AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 11617–11628.
DOI: 10.3934/math.2023588
Received: 06 January 2023
Revised: 06 March 2023
Accepted: 12 March 2023
Published: 16 March 2023

Research article

An application of p-adic Baker method to a special case of Jeśmanowicz' conjecture

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Abstract: In 1956, Jeśmanowicz conjectured that, for any positive integer n, the Diophantine equation $\left((f^2-g^2)n\right)^x+((2fg)n)^y=\left((f^2+g^2)n\right)^z$ has only the positive integral solution $(x,y,z)=(2,2,2)$, where f and g are positive integers with $f>g$, $\gcd(f,g)=1$, and $f\not\equiv g \pmod 2$. Let $r=6k+2$, $k\in\mathbb{N}$, $k\geq 25$. In this paper, combining p-adic form of Baker method with some detailed computation, we prove that if n satisfies $n\equiv 0,6,9 \pmod{12}$, $f=g+1$ and $g=2^r-1$, then the conjecture is true.

Keywords: exponential Diophantine equation; Pythagorean triple; Jeśmanowicz' conjecture
Mathematics Subject Classification: 11D61, 11J86

1. Introduction

Let a,b,c be relatively prime positive integers such that $a^2+b^2=c^2$. Such a triple (a,b,c) is called a primitive Pythagorean triple. In 1956, Jeśmanowicz [9] conjectured that, for any positive integer n, if (a,b,c) is a primitive Pythagorean triple, then the exponential Diophantine equation

$$(an)^x+(bn)^y=(cn)^z \tag{1.1}$$

has only the positive integral solution $(x,y,z)=(2,2,2)$. It is well-known that any primitive Pythagorean triple (a,b,c) can be parameterized by

$$a=f^2-g^2, b=2fg, c=f^2+g^2,$$

where f and g are positive integers with $f>g$, $\gcd(f,g)=1$ and $f\not\equiv g \pmod 2$. Then, (1.1) can be rewritten as follows:

$$\left((f^2-g^2)n\right)^x+((2fg)n)^y=\left((f^2+g^2)n\right)^z. \tag{1.2}$$

Jeśmanowicz' conjecture has been proved to be true in many cases. In 2013, using some properties of Pell equation, Miyazaki [11] proved that if $n=1$ and $a\equiv\pm1 \pmod b$ or $c\equiv 1 \pmod b$, then

Jeśmanowicz' conjecture is true. Similar to the above result, by using Baker method with various elementary arguments through rational and quadratic numbers, Miyazaki and Pink [13,14] showed that if $n = 1$, and $a \equiv \pm 1 \pmod{c}$ or $b \equiv \pm 1 \pmod{c}$, then Jeśmanowicz' conjecture is true. Combining a lower bound for linear forms in two logarithms due to Laurent [10] with some elementary methods, Terai [15] showed that Jeśmanowicz' conjecture is true for $n = 1$ and $g = 2$. By using some results of Diophantine equations in [1,7] and detailed calculations on 2-adic valuation, in 2015, Miyazaki [12] proved that if $n \geq 1$ and $(f, g) = (2^r, 1)$, where r is a positive integer, then Jeśmanowicz' conjecture is true. Recently, together Baker method with an elementary computation, Yang and Fu [17] proved that if $n = 1$, $fg \equiv 2 \pmod{4}$ and $f > 17.8g$, then Jeśmanowicz' conjecture holds.

In this paper, we pay our attention to the special case $f = g + 1$. It is easy to see that (1.2) can be rewritten as following in this case:

$$((2g+1)n)^x + \left((2g^2+2g)n\right)^y = \left((2g^2+2g+1)n\right)^z. \tag{1.3}$$

In 1965, Dem'janenko [4] proved that if $n = 1$, then exponential Diophantine equation (1.3) has only the positive integral solution $(x, y, z) = (2, 2, 2)$. In 2010, using a deep result in [2] on the existence of primitive divisors of Lucas numbers and Lehmer numbers, Hu and Yuan [8] provided a proof of Dem'janenko's result. In 2021, using some properties on the representation of integers by binary quadratic primitive forms, Fujita and Le [6] gave a new proof of Dem'janenko's result, which is more elementary than that in [8].

However, the result concerning with (1.3) in the case $n > 1$ is scarce. In 2017, Yang and Fu [16] proved that if $n > 1$, $g = 2^r$ and $2^r + 1$ is an odd prime, then Eq (1.3) has no positive solution other than $(2, 2, 2)$. Recently, Fujita and Le [6] proved that if $n > 1$ and $g = 2^r$, where $r \geq 80$ and $r + 1$ is an odd prime, then (1.3) has only the positive integral solution $(2, 2, 2)$. In this paper , we focus our attention on the case $f = g + 1$ and $g = 2^r - 1$, where $r = 6k + 2$, $k \in \mathbb{N}$. With the aid of p-adic form of Baker method, we obtain some interesting results about Jeśmanowicz' conjecture by some detailed computation on 2-adic valuation.

2. Preliminary results

This section is devoted to providing some known results which will be used in the next section. The following conclusion is clear in elementary number theory.

Lemma 2.1. *Let r, l be two positive integers. Then $2^r - 1 \mid 2^l - 1$ if and only if $r \mid l$.*

Let p be a prime and v_p the standard p-adic valuation normalized by $v_p(p) = 1$. Suppose that $l > 5$ is a positive integer and k is a divisor of $2^l + 1$ with $1 < k < 2^l + 1$. Denote by ζ the integer $(-1)^{(k-1)/2}$. Fujita and Le showed in [6] that $v_2(k - \zeta) \leq l/3$. Without restriction on l, we can obtain a similar result by the method in [6].

Lemma 2.2. *Let l be a positive integer, k a divisor of $2^l - 1$ with $1 < k < 2^l - 1$. Denote by ζ the integer $(-1)^{(k-1)/2}$. Then $v_2(k - \zeta) \leq l/2$.*

Proof. To ease the notation, denote by s the integer $v_2(k - \zeta)$. We may assume that

$$k = 2^s h + \zeta, s, h \in \mathbb{N}, s \geq 2, 2 \nmid h,$$

and

$$2^l - 1 = kk', k' \in \mathbb{N}, 1 < k' < 2^l - 1,$$

where

$$k' = 2^{s'}h' + \zeta', \zeta' = (-1)^{(k'-1)/2}, s', h' \in \mathbb{N}, s' \geq 2, 2 \nmid h'.$$

According to the above assumptions, we get

$$2^l - 1 = (2^s h + \zeta)(2^{s'}h' + \zeta') = 2^{s+s'}hh' + 2^s h\zeta' + 2^{s'}h'\zeta + \zeta\zeta'. \tag{2.1}$$

Since $\min\{s, s'\} \geq 2$, we see from (2.1) that $\zeta\zeta' = -1$. It follows that $\zeta' = -\zeta$ and

$$2^l = 2^{s+s'}hh' - 2^s h\zeta + 2^{s'}h'\zeta. \tag{2.2}$$

Suppose that $l < 2s$. We shall deduce contradictions from the following arguments in two cases.

Case 1: $h' \geq h$. Since $\min\{l, s+s'\} > \max\{s, s'\}$, by (2.2), we get $s = s'$ and

$$2^{l-s} = 2^s hh' + (h' - h)\zeta. \tag{2.3}$$

Since $l < 2s$, by (2.3), we have $(h - h') \equiv 0 \pmod{2^{l-s}}, \zeta = -1$ and

$$\frac{h' - h}{2^{l-s}} + 1 = 2^{2s-l}hh'. \tag{2.4}$$

Notice that $hh' > \frac{h'-h+1}{2}$. This implies in turn that (2.4) is false, and we get a contradiction.

Case 2: $h' < h$. A contradiction can also be established by the above method. □

The following result plays an important role in the proof of Proposition 3.9 in Section 3.

Lemma 2.3. [6, Lemma 4.1] *If b, c, m are positive integers such that $b > 1$, $2 \nmid b$ and $b = c^m$, then*

$$v_2\left(b - (-1)^{(b-1)/2}\right) \geq v_2\left(c - (-1)^{(c-1)/2}\right).$$

In order to obtain an upper bound for z in Section 3, a result due to Bugeaed [3] is essential. Now we introduce some notation. Let α_1, α_2 be integers such that $\min\{|\alpha_1|, |\alpha_2|\} \geq 2$. We consider the upper bound for p-adic valuation of the following number

$$\Lambda = \alpha_1^{\beta_1} - \alpha_2^{\beta_2},$$

where β_1, β_2 are positive integers. Let p be a prime with $p \nmid \alpha_1\alpha_2$. Denote by h_0 the smallest positive integer such that

$$v_p\left(\alpha_1^{h_0} - 1\right) > 0, v_p\left(\alpha_2^{h_0} - 1\right) > 0.$$

Assume that there exists a real E such that

$$v_p\left(\alpha_1^{h_0} - 1\right) \geq E > \frac{1}{p-1}, v_p\left(\alpha_2^{h_0} - 1\right) \geq E > \frac{1}{p-1},$$

and $A_1 > 1, A_2 > 1$ are real numbers with

$$\log A_i \geq \max\{\log|\alpha_i|, E\log p\}, (i = 1, 2).$$

With these notation in hand, we now present the result due to Bugeaed in [3].

Lemma 2.4. [3, Theorem 2] *With the notation as above, if α_1 and α_2 are multiplicatively independent, then we have*

$$v_p(\Lambda) \leqslant \frac{36.1h_0}{E^3(\log p)^4}\left(\max\{\log B + \log(E\log p) + 0.4,\ 6E\log p,\ 5\}\right)^2 \log A_1 \log A_2,$$

where $B = \frac{\beta_1}{\log A_2} + \frac{\beta_2}{\log A_1}$.

Denote by $P(n)$ the product of distinct prime factors of n. It has to be pointed out that the following conclusion, which is needed in the proof of Proposition 3.1, plays an important role in the research on Jeśmanowicz' conjecture.

Lemma 2.5. [5, Corollary 2.4] *Let (a, b, c) be a primitive Pythagorean triple such that the exponential Diophantine equation $a^x + b^y = c^z$ has the unique positive solution $(x, y, z) = (2, 2, 2)$. If $(x, y, z) \neq (2, 2, 2)$ is a solution of (1.1), then one of the following assertions holds:*
(1) $x > z > y$, $P(n) \mid b$;
(2) $y > z > x$, $P(n) \mid a$.

It is easy to see from Lemma 2.5 that, the case $n = 1$ is essential to the case $n > 1$ on the study of Jeśmanowicz' conjecture. Hence, the following lemma is necessary.

Lemma 2.6. [8] *If $n = 1$, then the exponential Diophantine equation (1.3) has only the positive integral solution $(x, y, z) = (2, 2, 2)$.*

The following two lemmas will be used to determine the relationship of size between x, y, z, where (x, y, z) is a positive integral solution of (1.3).

Lemma 2.7. [6, Theorem 1.3] *If $n > 1$ and $g > 48$, then the exponential Diophantine equation (1.3) has no solution (x, y, z) with $y > z > x$.*

Lemma 2.8. [6, Proposition 4.5] *If $(x, y, z) \neq (2, 2, 2)$ is a positive solution of (1.3) such that $x > z > y$, then $z > x - z$.*

3. Main results

As we all known, for any odd integer b with $b > 1$, and any positive integer m, we have

$$v_2(b^m - 1) = \begin{cases} v_2(b-1), & if 2 \nmid m, \\ v_2\left(b - (-1)^{(b-1)/2}\right) + v_2(m), & if 2 \mid m. \end{cases} \tag{3.1}$$

When $g > 1$, $g \equiv 1 \pmod 3$ and $P\left(2g^2 + 2g\right) \mid n$, Fujita and Le showed in [6] that (1.3) has no positive solution other than $(2, 2, 2)$. By similar arguments, it is not hard to get the following conclusion.

Proposition 3.1. *Let $g = 2^r - 1$, where r is even. If there exists a prime p such that $v_p\left(2g^2 + 2g\right) = 1$, and the positive integers n, g satisfy $P\left(2g^2 + 2g\right) \mid n$, then the Eq (1.3) has no positive solution other than $(2, 2, 2)$.*

Proof. Let (x, y, z) be a positive solution of (1.3). Suppose that $(x, y, z) \neq (2, 2, 2)$. Since $P\left(2g^2 + 2g\right) \mid n$ and $\gcd\left(2g^2 + 2g, 2g + 1\right) = 1$, then $P(n) \nmid 2g + 1$. By Lemma 2.5, we have $x > z > y$ and

$P(n) \mid 2g^2+2g$. This means that $P\left(2g^2+2g\right)=P(n)$, and

$$\left(2g^2+2g\right)^y = n^{z-y}\left((2g^2+2g+1)^z-(2g+1)^x n^{x-z}\right). \tag{3.2}$$

Combining $P\left(2g^2+2g\right)=P(n)$, $\gcd\left(2g^2+2g,2g^2+2g+1\right)=1$ with $g=2^r-1$, it follows from (3.2) that $(2g^2+2g)^y=n^{z-y}$, and

$$(2^{r+1}-1)^x n^{x-z}+1=(2^{2r+1}-2^{r+1}+1)^z. \tag{3.3}$$

Note here that $2^{2r+1}-2^{r+1}+1 \equiv 1 \pmod 4$. We can get that

$$\begin{aligned}(x-z)v_2(n) &= v_2\left((2^{2r+1}-2^{r+1}+1)^z-1\right)\\ &= v_2(2^{2r+1}-2^{r+1}+1-1)+v_2(z)\\ &= r+1+v_2(z), \end{aligned} \tag{3.4}$$

by (3.1) and (3.3). Since $\left(2^{2r+1}-2^{r+1}\right)^y=n^{z-y}$, we have

$$n=\left(2^{2r+1}-2^{r+1}\right)^{\frac{y}{z-y}},\, v_2(n)=(r+1)\frac{y}{z-y}. \tag{3.5}$$

It follows from $v_p\left(2^{2r+1}-2^{r+1}\right)=1$ and (3.5) that $\frac{y}{z-y}$ must be a positive integer. If $\frac{y}{z-y}=1$, then $z=2y$. Hence, $2 \mid z$. If $\frac{y}{z-y}>1$, by (3.4), we get

$$(r+1)(x-z)-(r+1)<v_2(z).$$

Hence, $v_2(z)>0$. Therefore, we always have $2 \mid z$.

It is clear from $2 \mid z$ that $2^{2r}-2^r+1 \mid \left(2^{2r+1}-2^{r+1}+1\right)^z-1$. Together this relation with (3.3), the relation

$$2^{2r}-2^r+1 \mid (2^{r+1}-1)^x n^{x-z} \tag{3.6}$$

can be easily established. By $P\left(2^{2r+1}-2^{r+1}\right)=P(n)$ and $\gcd\left(2^{2r+1}-2^{r+1},2^{2r}-2^r+1\right)=1$, we have $\gcd\left(2^{2r}-2^r+1,n^{x-z}\right)=1$, and by (3.6),

$$2^{2r}-2^r+1 \mid \left(2^{r+1}-1\right)^x.$$

Suppose that p_0 is a common prime divisor of $2^{2r}-2^r+1$ and $2^{r+1}-1$. Since $2^r-1 \equiv -1/2 \pmod{p_0}$, we have $2^{2r}-2^r+1 \equiv 3/4 \pmod{p_0}$. Hence, $p_0=3$. On the other hand, since r is even, $2^{2r}-2^r+1 \equiv 1 \pmod 3$, and this is absurd, and in turn completes our proof. □

By Proposition 3.1, the following result can be obtained easily.

Proposition 3.2. *Let $r=6k+2$, $k\in\mathbb{N}$. If $g=2^r-1$ and the positive integers n,g satisfy $P(2g^2+2g) \mid n$, then the Eq (1.3) has no positive solution other than* $(2,2,2)$.

Proof. It is clear from Proposition 3.1 that we just have to show that there exists a prime p such that $v_p\left(2g^2+2g\right)=1$. In fact, we claim that 3 is the one we are looking for. Since r is even, $3 \mid 2^{2r+1}-2^{r+1}$. Furthermore, notice that $r\equiv 2 \pmod 3$, $9 \nmid 2^{2r+1}-2^{r+1}$. □

Combining p-adic form of Baker method with some elementary computation, Fujita and Le proved in [6] that (1.3) has no positive solution other than $(2,2,2)$ for $g = 2^r$, where $r \geq 80$ and $r+1$ is a prime. With the help of Proposition 3.2 and ideas in [6], the following conclusion, which is our main result in this paper, can be obtained.

Theorem 3.3. *Let $r = 6k+2, k \in \mathbb{N}$, $k \geq 25$. If $g = 2^r - 1$ and the positive integer n satisfies $n \equiv 0,6,9 \pmod{12}$, then the Eq (1.3) has no positive solution other than $(2,2,2)$.*

In order to prove Theorem 3.3, in the rest of this section, we assume from now on that $g = 2^r - 1$, $r = 6k+2, k \in \mathbb{N}, k \geq 25$, the positive integer n satisfies $n \equiv 0,6,9 \pmod{12}$, and $(x,y,z) \neq (2,2,2)$ is a solution of (1.3). It is easy to see from Lemmas 2.5–2.7 that $x > z > y$. The above arguments imply that

$$(2^{2r+1} - 2^{r+1})^y = n^{z-y}\left((2^{2r+1} - 2^{r+1} + 1)^z - (2^{r+1} - 1)^x n^{x-z}\right). \tag{3.7}$$

Let

$$2^{2r+1} - 2^{r+1} = b_1 b_2,\ b_1, b_2 \in \mathbb{N}, \tag{3.8}$$

where

$$b_1^y = n^{z-y}. \tag{3.9}$$

It is obvious that $\gcd(b_1, b_2) = 1$ and (3.7) can be rewritten as

$$(2^{r+1} - 1)^x n^{x-z} + b_2^y = (2^{2r+1} - 2^{r+1} + 1)^z. \tag{3.10}$$

Now, in order to prove Theorem 3.3, we just have to prove that (3.10) is false.

Remark 3.4. *It is worth pointing out that the conclusion of Proposition 3.2 shows that if $b_1 = 2^{2r+1} - 2^{r+1}$, then (3.10) is false.*

When $b_1 < 2^{2r+1} - 2^{r+1}$, an upper bound for z can be obtained by using p-adic form of Baker method. With the similar arguments as in [6], we can prove the following proposition.

Proposition 3.5. If $b_1 < 2^{2r+1} - 2^{r+1}$, then

$$z < 360(\log b_2)\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2.$$

Proof. Notice that $r \equiv 2 \pmod 3$. Then

$$7 \nmid 2^{2r+1} - 2^{r+1} + 1, 7 \nmid 2^{2r+1} - 2^{r+1} = b_1 b_2.$$

Put

$$\alpha_1 = 2^{2r+1} - 2^{r+1} + 1,\ \alpha_2 = b_2,\ \beta_1 = z,\ \beta_2 = y.$$

Such α_1, α_2 are positive integers satisfying $\min\{\alpha_1, \alpha_2\} \geq 2$ and $7 \nmid \alpha_1\alpha_2$. It is clear that α_1 and α_2 are multiplicatively independent. Let $\Lambda = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$. Observe here that $r+1 = 6k+3$, $k \in \mathbb{N}$. It follows from (3.10) that $\Lambda = (2^{r+1} - 1)^x n^{x-z}$ and

$$v_7(\Lambda) \geq x. \tag{3.11}$$

When $b_1 \equiv 3,5,6 \pmod 7$, it is clear from (3.8) that $b_2 \equiv 1,2,4 \pmod 7$. Since $\gcd(b_1, b_2) = 1$, $b_1 < 2^{2r+1} - 2^{r+1}$, and $r \geq 152$, by (3.8), we have $b_2 > 7$. Put $h_0 = 3$, $E = 1$, $A_1 = \alpha_1$, $A_2 = \alpha_2$. Then, combining Lemma 2.4 with (3.11), it follows that

$$x \leqslant \frac{36.1 \times 3}{(\log 7)^4}\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)(\log b_2)(\max\{6\log 7, 0.4 + \log\log 7 + \log B\})^2, \tag{3.12}$$

where

$$B = \frac{z}{\log b_2} + \frac{y}{\log(2^{2r+1} - 2^{r+1} + 1)}. \tag{3.13}$$

If $6 \log 7 \geq 0.4 + \log\log 7 + \log B$, then we get from (3.13)

$$z \leqslant 7^6 \log b_2. \tag{3.14}$$

Notice that $r \geq 152$, it follows from (3.14) that

$$z \leqslant 7^6 \log b_2 < 360(\log b_2)(\log(2^{2r+1} - 2^{r+1} + 1))\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2.$$

If $6 \log 7 < 0.4 + \log\log 7 + \log B$, then, by (3.12),

$$x \leqslant 8.4(\log(2^{2r+1} - 2^{r+1} + 1))(\log b_2)(0.4 + \log\log 7 + \log B)^2. \tag{3.15}$$

Since $x > z > y$ and $b_2 = b/b_1 < b$ by (3.8), we see from (3.13) that

$$B < \frac{2z}{\log b_2}. \tag{3.16}$$

Combining (3.15) with (3.16), we get that

$$\frac{z}{\log b_2} < \frac{x}{\log b_2} < 8.4\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(2 + \log\left(\frac{z}{\log b_2}\right)\right)^2. \tag{3.17}$$

Let $F(t) = t - 8.4\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)(2 + \log t)^2$. The above inequalities mean that

$$F\left(\frac{z}{\log b_2}\right) < 0. \tag{3.18}$$

Since $r \geq 152$, $F(t) > 0$ for

$$t = 360\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2.$$

Furthermore,

$$F'(t) = 1 - 16.8\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)(2 + \log t)/t,$$

and $F'(t) > 0$ for

$$t > 360\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2,$$

where $F'(t)$ is the derivative of $F(t)$. Therefore, by (3.18), we get

$$z < 360(\log b_2)\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2.$$

When $b_1 \equiv 1, 2, 4 \pmod 7$, it is clear from (3.8) that $b_2 \equiv 3, 5, 6 \pmod 7$. Since $n \equiv 0 \pmod 3$, and $\gcd(b_1, b_2) = 1$, by (3.9), we have $b_2 \geq 5$. Put $h_0 = 6$, $E = \frac{\log 5}{\log 7}$, $A_1 = \alpha_1$, $A_2 = \alpha_2$. Then, combining Lemma 2.4 with (3.11) , it follows that

$$x \leqslant \frac{36.1 \times 6}{(\log 5)^3 \times \log 7}\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)(\log b_2)(\max\{6\log 5, 0.4 + \log\log 5 + \log B\})^2,$$

where

$$B = \frac{z}{\log b_2} + \frac{y}{\log(2^{2r+1} - 2^{r+1} + 1)}.$$

The rest of the proof omitted here is similar to the situation in which $b_1 \equiv 3, 5, 6 \pmod 7$. Thus, the proposition is proved. □

Using the above result, it is not difficult to show the following conclusion.

Proposition 3.6. *If $b_1 = 2^r - 1$, then (3.10) is false.*

Proof. It is easily deduced from (3.8) that $b_2 = 2^{r+1}$ when $b_1 = 2^r - 1$. Hence, by (3.10), we get

$$(2^{r+1} - 1)^x n^{x-z} + 2^{(r+1)y} = (2^{2r+1} - 2^{r+1} + 1)^z. \tag{3.19}$$

Notice that $x > 2$, then

$$2^{(r+1)y} = \left((2^{r+1} - 1) + 1\right)^y \equiv 1 + (2^{r+1} - 1)y \pmod{(2^{r+1} - 1)^2},$$

and

$$(2^{2r+1} - 2^{r+1} + 1)^z = \left(\frac{1}{2}\left((2^{r+1} - 1)^2 + 1\right)\right)^z \equiv \frac{1}{2^z} \pmod{(2^{r+1} - 1)^2}.$$

It follows from (3.19) and (3.20) respectively that

$$2^z(2^{r+1} - 1)y \equiv 1 - 2^z \pmod{(2^{r+1} - 1)^2}, \tag{3.20}$$

and

$$2^z - 1 \equiv 0 \pmod{2^{r+1} - 1}. \tag{3.21}$$

Applying Lemma 2.1 to (3.21), we get $r + 1 \mid z$ and

$$z = (r + 1)m_1, m_1 \in \mathbb{N}. \tag{3.22}$$

Combining (3.20) with (3.22), we have

$$-2^z y \equiv \frac{2^z - 1}{2^{r+1} - 1} = \frac{2^{(r+1)m_1} - 1}{2^{r+1} - 1} = \sum_{i=0}^{m_1-1} 2^{(r+1)i} \equiv m_1 \pmod{2^{r+1} - 1}, \tag{3.23}$$

and this implies in turn that $m_1 \equiv -y \pmod{2^{r+1} - 1}$ and

$$\frac{z}{r + 1} \equiv -y \pmod{2^{r+1} - 1}. \tag{3.24}$$

According to (3.24), we have

$$\left(\frac{r + 2}{r + 1}\right)z > \frac{z}{r + 1} + y > 2^{r+1} - 1. \tag{3.25}$$

On the other hand, combining the facts that $r \geq 150$, $r \equiv 2 \pmod 3$ and $b_1 = 2^r - 1 < 2^{2r+1} - 2^{r+1}$ with Proposition 3.5 and (3.25) , we get

$$2^{r+1} - 1 < \frac{152}{151}z < \frac{152}{151}360(\log b_2)\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2,$$

which is impossible under the condition $r \geq 152$. Thus, our proof is completed. □

Let k_1, k_2 be positive integers such that

$$2^r - 1 = k_1 k_2, 1 < k_1, k_2 < 2^r - 1, \gcd(k_1, k_2) = 1. \tag{3.26}$$

Before continuing our discussion, a remark is in order.

Remark 3.7. *Combining (3.8), Remark 3.4 with Proposition 3.6 yield that if $b_1 = 2^{r+1}(2^r - 1)$ or $b_1 = 2^r - 1$, then (3.10) is false. This implies that we just have to show that (3.10) is false in the case $(b_1, b_2) = (2^{r+1}k_1, k_2)$ or $(b_1, b_2) = (k_1, 2^{r+1}k_2)$.*

With the above preparations, we are now in a position to prove Theorem 3.3.

Proposition 3.8. *If $n \equiv 0, 6 \pmod{12}$, then (3.10) is false.*

Proof. When $n \equiv 0, 6 \pmod{12}$, it follows immediately from (3.9) that $6 \mid b_1$. Then, by Remark 3.7, we assume that $(b_1, b_2) = (2^{r+1}k_1, k_2)$. Hence, (3.10) can be represented as

$$(2^{r+1} - 1)^x n^{x-z} + k_2^y = (2^{2r+1} - 2^{r+1} + 1)^z, \tag{3.27}$$

where n satisfies

$$n^{z-y} = (2^{r+1}k_1)^y. \tag{3.28}$$

In other words, n can be rewritten as

$$n = (2^{r+1}k_1)^{\frac{y}{z-y}}, \tag{3.29}$$

and this means in turn that (3.27) can be rewritten as

$$(2^{r+1} - 1)^x (2^{r+1}k_1)^{(x-z)\frac{y}{z-y}} + k_2^y = (2^{2r+1} - 2^{r+1} + 1)^z. \tag{3.30}$$

Since $r + 1 = 6k + 3$, $k \in \mathbb{N}$, we have $3^2 \nmid 2^r - 1$. Notice that $3 \mid n$, it follows from (3.26) and (3.29) that $3 \mid k_1$ and $3^2 \nmid k_1$, which implies that $\frac{y}{z-y}$ is a positive integer. Taking (3.30) module 2^{r+1} yields that

$$k_2^y - 1 \equiv 0 \pmod{2^{r+1}}. \tag{3.31}$$

Let $s = v_2\left(k_2 - (-1)^{(k_2-1)/2}\right)$. Since $k_2 \mid 2^r - 1$, we see from (3.1) and (3.31) that

$$s + v_2(y) \geq r + 1. \tag{3.32}$$

Combining Lemma 2.2 with (3.26) yield that $s \leqslant r/2$. Hence, by (3.32), we have $v_2(y) \geq r/2 + 1$, and thus $y \geq 2^{r/2+1}$. Since $z > y$, we get

$$z > 2^{r/2+1}. \tag{3.33}$$

According to Proposition 3.5 and (3.33), we have

$$2^{r/2+1} < 360\left(\log(2^{2r+1} - 2^{r+1})\right)\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2,$$

whence we get $r < 152$, which yields a contradiction. Thus, our proof is completed. □

Proposition 3.9. *If $n \equiv 9 \pmod{12}$, then (3.10) is false.*

Proof. Since $n \equiv 9 \pmod{12}$, we have b_1 is odd by (3.9). Similar to the above proposition, we assume that $(b_1, b_2) = (k_1, 2^{r+1}k_2)$. Thus, equality (3.10) can be rewritten as

$$(2^{r+1} - 1)^x n^{x-z} + (2^{r+1}k_2)^y = (2^{2r+1} - 2^{r+1} + 1)^z, \tag{3.34}$$

where n satisfies

$$k_1^y = n^{z-y}. \tag{3.35}$$

Since $n \equiv 9 \pmod{12}$, taking (3.34) module 4, we get $(-1)^x \equiv 1 \pmod{12}$, which implies that $2 \mid x$. Furthermore, taking (3.34) module 2^{r+1}, it follows from (3.35) and $2 \mid x$ that

$$n^{x-z} - 1 \equiv k_1^{y(x-z)/(z-y)} - 1 \equiv 0 \pmod{2^{r+1}}. \tag{3.36}$$

Notice that $r + 1 = 6k + 3$, $k \in \mathbb{N}$, we get that $3^2 \nmid 2^r - 1$. Since $n \equiv 9 \pmod{12}$, we assert by (3.26) and (3.35) that $\frac{y}{z-y}$ is a positive integer. Combining (3.1), (3.26) with (3.36) yield

$$v_2\left(k_1^{y(x-z)/(z-y)} - 1\right) = v_2\left(k_1 - (-1)^{(k_1-1)/2}\right) + v_2\left(\frac{y(x-z)}{z-y}\right) \geq r + 1. \tag{3.37}$$

Let $s' = v_2\left(k_1 - (-1)^{(k_1-1)/2}\right)$. Applying (3.1), (3.26) and Lemma 2.3, we see by (3.37) that

$$v_2\left(k_1^{y(x-z)} - 1\right) = s' + v_2\left(y(x-z)\right) \geq r + 1, \tag{3.38}$$

which in turn means $s' \leqslant r/2$. Hence, by (3.38), we have $v_2\left(y(x-z)\right) \leqslant r/2 + 1$, and thus

$$y(x-z) \leqslant 2^{r/2+1}. \tag{3.39}$$

Lemma 2.8 tells us that $z > x - z$. Combining this fact with $z > y$ and (3.39), we have

$$z^2 > y(x-z) > 2^{r/2+1},$$

and thus

$$2^{r/4+1/2} < 360\left(\log(2^{2r+1} - 2^{r+1})\right)\left(\log(2^{2r+1} - 2^{r+1} + 1)\right)\left(\log\log(2^{2r+1} - 2^{r+1} + 1)\right)^2,$$

which implies that $r < 152$. So we get a contradiction, and complete the proof. □

We conclude the proof of Theorem 3.3 by bringing together the above two propositions.

4. Conclusions

In this paper, our attention is focused on a special case of Jeśmanowicz' conjecture, in which $f = g + 1$ and $g = 2^r - 1$, where $r = 6k + 2$, $k \in \mathbb{N}$. Using p-adic Baker method with some detailed computation on 2-adic valuation, we show that if $k \geq 25$ and the positive integer n satisfies $n \equiv 0, 6, 9 \pmod{12}$, then Jeśmanowicz' conjecture is true. Notice that our result is based on a condition of the value of n, we will try to promote our result for any positive integer n.

Acknowledgments

The authors were supported by National Natural Science Foundation of China (No. 11971382) and Anhui Professional Top-notch Project for University Research Talents.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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