset\ of\ (Ϙ_2, V)$. Thereby, $\beta Л(v) \subseteq \mathbb{Q}(v)$ and $xЛ(v) \subseteq \mathcal{M}(v)$. Since $(Л, V)$ is a SCMPR from Q_1 to Q_2, therefore, $(\beta \star_1 x)Л(v) = \beta Л(v) \star_2 xЛ(v) \subseteq \mathbb{Q}(v) \star_2 \mathcal{M}(v)$. As $(\mathcal{M}, V)$ is soft $Ϙ_I$ of Q_2 so, $(\beta \star_1 x)Л(v) \subseteq \mathbb{Q}(v) \star_2 \mathcal{M}(v) \subseteq \mathcal{M}(v)$. Thus, $(\beta \star_1 x)Л(v) \subseteq \mathcal{M}(v)$. Hence, $(\beta \star_1 x) \in \underline{Л}^{\mathcal{M}}(v)$.

Theorem 4.17. Let $(Л, V)$ be a SCMPR from Q_1 to Q_2. If $(\mathcal{N}, V)$ is a soft $Ϙ_I$ of Q_1, then $(\mathcal{N}, V)$ is a GL$_r$S $Ϙ_I$ of Q_2 w.r.t the foresets.

Proof. Similar to proof of Theorem 4.16.

The converse of above Theorem is not true in general. To substantiate our claim, we will provide the following example:

Example 4.18. Consider $Q_1 = \{\mathcal{L}, a, \mathscr{b}, c, d, Ŧ\}$ and $Q_2 = \{\mathcal{L}', e, \mathscr{f}, \mathscr{g}, \mathscr{h}, Ŧ'\}$ be two quantale modules described in Example 4.15. Let $V = \{v_1, v_2\}$ and the SBIR $(Л, V)$ from $Q_1\ to\ Q_2$ be defined by,

$$Л(v_1)=\begin{Bmatrix}(Ŧ,\hbar),(c,f),(\mathcal{L},g),(b,f),(d,\mathcal{L}'),(Ŧ,\mathcal{L}'),\\(\mathcal{L},f),(b,\mathcal{L}'),(c,\mathcal{L}'),(a,\mathcal{L}'),(d,f),(b,g)\\(a,f),(a,g),(Ŧ,f),(c,g),(d,g),(\mathcal{L},\mathcal{L}')\end{Bmatrix},$$

$$Л(v_2)=\begin{Bmatrix}(a,g),(a,f),(b,f),(b,g),(d,f),(c,\hbar),\\(\mathcal{L},\hbar),(Ŧ,g),(\mathcal{L},g),(Ŧ,f),(d,\hbar),(d,g)\\(b,\hbar),(\mathcal{L},f),(c,f),(c,g),(Ŧ,\hbar),(a,\hbar)\end{Bmatrix}.$$

Then $(Л,V)$ is SCMR. Following are the aftersets corresponding to $Л(v_1)$ and $Л(v_2)$. $\mathcal{L}Л(v_1)=\{\mathcal{L}',f,g\}, aЛ(v_1)=\{\mathcal{L}',f,g\}, bЛ(v_1)=\{\mathcal{L}',f,g\}$, $cЛ(v_1)=\{\mathcal{L}',f,g\}, dЛ(v_1)=\{\mathcal{L}',f,g\}, ŦЛ(v_1)=\{\mathcal{L}',f,g\}$,
$\mathcal{L}Л(v_2)=\{g,f,\hbar\}, aЛ(v_2)=\{g,f,\hbar\}$, $bЛ(v_2)=\{g,f,\hbar\}$ $cЛ(v_2)=\{g,f,\hbar\}, dЛ(v_2)=\{g,f,\hbar\}, ŦЛ(v_2)=\{g,f,\hbar\}$.
Then $(Л,V)$ is SCMPR from $Q_1$to Q_2 w.r.t aftersets. A soft set $(\mathcal{M},V)$ over Q_2 is defined by,
$\mathcal{M}(v_1)=\{\mathcal{L},f,g,\hbar\}$ and $\mathcal{M}(v_2)=\{e,f,g,\hbar\}$. Then $(\mathcal{M},V)$ is not a soft Q_I of Q_2. But $\underline{Л}^{\mathcal{M}}(v_1)=\{\mathcal{L},a,b,c,d,Ŧ\}$ and $\underline{Л}^{\mathcal{M}}(v_2)=\{\mathcal{L},a,b,c,d,Ŧ\}$ are Q_I of Q_1. So, $(\mathcal{M},V)$ is a GL$_r$S Q_I of Q_1w.r.t aftersets.

5. Relationship between soft quantale module homomorphism and their approximation

We define soft weak quantale module homomorphism (SWMH) in this section and then by using SBIR, we established relationship between homomorphic images and their approximation.

Definition 5.1. [20] Consider two quantale modules $(Q_1,\star_1)$ and $(Q_2,\star_2)$ over Q. A function $ƒ: Q_1\to Q_2$ is called weak quantale module homomorphism (WMH) if
(1) $ƒ(m\vee n)=ƒ(m)\vee ƒ(n)$;
(2) $ƒ(\gamma\star_1 m)=\gamma\star_2 ƒ(m)$. For any $\gamma\in Q$, $m,n\in Q_1$.
If ƒ is one-one then ƒ is monomorphism. If ƒ is onto then ƒ is called epimorphism and if ƒ is bijective then ƒ is called isomorphism between $(Q_1,\star_1)$ and $(Q_2,\star_2)$ over Q.

Definition 5.2. Let $(\mathcal{M},V_1)$ be a soft quantale module over Q_1 and $(\mathcal{N},V_2)$ be a soft quantale module over Q_2. If there exist an ordered pair of functions $(ƒ,\xi)$ satisfies the following,
(1) $ƒ: Q_1\to Q_2$ is onto WMH.
(2) $\xi: V_1\to V_2$ is surjective,
(3) $ƒ(\mathcal{M}(v_1))=(\mathcal{N}(\xi(v_1))\ \forall\ v_1\in V_1$.
Then $(\mathcal{M},V_1)$ is said to be soft weak homomorphic to $(\mathcal{N},V_2)$. The ordered pair $(ƒ,\xi)$ of functions is SWMH. The pair $(ƒ,\xi)$ is called soft week quantale module isomorphism (SWMI) and $(\mathcal{M},V_1)$ is said to soft weak isomorphic to $(\mathcal{N},V_2)$, if in ordered pair $(ƒ,\xi)$ both ƒ and ξ are one-to-one functions.

Lemma 5.3. Let $(\mathcal{M},V_1)$ be soft weak homomorphic to $(\mathcal{N},V_2)$ with SWMH $(ƒ,\zeta)$. Let $(Л_2,V_3)$ be a SBIR over Q_2 and $(\mathcal{M}_1,V_3)\subseteq(\mathcal{M},V_1)$. Define $Л_1(v_3)=\{(a,b)\in Q_1\times Q_1: (ƒ(a),ƒ(b))\in Л_2(v_3)\}$ be a SBIR over Q_1. Then the following holds,
(1) $(Л_1,V_3)$ is SCMR if $(Л_2,V_3)$ is SCMR.
(2) If $(ƒ,\xi)$ is SWMI and $(Л_2,V_3)$ is SCMR w.r.t the aftersets (w.r.t the foresets), then $(Л_1,V_3)$ is SCMR w.r.t the aftersets (w.r.t the foresets).
(3) $ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)=\overline{Л}_2^{ƒ(\mathcal{M}_1)}(v_3)$.

(4) $\text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right) \subseteq \underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$ and if $(\text{ƒ},\xi)$ is SWMI, then $\text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right) = \underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$.

(5) Let $(\text{ƒ},\xi)$ be a SWMI. Then $\text{ƒ}(a) \in \text{ƒ}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) \Leftrightarrow a \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ and $\text{ƒ}(a) \in \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right) \Leftrightarrow a \in \underline{Л}_1^{\mathcal{M}_1}(v_3)$.

Proof. **(**1) and (2) are obvious.

(3) Suppose $(\mathcal{M}_1, V_3) \subseteq (\mathcal{M}, V_1)$. For any $v_3 \in V_3$, let $p \in \text{ƒ}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$ for some $p \in \mathcal{Q}_2$. Then there exists $q \in \mathcal{Q}_1$ such that, $q \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ and $\text{ƒ}(q) = p$. Thereby, $qЛ_1(v_3) \cap \mathcal{M}_1(v_3) \neq \emptyset$. So, we have $k \in qЛ_1(v_3) \cap \mathcal{M}_1(v_3)$ i.e., $k \in qЛ_1(v_3)$ *and* $k \in \mathcal{M}_1(v_3)$ which means, $(q,k) \in Л_1(v_3)$ $\Rightarrow$ $(\text{ƒ}(q),\text{ƒ}(k)) \in Л_2(v_3)$ .Thereby, $\text{ƒ}(k) \in \text{ƒ}(q)Л_2(v_3)$ *and* $\text{ƒ}(k) \in \text{ƒ}(\mathcal{M}_1(v_3))$. Thus, $\text{ƒ}(k) \in \text{ƒ}(q)Л_2(v_3) \cap \text{ƒ}(\mathcal{M}_1(v_3)) \Rightarrow \text{ƒ}(q)Л_2(v_3) \cap \text{ƒ}(\mathcal{M}_1(v_3)) \neq \emptyset$. Hence, $p = \text{ƒ}(q) \in \overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$. Consequently, $\text{ƒ}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) \subseteq \overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$. Conversely, let $r \in \overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3) \subseteq \mathcal{Q}_2 \Rightarrow rЛ_2(v_3) \cap \text{ƒ}(\mathcal{M}_1(v_3)) \neq \emptyset$. So, there exists $t \in rЛ_2(v_3) \cap \text{ƒ}(\mathcal{M}_1(v_3))$ such that $t \in rЛ_2(v_3)$ *and* $t \in \text{ƒ}(\mathcal{M}_1(v_3))$ i.e $(r,t) \in Л_2(v_3)$. Let $u \in \mathcal{M}_1(v_3) \subseteq \mathcal{M}(v_1) \subseteq \mathcal{Q}_1$.. So, $u \in \mathcal{Q}_1$ and $u' \in \mathcal{Q}_1$ such that $\text{ƒ}(u) = t$ and $\text{ƒ}(u') = r$ we have $(r,t) = (\text{ƒ}(u),\text{ƒ}(u')) \in Л_2(v_3)$. By above definition, $(u,u') \in Л_1(v_3) \Rightarrow u \in u'Л_1(v_3)$. Thus, $u \in u'Л_1(v_3) \cap \mathcal{M}_1(v_3) \Rightarrow u'Л_1(v_3) \cap \mathcal{M}_1(v_3) \neq \emptyset$. So, $u' \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Hence $r = \text{ƒ}(u') \in \text{ƒ}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Consequently, $\overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3) \subseteq \text{ƒ}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$.

(4) Suppose $(\mathcal{M}_1, V_3) \subseteq (\mathcal{M}, V_1)$. For any $v_3 \in V_3$, let $p \in \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ for some $p \in \mathcal{Q}_2$ then there exists $q \in \mathcal{Q}_1$, such that $q \in \underline{Л}_1^{\mathcal{M}_1}(v_3)$ and $\text{ƒ}(q) = p$. Thereby, $qЛ_1(v_3) \subseteq \mathcal{M}_1(v_3)$. Assume, $k \in pЛ_2(v_3) \subseteq \mathcal{Q}_2$ there exists, $\ell \in \mathcal{Q}_1$ such that $\text{ƒ}(\ell) = k$. Thereby, $\text{ƒ}(\ell) \in \text{ƒ}(q)Л_2(v_3) \Rightarrow (\text{ƒ}(q),\text{ƒ}(\ell)) \in Л_2(v_3)$. By given definition $(q,\ell) \in Л_1(v_3) \Rightarrow \ell \in qЛ_1(v_3) \subseteq \mathcal{M}_1(v_3) \Rightarrow \text{ƒ}(\ell) \in \text{ƒ}(\mathcal{M}_1(v_3)$. Thus, $\text{ƒ}(q)Л_2(v_3) \subseteq \text{ƒ}(\mathcal{M}_1(v_3)$. Hence, $p = \text{ƒ}(q) \in \underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$. Consequently, $\text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right) \subseteq \underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$. Conversely, let $r \in \underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3) \subseteq \mathcal{Q}_2$ then there exists $s \in \mathcal{Q}_1$ such that, $\text{ƒ}(s)= r$ and $rЛ_2(v_3) \subseteq \text{ƒ}(\mathcal{M}_1(v_3))$. So, $\text{ƒ}(s)Л_2(v_3) \subseteq \text{ƒ}(\mathcal{M}_1(v_3))$. Let $t \in sЛ_1(v_3)$ i.e., $(s,t) \in Л_1(v_3)$. Thereby, $(\text{ƒ}(s),\text{ƒ}(t)) \in Л_2(v_3) \Rightarrow \text{ƒ}(t) \in \text{ƒ}(s)Л_2(v_3) \subseteq \text{ƒ}(\mathcal{M}_1(v_3))$. So, $\text{ƒ}(t) \in \text{ƒ}(\mathcal{M}_1(v_3)) \Rightarrow t \in (\mathcal{M}_1(v_3))$. Thus, $sЛ_1(v_3) \subseteq \mathcal{M}_1(v_3) \Rightarrow s \in \underline{Л}_1^{\mathcal{M}_1}(v_3) \Rightarrow \text{ƒ}(s) \in \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Thereby, $r = \text{ƒ}(s) \in \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Consequently, $\underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3) \subseteq \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Hence, $\underline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3) = \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right)$

(5) Let $r \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ for any $v_3 \in V_3$. Then, $\text{ƒ}(r) \in \text{ƒ}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Conversely, suppose that $\text{ƒ}(r) \in \text{ƒ}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. As ƒ is SWMI so, $r \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Similarly, we can show that $\text{ƒ}(a) \in \text{ƒ}\left(\underline{Л}_1^{\mathcal{M}_1}(v_3)\right) \Leftrightarrow a \in \underline{Л}_1^{\mathcal{M}_1}(v_3)$.

Theorem 5.4. Let $(\mathcal{M}, V_1)$ be soft weak isomorphic to $(\mathcal{N}, V_2)$ with SWMI $(\text{ƒ},\xi)$. Let $(Л_2, V_3)$ be a SCMR over $\mathcal{Q}_2$ and $(\mathcal{M}_1, V_3) \subseteq (\mathcal{M}, V_1)$. Define $Л_1(v_3) = \{(a,b) \in \mathcal{Q}_1 \times \mathcal{Q}_1 : (\text{ƒ}(a),\text{ƒ}(b)) \in Л_2(v_3)\}$ for any $v_3 \in V_3$. Then the following holds,

(1) $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_I of $\mathcal{Q}_1 \Leftrightarrow \overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$ is Q_I of $\mathcal{Q}_2$ $\forall\, v_3 \in V_3$.

(2) $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_S of $\mathcal{Q}_1 \Leftrightarrow \overline{Л}_2^{\text{ƒ}(\mathcal{M}_1)}(v_3)$ is Q_S of $\mathcal{Q}_2$ $\forall\, v_3 \in V_3$.

Proof. (1) Let $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ be Q_I of Q_1 for any $v_3 \in V_3$, we will show that $\overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$ is an Q_I of Q_2. By Lemma 5.3 we have $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) = \overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$.

i. Let $p, q \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Then there exists $r, s \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ such that $\mathfrak{f}(r) = p$ and $\mathfrak{f}(s) = q$. Since $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is an Q_I of Q_1 so, $r \vee s \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. By Lemma 5.3 (5) we have $\mathfrak{f}(r \vee s) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. As $(\mathfrak{f}, \xi)$ is SWMI, thus $p \vee q = \mathfrak{f}(r) \vee \mathfrak{f}(s) = \mathfrak{f}(r \vee s) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Thus, $p \vee q \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$.

ii. Let $p, q \in Q_2$Consider $p \leq q$ and $q \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Then there exists $s \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ and $r \in Q_1$ such that $\mathfrak{f}(r) = p$ and $\mathfrak{f}(s) = q$. Thus, $\mathfrak{f}(r) \leq \mathfrak{f}(s)$. As $(\mathfrak{f}, \xi)$ is SWMI so, $\mathfrak{f}(r \vee s) = \mathfrak{f}(r) \vee \mathfrak{f}(s) = \mathfrak{f}(s) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) \Rightarrow \mathfrak{f}(r \vee s) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. By Lemma 5.3 (5), $(r \vee s) \in \left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Thereby, $r \leq r \vee s$. As $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_I, so $r \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Thus, $\mathfrak{f}(r) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Consequently, $p \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$.

iii. Let $\gamma \in Q$ and $p \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Then there exists $r \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ such that $\mathfrak{f}(r) = p$. As $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_I of Q_1 so, $\gamma \star r \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Then by Lemma 5.3 (5) we have $\mathfrak{f}(\gamma \star r) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Since $(\mathfrak{f}, \xi)$ is SWMI thus, $\gamma \star \mathfrak{f}(r) = \mathfrak{f}(\gamma \star r) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Thereby, $\gamma \star \mathfrak{f}(r) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3)$. Hence, $\gamma \star p \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$.

Conversely, let $\overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$ be a Q_I of Q_2 for any $v_3 \in V_3$, we will show that $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is a Q_I of Q_1. By Lemma 5.3 we have $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) = \overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$.

i. Let $p, q \in \overline{Л}_1^{\mathcal{M}_1}(v_3) \Rightarrow \mathfrak{f}(p), \mathfrak{f}(q) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ is an Q_I of Q_2 so, $\mathfrak{f}(p) \vee \mathfrak{f}(q) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. As $(\mathfrak{f}, \xi)$ is SWMI, thus $\mathfrak{f}(p) \vee \mathfrak{f}(q) = \mathfrak{f}(p \vee q) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Thereby, $\mathfrak{f}(p \vee q) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. By Lemma 5.3 (5), $p \vee q \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$.

ii. Let $p, q \in Q_1$. Consider, $p \leq q$ and $q \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Thus, $\mathfrak{f}(p) \leq \mathfrak{f}(q) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ is an Q_I of Q_2 so, $\mathfrak{f}(p) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. By Lemma 5.3 (5) $p \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$.

iii. Let $\gamma \in Q$ and $p \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Thereby, $\mathfrak{f}(p) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ is Q_I of Q_2 so, $\gamma \star \mathfrak{f}(p) \in \mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $(\mathfrak{f}, \xi)$ is SWMI, thus, $\gamma \star \mathfrak{f}(p) = \mathfrak{f}(\gamma \star p) \in \mathfrak{f}(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Then by Lemma 5.3 (5) we have $\gamma \star p \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$.

(2) Let $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ be a Q_S of Q_1 for any $v_3 \in V_3$ we will show that $\overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$ is Q_S of Q_2. By Lemma 5.3 we have $\mathfrak{f}\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) = \overline{Л}_2^{\mathfrak{f}(\mathcal{M}_1)}(v_3)$.

i. Let $q_l \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$ for $l \in L$. Then there exists $s_l \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ such that $ƒ(s_l) = q_l$. Since $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is an Q_S of Q_1 so, $\vee_{l\in L} s_l \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. As $(ƒ, \xi)$ is SWMI, thus $\vee_{l\in L} q_l = \vee_{l\in L} ƒ(s_l) = ƒ(\vee_{l\in L} s_l) \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) \Rightarrow \vee_{l\in L} q_l \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$.

ii. Let $\beta \in Q$ and $q \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Then there exists $s \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ such that $ƒ(s) = q$. As $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_S of Q_1 so, $\beta \star s \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Then by Lemma 5.3 (5) we have $ƒ(\beta \star s) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Since $(ƒ, \xi)$ is SWMI thus, $\beta \star ƒ(s) = ƒ(\beta \star s) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Thereby, $\beta \star ƒ(s) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Hence, $\beta \star q \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$.

Conversely, let $\overline{Л}_2^{ƒ(\mathcal{M}_1)}(v_3)$ be a Q_S of Q_2 for any $v_3 \in V_3$, we will show that $\overline{Л}_1^{\mathcal{M}_1}(v_3)$ is a Q_S of Q_1. By Lemma 5.3 we have $ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right) = \overline{Л}_2^{ƒ(\mathcal{M}_1)}(v_3)$.

i. Let $q_l \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$ for $l \in L \Rightarrow ƒ(q_l) \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ is an Q_S of Q_2 so, $\vee_{l\in L} ƒ(q_l) \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. As $(ƒ, \xi)$ is SWMI, thus $\vee_{l\in L} ƒ(q_l) = ƒ(\vee_{l\in L} q_l) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Thereby, $ƒ(\vee_{l\in L} q_l) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. By Lemma 5.3 (5), $\vee_{l\in L} q_l \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$.

ii. Let $\beta \in Q$ and $q \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$. Thereby, $ƒ(q) \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$ is Q_S of Q_2 so, $\beta \star ƒ(q) \in ƒ\left(\overline{Л}_1^{\mathcal{M}_1}(v_3)\right)$. Since $(ƒ, \xi)$ is SWMI thus, $\beta \star ƒ(q) = ƒ(\beta \star q) \in ƒ(\overline{Л}_1^{\mathcal{M}_1}(v_3))$. Then by Lemma 5.3 (5), we have $\beta \star q \in \overline{Л}_1^{\mathcal{M}_1}(v_3)$.

Theorem 5.5. Let $(\mathcal{M}, V_1)$ be soft weak isomorphic to $(\mathcal{N}, V_2)$ with SWMI $(ƒ, \xi)$. Let $(Л_2, V_3)$ be a SCMPR over Q_2 and $(\mathcal{M}_1, V_3) \subseteq (\mathcal{M}, V_1)$. Define $Л_1(v_3) = \{(a, b) \in Q_1 \times Q_1 : (ƒ(a), ƒ(b)) \in Л_2(v_3)\}$ for any $v_3 \in V_3$. Then the following holds,

(1) $\underline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_I of $Q_1 \Leftrightarrow \underline{Л}_2^{ƒ(\mathcal{M}_1)}(v_3)$ is Q_I of Q_2 $\forall v_3 \in V_3$.

(2) $\underline{Л}_1^{\mathcal{M}_1}(v_3)$ is Q_S of $Q_1 \Leftrightarrow \underline{Л}_2^{ƒ(\mathcal{M}_1)}(v_3)$ is Q_S of Q_2 $\forall v_3 \in V_3$.

Proof. Similar to proof of Theorem 5.4.

6. An application of the decision-making approach

This section proposes decision-making techniques based on soft rough set theory based on soft binary relations. This strategy makes it possible to utilize the data provided by decision-makers without the need for additional information.

We obtain two values $\underline{Л}^{\mathcal{M}}(v_i)$ and $\overline{Л}^{\mathcal{M}}(v_i)$ which are most closed with respect to the aftersets by the soft lower and upper approximations of the soft set $\mathcal{M}$. Therefore, the choice value δ_i is redefined with respect to the aftersets as follows:

$$\delta_i = \sum_{j=1}^{n} \underline{d}_{ij} + \sum_{j=1}^{n} \overline{d}_{ij}.$$

In a decision making problem, the maximum choice value δ_i is the optimum decision for the object $x_i \in U$ and the minimum choice value δ_i is the worst decision for the object $x_i \in U$. For the given decision making problem, if the same maximum choice value δ_i belongs to two or more objects $x_i \in U$, then take one of them as the optimum decision randomly.

Algorithm

An algorithm is designed to approach a decision-making problem with respect to the aftersets is provided below. The decision algorithm is as follows:

(1) Compute the lower soft set approximation $\underline{Л}^{\mathcal{M}}$ and upper soft set approximation $\overline{Л}^{\mathcal{M}}$ of a soft set $\mathcal{M}$ with respect to the aftersets.

(2) Corresponding to each $x_i \in U$, we calculate $\underline{d}_{ij}$ which is 0 if $x_i \notin \underline{Л}^{\mathcal{M}}(v_j)$ and is 1 if $x_i \in \underline{Л}^{\mathcal{M}}(v_j)$. Similarly, we calculate $\overline{d}_{ij}$ which is 0 if $x_i \notin \overline{Л}^{\mathcal{M}}(v_j)$ and is 1 if $x_i \in \overline{Л}^{\mathcal{M}}(v_j)$.

(3) Compute the choice value $\delta_i = \sum_{j=1}^{n} \underline{d}_{ij} + \sum_{j=1}^{n} \overline{d}_{ij}$ with respect to the aftersets.

(4) The best decision is $x_k \in U$ if $\delta_k = max_i\ \delta_i$, i=1, 2,…, $|U|$.

(5) The worst decision is $x_k \in U$ if $\delta_k = min_i\ \delta_i$, i=1, 2,…, $|U|$.

(6) If the value of k is more than one, then we can choose any one of x_k. In a similar way, we can define an algorithm for foresets.

By an example in this subsection, an application of the decision-making approach is given.

Example 6.1. Suppose that Mr. X wants to buy a shirt for his own use. Let $U = \{the\ set\ of\ all\ shirts\ designs\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $W = \{the\ colors\ of\ all\ designs\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $V = \{v_1, v_2, v_3\} = \{the\ set\ of\ stores\ near\ his\ house\}$. Define $Л: V \longrightarrow P(U \times W)$ by

$$Л(v_1) = \begin{Bmatrix} (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \\ (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \end{Bmatrix},$$

$$Л(v_2) = \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\},$$

$$Л(v_3) = \{(d_3, c_3), (d_3, c_1), (d_2, c_4), (d_5, c_3), (d_5, c_4)\}.$$

Represents the relation between designs and colors available on store v_i for $1 \leq i \leq 3$.

Then

$$d_1Л(v_1) = \{c_1, c_2, c_3\}, d_2Л(v_1) = \{c_2, c_4\}, d_3Л(v_1) = \emptyset, d_4Л(v_1) = \{c_2, c_3\},$$

$$d_5Л(v_1) = \{c_4, c_3\}, d_6Л(v_1) = \{c_1\}, and\ d_1Л(v_2) = \{c_3\}, d_2Л(v_2) = \{c_3\},$$

$$d_3Л(v_2) = \{\emptyset\}, d_4Л(v_2) = \{c_1\}, d_5Л(v_2) = \{c_1\}, d_6Л(v_2) = \{c_2, c_3\}\ and$$

$$d_1Л(v_3) = \{\emptyset\}, d_2Л(v_3) = \{c_4\}, d_3Л(v_3) = \{c_1, c_3\}, d_4Л(v_3) = \emptyset,$$

$$d_5Л(v_3) = \{c_4, c_3\}, d_6Л(v_3) = \{\emptyset\}.$$

Where $d_iЛ(v_j)$ represents the color of the design d_i available on store v_j. Further

$$Л(v_1)c_1 = \{d_1, d_6\}, Л(v_1)c_2 = \{d_1, d_2, d_4\}, Л(v_1)c_3 = \{d_1, d_5, d_4\},$$

$$Л(v_1)c_4 = \{d_2, d_5\}, and\ Л(v_2)c_1 = \{d_4, d_5\}, Л(v_2)c_2 = \{d_6\},$$

$$Л(v_2)c_3 = \{d_1, d_2, d_6\}, Л(v_2)c_4 = \{\emptyset\}, and\ Л(v_3)c_1 = \{d_3\}, Л(v_3)c_2 = \{\emptyset\},$$

$$Л(v_3)c_3 = \{d_3, d_5\}, Л(v_3)c_4 = \{ d_2, d_5\}.$$

Where $Л(v_j)c_i$ represents the design of the color c_i available on store v_j.
Define $\mathcal{M} : V \to P(W)$ which represents the preference of the color given by Mr. X such that $\mathcal{M}(v_1) = \{c_1, c_4\}, \mathcal{M}(v_2) = \{c_2, c_5\}, \mathcal{M}(v_3) = \{c_2, c_3, c_4\}$ *and* define $H : V \to P(U)$ which represents the preference of the design given by Mr. X such that $H(v_1) = \{d_2, d_3, d_6\}, H(v_2) = \{d_1, d_3\}, H(v_3) = \{d_1, d_2, d_5, d_6\}$.
Consider the following table after applying the above algorithm (see Table 9).

Table 9. The results of the decision algorithm with respect to aftersets.

	$\underline{d}_{i1}$	$\underline{d}_{i2}$	$\underline{d}_{i3}$	$\overline{d}_{i1}$	$\overline{d}_{i2}$	$\overline{d}_{i3}$	Choice value δ_i
d_1	0	0	1	1	0	0	2
d_2	0	0	1	1	0	1	3
d_3	1	1	0	0	0	1	3
d_4	0	0	1	0	0	0	1
d_5	0	0	1	1	0	1	3
d_6	1	0	1	1	1	0	4

Here the choice value $\delta_i = \sum_{j=1}^{3} \underline{d}_{ij} + \sum_{j=1}^{3} \overline{d}_{ij}$ is calculated with respect to aftersets. The shirt of design d_6 scores the maximum choice value $\delta_k = 4 = \delta_6$, and the decision is in favor of the shirt of design d_6 for selection. Moreover, the shirts of designs d_4 are totally ignored. Hence, Mr. X will choose the shirt of design d_6 for his personal use and he will not select the shirt of design d_4 with respect to the aftersets.

7. Conclusions

There are many applications of soft set and rough set theories. Their combination involved many researchers to develop many concepts in mathematics. In this paper, major role of rough soft sets with substructures of quantale module are discussed. Some characterizations of soft substructures of quantale modules are introduced. The detailed study of approximations of soft substructure in quantale module are presented. During this process, we have made further detailed discussion how soft substructures of one quantale module can be related to that of another quantale module under soft quantale homomorphism. Additionally, we describe the algebraic relationships between the upper (lower) approximations of soft substructures of quantale modules and the upper (lower) approximations of their homomorphic images using the concept of soft quantale module homomorphism. For further work, one can proceed this type of approximations to different algebraic structures especially to substructures of hyperquantales and fuzzy hypersubstructures of hyperquantales.

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Conflict of interest

The authors state that they are not involved in any conflict of interest.

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