AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11629–11650.
DOI: 10.3934/math.2023589
Received: 30 October 2022
Revised: 28 February 2023
Accepted: 02 March 2023
Published: 16 March 2023

Research article

Hyperbolic inequalities with a Hardy potential singular on the boundary of an annulus

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Abstract: We are concerned with the study of existence and nonexistence of weak solutions for a class of hyperbolic inequalities with a Hardy potential singular on the boundary ∂B_1 of the annulus $A = \left\{x \in \mathbb{R}^3 : 1 < |x| \leq 2\right\}$, where $\partial B_1 = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$. A singular potential function of the form $(|x| - 1)^{-\rho}$, $\rho \geq 0$, is considered in front of the power nonlinearity. Two types of inhomogeneous boundary conditions on $(0, \infty) \times \partial B_2$, $\partial B_2 = \left\{x \in \mathbb{R}^3 : |x| = 2\right\}$, are studied: Dirichlet and Neumann. We use a unified approach to show the optimal criteria of Fujita-type for each case.

Keywords: hyperbolic inequalities; Hardy potential; singularities; nonexistence
Mathematics Subject Classification: 35L70, 35A01, 35B44, 35B33

1. Introduction

Let $A = \left\{x \in \mathbb{R}^3 : 1 < |x| \leq 2\right\}$ and $\partial B_2 = \left\{x \in \mathbb{R}^3 : |x| = 2\right\}$. This work is devoted to the study of existence and nonexistence of weak solutions to hyperbolic inequalities of the form

$$u_{tt} - \Delta u + \frac{\mu}{(|x| - 1)^2} u \geq (|x| - 1)^{-\rho} |u|^p \quad \text{in } (0, \infty) \times A, \tag{1.1}$$

where $u = u(t, x)$, $\mu \geq \mu^* = -\frac{1}{4}$, $\rho \geq 0$ and $p > 1$. Problem (1.1) is studied under two types of inhomogeneous boundary conditions on ∂B_2: the Dirichlet-type boundary condition

$$u \geq f \quad \text{on } (0, \infty) \times \partial B_2 \tag{1.2}$$

and the Neumann-type boundary condition

$$\frac{\partial u}{\partial \nu} \geq f \quad \text{on } (0, \infty) \times \partial B_2, \tag{1.3}$$

where $f = f(x) \in L^1(\partial B_2)$ and ν is the outward unit normal vector on ∂B_2, relative to A. Notice that the constant μ^* appears in the Hardy inequality that involves the Hardy potential corresponding to the distance to the boundary of an annulus, see Marcus, Mizel and Pinchover [1].

Several authors contributed in the study of the large-time behavior of solutions to the nonlinear wave equation

$$u_{tt} - \Delta u = |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.4}$$

where $p > 1$, see e.g., [2–10]. Thanks to these contributions, it is known that for all $N \geq 2$, there exists a Fujita-type critical exponent $p_S(N)$ for the global existence question to (1.4) with compactly supported data, that is the positive root of the polynomial

$$(N-1)p^2 - (N+1)p - 2 = 0.$$

More precisely,

(i) for any $(u, u_t)|_{t=0}$ compactly supported with positive average, if $1 < p \leq p_S(N)$, then the solution to (1.4) blows-up in a finite time;
(ii) if $p > p_S(N)$, then there are compactly supported initial conditions $(u, u_t)|_{t=0}$ such that the solution to (1.4) exists globally in time.

In [11], Kato considered for the first time the wave inequality

$$u_{tt} - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.5}$$

where $p > 1$. He found another critical exponent $p_K(N) = \frac{N+1}{N-1}$, $N \geq 2$. Pohozaev and Véron [12] generalized Kato's work and established the sharpness of $p_K(N)$ for problem (1.5). Namely, it was shown that,

(i) if $1 < p \leq p_K(N)$, there is no global weak solution to (1.5) satisfying

$$\int_{\mathbb{R}^N} u_t(0,x)\,dx > 0; \tag{1.6}$$

(ii) if $p > p_K(N)$, then (1.5) admits global positive solutions satisfying (1.6).

Hamidi and Laptev [13] investigated the hyperbolic inequality

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\lambda}{|x|^2}u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ \\ u_t(0,x) \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $N \geq 3$ and $\lambda \geq -\left(\frac{N-2}{2}\right)^2$. It was proven that, if one of the following assumptions is satisfied:

$$\lambda \geq 0,\ 1 < p \leq 1 + \frac{2}{1+s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0,\ 1 < p \leq 1 + \frac{2}{1-s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2},\ s_* = s^* + 2 - N,$$

then (1.7) admits no nontrivial weak solution. Other contributions related to existence and nonexistence of solutions to wave inequalities in $\mathbb{R}^N$ with different types of nonlinearities can be found in [12,14–18]. We also refer to [19–21], where the issue of nonexistence for wave inequalities has been investigated in Riemannian manifolds.

It is natural to ask about the behavior of solutions to wave equations or inequalities in other unbounded domains of $\mathbb{R}^N$. For instance, in [22], the authors investigated hyperbolic inequalities of the form

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\lambda}{|x|^2} u \geq |u|^p & \text{in } (0,\infty) \times \mathbb{R}^N \backslash B_1, \\ \alpha \dfrac{\partial u}{\partial \nu}(t,x) + \beta u(t,x) \geq w(x) & \text{on } (0,\infty) \times \partial B_1, \end{cases} \tag{1.8}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geq 2$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0,0)$ and ν is the outward unit normal vector on ∂B_1, relative to $\Omega = \mathbb{R}^N \backslash B_1$. Namely, it was shown that (1.8) admits a Fujita-type critical exponent

$$p_c(\lambda, N) = \begin{cases} \infty & \text{if } \ N - 2 + 2\lambda_N = 0, \\ 1 + \frac{4}{N-2+2\lambda_N} & \text{if } \ N - 2 + 2\lambda_N > 0, \end{cases}$$

where

$$\lambda_N = \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2}.$$

More precisely, it was proven that,

(i) if $1 < p < p_c(\lambda, N)$ and $\displaystyle\int_{\partial B_1} w(x)\, dS_x > 0$, then (1.8) admits no global weak solution;
(ii) if $p > p_c(\lambda, N)$, then (1.8) admits global solutions for some $w > 0$.

For other results related to evolution inequalities in an exterior domain of $\mathbb{R}^N$, see e.g., [23–27] and the references therein.

Very recently, in [28], we investigated problem (1.1) in the case $\mu = 0$, $\rho > 2$ and $A \subset \mathbb{R}^N$, $N \geq 2$. Namely, among author results, we proved that,

(i) if $1 < p < \rho - 1$, then problem (1.1) (with $\mu = 0$) under the Dirichlet-type boundary condition (1.2), or the Neumann-type boundary condition (1.3), admits no weak solution, provided $f \in L^1(\partial B_2)$ and $\int_{\partial B_2} f(x)\, dS_x > 0$;
(ii) if $p > \rho - 1$, then both problems (1.1) (with $\mu = 0$), (1.2) and (1.1) (with $\mu = 0$), (1.3) admit (sationary) solutions for some $f > 0$.

To the best of our knowledge, no other results dealing with hyperbolic inequalities in an annulus have been obtained in the literature. Our aim in this paper is to study the influence of the parameter μ as well as the singularity on the boundary appearing in the Hardy potential on the critical behavior of problem (1.1) with $\mu = 0$, previously studied in [28].

Before presenting our main results, let us define weak solutions to the considered problems. Let

$$\Omega = (0,\infty) \times A, \quad \Gamma = (0,\infty) \times \partial B_2.$$

Notice that $\Gamma \subset \Omega$.

For problem (1.1) under the Dirichlet-type boundary condition (1.2), we define admissible trial functions as follows.

Definition 1.1. We say that $\varphi = \varphi(t,x)$ is an admissible trial function, if

(i) $\varphi \in C^2(\Omega)$, $\mathrm{supp}(\varphi) \subset\subset \Omega$, $\varphi \geq 0$;
(ii) $\varphi = 0$ on Γ;
(iii) $\dfrac{\partial \varphi}{\partial \nu} \leq 0$ on Γ.

By standard integration by parts, we define weak solutions to (1.1), (1.2) as follows.

Definition 1.2. We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), (1.2), if

$$\int_\Omega (|x|-1)^{-\rho}|u|^p \varphi\, dx\, dt - \int_\Gamma \frac{\partial \varphi}{\partial \nu} f(x)\, dS_x\, dt \leq \int_\Omega u \left(\varphi_{tt} - \Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi \right) dx\, dt \tag{1.9}$$

for every admissible trial function $\varphi = \varphi(t,x)$.

For problem (1.1) under the Neumann-type boundary condition (1.3), admissible trial functions are defined as follows.

Definition 1.3. We say that $\psi = \psi(t,x)$ is an admissible trial function, if

(i) $\psi \in C^2(\Omega)$, $\mathrm{supp}(\psi) \subset\subset \Omega$, $\psi \geq 0$;
(ii) $\dfrac{\partial \psi}{\partial \nu} = 0$ on Γ.

Weak solutions to (1.1), (1.3) are defined as follows.

Definition 1.4. We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), (1.3), if

$$\int_\Omega (|x|-1)^{-\rho}|u|^p \psi\, dx\, dt + \int_\Gamma f(x)\psi\, dS_x\, dt \leq \int_\Omega u \left(\psi_{tt} - \Delta\psi + \frac{\mu}{(|x|-1)^2}\psi \right) dx\, dt \tag{1.10}$$

for every admissible trial function $\psi = \psi(t,x)$.

For $\mu \geq -\frac{1}{4}$, we introduce the parameter

$$\sigma = \frac{1}{2} - \sqrt{\mu + \frac{1}{4}}. \tag{1.11}$$

Our main results are stated in the following theorem.

Theorem 1.5. *Let $\mu \geq -\frac{1}{4}$ and $\rho \geq 0$.*

(I) *Let* $f \in L^1(\partial B_2)$ *and* $\int_{\partial B_2} f(x)\, dS_x > 0$. *If*

$$\rho > 2 \quad \textit{and} \quad 1 < p < 1 + \frac{\rho - 2}{1 - \sigma}, \tag{1.12}$$

then problem (1.1) *under the Dirichlet-type boundary condition* (1.2)*, or the Neumann-type boundary condition* (1.3)*, admits no weak solution.*

(II) *If*

$$p > \max\left\{1, 1 + \frac{\rho - 2}{1 - \sigma}\right\}, \tag{1.13}$$

then both problems (1.1)*,* (1.2) *and* (1.1)*,* (1.3) *admit (sationary) solutions for some* $f > 0$.

Remark 1.6. By a stationary solution to (1.1), (1.2) (resp. (1.1), (1.3)), we mean a function $u = u(x)$ that verifies (1.1), (1.2) (resp. (1.1), (1.3)) for almost everywhere x.

The proofs of the nonexistence results provided by part (I) of Theorem 1.5 are based on nonlinear capacity estimates specifically adapted to the operator

$$-\Delta + \frac{\mu}{(|x| - 1)^2},$$

the domain and the considered boundary conditions. Namely, for each problem, we derive integral estimates involving a specially constructed class of admissible trial functions in the sense of Definitions 1.1 and 1.3. We refer to [17, 27] for a general account of these methods. The existence results are established by the construction of explicit solutions.

Remark 1.7. (i) In the case $\rho > 2$, we deduce that both problems (1.1), (1.2) and (1.1), (1.3) have the same Fujita critical exponent given by

$$p^*(\mu, \rho) = 1 + \frac{\rho - 2}{1 - \sigma}.$$

In the special case $\mu = 0$, one has $p^*(0, \rho) = \rho - 1$, which is the Fujita critical exponent for problems (1.1) (with $\mu = 0$), (1.2) and (1.1) (with $\mu = 0$), (1.3), recently obtained in [28].

(ii) By (1.13), we deduce that in the case $0 \leq \rho \leq 2$, both problems (1.1), (1.2) and (1.1), (1.3) admit no critical behaviors (solutions exist for all $p > 1$).

Remark 1.8. (i) The critical case

$$\rho > 2, \qquad p = 1 + \frac{\rho - 2}{1 - \sigma}$$

for problem (1.1) is not investigated here. It should be interesting to decide whether this case belongs to the blow-up situation.

(ii) In this work, problem (1.1) is investigated in the three dimensional case. It will be interesting to study the N-dimensional case for any $N \geq 2$.

Clearly, Theorem 1.5 yields existence and nonexistence results for the corresponding elliptic inequality

$$-\Delta u + \frac{\mu}{(|x| - 1)^2} u \geq (|x| - 1)^{-\rho} |u|^p \quad \text{in } A \tag{1.14}$$

under the Dirichlet-type boundary condition

$$u(x) \geq f(x) \quad \text{on } \partial B_2 \tag{1.15}$$

or the Neumann-type boundary condition

$$\frac{\partial u}{\partial \nu}(x) \geq f(x) \quad \text{on } \partial B_2. \tag{1.16}$$

Corollary 1.9. *Let $\mu \geq -\frac{1}{4}$ and $\rho \geq 0$.*

(I) Let $f \in L^1(\partial B_2)$ and $\displaystyle\int_{\partial B_2} f(x)\, dS_x > 0$. If (1.12) *holds, then problem* (1.14) *under the Dirichlet-type boundary condition* (1.15)*, or the Neumann-type boundary condition* (1.16)*, admits no weak solution.*

(II) If (1.13) *holds, then both problems* (1.14), (1.15) *and* (1.14), (1.16) *admit solutions for some $f > 0$.*

The rest of the paper is organized as follows. In Section 2, we study the Dirichlet-type boundary condition (1.2). Namely, we introduce a class of admissible trial functions in the sense of Definition 1.1 and establish some useful integral estimates involving such functions. Section 3 is devoted to the study of the Neumann-type boundary condition (1.3). As in Section 2, a class of admissible trial functions in the sense of Definition 1.3 is introduced and some useful estimates are provided. The proof of Theorem 1.5 is given in Section 4.

Throughout this paper, C denotes always a generic positive constant, which is independent of the scaling parameters T and R, and the solution u. Its value could be changed from one line to another.

2. The Dirichlet-type boundary condition

Let $\mu \geq -\frac{1}{4}$, $\rho \geq 0$ and $p > 1$. For $0 < \delta_1 < \delta_2$, we denote by $A(\delta_1, \delta_2)$ the subset of $\mathbb{R}^3$ defined by

$$A(\delta_1, \delta_2) = \left\{ x \in \mathbb{R}^3 : \delta_1 < |x| \leq \delta_2 \right\}.$$

2.1. Admissible trial functions

We introduce the nonnegative function $\mathcal{D}$ defined in A by

$$\mathcal{D}(x) = \begin{cases} |x|^{-1}\sqrt{|x|-1}\,\ln\left(\frac{1}{|x|-1}\right) & \text{if } \mu = -\frac{1}{4}, \\ |x|^{-1}(|x|-1)^{\sigma}\left(1-(|x|-1)^{1-2\sigma}\right) & \text{if } \mu > -\frac{1}{4}, \end{cases} \tag{2.1}$$

where the parameter σ is given by (1.11). Elementary calculations show that

$$-\Delta\mathcal{D} + \frac{\mu}{(|x|-1)^2}\mathcal{D} = 0 \text{ in } A, \quad \mathcal{D} = 0 \text{ on } \partial B_2. \tag{2.2}$$

Let $\zeta \in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$\zeta \geq 0, \quad \operatorname{supp}(\zeta) \subset\subset (0,1). \tag{2.3}$$

For $T > 0$ and sufficiently large ℓ, let

$$\zeta_T(t) = \zeta^\ell\left(\frac{t}{T}\right), \quad t > 0. \tag{2.4}$$

Let $\xi \in C^\infty([0,\infty))$ be an increasing function satisfying

$$\xi(s) = 0 \text{ if } 0 \le s \le \frac{1}{2}, \quad \xi(s) = 1 \text{ if } s \ge 1. \tag{2.5}$$

For sufficiently large R, let

$$\xi_R(x) = \mathcal{D}(x)\xi^\ell(R(|x|-1)), \quad x \in A. \tag{2.6}$$

We introduce functions of the form

$$\varphi(t,x) = \zeta_T(t)\xi_R(x), \quad (t,x) \in \Omega. \tag{2.7}$$

Lemma 2.1. *For $T > 0$ and sufficiently large R and ℓ, the function φ defined by* (2.7) *is an admissible trial function for problems* (1.1) *and* (1.2).

Proof. The properties (i) and (ii) of Definition 1.1 follow immediately from (2.1)–(2.7). On the other hand, by (2.5) and (2.6), one has

$$\xi_R(x) = \mathcal{D}(x), \quad x \in A\left(1+\frac{1}{R}, 2\right). \tag{2.8}$$

Moreover, in view of (2.1), (2.7) and (2.8), for all $(t,x) \in \Gamma$, we obtain

$$\frac{\partial\varphi}{\partial\nu}(t,x) = -\frac{1}{2}\zeta_T(t) \times \begin{cases} 1 & \text{if } \mu = -\frac{1}{4}, \\ (1-2\sigma) & \text{if } \mu > -\frac{1}{4}. \end{cases} \tag{2.9}$$

Since $\zeta_T \ge 0$ and $1 - 2\sigma > 0$, we deduce by (2.9) that $\dfrac{\partial\varphi}{\partial\nu} \le 0$ on Γ, which shows that the property (iii) of Definition 1.1 is satisfied. □

2.2. *Preliminary estimates*

For $T > 0$ and sufficiently large R and ℓ, let φ be the function defined by (2.7). In this subsection, some useful integral estimates involving the function φ are provided. We consider separately the cases $\mu = -\frac{1}{4}$ and $\mu > -\frac{1}{4}$.

2.2.1. The case $\mu = -\frac{1}{4}$

Lemma 2.2. *Let $\mu = -\frac{1}{4}$. The following estimate holds:*

$$\int_{\text{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} \, dx\, dt \le C T^{1-\frac{2p}{p-1}} \ln R. \tag{2.10}$$

Proof. By (2.3)–(2.7), we obtain

$$\begin{aligned}
&\int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}}\, dx\, dt \\
&= \left(\int_0^T \zeta_T^{\frac{-1}{p-1}}(t) |\zeta_T''(t)|^{\frac{p}{p-1}}\, dt \right) \left(\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R(x)\, dx \right).
\end{aligned} \tag{2.11}$$

On the other hand, one has

$$|\zeta_T''(t)| \le CT^{-2}\zeta^{\ell-2}\left(\frac{t}{T}\right), \quad 0 < t < T,$$

which yields

$$\begin{aligned}
\int_0^T \zeta_T^{\frac{-1}{p-1}}(t) |\zeta_T''(t)|^{\frac{p}{p-1}}\, dt &\le CT^{\frac{-2p}{p-1}} \int_0^T \zeta^{\ell-\frac{2p}{p-1}}\left(\frac{t}{T}\right) dt \\
&= CT^{1-\frac{2p}{p-1}} \int_0^1 \zeta^{\ell-\frac{2p}{p-1}}(s)\, ds,
\end{aligned}$$

that is,

$$\int_0^T \zeta_T^{\frac{-1}{p-1}}(t) |\zeta_T''(t)|^{\frac{p}{p-1}}\, dt \le CT^{1-\frac{2p}{p-1}}. \tag{2.12}$$

Moreover, by (2.1) and (2.6), we get

$$\begin{aligned}
&\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R(x)\, dx \\
&= \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \mathcal{D}(x) \xi^{\ell}(R(|x|-1))\, dx \\
&\le \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} |x|^{-1} \sqrt{|x|-1} \ln\left(\frac{1}{|x|-1}\right) dx \\
&\le C \ln R.
\end{aligned} \tag{2.13}$$

Hence, (2.10) follows from (2.11)–(2.13). □

Lemma 2.3. *Let $\mu = -\frac{1}{4}$. The following estimate holds:*

$$\int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} \left| -\Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi \right|^{\frac{p}{p-1}} dx\, dt \le CTR^{\frac{p-2\rho+3}{2(p-1)}} \ln R. \tag{2.14}$$

Proof. By (2.3), (2.4) and (2.7), we obtain

$$\begin{aligned}
&\int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} \left| -\Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi \right|^{\frac{p}{p-1}} dx\, dt \\
&= \left(\int_0^T \zeta_T(t)\, dt \right) \left(\int_{\mathrm{supp}(\xi_R)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R^{\frac{-1}{p-1}}(x) \left| -\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R \right|^{\frac{p}{p-1}} dx \right).
\end{aligned} \tag{2.15}$$

On the other hand, we have

$$\begin{aligned}\int_0^T \zeta_T(t)\,dt &= \int_0^T \zeta^\ell\left(\frac{t}{T}\right)dt\\ &= T\int_0^1 \zeta^\ell(s)\,ds,\end{aligned}$$

that is,

$$\int_0^T \zeta_T(t)\,dt = CT. \tag{2.16}$$

Moreover, by (2.6), for $x \in \operatorname{supp}(\xi_R)$, one has

$$\begin{aligned}&-\Delta(\xi_R(x)) + \frac{\mu}{(|x|-1)^2}\xi_R(x)\\ &= -\Delta\left[\mathcal{D}(x)\xi^\ell(R(|x|-1))\right] + \frac{\mu}{(|x|-1)^2}\xi_R(x)\\ &= -\xi^\ell(R(|x|-1))\Delta\mathcal{D}(x) - \mathcal{D}(x)\Delta\left[\xi^\ell(R(|x|-1))\right] - 2\nabla\mathcal{D}(x)\cdot\nabla\left[\xi^\ell(R(|x|-1))\right]\\ &\quad + \frac{\mu}{(|x|-1)^2}\mathcal{D}(x)\xi^\ell(R(|x|-1))\\ &= \xi^\ell(R(|x|-1))\left(-\Delta\mathcal{D}(x) + \frac{\mu}{(|x|-1)^2}\mathcal{D}(x)\right) - \mathcal{D}(x)\Delta\left[\xi^\ell(R(|x|-1))\right]\\ &\quad - 2\nabla\mathcal{D}(x)\cdot\nabla\left[\xi^\ell(R(|x|-1))\right].\end{aligned}$$

Taking in consideration (2.2), we get

$$-\Delta(\xi_R(x)) + \frac{\mu}{(|x|-1)^2}\xi_R(x) = -\mathcal{D}(x)\Delta\left[\xi^\ell(R(|x|-1))\right] - 2\nabla\mathcal{D}(x)\cdot\nabla\left[\xi^\ell(R(|x|-1))\right], \tag{2.17}$$

which implies par (2.5) that

$$\begin{aligned}&\int_{\operatorname{supp}(\xi_R)} (|x|-1)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}}dx\\ &= \int_{A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)} (|x|-1)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}}dx.\end{aligned} \tag{2.18}$$

On the other hand, by (2.1) and (2.5), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we obtain

$$\mathcal{D}(x) \le CR^{\frac{-1}{2}}\ln R,\ \ |\nabla\mathcal{D}(x)| \le CR^{\frac{1}{2}}\ln R \tag{2.19}$$

and

$$\left|\Delta\left[\xi^\ell(R(|x|-1))\right]\right| \le CR^2\xi^{\ell-2}(R(|x|-1)),\ \ \left|\nabla\left[\xi^\ell(R(|x|-1))\right]\right| \le CR\xi^{\ell-1}(R(|x|-1)). \tag{2.20}$$

Then, (2.17), (2.19), (2.20) and Cauchy-Schwarz inequality yield

$$\left|-\Delta(\xi_R(x)) + \frac{\mu}{(|x|-1)^2}\xi_R(x)\right| \le CR^{\frac{3}{2}}\ln R\,\xi^{\ell-2}(R(|x|-1)). \tag{2.21}$$

Next, using (2.1), (2.6) and (2.21), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we obtain

$$\begin{aligned}
\xi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}} &\leq CR^{\frac{3p}{2(p-1)}}(\ln R)^{\frac{p}{p-1}}\mathcal{D}^{\frac{-1}{p-1}}(x)\xi^{\ell-\frac{2p}{p-1}}(R(|x|-1)) \\
&\leq CR^{\frac{3p}{2(p-1)}}(\ln R)^{\frac{p}{p-1}}R^{\frac{1}{2(p-1)}}(\ln R)^{\frac{-1}{p-1}} \\
&= CR^{\frac{3p+1}{2(p-1)}}\ln R.
\end{aligned}$$

Using (2.18) and integrating over $A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we get

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)}(|x|-1)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}}\,dx \\
&\leq CR^{\frac{3p+1}{2(p-1)}}\ln R\int_{1+\frac{1}{2R}<|x|<1+\frac{1}{R}}(|x|-1)^{\frac{\rho}{p-1}}\,dx \\
&= CR^{\frac{3p+1}{2(p-1)}}\ln R\int_{r=1+\frac{1}{2R}}^{1+\frac{1}{R}}(r-1)^{\frac{\rho}{p-1}}r^2\,dr \\
&\leq CR^{\frac{3p+1}{2(p-1)}}\ln R\,R^{-1-\frac{\rho}{p-1}} \\
&= CR^{\frac{p-2\rho+3}{2(p-1)}}\ln R.
\end{aligned} \tag{2.22}$$

Thus, (2.15), (2.16) and (2.22) yield (2.14). □

2.2.2. The case $\mu > -\frac{1}{4}$

Lemma 2.4. *Let $\mu > -\frac{1}{4}$. The following estimate holds:*

$$\int_{\mathrm{supp}(\varphi)}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt \leq CT^{1-\frac{2p}{p-1}}\left(\ln R + R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)}\right). \tag{2.23}$$

Proof. Following the proof of Lemma 2.2, we have

$$\int_{\mathrm{supp}(\varphi)}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt \leq CT^{1-\frac{2p}{p-1}}\int_{A\left(1+\frac{1}{2R},2\right)}(|x|-1)^{\frac{\rho}{p-1}}\xi_R(x)\,dx. \tag{2.24}$$

On the other hand, by (2.1) and (2.6), we get

$$
\begin{aligned}
&\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R(x)\,dx \\
&= \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \mathcal{D}(x) \xi^{\ell}(R(|x|-1))\,dx \\
&\le \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}+\sigma} |x|^{-1} \left(1-(|x|-1)^{1-2\sigma}\right) dx \\
&\le \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}+\sigma}\,dx \\
&\le C \int_{r=\frac{1}{2R}}^{1} r^{\frac{\rho}{p-1}+\sigma}\,dr \\
&\le C \times \begin{cases} \ln R & \text{if } \frac{\rho}{p-1}+\sigma=-1, \\ 1 & \text{if } \frac{\rho}{p-1}+\sigma>-1, \\ R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)} & \text{if } \frac{\rho}{p-1}+\sigma<-1, \end{cases}
\end{aligned}
$$

which yields

$$
\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)}\right). \tag{2.25}
$$

Combining (2.24) with (2.25), we obtain (2.23). □

Lemma 2.5. *Let $\mu > -\frac{1}{4}$. The following estimate holds:*

$$
\int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} \left|-\Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi\right|^{\frac{p}{p-1}} dx\,dt \le CTR^{\frac{(1-\sigma)p+\sigma-\rho+1}{p-1}}. \tag{2.26}
$$

Proof. Following the proof of Lemma 2.3, we obtain

$$
\begin{aligned}
&\int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}} \varphi^{\frac{-1}{p-1}} \left|-\Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi\right|^{\frac{p}{p-1}} dx\,dt \\
&\le CT \int_{A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}} dx.
\end{aligned} \tag{2.27}
$$

On the other hand, by (2.1), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we get

$$
\mathcal{D}(x) \le CR^{-\sigma}, \quad |\nabla\mathcal{D}(x)| \le CR^{1-\sigma}. \tag{2.28}
$$

Hence, using (2.17), (2.20) and (2.28), we obtain

$$
\left|-\Delta(\xi_R(x)) + \frac{\mu}{(|x|-1)^2}\xi_R(x)\right| \le CR^{2-\sigma}\xi^{\ell-2}(R(|x|-1)). \tag{2.29}
$$

Next, using (2.1), (2.6) and (2.28), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we obtain

$$
\xi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}} \quad \le \quad CR^{\frac{(2-\sigma)p}{p-1}} \mathcal{D}^{\frac{-1}{p-1}}(x) \xi^{\ell-\frac{2p}{p-1}}(R(|x|-1))
$$

$$\begin{aligned} &\leq CR^{\frac{(2-\sigma)p}{p-1}} R^{\frac{\sigma}{p-1}} \\ &= CR^{\frac{(2-\sigma)p+\sigma}{p-1}}. \end{aligned}$$

Integrating, we obtain

$$\begin{aligned} &\int_{A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)} (|x|-1)^{\frac{\rho}{p-1}} \xi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\xi_R + \frac{\mu}{(|x|-1)^2}\xi_R\right|^{\frac{p}{p-1}} dx \\ &\leq CR^{\frac{(2-\sigma)p+\sigma}{p-1}} \int_{A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)} (|x|-1)^{\frac{\rho}{p-1}} dx \\ &\leq CR^{\frac{(2-\sigma)p+\sigma}{p-1}} R^{-\left(\frac{\rho}{p-1}+1\right)} \\ &= CR^{\frac{(1-\sigma)p+\sigma-\rho+1}{p-1}}. \end{aligned} \tag{2.30}$$

Hence, in view of (2.27) and (2.30), we get (2.26). □

3. The Neumann-type boundary condition

Let $\mu \geq -\frac{1}{4}$, $\rho \geq 0$ and $p > 1$.

3.1. *Admissible trial functions*

We introduce the nonnegative function $\mathcal{N}$ defined in A by

$$\mathcal{N}(x) = \begin{cases} |x|^{-1}\sqrt{|x|-1} & \text{if} \quad \mu = -\frac{1}{4}, \\ |x|^{-1}(|x|-1)^{\sigma}\left(1+(|x|-1)^{1-2\sigma}\right) & \text{if} \quad \mu > -\frac{1}{4}, \end{cases} \tag{3.1}$$

where the parameter σ is given by (1.11). Elementary calculations show that

$$-\Delta\mathcal{N} + \frac{\mu}{(|x|-1)^2}\mathcal{N} = 0 \text{ in } A, \quad \frac{\partial\mathcal{N}}{\partial\nu} = 0 \text{ on } \partial B_2. \tag{3.2}$$

For $T > 0$ and sufficiently large R and ℓ, we introduce functions of the form

$$\psi(t,x) = \zeta_T(t)\chi_R(x), \quad (t,x) \in \Omega, \tag{3.3}$$

where the function ζ_T is defined by (2.4),

$$\chi_R(x) = \mathcal{N}(x)\xi^{\ell}(R(|x|-1)), \quad x \in A \tag{3.4}$$

and the function $\xi \in C^\infty([0,\infty))$ satisfies (2.5).

Lemma 3.1. *For $T > 0$ and sufficiently large R and ℓ, the function ψ defined by* (3.3) *is an admissible trial function for problem* (1.1)*,* (1.3)*.*

Proof. The property (i) of Definition 1.3 follow immediately from (2.3)–(2.5), (3.1), (3.3) and (3.4). On the other hand, by (2.5) and (3.4), one has

$$\chi_R(x) = \mathcal{N}(x), \quad x \in A\left(1+\frac{1}{R},2\right). \tag{3.5}$$

Thus, by (3.2) and (3.3), we obtain

$$\frac{\partial\psi}{\partial\nu}(t,x) = \zeta_T(t)\frac{\partial\mathcal{N}}{\partial\nu}(x) = 0, \quad (t,x) \in \Gamma,$$

which shows that the property (ii) of Definition 1.3 is satisfied. □

3.2. Preliminary estimates

For $T > 0$ and sufficiently large R and ℓ, let ψ be the function defined by (3.3). As for the Dirichlet case, some useful integral estimates involving the function ψ are provided in this subsection. The cases $\mu = -\frac{1}{4}$ and $\mu > -\frac{1}{4}$ are considered separately.

3.2.1. The case $\mu = -\frac{1}{4}$

Lemma 3.2. *Let $\mu = -\frac{1}{4}$. The following estimate holds:*

$$\int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\psi_{tt}|^{\frac{p}{p-1}} \, dx\, dt \le C T^{1-\frac{2p}{p-1}}. \tag{3.6}$$

Proof. By (2.3)–(2.5), (3.4) and (3.3), we obtain

$$\begin{aligned} &\int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\psi_{tt}|^{\frac{p}{p-1}} \, dx\, dt \\ &= \left(\int_0^T \zeta_T^{\frac{-1}{p-1}}(t) |\zeta_T''(t)|^{\frac{p}{p-1}} \, dt\right)\left(\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R(x)\, dx\right). \end{aligned} \tag{3.7}$$

Moreover, by (3.1) and (3.4), we get

$$\begin{aligned} &\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R(x)\, dx \\ &= \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \mathcal{N}(x) \xi^{\ell}(R(|x|-1))\, dx \\ &\le \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} |x|^{-1} \sqrt{|x|-1}\, dx \\ &\le C. \end{aligned} \tag{3.8}$$

Hence, in view of (2.12), (3.7) and (3.8), we obtain (3.6). □

Lemma 3.3. *Let $\mu = -\frac{1}{4}$. The following estimate holds:*

$$\int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} \left|-\Delta\psi + \frac{\mu}{(|x|-1)^2}\psi\right|^{\frac{p}{p-1}} \, dx\, dt \le CTR^{\frac{p-2\rho+3}{2(p-1)}}. \tag{3.9}$$

Proof. By (2.3), (2.4) and (3.3), we obtain

$$\begin{aligned} &\int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} \left|-\Delta\psi + \frac{\mu}{(|x|-1)^2}\psi\right|^{\frac{p}{p-1}} \, dx\, dt \\ &= \left(\int_0^T \zeta_T(t)\, dt\right)\left(\int_{\mathrm{supp}(\chi_R)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\chi_R + \frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}} \, dx\right). \end{aligned} \tag{3.10}$$

Moreover, by (3.2) and (3.4), for $x \in \mathrm{supp}(\chi_R)$, we get

$$-\Delta(\chi_R(x)) + \frac{\mu}{(|x|-1)^2}\chi_R(x) = -\mathcal{N}(x)\Delta\left[\xi^{\ell}(R(|x|-1))\right] - 2\nabla\mathcal{N}(x)\cdot\nabla\left[\xi^{\ell}(R(|x|-1))\right], \tag{3.11}$$

which implies par (2.5) that

$$\begin{aligned}
&\int_{\operatorname{supp}(\chi_R)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\chi_R + \frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}} dx \\
&= \int_{A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\chi_R + \frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}} dx.
\end{aligned} \tag{3.12}$$

On the other hand, by (2.5) and (3.1), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we obtain

$$\mathcal{N}(x) \le CR^{\frac{-1}{2}}, \quad |\nabla\mathcal{N}(x)| \le CR^{\frac{1}{2}}. \tag{3.13}$$

Using (2.20), (3.11) and (3.13), we deduce that

$$\left|-\Delta(\chi_R(x)) + \frac{\mu}{(|x|-1)^2}\chi_R(x)\right| \le CR^{\frac{3}{2}}\xi^{\ell-2}(R(|x|-1)). \tag{3.14}$$

Next, by (3.1), (3.4) and (3.14), for all $x \in A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we obtain

$$\chi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\chi_R + \frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}} \le CR^{\frac{3p+1}{2(p-1)}}.$$

Using (3.12) and integrating over $A\left(1+\frac{1}{2R}, 1+\frac{1}{R}\right)$, we get

$$\begin{aligned}
&\int_{\operatorname{supp}(\chi_R)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R^{\frac{-1}{p-1}}(x) \left|-\Delta\chi_R + \frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}} dx \\
&\le CR^{\frac{3p+1}{2(p-1)}} \int_{1+\frac{1}{2R}<|x|<1+\frac{1}{R}} (|x|-1)^{\frac{\rho}{p-1}}\, dx \\
&\le CR^{\frac{3p+1}{2(p-1)}} R^{-1-\frac{\rho}{p-1}} \\
&= CR^{\frac{p-2\rho+3}{2(p-1)}}.
\end{aligned} \tag{3.15}$$

In view of (2.16), (3.10) and (3.15), we obtain (3.9). □

3.2.2. The case $\mu > -\frac{1}{4}$

Lemma 3.4. *Let $\mu > -\frac{1}{4}$. The following estimate holds:*

$$\int_{\operatorname{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\psi_{tt}|^{\frac{p}{p-1}}\, dx\, dt \le CT^{1-\frac{2p}{p-1}} \left(\ln R + R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)}\right). \tag{3.16}$$

Proof. Following the proof of Lemma 3.2, we have

$$\int_{\operatorname{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\psi_{tt}|^{\frac{p}{p-1}}\, dx\, dt \le CT^{1-\frac{2p}{p-1}} \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R(x)\, dx. \tag{3.17}$$

On the other hand, by (3.1) and (3.4), we obtain

$$\int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}} \chi_R(x)\, dx \quad \le \quad \int_{A\left(1+\frac{1}{2R},2\right)} (|x|-1)^{\frac{\rho}{p-1}+\sigma} |x|^{-1} \left(1 + (|x|-1)^{1-2\sigma}\right) dx$$

$$
\begin{aligned}
&\leq \int_{A\left(1+\frac{1}{2R},2\right)}(|x|-1)^{\frac{\rho}{p-1}+\sigma}\,dx\\
&\leq C\left(\ln R+R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)}\right).
\end{aligned}\tag{3.18}
$$

Hence, (3.16) follows from (3.17) and (3.18). □

Lemma 3.5. *Let $\mu > -\frac{1}{4}$. The following estimate holds:*

$$
\int_{\mathrm{supp}(\psi)}(|x|-1)^{\frac{\rho}{p-1}}\psi^{\frac{-1}{p-1}}\left|-\Delta\psi+\frac{\mu}{(|x|-1)^2}\psi\right|^{\frac{p}{p-1}}\,dx\,dt \leq CTR^{\frac{(1-\sigma)p+\sigma-\rho+1}{p-1}}. \tag{3.19}
$$

Proof. Following the proof of Lemma 3.3, we obtain

$$
\begin{aligned}
&\int_{\mathrm{supp}(\psi)}(|x|-1)^{\frac{\rho}{p-1}}\psi^{\frac{-1}{p-1}}\left|-\Delta\psi+\frac{\mu}{(|x|-1)^2}\psi\right|^{\frac{p}{p-1}}\,dx\,dt\\
&\leq CT\int_{\mathrm{supp}(\chi_R)}(|x|-1)^{\frac{\rho}{p-1}}\chi_R^{\frac{-1}{p-1}}(x)\left|-\Delta\chi_R+\frac{\mu}{(|x|-1)^2}\chi_R\right|^{\frac{p}{p-1}}\,dx.
\end{aligned}
$$

On the other hand, by (3.1), for all $x\in A\left(1+\frac{1}{2R},1+\frac{1}{R}\right)$, we get

$$
\mathcal{N}(x)\leq CR^{-\sigma},\ \ |\nabla\mathcal{N}(x)|\leq CR^{1-\sigma}.
$$

The rest of the proof is similar to that of Lemma 2.5. □

4. Proof of the main results

4.1. *Problem* (1.1) *under the Dirichlet-type boundary condition* (1.2)

Proof of Theorem 1.5. (I) We use the contradiction argument. Namely, suppose that $u\in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), (1.2). Then, by (1.9) and Lemma 2.1, for sufficiently large T,R and ℓ, we deduce that

$$
\begin{aligned}
&\int_\Omega(|x|-1)^{-\rho}|u|^p\varphi\,dx\,dt-\int_\Gamma\frac{\partial\varphi}{\partial\nu}f(x)\,dS_x\,dt\\
&\leq\int_\Omega|u||\varphi_{tt}|\,dx\,dt+\int_\Omega|u|\left|-\Delta\varphi+\frac{\mu}{(|x|-1)^2}\varphi\right|\,dx\,dt,
\end{aligned}\tag{4.1}
$$

where $\varphi=\varphi(t,x)$ is the admissible trial function given by (2.7). On the other hand, making use of Young's inequality, we obtain

$$
\begin{aligned}
\int_\Omega|u||\varphi_{tt}|\,dx\,dt &= \int_\Omega\left[(|x|-1)^{\frac{-\rho}{p}}|u|\varphi^{\frac{1}{p}}\right]\left[(|x|-1)^{\frac{\rho}{p}}\varphi^{\frac{-1}{p}}|\varphi_{tt}|\right]\,dx\,dt\\
&\leq \frac{1}{2}\int_\Omega(|x|-1)^{-\rho}|u|^p\varphi\,dx\,dt+C\int_{\mathrm{supp}(\varphi)}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt.
\end{aligned}\tag{4.2}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_\Omega|u|\left|-\Delta\varphi+\frac{\mu}{(|x|-1)^2}\varphi\right|\,dx\,dt\\
&\leq\frac{1}{2}\int_\Omega(|x|-1)^{-\rho}|u|^p\varphi\,dx\,dt+C\int_{\mathrm{supp}(\varphi)}(|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}\left|-\Delta\varphi+\frac{\mu}{(|x|-1)^2}\varphi\right|^{\frac{p}{p-1}}\,dx\,dt.
\end{aligned}\tag{4.3}
$$

Hence, in view of (4.1)–(4.3), we get

$$\begin{aligned} -C\int_\Gamma \frac{\partial\varphi}{\partial\nu} f(x)\,dS_x\,dt &\leq \int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt \\ &\quad + \int_{\mathrm{supp}(\varphi)} (|x|-1)^{\frac{\rho}{p-1}}\varphi^{\frac{-1}{p-1}}\left|-\Delta\varphi + \frac{\mu}{(|x|-1)^2}\varphi\right|^{\frac{p}{p-1}}\,dx\,dt. \end{aligned} \tag{4.4}$$

Moreover, by (2.3), (2.4) and (2.9), there holds

$$\begin{aligned} -\int_\Gamma \frac{\partial\varphi}{\partial\nu} f(x)\,dS_x\,dt &= C\int_\Gamma \zeta_T(t) f(x)\,dS_x\,dt \\ &= C\left(\int_0^T \zeta_T(t)\,dt\right)\left(\int_{\partial B_2} f(x)\,dS_x\right). \end{aligned}$$

Hence, in view of (2.16), we deduce that

$$-\int_\Gamma \frac{\partial\varphi}{\partial\nu} f(x)\,dS_x\,dt = CT\int_{\partial B_2} f(x)\,dS_x. \tag{4.5}$$

Next, using Lemmas 2.2–2.5, (4.4) and (4.5), we obtain

$$T\int_{\partial B_2} f(x)\,dS_x \leq C\left[T^{1-\frac{2p}{p-1}}\left(\ln R + R^{-\left(\frac{\rho}{p-1}+\sigma+1\right)}\right) + TR^{\frac{(1-\sigma)p+\sigma-\rho+1}{p-1}}\ln R\right],$$

that is,

$$\int_{\partial B_2} f(x)\,dS_x \leq C\left(T^{-\frac{2p}{p-1}}\ln R + T^{-\frac{2p}{p-1}}R^{\lambda_1} + R^{\lambda_2}\ln R\right), \tag{4.6}$$

where

$$\lambda_1 = -\left(\frac{\rho}{p-1} + \sigma + 1\right) \tag{4.7}$$

and

$$\lambda_2 = \frac{(1-\sigma)p + \sigma - \rho + 1}{p-1}. \tag{4.8}$$

Taking $T = R^\eta$, where

$$\eta > \max\left\{0, \frac{\lambda_1(p-1)}{2p}\right\}, \tag{4.9}$$

(4.6) reduces to

$$\int_{\partial B_2} f(x)\,dS_x \leq C\left(R^{-\frac{2p\eta}{p-1}}\ln R + R^{\lambda_1-\frac{2p\eta}{p-1}} + R^{\lambda_2}\ln R\right). \tag{4.10}$$

Notice that due to the choice (4.9) of the parameter η, one has

$$\lambda_1 - \frac{2p\eta}{p-1} < 0.$$

Moreover, due to (1.12), one has $\lambda_2 < 0$. Hence, passing to the limit as $R \to \infty$ in (4.10), we get $\int_{\partial B_2} f(x)\,dS_x \leq 0$, which contradicts the positivity of $\int_{\partial B_2} f(x)\,dS_x$. Consequently, problems (1.1) and (1.2) admit no weak solution.

(II) We first consider the case $\mu = -\frac{1}{4}$. For $0 < \gamma < 1$, let

$$F(r) = r^{p-1}(r-1)^{\frac{2\rho-3-p}{2}}\left(1-\ln(r-1)\right)^{\gamma(1-p)-2}, \quad 1 < r \le 2.$$

Notice that due to (1.13), one has $2\rho - 3 - p < 0$. Moreover, $\gamma(1-p) - 2 < 0$. Hence,

$$\lim_{r \to 1^+} F(r) = +\infty,$$

which implies that there exists some constant $C_{\gamma,p,\rho} > 0$ such that

$$F(r) \ge C_{\gamma,p,\rho}, \quad 1 < r \le 2. \tag{4.11}$$

For

$$0 < \epsilon \le \left[\gamma(1-\gamma)C_{\gamma,p,\rho}\right]^{\frac{1}{p-1}}, \tag{4.12}$$

let

$$u_{\gamma,\epsilon}(x) = \epsilon|x|^{-1}\sqrt{|x|-1}\left(1-\ln(|x|-1)\right)^{\gamma}, \quad x \in A. \tag{4.13}$$

Elementary calculations show that

$$-\Delta u_{\gamma,\epsilon}(x) + \frac{\mu}{(|x|-1)^2}u_{\gamma,\epsilon}(x) = \epsilon\gamma(1-\gamma)|x|^{-1}(|x|-1)^{\frac{-3}{2}}\left(1-\ln(|x|-1)\right)^{\gamma-2}, \quad x \in A. \tag{4.14}$$

Hence, using (4.11)–(4.14), for all $x \in A$, we obtain

$$\begin{aligned}
&-\Delta u_{\gamma,\epsilon}(x) + \frac{\mu}{(|x|-1)^2}u_{\gamma,\epsilon}(x)\\
&= (|x|-1)^{-\rho}\left[\epsilon^p|x|^{-p}(|x|-1)^{\frac{p}{2}}\left(1-\ln(|x|-1)\right)^{\gamma p}\right]\epsilon^{1-p}\gamma(1-\gamma)F(|x|)\\
&\ge (|x|-1)^{-\rho}u^p_{\gamma,\epsilon}(x)\epsilon^{1-p}\gamma(1-\gamma)C_{\gamma,p,\rho}\\
&\ge (|x|-1)^{-\rho}u^p_{\gamma,\epsilon}(x).
\end{aligned}$$

Hence, we deduce that $u_{\gamma,\epsilon}$ is a stationary positive solution to (1.1), (1.2) with $f \equiv \frac{\epsilon}{2}$.

Next, we consider the case $\mu > -\frac{1}{4}$. For

$$\max\left\{\sigma, \frac{\rho-2}{p-1}\right\} < \delta < 1-\sigma \tag{4.15}$$

and

$$0 < \varepsilon \le \left[\delta(1-\delta)+\mu\right]^{\frac{1}{p-1}}, \tag{4.16}$$

let

$$u_{\delta,\varepsilon}(x) = \varepsilon|x|^{-1}(|x|-1)^{\delta}, \quad x \in A. \tag{4.17}$$

Notice that $1 - 2\sigma > 0$. Moreover, by (1.13), one has

$$\frac{\rho-2}{p-1} < 1-\sigma.$$

Thus, the set of δ satisfying (4.15) is nonempty. On the other hand, by (4.15), one has $\sigma < \delta < 1-\sigma$, which yields $\delta(1-\delta)+\mu > 0$. Elementary calculations show that

$$-\Delta u_{\delta,\varepsilon}(x) + \frac{\mu}{(|x|-1)^2} u_{\delta,\varepsilon}(x) = \varepsilon\left[\delta(1-\delta)+\mu\right]|x|^{-1}(|x|-1)^{\delta-2}, \quad x \in A. \tag{4.18}$$

Hence, using (4.15)–(4.18), for all $x \in A$, we obtain

$$\begin{aligned}
& -\Delta u_{\delta,\varepsilon}(x) + \frac{\mu}{(|x|-1)^2} u_{\delta,\varepsilon}(x) \\
&= (|x|-1)^{-\rho}\left[\varepsilon^p |x|^{-p}(|x|-1)^{\delta p}\right]\varepsilon^{1-p}\left[\delta(1-\delta)+\mu\right]|x|^{p-1}(|x|-1)^{\rho-2+\delta(1-p)} \\
&\geq (|x|-1)^{-\rho} u^p_{\delta,\varepsilon}(x)(|x|-1)^{\rho-2+\delta(1-p)} \\
&\geq (|x|-1)^{-\rho} u^p_{\delta,\varepsilon}(x).
\end{aligned}$$

Then, $u_{\delta,\varepsilon}$ is a stationary positive solution to (1.1), (1.2) with $f \equiv \frac{\varepsilon}{2}$. This completes the proof of Theorem 1.5 for problems (1.1) and (1.2). □

4.2. *Problem* (1.1) *under the Neumann-type boundary condition* (1.3)

Proof of Theorem 1.5. (I) We also argue by contradiction by supposing that $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1), (1.3). Then, by (1.10) and Lemma 3.1, for sufficiently large T, R and ℓ, we obtain

$$\begin{aligned}
& \int_\Omega (|x|-1)^{-\rho}|u|^p \psi \, dx\, dt + \int_\Gamma f(x)\psi \, dS_x \, dt \\
& \leq \int_\Omega |u||\psi_{tt}| \, dx\, dt + \int_\Omega |u| \left| -\Delta\psi + \frac{\mu}{(|x|-1)^2}\psi \right| dx\, dt,
\end{aligned}$$

where $\psi = \psi(t,x)$ is the admissible trial function given by (3.3). Proceeding as in the Dirichlet case, by means of Young's inequality, we get

$$\begin{aligned}
C\int_\Gamma f(x)\psi \, dS_x \, dt \quad \leq \quad & \int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\psi_{tt}|^{\frac{p}{p-1}} \, dx\, dt \\
& + \int_{\mathrm{supp}(\psi)} (|x|-1)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} \left| -\Delta\psi + \frac{\mu}{(|x|-1)^2}\psi \right|^{\frac{p}{p-1}} dx\, dt.
\end{aligned} \tag{4.19}$$

On the other hand, by (3.1), (3.3) and (3.5), one has

$$\begin{aligned}
\int_\Gamma f(x)\psi \, dS_x \, dt \quad &= \quad \left(\int_0^T \zeta_T(t)\, dt\right)\left(\int_{\partial B_2} f(x)\chi_R(x)\, dS_x\right) \\
&= \quad CT \int_{\partial B_2} f(x)\mathcal{N}(x)\, dS_x \\
&= \quad CT \int_{\partial B_2} f(x)\, dS_x.
\end{aligned} \tag{4.20}$$

Hence, using Lemmas 3.2–3.5, (4.19) and (4.20), we obtain

$$\int_{\partial B_2} f(x)\, dS_x \leq C\left(T^{-\frac{2p}{p-1}} \ln R + T^{-\frac{2p}{p-1}} R^{\lambda_1} + R^{\lambda_2}\right), \tag{4.21}$$

where λ_1 and λ_2 are given respectively by (4.7) and (4.8). Next, taking $T = R^\eta$, where η satisfies (4.9), passing to the limit as $R \to \infty$ in (4.21) and using (1.12), we obtain a contradiction with $\displaystyle\int_{\partial B_2} f(x)\,dS_x >$ 0. This shows that problems (1.1) and (1.3) admit no weak solution.

(II) We first consider the case $\mu = -\frac{1}{4}$. For

$$\frac{\rho - 2}{p - 1} < \alpha < \frac{1}{2} \tag{4.22}$$

and

$$-\left(\frac{1}{2} - \alpha\right)^{\frac{2}{p-1}} < \beta < 0, \tag{4.23}$$

let

$$u_{\alpha,\beta}(x) = \beta|x|^{-1}(|x| - 1)^{\alpha}, \quad x \in A. \tag{4.24}$$

Notice that due to (1.13), one has

$$\frac{\rho - 2}{p - 1} < \frac{1}{2}.$$

Then the set of α satisfying (4.22) is nonempty. On the other hand, elementary calculations show that

$$-\Delta u_{\alpha,\beta}(x) + \frac{\mu}{(|x| - 1)^2}u_{\alpha,\beta}(x) = -\beta\left(\frac{1}{2} - \alpha\right)^2 |x|^{-1}(|x| - 1)^{\alpha-2}, \quad x \in A. \tag{4.25}$$

Hence, using (4.22)–(4.25), for all $x \in A$, we obtain

$$\begin{aligned}
&-\Delta u_{\alpha,\beta}(x) + \frac{\mu}{(|x| - 1)^2}u_{\alpha,\beta}(x)\\
&= (|x| - 1)^{-\rho}\left[(-\beta)^p|x|^{-p}(|x| - 1)^{\alpha p}\right](-\beta)^{1-p}\left(\frac{1}{2} - \alpha\right)^2 |x|^{p-1}(|x| - 1)^{\alpha(1-p)+\rho-2}\\
&\geq (|x| - 1)^{-\rho}|u_{\alpha,\beta}(x)|^p(|x| - 1)^{\alpha(1-p)+\rho-2}\\
&\geq (|x| - 1)^{-\rho}|u_{\alpha,\beta}(x)|^p.
\end{aligned} \tag{4.26}$$

Moreover, by (4.22) and (4.23), we obtain

$$\frac{\partial u_{\alpha,\beta}}{\partial \nu}(x) = \frac{(2\alpha - 1)\beta}{4} > 0, \quad x \in \partial B_2. \tag{4.27}$$

Thus, by (4.26) and (4.27), we deduce that $u_{\alpha,\beta}$ is a stationary negative solution to (1.1), (1.3) with $f \equiv \frac{(2\alpha-1)\beta}{4}$.

Next, let us consider the case $\mu > -\frac{1}{4}$. For

$$\max\left\{\frac{1}{2}, \sigma, \frac{\rho - 2}{p - 1}\right\} < \delta < 1 - \sigma \tag{4.28}$$

and ε satisfying (4.16), let $u_{\delta,\varepsilon}$ be the function defined by (4.17). Notice that $\sigma < \frac{1}{2}$, so the set of δ satisfying (4.28) is nonempty. It was shown previously that

$$-\Delta u_{\delta,\varepsilon}(x) + \frac{\mu}{(|x| - 1)^2}u_{\delta,\varepsilon}(x) \geq (|x| - 1)^{-\rho}u_{\delta,\varepsilon}^p(x), \quad x \in A.$$

Moreover, using (4.17) and (4.28), we obtain

$$\frac{\partial u_{\delta,\varepsilon}}{\partial \nu}(x) = \frac{(2\delta-1)\varepsilon}{4} > 0, \quad x \in \partial B_2.$$

Hence, $u_{\delta,\varepsilon}$ is a stationary positive solution to (1.1), (1.3) with $f \equiv \frac{(2\delta-1)\varepsilon}{4}$. This completes the proof of Theorem 1.5 for problems (1.1) and (1.3). □

5. Conclusions

The existence and nonexistence of weak solutions to the hyperbolic inequality (1.1) have been investigated in this paper. We studied (1.1) under the Dirichlet-type boundary condition (1.2) and the Neumann-type boundary condition (1.3). Using nonlinear capacity estimates, we proved that when $\rho > 2$, both problems have the same Fujita critical exponent given by

$$p^*(\mu,\rho) = 1 + \frac{\rho-2}{1-\sigma}.$$

The critical case

$$\rho > 2, \qquad p = 1 + \frac{\rho-2}{1-\sigma}$$

is not investigated here. It should be interesting to decide whether this case belongs to the blow-up situation.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at the Imam Mohammad Ibn Saud Islamic University (IMSIU) for funding and supporting this work through Research Partnership Program no. RP-21-09-02.

Conflict of interest

The authors declare no conflict of interest.

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