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Research article

System decomposition-based stability criteria for Takagi-Sugeno fuzzy uncertain stochastic delayed neural networks in quaternion field

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Abstract: Stochastic disturbances often occur in real-world systems which can lead to undesirable system dynamics. Therefore, it is necessary to investigate stochastic disturbances in neural network modeling. As such, this paper examines the stability problem for Takagi-Sugeno fuzzy uncertain quaternion-valued stochastic neural networks. By applying Takagi-Sugeno fuzzy models and stochastic analysis, we first consider a general form of Takagi-Sugeno fuzzy uncertain quaternion-valued stochastic neural networks with time-varying delays. Then, by constructing suitable Lyapunov-Krasovskii functional, we present new delay-dependent robust and global asymptotic stability criteria for the considered networks. Furthermore, we present our results in terms of real-valued linear matrix inequalities that can be solved in MATLAB LMI toolbox. Finally, two numerical examples are presented with their simulations to demonstrate the validity of the theoretical analysis.

Keywords: quaternion-valued neural networks; robust stability; stochastic disturbance; Lyapunov-Krasovskii functional; Takagi-Sugeno fuzzy
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1. Introduction

Since the 1970s, different types of neural networks (NNs) have attracted substantial interest from researchers due to their potential applications in various fields including secure communications, parallel computing, artificial intelligence, signal and image processing, optimization, and others [1–5].

It is well known that real-world systems are subject to random factors, which influence the system dynamics. As pointed out in [6–8], a real nervous system is usually affected by external noise which is great uncertainty and hence may be regarded as a stochastic perturbation. Because of this, stochastic perturbations are inevitable in neural systems, and they should be considered in the modeling process [7, 9–12]. Meanwhile, NNs are classified into two groups: deterministic NNs and stochastic NNs. In the study of NNs, deterministic NNs are very effective for describing and analyzing the system when there is no external disturbance [3–5]. Conversely, deterministic NNs fail when external disturbances occur. While stochastic NNs are very effective for describing and analyzing the system when they are subjected to external disturbances [7–10, 13, 14]. In recent years, stochastic NNs have drawn increasing attention from researchers and several results have been published [15–21].

It has been shown recently that real-valued NNs and complex-valued NNs have been successfully applied to a variety of engineering applications [1, 2, 11, 12]. However, real-valued NNs and complex-valued NNs have some limitations when it comes to the problem of symmetry detection and high-dimensional data [22, 23]. In order to address these issues, some researchers have developed quaternion-valued neural networks (QVNNs) by incorporating quaternions into conventional NNs [24–27]. Moreover, QVNNs have shown superior performance compared to complex-valued NNs and real-valued NNs because of the general representation, as well as their ability to handle multidimensional data with high efficiency. As a result of these aspects, a variety of applications have been developed including color images [28, 29], signal processing [30], optimization [31], sparse representation [32], extreme learning machine [33], and so on. Recently, several research works have been published regarding various dynamics of stochastic QVNNs using the Lyapunov-Krasovskii functional (LKF) and linear matrix inequality (LMI) [34–38]. For example, by employing decomposition method in [34], discrete-time stochastic QVNNs with time-varying delays are discussed, and some sufficient conditions are obtained to ensure global asymptotic stability. In [35], stochastic QVNNs with event-triggered control are studied, as well as various criteria are derived for stochastic stability based on direct quaternion method. Recently, mean square exponential input-to-state stability criterion based on a real-valued decomposition was found in [38] for stochastic delayed QVNNs. There are similar results can also be found in [36, 37].

Furthermore, Takagi-Sugeno (T-S) fuzzy system is a powerful and convenient tool in functional approximations for complex nonlinear systems [39, 40]. The T-S fuzzy system has the advantage of being able to approximate a nonlinear system with a set of linear models. Unlike typical NN structures, T-S fuzzy NNs have fuzzy operations and they can preserve the direct connection among cells. Due to their good approximation properties, T-S fuzzy NNs have proved to be an important research topic. Many scientific papers have been proposed the idea of incorporating fuzzy logic into the NNs in order to enhance their performance [41–46]. For example, using LKFs and matrix inequality, the authors of [45] have demonstrated exponential convergence for T-S fuzzy complex-valued NNs with impulsive effects and time delays. By decomposing Clifford-valued NNs into $2^m n$-dimensional real-valued NNs, the authors of [46] have derived the global asymptotic stability criteria for T-S fuzzy Clifford-valued NNs with time-varying delays and impulses.

As we all know, the stability issue is the most significant problem in the field of NNs because it is a precondition for an actual system to be able to function normally, which is fundamental for solving any other issues [2, 4, 6]. Unfortunately, time delays are often observed when implementing NNs due to the limited switching speed of amplifiers or information processing, which may result in oscillations, divergences, and even instability in the designing systems [34,37,39]. Therefore, it is essential to study how delays affect the system's dynamics. Several theoretical studies on the stability of NNs with time delays can be found in [46–48]. On the other hand, parameter uncertainties also occur in real systems, as well as NNs, as a result of modeling inaccuracies and/or environmental changes, which can lead to undesirable dynamic behaviours. In this regard, it is important to ensure that the system is stable with respect to uncertainties. Recently, the robustness analysis of various uncertain systems has gained an increasing amount of attention [11, 12, 18, 21, 50].

As far as we know, no papers have been published on Takagi-Sugeno fuzzy uncertain quaternion-valued stochastic neural networks (T-S FUQVSNNs) with time-varying delays. The purpose of this study is to fill such gaps by investigating the robust and global asymptotic stability criteria for T-S FUQVSNNs. Recently, several results have been published regarding the stability of stochastic QVNNs; however, T-S FUQVSNNs have not been thoroughly explored and have not received much attention, which motivates us to investigate this topic. The main merits of this paper are:

(1) To represent more realistic dynamics of QVNNs, we present a general form of T-S FUQVSNNs with time-varying delays.
(2) We analyze the robust and global asymptotic stability criteria for T-S FUQVSNNs by employing the system decomposition method.
(3) By constructing suitable LKFs and employing integral inequalities, enhanced stability conditions for the T-S FUQVSNNs are derived in terms of real-valued LMIs, which could be verified directly by MATLAB LMI toolbox.

The paper is structured as follows: Section 2 provides the problem model, definitions of robust asymptotic stability, assumptions about activation functions and time delays, and helpful lemmas. The main results of this study are stated in Section 3; Theorem (3.1) presents the robust and global asymptotic stability criteria; Theorem (3.5) provides the global asymptotic stability criteria for the considered networks. In Corollary (3.3), (3.7), the results of stability criteria are discussed in a particular case. Section 4 discusses two numerical case studies. Section 5 shows the conclusion of this paper.

2. Mathematical formulation and problem definition

2.1. Notations

This paper uses the following notations. Let the quaternion, complex and real numbers are denoted by $\mathbf{H}, \mathbf{C}$ and $\mathbf{R}$, respectively. The n-dimensional quaternion, complex and real vectors are denoted by $\mathbf{H}^n, \mathbf{C}^n$ and $\mathbf{R}^n$, respectively. The quaternion, complex and real matrices of size $n \times n$ are represented by $\mathbf{H}^{n\times n}, \mathbf{C}^{n\times n}$ and $\mathbf{R}^{n\times n}$, respectively. Let the matrix $\mathcal{P} < 0$ ($\mathcal{P} > 0$) means $\mathcal{P}$ is negative (positive) definite matrix. The block diagonal matrix is shown in $diag\{\cdot\}$. $\mathcal{P}^T$ denotes the transpose of matrix $\mathcal{P}$ and $\mathcal{P}^*$ denotes the Hermitian transpose of matrix $\mathcal{P}$. $\mathcal{I}$ denotes the identity matrix with appropriate dimensions. For $\ell > 0$, $\mathscr{C}([-\ell, 0], \mathbf{H}^n)$ denotes the family of continuous functions from φ to $\mathbf{H}^n$ with the norm $\|\varphi\| = \sup_{-\ell\leq t\leq 0} |\varphi(t)|$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, \mathscr{P})$ be a complete probability space with a filtration $\{\mathscr{F}\}_{t\geq 0}$ satisfying the usual conditions. The symmetric term in a matrix is showed by ✠.

2.2. Quaternion algebra

The quaternion was first invented by Hamilton in 1843. The skew field of a quaternion is denoted by

$$z = z^R + iz^I + jz^J + kz^K \in \mathbf{H},$$

where $z^R, z^I, z^J, z^K \in \mathbf{R}$, z is the quaternion-valued input and i, j, k are the quaternion basis which subjects to Hamilton's multiplication rules as follows:

$$\begin{aligned}
&k^2 = j^2 = i^2 = -1\\
&jk = -kj = i,\ ki = -ik = j,\ ij = -ji = k.
\end{aligned}$$

The following are some fundamental operation rules for quaternions and quaternion matrices [24, 49].

(1) The conjugate of the quaternion as follows:

$$\bar{z} = z^R - iz^I - jz^J - kz^K \in \mathbf{H}.$$

(2) The modulus of the quaternion as follows:

$$|z| = \sqrt{z\bar{z}} = \sqrt{(z^R)^2 + (z^I)^2 + (z^J)^2 + (z^K)^2}.$$

(3) Let $x = x^R + ix^I + jx^J + kx^K \in \mathbf{H}$ and $y = y^R + iy^I + jy^J + ky^K \in \mathbf{H}$. The addition and multiplication of two quaternions can be accomplished as follows:

$$\begin{aligned}
x + y &= (x^R + y^R) + i(x^I + y^I) + j(x^J + y^J) + k(x^K + y^K),\\
xy &= (x^R y^R - x^I y^I - x^J y^J - x^K y^K) + i(x^R y^I + x^I y^R + x^J y^K - x^K y^J)\\
&\quad + j(x^R y^J + x^J y^R - x^I y^K + x^K y^I) + k(x^R y^K + x^K y^I + x^I y^J - x^J y^I).
\end{aligned}$$

2.3. Problem formulation

In this section, we consider the following uncertain stochastic QVNNs with time-varying delays:

$$\begin{aligned}
dz(t) = {} & [-(\overline{\mathcal{D}} + \triangle\overline{\mathcal{D}}(t))z(t) + (\overline{\mathcal{A}} + \triangle\overline{\mathcal{A}}(t))g(z(t)) + (\overline{\mathcal{B}} + \triangle\overline{\mathcal{B}}(t))g(z(t - \ell(t)))]dt\\
& + \sigma(t, z(t), z(t - \ell(t)))d\omega(t),
\end{aligned} \tag{2.1}$$

where $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T \in \mathbf{H}^n$ and $g(z(\cdot)) = (g_1(z_1(\cdot)), g_2(z_2(\cdot)), ..., g_n(z_n(\cdot)))^T \in \mathbf{H}^n$ are the state vector and neuron activation functions, respectively. $\overline{\mathcal{D}} = [\overline{d}_i]_{n\times n} \in \mathbf{R}^{n\times n}$, $\overline{\mathcal{A}} = [\overline{a}_{ij}]_{n\times n} \in \mathbf{H}^{n\times n}$, $\overline{\mathcal{B}} = [\overline{b}_{ij}]_{n\times n} \in \mathbf{H}^{n\times n}$ are known matrices with appropriate dimensions. $\omega(t) = (\omega_1(t), \omega_2(t), ..., \omega_n(t))^T$ is n-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, \mathscr{P})$ with $\mathbf{E}[\omega(t)] = 0$ and $\mathbf{E}[\omega(t)^2] = t$. We also suppose that the stochastic disturbance $\sigma(t, z(t), z(t - \ell(t))) : \mathbf{R} \times \mathbf{H}^n \times \mathbf{H}^n \to \mathbf{H}^{n\times m}$ is locally Lipschitz continuous and satisfies the linear growth condition, i.e., $\sigma(t, 0, 0) = 0$, see [6–10] and the references therein.

This paper also makes the following assumptions about the transmission delays $\ell(t)$ and activation functions $g(z(\cdot))$.

Assumption 1: $\ell(t)$ is bounded on **R**, that is, $0 \leq \ell(t) \leq \ell$ and is differentiable function with $\dot{\ell}(t) \leq \mu < 1$, where ℓ, μ are constants.
Assumption 2: For any $z, z' \in \mathbf{H}$, there exists a positive constant l_α^g such that

$$|g_\alpha(z) - g_\alpha(z')| \leq l_\alpha^g |z - z'|, \ \alpha = 1, 2, ..., n.$$

We also assume that $g(0) = 0$.
From Assumption 2, we have

$$(g(z) - g(z'))^*(g(z) - g(z')) \leq (z - z')^* \mathcal{L}_g^T \mathcal{L}_g (z - z'), \tag{2.2}$$

where $\mathcal{L}_g = diag\{l_1^g, l_2^g, ..., l_n^g\}$.
Assumption 3: The parameter uncertainties $\triangle\mathcal{D}(t)$, $\triangle\overline{\mathcal{A}}(t) = \triangle\overline{\mathcal{A}}^R(t) + i\triangle\overline{\mathcal{A}}^I(t) + j\triangle\overline{\mathcal{A}}^J(t) + k\triangle\overline{\mathcal{A}}^K(t)$, $\triangle\overline{\mathcal{B}}(t) = \triangle\overline{\mathcal{B}}^R(t) + i\triangle\overline{\mathcal{B}}^I(t) + j\triangle\overline{\mathcal{B}}^J(t) + k\triangle\overline{\mathcal{B}}^K(t)$ in (2.1) are assumed to satisfy: $\triangle\overline{\mathcal{D}}(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^1$, $\triangle\overline{\mathcal{A}}^R(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^2$, $\triangle\overline{\mathcal{A}}^I(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^3$, $\triangle\overline{\mathcal{A}}^J(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^4$, $\triangle\overline{\mathcal{A}}^K(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^5$, $\triangle\overline{\mathcal{B}}^R(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^6$, $\triangle\overline{\mathcal{B}}^I(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^7$, $\triangle\overline{\mathcal{B}}^J(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^8$, $\triangle\overline{\mathcal{B}}^K(t) = \mathcal{G}\mathcal{F}(t)\mathcal{H}^9$, where $\mathcal{G}$ and $\mathcal{H}^\alpha$, $\alpha = 1, 2, ..., 9$ are constant matrices and $\mathcal{F}(t)$ is the time-varying uncertain matrix satisfies $\mathcal{F}^T(t)\mathcal{F}(t) \leq \mathcal{I}$.

The initial condition of the NNs (2.1) is given by

$$z(t) = \varphi(t), \ \ t \in [-\ell, 0], \tag{2.3}$$

where $\varphi(t)$ is continuously differential on $t \in [-\ell, 0]$.

As shown in [39–42], this paper presents a class of T-S FUQVSNNs with time-varying delays based on the T-S fuzzy models as follows.
Plant Rule a:
IF $\vartheta_1(t)$ is η_1^a and $\vartheta_2(t)$ is η_2^a and ... and $\vartheta_g(t)$ is η_g^a, THEN

$$\begin{aligned} dz(t) = & [-(\overline{\mathcal{D}}_a + \triangle\overline{\mathcal{D}}_a(t))z(t) + (\overline{\mathcal{A}}_a + \triangle\overline{\mathcal{A}}_a(t))g(z(t)) + (\overline{\mathcal{B}}_a + \triangle\overline{\mathcal{B}}_a(t))g(z(t - \ell(t)))]dt \\ & + \sigma_a(t, z(t), z(t - \ell(t)))d\omega(t), \end{aligned} \tag{2.4}$$

where $\vartheta_c(t)$ $(c = 1, ..., g)$ is the premise variables vector; η_c^a $(a = 1, ..., m; \ c = 1, ..., g)$ is the fuzzy set, and m is the number of If-Then rules.

By inferring from the fuzzy models, the final output of T-S FUQVSNNs can be obtained as follows

$$dz(t) = \frac{\sum_{a=1}^{m} \psi_a(\vartheta(t)) \left\{ \begin{aligned} & -(\overline{\mathcal{D}}_a + \triangle\overline{\mathcal{D}}_a(t))z(t) + (\overline{\mathcal{A}}_a + \triangle\overline{\mathcal{A}}_a(t))g(z(t)) \\ & + (\overline{\mathcal{B}}_a + \triangle\overline{\mathcal{B}}_a(t))g(z(t - \ell(t)))dt + \sigma_a(t, z(t), z(t - \ell(t)))d\omega(t) \end{aligned} \right\}}{\sum_{a=1}^{m} \psi_a(\vartheta(t))}, \tag{2.5}$$

or equivalently

$$dz(t) = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{ -(\overline{\mathcal{D}}_a + \triangle\overline{\mathcal{D}}_a(t))z(t) + (\overline{\mathcal{A}}_a + \triangle\overline{\mathcal{A}}_a(t))g(z(t)) \\ + (\overline{\mathcal{B}}_a + \triangle\overline{\mathcal{B}}_a(t))g(z(t-\ell(t)))dt + \sigma_a(t, z(t), z(t-\ell(t)))d\omega(t)\Big\}, \tag{2.6}$$

where $\vartheta(t) = (\vartheta_1(t), ..., \vartheta_g(t))^T$, $\chi_a(\vartheta(t)) = \frac{\psi_a(\vartheta(t))}{\sum_{a=1}^{m}\psi_a(\vartheta(t))}$, and $\psi_a(\vartheta(t)) = \prod_{c=1}^{g}\eta_c^a(\vartheta(t))$. The term $\eta_c^a(\vartheta_c(t))$ is the grade membership of $\vartheta_c(t)$ in η_c^a. It is stated that $\psi_a(\vartheta(t)) \geq 0,\ a = 1, ..., m$ and $\sum_{a=1}^{m}\psi_a(\vartheta(t)) > 0$ for all $t \geq 0$. By fuzzy set theory, we have $\chi_a(\vartheta(t)) \geq 0,\ a = 1, ..., m$ and $\sum_{a=1}^{m}\chi_a(\vartheta(t)) = 1$ for all $t \geq 0$.

Assumption 4: For $z = z^R + iz^I + jz^J + kz^K, \widehat{z} = \widehat{z}^R + i\widehat{z}^I + j\widehat{z}^J + k\widehat{z}^K$, with $z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K \in \mathbf{R}^n$, $\sigma_a(t, z, \widehat{z})$ is defined as

$$\sigma_a(t, z, \widehat{z}) = \sigma_a^R(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K) + i\sigma_a^I(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K) \\ + j\sigma_a^J(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K) + k\sigma_a^K(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K), \tag{2.7}$$

where $\sigma_a^R, \sigma_a^I, \sigma_a^J, \sigma_a^K : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^{n\times m}$. There exist matrices $\overline{\mathcal{U}}_a^\alpha \geq 0, \overline{\mathcal{V}}_a^\alpha \geq 0, \overline{\mathcal{M}}_a^\alpha \geq 0, \overline{\mathcal{N}}_a^\alpha \geq 0,\ \alpha = 1, 2, ..., 8$ such that

$$\begin{aligned}
&\mathrm{trace}\{\sigma_a^R(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)^T\sigma_a^R(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)\} \\
&\quad \leq (z^R)^T\overline{\mathcal{U}}_a^1(z^R) + (z^I)^T\overline{\mathcal{U}}_a^2(z^I) + (z^J)^T\overline{\mathcal{U}}_a^3(z^J) + (z^K)^T\overline{\mathcal{U}}_a^4(z^K) \\
&\qquad + (\widehat{z}^R)^T\overline{\mathcal{U}}_a^5(\widehat{z})^R + (\widehat{z}^I)^T\overline{\mathcal{U}}_a^6(\widehat{z}^I) + (\widehat{z}^J)^T\overline{\mathcal{U}}_a^7(\widehat{z}^J) + (\widehat{z}^K)^T\overline{\mathcal{U}}_a^8(\widehat{z}^K), \\
&\mathrm{trace}\{\sigma_a^I(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)^T\sigma_a^I(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)\} \\
&\quad \leq (z^R)^T\overline{\mathcal{V}}_a^1(z^R) + (z^I)^T\overline{\mathcal{V}}_a^2(z^I) + (z^J)^T\overline{\mathcal{V}}_a^3(z^J) + (z^K)^T\overline{\mathcal{V}}_a^4(z^K) \\
&\qquad + (\widehat{z}^R)^T\overline{\mathcal{V}}_a^5(\widehat{z})^R + (\widehat{z}^I)^T\overline{\mathcal{V}}_a^6(\widehat{z}^I) + (\widehat{z}^J)^T\overline{\mathcal{V}}_a^7(\widehat{z}^J) + (\widehat{z}^K)^T\overline{\mathcal{V}}_a^8(\widehat{z}^K), \\
&\mathrm{trace}\{\sigma_a^J(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)^T\sigma_a^J(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)\} \\
&\quad \leq (z^R)^T\overline{\mathcal{M}}_a^1(z^R) + (z^I)^T\overline{\mathcal{M}}_a^2(z^I) + (z^J)^T\overline{\mathcal{M}}_a^3(z^J) + (z^K)^T\overline{\mathcal{M}}_a^4(z^K) \\
&\qquad + (\widehat{z}^R)^T\overline{\mathcal{M}}_a^5(\widehat{z})^R + (\widehat{z}^I)^T\overline{\mathcal{M}}_a^6(\widehat{z}^I) + (\widehat{z}^J)^T\overline{\mathcal{M}}_a^7(\widehat{z}^J) + (\widehat{z}^K)^T\overline{\mathcal{M}}_a^8(\widehat{z}^K), \\
&\mathrm{trace}\{\sigma_a^K(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)^T\sigma_a^K(t, z^R, z^I, z^J, z^K, \widehat{z}^R, \widehat{z}^I, \widehat{z}^J, \widehat{z}^K)\} \\
&\quad \leq (z^R)^T\overline{\mathcal{N}}_a^1(z^R) + (z^I)^T\overline{\mathcal{N}}_a^2(z^I) + (z^J)^T\overline{\mathcal{N}}_a^3(z^J) + (z^K)^T\overline{\mathcal{N}}_a^4(z^K) \\
&\qquad + (\widehat{z}^R)^T\overline{\mathcal{N}}_a^5(\widehat{z})^R + (\widehat{z}^I)^T\overline{\mathcal{N}}_a^6(\widehat{z}^I) + (\widehat{z}^J)^T\overline{\mathcal{N}}_a^7(\widehat{z}^J) + (\widehat{z}^K)^T\overline{\mathcal{N}}_a^8(\widehat{z}^K).
\end{aligned}$$

To further investigate, we define $z(t) = z^R(t) + iz^I(t) + jz^J(t) + kz^K(t)$, $\overline{\mathcal{A}} = \overline{\mathcal{A}}^R + i\overline{\mathcal{A}}^I + j\overline{\mathcal{A}}^J + k\overline{\mathcal{A}}^K$, $\overline{\mathcal{B}} = \overline{\mathcal{B}}^R + i\overline{\mathcal{B}}^I + j\overline{\mathcal{B}}^J + k\overline{\mathcal{B}}^K$, $g(z(t)) = g^R(z^R(t), z^I(t), z^J(t), z^K(t)) + ig^I(z^R(t), z^I(t), z^J(t), z^K(t)) + jg^J(z^R(t), z^I(t), z^J(t), z^K(t)) + kg^K(z^R(t), z^I(t), z^J(t), z^K(t))$, $g(z(t-\ell(t))) = g^R(z^R(t-\ell(t)), z^I(t-\ell_1(t)), z^J(t-\ell(t)), z^K(t-\ell(t))) + ig^I(z^R(t-\ell(t)), z^I(t-\ell(t)), z^J(t-\ell(t)), z^K(t-\ell(t))) + jg^J(z^R(t-\ell(t)), z^I(t-\ell(t)), z^J(t-\ell(t)), z^K(t-\ell(t))) + kg^K(z^R(t-\ell(t)), z^I(t-\ell(t)), z^J(t-\ell(t)), z^K(t-\ell(t)))$.

In order to simplify the resulting parts, the following notations are used:
$z^R = z^R(t)$, $z^I = z^I(t)$, $z^J = z^J(t)$, $z^K = z^K(t)$, $z^R_{\ell(t)} = z^R(t-\ell(t))$, $z^I_{\ell(t)} = z^I(t-\ell(t))$, $z^J_{\ell(t)} = z^J(t-\ell(t))$, $z^K_{\ell(t)} = z^K(t-\ell(t))$.

Hence, the T-S FUQVSNNs (2.6) can be splitting into real and imaginary parts as

$$\begin{cases} dz^R = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\big[-(\overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t))z^R+(\overline{\mathcal{A}}^R_a+\triangle\overline{\mathcal{A}}^R_a(t))g^R(z^R,z^I,z^J,z^K) \\ -(\overline{\mathcal{A}}^I_a+\triangle\overline{\mathcal{A}}^I_a(t))g^I(z^R,z^I,z^J,z^K)-(\overline{\mathcal{A}}^J_a+\triangle\overline{\mathcal{A}}^J_a(t))g^J(z^R,z^I,z^J,z^K) \\ -(\overline{\mathcal{A}}^K_a+\triangle\overline{\mathcal{A}}^K_a(t))g^K(z^R,z^I,z^J,z^K)+(\overline{\mathcal{B}}^R_a+\triangle\overline{\mathcal{B}}^R_a(t))g^R(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ -(\overline{\mathcal{B}}^I_a+\triangle\overline{\mathcal{B}}^I_a(t))g^I(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})-(\overline{\mathcal{B}}^J_a+\triangle\overline{\mathcal{B}}^J_a(t))g^J(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ -(\overline{\mathcal{B}}^K_a+\triangle\overline{\mathcal{B}}^K_a(t))g^K(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\big]dt \\ +\sigma^R_a(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})d\omega(t)\Big\} \\ dz^I = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\big[-(\overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t))z^I+(\overline{\mathcal{A}}^R_a+\triangle\overline{\mathcal{A}}^R_a(t))g^I(z^R,z^I,z^J,z^K) \\ +(\overline{\mathcal{A}}^I_a+\triangle\overline{\mathcal{A}}^I_a(t))g^R(z^R,z^I,z^J,z^K)+(\overline{\mathcal{A}}^J_a+\triangle\overline{\mathcal{A}}^J_a(t))g^K(z^R,z^I,z^J,z^K) \\ -(\overline{\mathcal{A}}^K_a+\triangle\overline{\mathcal{A}}^K_a(t))g^J(z^R,z^I,z^J,z^K)+(\overline{\mathcal{B}}^R_a+\triangle\overline{\mathcal{B}}^R_a(t))g^I(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ +(\overline{\mathcal{B}}^I_a+\triangle\overline{\mathcal{B}}^I_a(t))g^R(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})+(\overline{\mathcal{B}}^J_a+\triangle\overline{\mathcal{B}}^J_a(t))g^K(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ -(\overline{\mathcal{B}}^K_a+\triangle\overline{\mathcal{B}}^K_a(t))g^J(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\big]dt \\ +\sigma^I_a(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})d\omega(t)\Big\} \\ dz^J = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\big[-(\overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t))z^J+(\overline{\mathcal{A}}^R_a+\triangle\overline{\mathcal{A}}^R_a(t))g^J(z^R,z^I,z^J,z^K) \\ +(\overline{\mathcal{A}}^J_a+\triangle\overline{\mathcal{A}}^J_a(t))g^R(z^R,z^I,z^J,z^K)-(\overline{\mathcal{A}}^I_a+\triangle\overline{\mathcal{A}}^I_a(t))g^K(z^R,z^I,z^J,z^K) \\ +(\overline{\mathcal{A}}^K_a+\triangle\overline{\mathcal{A}}^K_a(t))g^I(z^R,z^I,z^J,z^K)+(\overline{\mathcal{B}}^R_a+\triangle\overline{\mathcal{B}}^R_a(t))g^J(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ +(\overline{\mathcal{B}}^J_a+\triangle\overline{\mathcal{B}}^J_a(t))g^R(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})-(\overline{\mathcal{B}}^I_a+\triangle\overline{\mathcal{B}}^I_a(t))g^K(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ +(\overline{\mathcal{B}}^K_a+\triangle\overline{\mathcal{B}}^K_a(t))g^I(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\big]dt \\ +\sigma^J_a(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})d\omega(t)\Big\} \\ dz^K = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\big[-(\overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t))z^K+(\overline{\mathcal{A}}^R_a+\triangle\overline{\mathcal{A}}^R_a(t))g^K(z^R,z^I,z^J,z^K) \\ +(\overline{\mathcal{A}}^K_a+\triangle\overline{\mathcal{A}}^K_a(t))g^R(z^R,z^I,z^J,z^K)+(\overline{\mathcal{A}}^I_a+\triangle\overline{\mathcal{A}}^I_a(t))g^J(z^R,z^I,z^J,z^K) \\ -(\overline{\mathcal{A}}^J_a+\triangle\overline{\mathcal{A}}^J_a(t))g^I(z^R,z^I,z^J,z^K)+(\overline{\mathcal{B}}^R_a+\triangle\overline{\mathcal{B}}^R_a(t))g^K(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ +(\overline{\mathcal{B}}^K_a+\triangle\overline{\mathcal{B}}^K_a(t))g^R(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})+(\overline{\mathcal{B}}^I_a+\triangle\overline{\mathcal{B}}^I_a(t))g^J(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ -(\overline{\mathcal{B}}^J_a+\triangle\overline{\mathcal{B}}^J_a(t))g^I(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\big]dt \\ +\sigma^K_a(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})d\omega(t)\Big\}. \end{cases} \tag{2.8}$$

Based on (2.8), the following NNs can be written as:

$$
\begin{cases}
\begin{bmatrix} dz^R \\ dz^I \\ dz^J \\ dz^K \end{bmatrix} = \sum_{a=1}^{m} \chi_a(\vartheta(t)) \Bigg\{ \Bigg(- \begin{bmatrix} \overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t) & 0 & 0 & 0 \\ 0 & \overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t) & 0 & 0 \\ 0 & 0 & \overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t) & 0 \\ 0 & 0 & 0 & \overline{\mathcal{D}}_a+\triangle\overline{\mathcal{D}}_a(t) \end{bmatrix} \begin{bmatrix} z^R \\ z^I \\ z^J \\ z^K \end{bmatrix} \\
+ \begin{bmatrix} \overline{\mathcal{A}}_a^R+\triangle\overline{\mathcal{A}}_a^R(t) & -\overline{\mathcal{A}}_a^I-\triangle\overline{\mathcal{A}}_a^I(t) & -\overline{\mathcal{A}}_a^J-\triangle\overline{\mathcal{A}}_a^J(t) & -\overline{\mathcal{A}}_a^K-\triangle\overline{\mathcal{A}}_a^K(t) \\ \overline{\mathcal{A}}_a^I+\triangle\overline{\mathcal{A}}_a^I(t) & \overline{\mathcal{A}}_a^R+\triangle\overline{\mathcal{A}}_a^R(t) & -\overline{\mathcal{A}}_a^K-\triangle\overline{\mathcal{A}}_a^K(t) & \overline{\mathcal{A}}_a^J+\triangle\overline{\mathcal{A}}_a^J(t) \\ \overline{\mathcal{A}}_a^J+\triangle\overline{\mathcal{A}}_a^J(t) & \overline{\mathcal{A}}_a^K+\triangle\overline{\mathcal{A}}_a^K(t) & \overline{\mathcal{A}}_a^R+\triangle\overline{\mathcal{A}}_a^R(t) & -\overline{\mathcal{A}}_a^I-\triangle\overline{\mathcal{A}}_a^I(t) \\ \overline{\mathcal{A}}_a^K+\triangle\overline{\mathcal{A}}_a^K(t) & -\overline{\mathcal{A}}_a^J-\triangle\overline{\mathcal{A}}_a^J(t) & \overline{\mathcal{A}}_a^I+\triangle\overline{\mathcal{A}}_a^I(t) & \overline{\mathcal{A}}_a^R+\triangle\overline{\mathcal{A}}_a^R(t) \end{bmatrix} \begin{bmatrix} g^R\big(z^R,z^I,z^J,z^K\big) \\ g^I\big(z^R,z^I,z^J,z^K\big) \\ g^J\big(z^R,z^I,z^J,z^K\big) \\ g^K\big(z^R,z^I,z^J,z^K\big) \end{bmatrix} \\
+ \begin{bmatrix} \overline{\mathcal{B}}_a^R+\triangle\overline{\mathcal{B}}_a^R(t) & -\overline{\mathcal{B}}_a^I-\triangle\overline{\mathcal{B}}_a^I(t) & -\overline{\mathcal{B}}_a^J-\triangle\overline{\mathcal{B}}_a^J(t) & -\overline{\mathcal{B}}_a^K-\triangle\overline{\mathcal{B}}_a^K(t) \\ \overline{\mathcal{B}}_a^I+\triangle\overline{\mathcal{B}}_a^I(t) & \overline{\mathcal{B}}_a^R+\triangle\overline{\mathcal{B}}_a^R(t) & -\overline{\mathcal{B}}_a^K-\triangle\overline{\mathcal{B}}_a^K(t) & \overline{\mathcal{B}}_a^J+\triangle\overline{\mathcal{B}}_a^J(t) \\ \overline{\mathcal{B}}_a^J+\triangle\overline{\mathcal{B}}_a^J(t) & \overline{\mathcal{B}}_a^K+\triangle\overline{\mathcal{B}}_a^K(t) & \overline{\mathcal{B}}_a^R+\triangle\overline{\mathcal{B}}_a^R(t) & -\overline{\mathcal{B}}_a^I-\triangle\overline{\mathcal{B}}_a^I(t) \\ \overline{\mathcal{B}}_a^K+\triangle\overline{\mathcal{B}}_a^K(t) & -\overline{\mathcal{B}}_a^J-\triangle\overline{\mathcal{B}}_a^J(t) & \overline{\mathcal{B}}_a^I+\triangle\overline{\mathcal{B}}_a^I(t) & \overline{\mathcal{B}}_a^R+\triangle\overline{\mathcal{B}}_a^R(t) \end{bmatrix} \begin{bmatrix} g^R\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^I\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^J\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^K\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \end{bmatrix} \Bigg) dt \\
+ \begin{bmatrix} \sigma_a^R\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^I\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^J\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^K\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \end{bmatrix} d\omega(t) \Bigg\},
\end{cases}
\tag{2.9}
$$

which is equivalent to

$$
\begin{cases}
\begin{bmatrix} dz^R \\ dz^I \\ dz^J \\ dz^K \end{bmatrix} = \sum_{a=1}^{m} \chi_a(\vartheta(t)) \Bigg\{ \Bigg(- \Bigg(\begin{bmatrix} \overline{\mathcal{D}}_a & 0 & 0 & 0 \\ 0 & \overline{\mathcal{D}}_a & 0 & 0 \\ 0 & 0 & \overline{\mathcal{D}}_a & 0 \\ 0 & 0 & 0 & \overline{\mathcal{D}}_a \end{bmatrix} + \begin{bmatrix} \triangle\overline{\mathcal{D}}_a(t) & 0 & 0 & 0 \\ 0 & \triangle\overline{\mathcal{D}}_a(t) & 0 & 0 \\ 0 & 0 & \triangle\overline{\mathcal{D}}_a(t) & 0 \\ 0 & 0 & 0 & \triangle\overline{\mathcal{D}}_a(t) \end{bmatrix} \Bigg) \begin{bmatrix} z^R \\ z^I \\ z^J \\ z^K \end{bmatrix} \\
+ \Bigg(\begin{bmatrix} \overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^I & -\overline{\mathcal{A}}_a^J & -\overline{\mathcal{A}}_a^K \\ \overline{\mathcal{A}}_a^I & \overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^K & \overline{\mathcal{A}}_a^J \\ \overline{\mathcal{A}}_a^J & \overline{\mathcal{A}}_a^K & \overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^I \\ \overline{\mathcal{A}}_a^K & -\overline{\mathcal{A}}_a^J & \overline{\mathcal{A}}_a^I & \overline{\mathcal{A}}_a^R \end{bmatrix} + \begin{bmatrix} \triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^I(t) & -\triangle\overline{\mathcal{A}}_a^J(t) & -\triangle\overline{\mathcal{A}}_a^K(t) \\ \triangle\overline{\mathcal{A}}_a^I(t) & \triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^K(t) & \triangle\overline{\mathcal{A}}_a^J(t) \\ \triangle\overline{\mathcal{A}}_a^J(t) & \triangle\overline{\mathcal{A}}_a^K(t) & \triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^I(t) \\ \triangle\overline{\mathcal{A}}_a^K(t) & -\triangle\overline{\mathcal{A}}_a^J(t) & \triangle\overline{\mathcal{A}}_a^I(t) & \triangle\overline{\mathcal{A}}_a^R(t) \end{bmatrix} \Bigg) \begin{bmatrix} g^R\big(z^R,z^I,z^J,z^K\big) \\ g^I\big(z^R,z^I,z^J,z^K\big) \\ g^J\big(z^R,z^I,z^J,z^K\big) \\ g^K\big(z^R,z^I,z^J,z^K\big) \end{bmatrix} \\
+ \Bigg(\begin{bmatrix} \overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^I & -\overline{\mathcal{B}}_a^J & -\overline{\mathcal{B}}_a^K \\ \overline{\mathcal{B}}_a^I & \overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^K & \overline{\mathcal{B}}_a^J \\ \overline{\mathcal{B}}_a^J & \overline{\mathcal{B}}_a^K & \overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^I \\ \overline{\mathcal{B}}_a^K & -\overline{\mathcal{B}}_a^J & \overline{\mathcal{B}}_a^I & \overline{\mathcal{B}}_a^R \end{bmatrix} + \begin{bmatrix} \triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^I(t) & -\triangle\overline{\mathcal{B}}_a^J(t) & -\triangle\overline{\mathcal{B}}_a^K(t) \\ \triangle\overline{\mathcal{B}}_a^I(t) & \triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^K(t) & \triangle\overline{\mathcal{B}}_a^J(t) \\ \triangle\overline{\mathcal{B}}_a^J(t) & \triangle\overline{\mathcal{B}}_a^K(t) & \triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^I(t) \\ \triangle\overline{\mathcal{B}}_a^K(t) & -\triangle\overline{\mathcal{B}}_a^J(t) & \triangle\overline{\mathcal{B}}_a^I(t) & \triangle\overline{\mathcal{B}}_a^R(t) \end{bmatrix} \Bigg) \begin{bmatrix} g^R\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^I\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^J\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ g^K\big(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \end{bmatrix} \Bigg) dt \\
+ \begin{bmatrix} \sigma_a^R\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^I\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^J\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \\ \sigma_a^K\big(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}\big) \end{bmatrix} d\omega(t) \Bigg\}.
\end{cases}
\tag{2.10}
$$

Let

$$
\widehat{g}(\pi(t)) = \begin{bmatrix} g^R(z^R,z^I,z^J,z^K) \\ g^I(z^R,z^I,z^J,z^K) \\ g^J(z^R,z^I,z^J,z^K) \\ g^K(z^R,z^I,z^J,z^K) \end{bmatrix}, \ \widehat{g}(\pi(t-\ell(t))) = \begin{bmatrix} g^R(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ g^I(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ g^J(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \\ g^K(z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)}) \end{bmatrix},
$$

$$\pi(t)=\begin{bmatrix} z^R\\ z^I\\ z^J\\ z^K\end{bmatrix},\ \widehat{\mathcal{D}}_a=\begin{bmatrix}\overline{\mathcal{D}}_a & 0 & 0 & 0\\ 0 & \overline{\mathcal{D}}_a & 0 & 0\\ 0 & 0 & \overline{\mathcal{D}}_a & 0\\ 0 & 0 & 0 & \overline{\mathcal{D}}_a\end{bmatrix},\ \widehat{\mathcal{A}}_a=\begin{bmatrix}\overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^I & -\overline{\mathcal{A}}_a^J & -\overline{\mathcal{A}}_a^K\\ \overline{\mathcal{A}}_a^I & \overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^K & \overline{\mathcal{A}}_a^J\\ \overline{\mathcal{A}}_a^J & \overline{\mathcal{A}}_a^K & \overline{\mathcal{A}}_a^R & -\overline{\mathcal{A}}_a^I\\ \overline{\mathcal{A}}_a^K & -\overline{\mathcal{A}}_a^J & \overline{\mathcal{A}}_a^I & \overline{\mathcal{A}}_a^R\end{bmatrix},$$

$$\widehat{\mathcal{B}}_a=\begin{bmatrix}\overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^I & -\overline{\mathcal{B}}_a^J & -\overline{\mathcal{B}}_a^K\\ \overline{\mathcal{B}}_a^I & \overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^K & \overline{\mathcal{B}}_a^J\\ \overline{\mathcal{B}}_a^J & \overline{\mathcal{B}}_a^K & \overline{\mathcal{B}}_a^R & -\overline{\mathcal{B}}_a^I\\ \overline{\mathcal{B}}_a^K & -\overline{\mathcal{B}}_a^J & \overline{\mathcal{B}}_a^I & \overline{\mathcal{B}}_a^R\end{bmatrix},\ \varsigma_a(t)=\begin{bmatrix}\sigma_a^R(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\\ \sigma_a^I(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\\ \sigma_a^J(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\\ \sigma_a^K(t,z^R,z^I,z^J,z^K,z^R_{\ell(t)},z^I_{\ell(t)},z^J_{\ell(t)},z^K_{\ell(t)})\end{bmatrix},$$

$$\triangle\widehat{\mathcal{D}}_a=\begin{bmatrix}\triangle\overline{\mathcal{D}}_a(t) & 0 & 0 & 0\\ 0 & \triangle\overline{\mathcal{D}}_a(t) & 0 & 0\\ 0 & 0 & \triangle\overline{\mathcal{D}}_a(t) & 0\\ 0 & 0 & 0 & \triangle\overline{\mathcal{D}}_a(t)\end{bmatrix},$$

$$\triangle\widehat{\mathcal{A}}_a=\begin{bmatrix}\triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^I(t) & -\triangle\overline{\mathcal{A}}_a^J(t) & -\triangle\overline{\mathcal{A}}_a^K(t)\\ \triangle\overline{\mathcal{A}}_a^I(t) & \triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^K(t) & \triangle\overline{\mathcal{A}}_a^J(t)\\ \triangle\overline{\mathcal{A}}_a^J(t) & \triangle\overline{\mathcal{A}}_a^K(t) & \triangle\overline{\mathcal{A}}_a^R(t) & -\triangle\overline{\mathcal{A}}_a^I(t)\\ \triangle\overline{\mathcal{A}}_a^K(t) & -\triangle\overline{\mathcal{A}}_a^J(t) & \triangle\overline{\mathcal{A}}_a^I(t) & \triangle\overline{\mathcal{A}}_a^R(t)\end{bmatrix},$$

$$\triangle\widehat{\mathcal{B}}_a=\begin{bmatrix}\triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^I(t) & -\triangle\overline{\mathcal{B}}_a^J(t) & -\triangle\overline{\mathcal{B}}_a^K(t)\\ \triangle\overline{\mathcal{B}}_a^I(t) & \triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^K(t) & \triangle\overline{\mathcal{B}}_a^J(t)\\ \triangle\overline{\mathcal{B}}_a^J(t) & \triangle\overline{\mathcal{B}}_a^K(t) & \triangle\overline{\mathcal{B}}_a^R(t) & -\triangle\overline{\mathcal{B}}_a^I(t)\\ \triangle\overline{\mathcal{B}}_a^K(t) & -\triangle\overline{\mathcal{B}}_a^J(t) & \triangle\overline{\mathcal{B}}_a^I(t) & \triangle\overline{\mathcal{B}}_a^R(t)\end{bmatrix}.$$

Now, the system (2.10) is equivalent form as

$$\begin{aligned} d\pi(t)=\sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\big[-(\widehat{\mathcal{D}}_a+\triangle\widehat{\mathcal{D}}_a)\pi(t)+(\widehat{\mathcal{A}}_a+\triangle\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t))\\ +(\widehat{\mathcal{B}}_a+\triangle\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t-\ell(t)))\big]dt+\varsigma_a(t)d\omega(t)\Big\}. \end{aligned} \tag{2.11}$$

From (2.11), the parameter uncertainties $\triangle\widehat{\mathcal{D}}_a,\triangle\widehat{\mathcal{A}}_a,\triangle\widehat{\mathcal{B}}_a$, which satisfy:

$$\triangle\widehat{\mathcal{D}}_a=\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^1,\ \triangle\widehat{\mathcal{A}}_a=\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^2,\ \triangle\widehat{\mathcal{B}}_a=\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^3, \tag{2.12}$$

where

$$\widehat{\mathcal{G}}_a=\begin{bmatrix}\mathcal{G}_a & 0 & 0 & 0\\ 0 & \mathcal{G}_a & 0 & 0\\ 0 & 0 & \mathcal{G}_a & 0\\ 0 & 0 & 0 & \mathcal{G}_a\end{bmatrix},\ \widehat{\mathcal{F}}_a(t)=\begin{bmatrix}\mathcal{F}_a(t) & 0 & 0 & 0\\ 0 & \mathcal{F}_a(t) & 0 & 0\\ 0 & 0 & \mathcal{F}_a(t) & 0\\ 0 & 0 & 0 & \mathcal{F}_a(t)\end{bmatrix},$$

$$\widehat{\mathcal{H}}_a^1=\begin{bmatrix}\mathcal{H}_a^1 & 0 & 0 & 0\\ 0 & \mathcal{H}_a^1 & 0 & 0\\ 0 & 0 & \mathcal{H}_a^1 & 0\\ 0 & 0 & 0 & \mathcal{H}_a^1\end{bmatrix},\ \widehat{\mathcal{H}}_a^2=\begin{bmatrix}\mathcal{H}_a^2 & -\mathcal{H}_a^3 & -\mathcal{H}_a^4 & -\mathcal{H}_a^5\\ \mathcal{H}_a^3 & \mathcal{H}_a^2 & -\mathcal{H}_a^5 & \mathcal{H}_a^4\\ \mathcal{H}_a^4 & \mathcal{H}_a^5 & \mathcal{H}_a^2 & -\mathcal{H}_a^3\\ \mathcal{H}_a^5 & -\mathcal{H}_a^4 & \mathcal{H}_a^3 & \mathcal{H}_a^2\end{bmatrix},$$

$$\widehat{\mathcal{H}}_a^3 = \begin{bmatrix} \mathcal{H}_a^6 & -\mathcal{H}_a^7 & -\mathcal{H}_a^8 & -\mathcal{H}_a^9 \\ \mathcal{H}_a^7 & \mathcal{H}_a^6 & -\mathcal{H}_a^9 & \mathcal{H}_a^8 \\ \mathcal{H}_a^8 & \mathcal{H}_a^9 & \mathcal{H}_a^6 & -\mathcal{H}_a^7 \\ \mathcal{H}_a^9 & -\mathcal{H}_a^8 & \mathcal{H}_a^7 & \mathcal{H}_a^6 \end{bmatrix}.$$

The initial condition of the system (2.11) is given by

$$\pi(t) = \widehat{\varphi}(t), t \in [-\ell, 0], \tag{2.13}$$

where $\widehat{\varphi}(t) = (\varphi^R(t)\ \varphi^I(t)\ \varphi^J(t)\ \varphi^K(t))^T$.

To simplify the sequel, the following abbreviations are used

$$\theta_a(t) = -(\widehat{\mathcal{D}}_a + \triangle\widehat{\mathcal{D}}_a)\pi(t) + (\widehat{\mathcal{A}}_a + \triangle\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t)) + (\widehat{\mathcal{B}}_a + \triangle\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t - \ell(t))).$$

The system (2.11) read as

$$d\pi(t) = \sum_{a=1}^{m} \chi_a(\vartheta(t))\theta_a(t)dt + \sum_{a=1}^{m} \chi_a(\vartheta(t))\varsigma_a(t)d\omega(t). \tag{2.14}$$

In order to derive our main results, we present some definitions, lemmas.

Definition 2.1. [50] The NN model (2.1) is said to be mean-square stable if for any $\epsilon > 0$ there exists a scalar $\kappa(\epsilon) > 0$ such that $\mathbf{E}\{\|z(t)\|^2\} < \epsilon,\ t > 0$, whenever $\sup_{-\ell \le t \le 0} \mathbf{E}\{\|\varphi(t)\|^2\} < \kappa(\epsilon)$. In addition, if $\lim_{t\to\infty} \mathbf{E}\{\|z(t)\|^2\} = 0$ for any initial condition, then the NNs (2.1) is said to be mean-square robustly asymptotically stable.

Lemma 2.2. *[51] Let $\mathcal{M} \in \mathbf{R}^{n\times n}$ be a positive definite matrix, vector function $z(s) : [a, b] \to \mathbf{R}^n$ with scalars $a < b$, then*

$$-(b-a)\int_a^b z^T(s)\mathcal{M}z(s)ds \le -\Big[\int_a^b z(s)ds\Big]^T \mathcal{M}\Big[\int_a^b z(s)ds\Big].$$

Lemma 2.3. *[51] Let $\Omega = \Omega^T$, $\mathcal{J}_1$ and $\mathcal{J}_2$ be real matrices, $\mathcal{F}(t)$ satisfies $\mathcal{F}^T(t)\mathcal{F}(t) \le \mathcal{I}$. Then $\Omega + (\mathcal{J}_1\mathcal{F}(t)\mathcal{J}_2) + (\mathcal{J}_1\mathcal{F}(t)\mathcal{J}_2)^T < 0$, iff there exist a scalar $\epsilon > 0$ such that $\Omega + \epsilon\mathcal{J}_1\mathcal{J}_1^T + \epsilon^{-1}\mathcal{J}_2^T\mathcal{J}_2 < 0$.*

Lemma 2.4. *[51] Given constant matrices $\mathcal{M}, \mathcal{N}$ and $\mathcal{O}$ with $0 < \mathcal{M} = \mathcal{M}^T$ and $0 < \mathcal{N} = \mathcal{N}^T$, then $\begin{bmatrix} \mathcal{M} & \mathcal{O} \\ \mathcal{O}^T & \mathcal{N} \end{bmatrix} < 0$, is equivalent to one of the following conditions: (i) $\mathcal{N} < 0,\ \mathcal{M} - \mathcal{O}\mathcal{N}^{-1}\mathcal{O}^T < 0$, (ii) $\mathcal{M} < 0,\ \mathcal{N} - \mathcal{O}^T\mathcal{M}^{-1}\mathcal{O} < 0$.*

Lemma 2.5. *[52] Let $\mathcal{M} \in \mathbf{R}^{n\times n}$ be a positive definite matrix, two matrices $\Lambda_1, \Lambda_2 \in \mathbf{R}^{n\times m}$, positive integers n and m, scalar $\zeta \in (0, 1)$, any vector $\xi \in \mathbf{R}^m$, denote the function $\Xi(\zeta, \mathcal{M})$ with the following form:*

$$\Xi(\zeta, \mathcal{M}) = \frac{1}{\zeta}\xi^T\Lambda_1^T\mathcal{M}\Lambda_1\xi + \frac{1}{1-\zeta}\xi^T\Lambda_2^T\mathcal{M}\Lambda_2\xi.$$

There exists a matrix $\mathcal{N} \in \mathbf{R}^{n\times n}$ satisfying $\begin{bmatrix} \mathcal{M} & \mathcal{N} \\ \mathcal{N}^T & \mathcal{M} \end{bmatrix} > 0$, then

$$\min_{\zeta\in(0,1)} \Xi(\zeta, \mathcal{M}) \ge \begin{bmatrix} \Lambda_1\xi \\ \Lambda_2\xi \end{bmatrix}^T \begin{bmatrix} \mathcal{M} & \mathcal{N} \\ \mathcal{N}^T & \mathcal{M} \end{bmatrix} \begin{bmatrix} \Lambda_1\xi \\ \Lambda_2\xi \end{bmatrix}.$$

3. Main results

This section derives sufficient criteria for the robust and global asymptotic stability criteria for T-S FUQVSNNs using the Lyapunov stability theory and LMI method.

3.1. Robust stability analysis

In the following Theorem (3.1), we derive the mean square robust asymptotic stability criteria for T-S FUQVSNNs (2.11).

Theorem 3.1. *Suppose Assumptions 1-4 hold. If there exist positive symmetric matrices* $\mathcal{P} \in \mathbf{R}^{4n\times 4n}, \mathcal{Q} \in \mathbf{R}^{4n\times 4n}, \mathcal{R} \in \mathbf{R}^{4n\times 4n}, \mathcal{S} \in \mathbf{R}^{4n\times 4n}$*, any matrix* $\mathcal{T} \in \mathbf{R}^{4n\times 4n}$ *and positive scalars* $\epsilon_1 \in \mathbf{R}^n, \epsilon_2 \in \mathbf{R}^n, \epsilon_3 \in \mathbf{R}^n, \gamma_1 \in \mathbf{R}^n, \gamma_2 \in \mathbf{R}^n, \lambda \in \mathbf{R}^n$*, such that the following LMIs hold for all* $a = 1, 2, ..., m$

$$\mathcal{P} \leq \lambda \mathcal{I}, \tag{3.1}$$

$$\begin{bmatrix} \widehat{\Phi}_a & \Upsilon_a \\ \maltese & \Omega_a \end{bmatrix} < 0, \tag{3.2}$$

where $\widehat{\Phi}_a = (\widehat{\Phi}_{i,j,a})_{7\times 7}$, $\Upsilon_a = (\Upsilon_{i,j,a})_{7\times 3}$, $\Omega_a = (\Omega_{i,j,a})_{3\times 3}$ *with* $\widehat{\Phi}_{1,1,a} = -\mathcal{P}\widehat{\mathcal{D}}_a - \widehat{\mathcal{D}}_a\mathcal{P}^T + \epsilon_1 \widehat{\mathcal{H}}_a^{1^T} \widehat{\mathcal{H}}_a^1 + \mathcal{Q} + \mathcal{R} + \ell^2\mathcal{S} + \gamma_1\widehat{\mathcal{L}}_g + \lambda\Pi_1$, $\widehat{\Phi}_{1,4,a} = \mathcal{P}\widehat{\mathcal{A}}_a$, $\widehat{\Phi}_{1,5,a} = \mathcal{P}\widehat{\mathcal{B}}_a$, $\widehat{\Phi}_{2,2,a} = -(1-\mu)\mathcal{Q} + \gamma_2\widehat{\mathcal{L}}_g + \lambda\Pi_2$, $\widehat{\Phi}_{3,3,a} = -\mathcal{R}$, $\widehat{\Phi}_{4,4,a} = -\gamma_1 + \epsilon_2 \widehat{\mathcal{H}}_a^{2^T} \widehat{\mathcal{H}}_a^2$, $\widehat{\Phi}_{5,5,a} = -\gamma_2 + \epsilon_3 \widehat{\mathcal{H}}_a^{3^T} \widehat{\mathcal{H}}_a^3$, $\widehat{\Phi}_{6,6,a} = -\mathcal{S}$, $\widehat{\Phi}_{6,7,a} = -\mathcal{T}$, $\widehat{\Phi}_{7,7,a} = -\mathcal{S}$, $\Upsilon_{1,1,a} = \mathcal{P}\widehat{\mathcal{G}}_a$, $\Upsilon_{1,2,a} = \mathcal{P}\widehat{\mathcal{G}}_a$, $\Upsilon_{1,3,a} = \mathcal{P}\widehat{\mathcal{G}}_a$, $\Omega_{1,1,a} = -\epsilon_1$, $\Omega_{2,2,a} = -\epsilon_2$, $\Omega_{3,3,a} = -\epsilon_3$*, then the NN model* (2.11) *is robustly asymptotically stable in the mean square.*

Proof: Take the following LKF (3.3) for the NNs (2.11)

$$\begin{aligned} \mathbf{V}(t, \pi(t), a) = \pi^T(t)\mathcal{P}\pi(t) + \int_{t-\ell(t)}^{t} \pi^T(s)\mathcal{Q}\pi(s)ds + \int_{t-\ell}^{t} \pi^T(s)\mathcal{R}\pi(s)ds \\ + \ell \int_{-\ell}^{0}\int_{t+u}^{t} \pi^T(s)\mathcal{S}\pi(s)dsdu. \end{aligned} \tag{3.3}$$

Suppose $\mathcal{L}$ is the weak infinitesimal generator. By Ito's formula, the time derivative of $\mathbf{V}(t, \pi(t), a)$ can be calculated along the trajectories of the system (2.11) is given by

$$\begin{aligned} \mathcal{L}\mathbf{V}(t, \pi(t), a) = & \sum_{a=1}^{m} \chi_a(\vartheta(t)) \Big\{ 2\pi^T(t)\mathcal{P}\theta_a(t) + \text{trace}\{\varsigma_a^T(t)\mathcal{P}\varsigma_a(t)\} + \pi^T(t)\mathcal{Q}\pi(t) \\ & - (1 - \dot{\ell}(t))\pi^T(t-\ell(t))\mathcal{Q}\pi(t-\ell(t)) + \pi^T(t)\mathcal{R}\pi(t) \\ & - \pi^T(t-\ell)\mathcal{R}\pi(t-\ell) + \ell^2\pi^T(t)\mathcal{S}\pi(t) - \ell\int_{t-\ell}^{t} \pi^T(u)\mathcal{S}\pi(u)du \Big\} \\ = & \sum_{a=1}^{m} \chi_a(\vartheta(t)) \Big\{ 2\pi^T(t)\mathcal{P}[-(\widehat{\mathcal{D}}_a + \triangle\widehat{\mathcal{D}}_a)\pi(t) + (\widehat{\mathcal{A}}_a + \triangle\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t)) \\ & + (\widehat{\mathcal{B}}_a + \triangle\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t-\ell(t)))] + \text{trace}\{\varsigma_a^T(t)\mathcal{P}\varsigma_a(t)\} + \pi^T(t)\mathcal{Q}\pi(t) \\ & - (1 - \dot{\ell}(t))\pi^T(t-\ell(t))\mathcal{Q}\pi(t-\ell(t)) + \pi^T(t)\mathcal{R}\pi(t) \end{aligned}$$

$$- \pi^T(t-\ell)\mathcal{R}\pi(t-\ell) + \ell^2\pi^T(t)\mathcal{S}\pi(t) - \ell\int_{t-\ell}^{t}\pi^T(u)\mathcal{S}\pi(u)du\Big\}$$
$$\leq \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{-2\pi^T(t)(\mathcal{P}\widehat{\mathcal{D}}_a)\pi(t) - 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^1)\pi(t)$$
$$+ 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t)) + 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^2)\widehat{g}(\pi(t))$$
$$+ 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t-\ell(t))) + 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{F}}_a(t)\widehat{\mathcal{H}}_a^3)\widehat{g}(\pi(t-\ell(t)))$$
$$+ \text{trace}\{\varsigma_a^T(t)\mathcal{P}\varsigma_a(t)\} + \pi^T(t)\mathcal{Q}\pi(t) - (1-\mu)\pi^T(t-\ell(t))\mathcal{Q}$$
$$\times\pi(t-\ell(t)) + \pi^T(t)\mathcal{R}\pi(t) - \pi^T(t-\ell)\mathcal{R}\pi(t-\ell) + \ell^2\pi^T(t)\mathcal{S}\pi(t)$$
$$- \ell\int_{t-\ell}^{t}\pi^T(u)\mathcal{S}\pi(u)du\Big\}. \tag{3.4}$$

By using Lemma (2.3), we can get

$$\mathcal{L}\mathbf{V}(t,\pi(t),a) \leq \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{-2\pi^T(t)(\mathcal{P}\widehat{\mathcal{D}}_a)\pi(t) + \epsilon_1^{-1}\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T)\pi(t)$$
$$+ \epsilon_1\pi^T(t)(\widehat{\mathcal{H}}_a^{1^T}\widehat{\mathcal{H}}_a^1)\pi(t) + 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t)) + \epsilon_2^{-1}\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T)$$
$$\times\pi(t) + \epsilon_2\widehat{g}^T(\pi(t))(\widehat{\mathcal{H}}_a^{2^T}\widehat{\mathcal{H}}_a^2)\widehat{g}(\pi(t)) + 2\pi^T(t)(\mathcal{P}\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t-\ell(t)))$$
$$+ \epsilon_3^{-1}\pi^T(t)(\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T)\pi(t) + \epsilon_3\widehat{g}^T(\pi(t-\ell(t)))(\widehat{\mathcal{H}}_a^{3^T}\widehat{\mathcal{H}}_a^3)\widehat{g}(\pi(t-\ell(t)))$$
$$+ \text{trace}\{\varsigma_a^T(t)\mathcal{P}\varsigma_a(t)\} + \pi^T(t)\mathcal{Q}\pi(t) - (1-\mu)\pi^T(t-\ell(t))\mathcal{Q}$$
$$\times\pi(t-\ell(t)) + \pi^T(t)\mathcal{R}\pi(t) - \pi^T(t-\ell)\mathcal{R}\pi(t-\ell) + \ell^2\pi^T(t)\mathcal{S}\pi(t)$$
$$- \ell\int_{t-\ell}^{t}\pi^T(u)\mathcal{S}\pi(u)du\Big\}. \tag{3.5}$$

By using Assumption 1 and Lemma (2.2), we have

$$-\ell\int_{t-\ell}^{t}\pi^T(u)\mathcal{S}\pi(u)du = -\ell\int_{t-\ell}^{t-\ell(t)}\pi^T(u)\mathcal{S}\pi(u)du - \ell\int_{t-\ell(t)}^{t}\pi^T(u)\mathcal{S}\pi(u)du$$
$$-\ell\int_{t-\ell}^{t}\pi^T(u)\mathcal{S}\pi(u)du = -\frac{\ell}{\ell-\ell(t)}\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]$$
$$-\frac{\ell}{\ell(t)}\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]$$
$$= -\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]$$
$$-\frac{\ell(t)}{\ell-\ell(t)}\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell}^{t-\ell(t)}\pi(u)du\Big]$$
$$-\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]$$
$$-\frac{\ell-\ell(t)}{\ell(t)}\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]^T\mathcal{S}\Big[\int_{t-\ell(t)}^{t}\pi(u)du\Big]. \tag{3.6}$$

If $\begin{bmatrix} \mathcal{S} & \mathcal{T} \\ \mathcal{T}^T & \mathcal{S} \end{bmatrix} \geq 0$ by Lemma (2.5), the following inequality is true:

$$\begin{bmatrix} \sqrt{\frac{\ell(t)}{\ell-\ell(t)}}\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big] \\ \sqrt{\frac{\ell-\ell(t)}{\ell(t)}}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] \end{bmatrix}^T \begin{bmatrix} \mathcal{S} & \mathcal{T} \\ \mathcal{T}^T & \mathcal{S} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\ell(t)}{\ell-\ell(t)}}\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big] \\ \sqrt{\frac{\ell-\ell(t)}{\ell(t)}}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] \end{bmatrix} \geq 0, \tag{3.7}$$

which implies

$$\begin{aligned} &-\frac{\ell(t)}{\ell-\ell(t)}\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]^T \mathcal{S}\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big] - \frac{\ell-\ell(t)}{\ell(t)}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big]^T \mathcal{S}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] \\ &\quad \leq -\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]^T \mathcal{T}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] - \Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big]^T \mathcal{T}^T\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]. \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), one can obtain that

$$\begin{aligned} -\ell\int_{t-\ell}^{t} \pi^T(u)\mathcal{S}\pi(u)du \leq &-\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]^T \mathcal{S}\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big] \\ &-\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big]^T \mathcal{S}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] \\ &-\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]^T \mathcal{T}\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big] \\ &-\Big[\int_{t-\ell(t)}^{t} \pi(u)du\Big]^T \mathcal{T}^T\Big[\int_{t-\ell}^{t-\ell(t)} \pi(u)du\Big]. \end{aligned} \tag{3.9}$$

From the Assumption 4, one has

$$\begin{aligned} \mathrm{trace}(\varsigma_a^T(t)\varsigma_a(t)) \leq\ & (z^R)^T(\overline{\mathcal{U}}_a^1 + \overline{\mathcal{V}}_a^1 + \overline{\mathcal{M}}_a^1 + \overline{\mathcal{N}}_a^1)(z^R) + (z^R)_{\ell(t)}^T(\overline{\mathcal{U}}_a^5 + \overline{\mathcal{V}}_a^5 + \overline{\mathcal{M}}_a^5 + \overline{\mathcal{N}}_a^5)(z^R)_{\ell(t)} \\ &+ (z^I)^T(\overline{\mathcal{U}}_a^2 + \overline{\mathcal{V}}_a^2 + \overline{\mathcal{M}}_a^2 + \overline{\mathcal{N}}_a^2)(z^I) + (z^I)_{\ell(t)}^T(\overline{\mathcal{U}}_a^6 + \overline{\mathcal{V}}_a^6 + \overline{\mathcal{M}}_a^6 + \overline{\mathcal{N}}_a^6)(z^I)_{\ell(t)} \\ &+ (z^J)^T(\overline{\mathcal{U}}_a^3 + \overline{\mathcal{V}}_a^3 + \overline{\mathcal{M}}_a^3 + \overline{\mathcal{N}}_a^3)(z^J) + (z^J)_{\ell(t)}^T(\overline{\mathcal{U}}_a^7 + \overline{\mathcal{V}}_a^7 + \overline{\mathcal{M}}_a^7 + \overline{\mathcal{N}}_a^7)(z^J)_{\ell(t)} \\ &+ (z^K)^T(\overline{\mathcal{U}}_a^4 + \overline{\mathcal{V}}_a^4 + \overline{\mathcal{M}}_a^4 + \overline{\mathcal{N}}_a^4)(z^K) + (z^K)_{\ell(t)}^T(\overline{\mathcal{U}}_a^8 + \overline{\mathcal{V}}_a^8 + \overline{\mathcal{M}}_a^8 + \overline{\mathcal{N}}_a^8)(z^K)_{\ell(t)} \\ \leq\ & \pi^T(t)\Pi_1\pi(t) + \pi^T(t-\ell(t))\Pi_2\pi(t-\ell(t)), \end{aligned} \tag{3.10}$$

where

$$\Pi_1 = \begin{bmatrix} \overline{\mathcal{U}}_a^1 + \overline{\mathcal{V}}_a^1 + \overline{\mathcal{M}}_a^1 + \overline{\mathcal{N}}_a^1 & 0 & 0 & 0 \\ 0 & \overline{\mathcal{U}}_a^2 + \overline{\mathcal{V}}_a^2 + \overline{\mathcal{M}}_a^2 + \overline{\mathcal{N}}_a^2 & 0 & 0 \\ 0 & 0 & \overline{\mathcal{U}}_a^3 + \overline{\mathcal{V}}_a^3 + \overline{\mathcal{M}}_a^3 + \overline{\mathcal{N}}_a^3 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{U}}_a^4 + \overline{\mathcal{V}}_a^4 + \overline{\mathcal{M}}_a^4 + \overline{\mathcal{N}}_a^4 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} \overline{\mathcal{U}}_a^5 + \overline{\mathcal{V}}_a^5 + \overline{\mathcal{M}}_a^5 + \overline{\mathcal{N}}_a^5 & 0 & 0 & 0 \\ 0 & \overline{\mathcal{U}}_a^6 + \overline{\mathcal{V}}_a^6 + \overline{\mathcal{M}}_a^6 + \overline{\mathcal{N}}_a^6 & 0 & 0 \\ 0 & 0 & \overline{\mathcal{U}}_a^7 + \overline{\mathcal{V}}_a^7 + \overline{\mathcal{M}}_a^7 + \overline{\mathcal{N}}_a^7 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{U}}_a^8 + \overline{\mathcal{V}}_a^8 + \overline{\mathcal{M}}_a^8 + \overline{\mathcal{N}}_a^8 \end{bmatrix}.$$

From (3.10) and (3.1), one can obtain

$$\mathrm{trace}(\varsigma_a^T(t)\mathcal{P}\varsigma_a(t)) \leq \pi^T(t)(\lambda\Pi_1)\pi(t) + \pi^T(t-\ell(t))(\lambda\Pi_2)\pi(t-\ell(t)). \tag{3.11}$$

Moreover, from Assumption 2 it follows that

$$0 \leq \gamma_1[\pi^T(t)\widehat{\mathcal{L}}_g\pi(t) - \widehat{g}^T(\pi(t))\widehat{g}(\pi(t))], \tag{3.12}$$

$$0 \leq \gamma_2[\pi^T(t-\ell(t))\widehat{\mathcal{L}}_g\pi(t-\ell(t)) - \widehat{g}^T(\pi(t-\ell(t)))\widehat{g}(\pi(t-\ell(t)))]. \tag{3.13}$$

Adding from (3.5)-(3.13), we get

$$\mathcal{L}\mathbf{V}(t,\pi(t),a) \leq \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{\xi^T(t)\Phi_a\xi(t)\Big\}, \tag{3.14}$$

where

$$\xi(t) = \Big[\pi^T(t)\ \pi^T(t-\ell(t))\ \pi^T(t-\ell)\ \widehat{g}^T(\pi(t))\ \widehat{g}^T(\pi(t-\ell(t)))\ \int_{t-\ell}^{t-\ell(t)}\pi^T(u)du\ \int_{t-\ell(t)}^{t}\pi^T(u)du\Big]^T,$$

and $\Phi_a = (\Phi_{i,j,a})_{7\times 7}$ with $\Phi_{1,1,a} = -\mathcal{P}\widehat{\mathcal{D}}_a - \widehat{\mathcal{D}}_a\mathcal{P}^T + \epsilon_1^{-1}\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T + \epsilon_1\widehat{\mathcal{H}}_a^{1\,T}\widehat{\mathcal{H}}_a^1 + \epsilon_2^{-1}\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T + \epsilon_3^{-1}\mathcal{P}\widehat{\mathcal{G}}_a\widehat{\mathcal{G}}_a^T\mathcal{P}^T + \mathcal{Q} + \mathcal{R} + \ell^2\mathcal{S} + \gamma_1\widehat{\mathcal{L}}_g + \lambda\Pi_1$, $\Phi_{1,4,a} = \mathcal{P}\widehat{\mathcal{A}}_a$, $\Phi_{1,5,a} = \mathcal{P}\widehat{\mathcal{B}}_a$, $\Phi_{2,2,a} = -(1-\mu)\mathcal{Q} + \gamma_2\widehat{\mathcal{L}}_g + \lambda\Pi_2$, $\Phi_{3,3,a} = -\mathcal{R}$, $\Phi_{4,4,a} = -\gamma_1 + \epsilon_2\widehat{\mathcal{H}}_a^{2\,T}\widehat{\mathcal{H}}_a^2$, $\Phi_{5,5,a} = -\gamma_2 + \epsilon_3\widehat{\mathcal{H}}_a^{3\,T}\widehat{\mathcal{H}}_a^3$, $\Phi_{6,6,a} = -\mathcal{S}$, $\Phi_{6,7,a} = -\mathcal{T}$, $\Phi_{7,7,a} = -\mathcal{S}$.

By Schur Complement Lemma (2.4), it is obvious that Φ_a is equivalent to $\begin{bmatrix}\widehat{\Phi}_a & \Upsilon_a \\ \maltese & \Omega_a\end{bmatrix} < 0$. Then taking mathematical expectation, we have

$$\begin{aligned}\mathbf{E}\{\mathcal{L}\mathbf{V}(t,\pi(t),a)\} &\leq \mathbf{E}\Big\{\xi^T(t)\begin{bmatrix}\widehat{\Phi}_a & \Upsilon_a \\ \maltese & \Omega_a\end{bmatrix}\xi(t)\Big\}, \\ &\leq -\epsilon\mathbf{E}\{\|\pi(t)\|^2\}.\end{aligned} \tag{3.15}$$

This implies that the NNs (2.11) is robustly asymptotically stable in the mean square. The proof is completed.

Remark 3.2. *Suppose there has no stochastic disturbance, then NNs* (2.6) *turns to*

$$\frac{dz(t)}{dt} = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{-(\mathcal{D}_a + \triangle\mathcal{D}_a(t))z(t) + (\mathcal{A}_a + \triangle\mathcal{A}_a(t))g(z(t)) + (\mathcal{B}_a + \triangle\mathcal{B}_a(t))g(z(t-\ell(t)))\Big\}. \tag{3.16}$$

At the same time, system (2.11) *turns to*

$$\frac{d\pi(t)}{dt} = \sum_{a=1}^{m}\chi_a(\vartheta(t))\Big\{-(\widehat{\mathcal{D}}_a + \triangle\widehat{\mathcal{D}}_a)\pi(t) + (\widehat{\mathcal{A}}_a + \triangle\widehat{\mathcal{A}}_a)\widehat{g}(\pi(t)) + (\widehat{\mathcal{B}}_a + \triangle\widehat{\mathcal{B}}_a)\widehat{g}(\pi(t-\ell(t)))\Big\}. \tag{3.17}$$

By setting the stochastic disturbance $\varsigma_a(t)d\omega(t) = 0$ in Theorem (3.1)*, Corollary* (3.3) *can be obtained.*

Corollary 3.3. *Suppose Assumptions 1-3 hold. If there exist positive symmetric matrices* $\mathcal{P} \in \mathbf{R}^{4n\times 4n}, \mathcal{Q} \in \mathbf{R}^{4n\times 4n}, \mathcal{R} \in \mathbf{R}^{4n\times 4n}, \mathcal{S} \in \mathbf{R}^{4n\times 4n}$*, any matrix* $\mathcal{T} \in \mathbf{R}^{4n\times 4n}$ *and positive scalars* $\epsilon_1 \in \mathbf{R}^n, \epsilon_2 \in \mathbf{R}^n, \epsilon_3 \in \mathbf{R}^n, \gamma_1 \in \mathbf{R}^n, \gamma_2 \in \mathbf{R}^n$*, such that the following LMI hold for all* $a = 1, 2, ..., m$

$$\begin{bmatrix} \tilde{\Phi}_a & \Upsilon_a \\ \maltese & \Omega_a \end{bmatrix} < 0, \tag{3.18}$$

where $\tilde{\Phi}_a = (\tilde{\Phi}_{i,j,a})_{7\times 7},\ \Upsilon_a = (\Upsilon_{i,j,a})_{7\times 3},\ \Omega_a = (\Omega_{i,j,a})_{3\times 3}$ *with* $\tilde{\Phi}_{1,1,a} = -\mathcal{P}\widehat{\mathcal{D}}_a - \widehat{\mathcal{D}}_a\mathcal{P}^T + \epsilon_1 \widehat{\mathcal{H}}_a^{1^T} \widehat{\mathcal{H}}_a^1 + \mathcal{Q} + \mathcal{R} + \ell^2\mathcal{S} + \gamma_1\widehat{\mathcal{L}}_g,\ \tilde{\Phi}_{1,4,a} = \mathcal{P}\widehat{\mathcal{A}}_a,\ \tilde{\Phi}_{1,5,a} = \mathcal{P}\widehat{\mathcal{B}}_a,\ \tilde{\Phi}_{2,2,a} = -(1-\mu)\mathcal{Q} + \gamma_2\widehat{\mathcal{L}}_g,\ \tilde{\Phi}_{3,3,a} = -\mathcal{R},\ \tilde{\Phi}_{4,4,a} = -\gamma_1 + \epsilon_2\widehat{\mathcal{H}}_a^{2^T}\widehat{\mathcal{H}}_a^2,\ \tilde{\Phi}_{5,5,a} = -\gamma_2 + \epsilon_3\widehat{\mathcal{H}}_a^{3^T}\widehat{\mathcal{H}}_a^3,\ \tilde{\Phi}_{6,6,a} = -\mathcal{S},\ \tilde{\Phi}_{6,7,a} = -\mathcal{T},\ \tilde{\Phi}_{7,7,a} = -\mathcal{S},\ \Upsilon_{1,1,a} = \mathcal{P}\widehat{\mathcal{G}}_a,\ \Upsilon_{1,2,a} = \mathcal{P}\widehat{\mathcal{G}}_a,\ \Upsilon_{1,3,a} = \mathcal{P}\widehat{\mathcal{G}}_a,\ \Omega_{1,1,a} = -\epsilon_1,\ \Omega_{2,2,a} = -\epsilon_2,\ \Omega_{3,3,a} = -\epsilon_3$*, then the NN model* (3.17) *is robustly asymptotically stable.*

Remark 3.4. *In Theorem* (3.1) *and Corollary* (3.3)*, sufficient conditions are obtained to ensure that the NN* (2.11) *model is robust asymptotic stability by decomposes QVNNs into real-valued NNs, but the result we achieve is actually about QVNNs themselves.*

3.2. Global stability analysis

If there are no uncertainties, then NNs (2.6) becomes

$$dz(t) = \sum_{a=1}^{m} \chi_a(\vartheta(t))\Big\{[-\mathcal{D}_a z(t) + \mathcal{A}_a g(z(t)) + \mathcal{B}_a g(z(t-\ell(t)))]dt + \sigma_a(t, z(t), z(t-\ell(t)))d\omega(t)\Big\}. \tag{3.19}$$

At the same time, the NNs (2.11) turns to

$$d\pi(t) = \sum_{a=1}^{m} \chi_a(\vartheta(t))\Big\{[-\widehat{\mathcal{D}}_a\pi(t) + \widehat{\mathcal{A}}_a\widehat{g}(\pi(t)) + \widehat{\mathcal{B}}_a\widehat{g}(\pi(t-\ell(t)))]dt + \varsigma_a(t)d\omega(t)\Big\}. \tag{3.20}$$

By setting $\triangle\widehat{\mathcal{D}}_a = \triangle\widehat{\mathcal{A}}_a = \triangle\widehat{\mathcal{B}}_a = 0$ in Theorem (3.1), Theorem (3.5) can be obtained.

Theorem 3.5. *Suppose Assumptions 1,2,4 hold. If there exist positive symmetric matrices* $\mathcal{P} \in \mathbf{R}^{4n\times 4n}, \mathcal{Q} \in \mathbf{R}^{4n\times 4n}, \mathcal{R} \in \mathbf{R}^{4n\times 4n}, \mathcal{S} \in \mathbf{R}^{4n\times 4n}$*, any matrix* $\mathcal{T} \in \mathbf{R}^{4n\times 4n}$ *and positive scalars* $\gamma_1 \in \mathbf{R}^n, \gamma_2 \in \mathbf{R}^n, \lambda \in \mathbf{R}^n$*, such that the following LMIs hold for all* $a = 1, 2, ..., m$

$$\mathcal{P} \leq \lambda\mathcal{I}, \tag{3.21}$$

$$\breve{\Phi}_a < 0, \tag{3.22}$$

where $\breve{\Phi}_a = (\breve{\Phi}_{i,j,a})_{7\times 7}$*, with* $\breve{\Phi}_{1,1,a} = -\mathcal{P}\widehat{\mathcal{D}}_a - \widehat{\mathcal{D}}_a\mathcal{P}^T + \mathcal{Q} + \mathcal{R} + \ell^2\mathcal{S} + \gamma_1\widehat{\mathcal{L}}_g + \lambda\Pi_1,\ \breve{\Phi}_{1,4,a} = \mathcal{P}\widehat{\mathcal{A}}_a,\ \breve{\Phi}_{1,5,a} = \mathcal{P}\widehat{\mathcal{B}}_a,\ \breve{\Phi}_{2,2,a} = -(1-\mu)\mathcal{Q} + \gamma_2\widehat{\mathcal{L}}_g + \lambda\Pi_2,\ \breve{\Phi}_{3,3,a} = -\mathcal{R},\ \breve{\Phi}_{4,4,a} = -\gamma_1,\ \breve{\Phi}_{5,5,a} = -\gamma_2,\ \breve{\Phi}_{6,6,a} = -\mathcal{S},\ \breve{\Phi}_{6,7,a} = -\mathcal{T},\ \breve{\Phi}_{7,7,a} = -\mathcal{S}$*, then the NN model* (3.20) *is globally asymptotically stable in the mean square.*

Remark 3.6. *When stochastic disturbance is not appear, then the NNs* (3.19) *turns to*

$$\frac{dz(t)}{dt} = \sum_{a=1}^{m} \chi_a(\vartheta(t))\Big\{-\mathcal{D}_a z(t) + \mathcal{A}_a g(z(t)) + \mathcal{B}_a g(z(t-\ell(t)))\Big\}. \tag{3.23}$$

At the same time, the NNs (3.20) *turns to*

$$\frac{d\pi(t)}{dt} = \sum_{a=1}^{m} \chi_a(\vartheta(t))\Big\{ -\widehat{\mathcal{D}}_a\pi(t) + \widehat{\mathcal{A}}_a\widehat{g}(\pi(t)) + \widehat{\mathcal{B}}_a\widehat{g}(\pi(t-\ell(t)))\Big\}. \tag{3.24}$$

By setting $\triangle\widehat{\mathcal{D}}_a = \triangle\widehat{\mathcal{A}}_a = \triangle\widehat{\mathcal{B}}_a = 0$ *and* $\varsigma_a(t)d\omega(t) = 0$ *in Theorem* (3.5)*, Corollary* (3.7) *can be obtained.*

Corollary 3.7. *Suppose Assumptions 1, 2 hold. If there exist positive symmetric matrices* $\mathcal{P} \in \mathbf{R}^{4n\times 4n}, \mathcal{Q} \in \mathbf{R}^{4n\times 4n}, \mathcal{R} \in \mathbf{R}^{4n\times 4n}, \mathcal{S} \in \mathbf{R}^{4n\times 4n}$*, any matrix* $\mathcal{T} \in \mathbf{R}^{4n\times 4n}$ *and positive scalars* $\gamma_1 \in \mathbf{R}^n, \gamma_2 \in \mathbf{R}^n$*, such that the following LMIs hold for all* $a = 1, 2, ..., m$

$$\check{\Phi}_a < 0, \tag{3.25}$$

where $\check{\Phi}_a = (\check{\Phi}_{i,j,a})_{7\times 7}$ *with* $\check{\Phi}_{1,1,a} = -\mathcal{P}\widehat{\mathcal{D}}_a - \widehat{\mathcal{D}}_a\mathcal{P}^T + \mathcal{Q} + \mathcal{R} + \ell^2\mathcal{S} + \gamma_1\widehat{\mathcal{L}}_g$, $\check{\Phi}_{1,4,a} = \mathcal{P}\widehat{\mathcal{A}}_a$, $\check{\Phi}_{1,5,a} = \mathcal{P}\widehat{\mathcal{B}}_a$, $\check{\Phi}_{2,2,a} = -(1-\mu)\mathcal{Q} + \gamma_2\widehat{\mathcal{L}}_g$, $\check{\Phi}_{3,3,a} = -\mathcal{R}$, $\check{\Phi}_{4,4,a} = -\gamma_1$, $\check{\Phi}_{5,5,a} = -\gamma_2$, $\check{\Phi}_{6,6,a} = -\mathcal{S}$, $\check{\Phi}_{6,7,a} = -\mathcal{T}$, $\check{\Phi}_{7,7,a} = -\mathcal{S}$*, then the NN model* (3.19) *is globally asymptotically stable.*

Remark 3.8. *As is well known, QVNNs are the extensions of real-valued and complex-valued NNs with quaternion-valued states, inputs, and connection weights. The activation functions in the real domain are generally assumed to be smooth and bounded, whereas these assumptions do not apply to the quaternion domain. Therefore, choosing the activation function is very important when dealing with quaternions. Because of this, quaternion-valued activation functions are generally examined in three different ways: (i) The real-valued decomposition method, (ii) The complex-valued decomposition method, (iii) The direct quaternion method. Based on these three approaches, several works have been published on the dynamics of QVNNs [28, 30, 34–38].*

Remark 3.9. *By using suitable LKF, T-S fuzzy system model and the theory of stochastic analysis, we obtain a novel set of sufficient conditions for T-S FUQVSNNs* (2.6) *to ascertain the robust and global asymptotic stability. As far as we are aware, no results have been published on the robust and global asymptotic stable for T-S FUQVSNNs with time-varying delays* (2.6)*.*

Remark 3.10. *As we all know, QVNNs are aimed for investigating new capabilities and improved accuracy to address issues that cannot be resolved using complex-valued and real-valued NN models. For instance, the global stability of complex-valued NN [18, 19, 45], and real-valued NNs [12–14] can be summarized as a particular case of the results of this work.*

Remark 3.11. *It is obvious that real-valued LMI can be solved straightforwardly with MATLAB LMI toolbox; however, solving quaternion-valued LMI is more challenging. Therefore, we establish the robust and global asymptotic stability criteria for T-S FUQVSNNs by decomposing n-dimensional quaternion-valued NNs into* 4*n-dimensional real-valued NNs. Based on that, sufficient criteria in this paper are derived in terms of real-valued LMIs.*

4. Numerical evaluations

In this section, two numerical examples illustrate the effectiveness of the theoretical results presented in the previous section.

Example 1: Consider the plant rules with $a = 1, 2$, the T-S FUQVSNNs is described as follows

$$dz(t) = \sum_{a=1}^{2} \chi_a(\vartheta(t))\Big\{\big[-(\overline{\mathcal{D}}_a + \triangle\overline{\mathcal{D}}_a(t))z(t) + (\overline{\mathcal{A}}_a + \triangle\overline{\mathcal{A}}_a(t))g(z(t)) + (\overline{\mathcal{B}}_a + \triangle\overline{\mathcal{B}}_a(t))g(z(t-\ell(t)))\big]dt + \sigma_a(t, z(t), z(t-\ell(t)))d\omega(t)\Big\}. \tag{4.1}$$

Plant Rule 1: IF $\vartheta_1(t)$ is η_1^1, THEN

$$dz(t) = \big[-(\overline{\mathcal{D}}_1 + \triangle\overline{\mathcal{D}}_1(t))z(t) + (\overline{\mathcal{A}}_1 + \triangle\overline{\mathcal{A}}_1(t))g(z(t)) + (\overline{\mathcal{B}}_1 + \triangle\overline{\mathcal{B}}_1(t))g(z(t-\ell(t)))\big]dt + \sigma_1(t, z(t), z(t-\ell(t)))d\omega(t).$$

Plant Rule 2: IF $\vartheta_1(t)$ is η_1^2, THEN

$$dz(t) = \big[-(\overline{\mathcal{D}}_2 + \triangle\overline{\mathcal{D}}_2(t))z(t) + (\overline{\mathcal{A}}_2 + \triangle\overline{\mathcal{A}}_2(t))g(z(t)) + (\overline{\mathcal{B}}_2 + \triangle\overline{\mathcal{B}}_2(t))g(z(t-\ell(t)))\big]dt + \sigma_2(t, z(t), z(t-\ell(t)))d\omega(t),$$

where η_1^1 is $z_1(t) \leq 1$, η_1^2 is $z_1(t) > 1$, and

$$\overline{\mathcal{D}}_1 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}, \ \overline{\mathcal{D}}_2 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix},$$

$$\overline{\mathcal{A}}_1 = \begin{bmatrix} 0.5 - 0.5i + 0.3j - 0.4k & 0.5 + 0.5i - 0.4j + 0.3k \\ 0.4 + 0.5i - 0.4j - 0.5k & 0.5 - 0.5i + 0.4j + 0.4k \end{bmatrix},$$

$$\overline{\mathcal{A}}_2 = \begin{bmatrix} 0.4 - 0.5i + 0.3j - 0.4k & 0.4 + 0.3i - 0.4j + 0.3k \\ 0.2 + 0.4i - 0.4j - 0.4k & 0.3 - 0.3i + 0.2j + 0.2k \end{bmatrix},$$

$$\overline{\mathcal{B}}_1 = \begin{bmatrix} 0.6 - 0.5i + 0.5j - 0.4k & 0.6 + 0.5i - 0.6j + 0.3k \\ 0.6 + 0.5i - 0.6j - 0.5k & 0.4 - 0.5i + 0.6j + 0.4k \end{bmatrix},$$

$$\overline{\mathcal{B}}_2 = \begin{bmatrix} 0.5 - 0.4i + 0.3j - 0.4k & 0.2 + 0.3i - 0.5j + 0.3k \\ 0.6 + 0.5i - 0.4j - 0.3k & 0.3 - 0.5i + 0.2j + 0.4k \end{bmatrix},$$

$$\mathcal{G}_1 = \mathcal{G}_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \mathcal{H}_1^1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \mathcal{H}_2^1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\mathcal{H}_1^2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_2^2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \mathcal{H}_1^3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_2^3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\mathcal{H}_1^4 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \mathcal{H}_2^4 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_1^5 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \ \mathcal{H}_2^5 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$\mathcal{H}_1^6 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_2^6 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \ \mathcal{H}_1^7 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \ \mathcal{H}_2^7 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$\mathcal{H}_1^8 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_2^8 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_1^9 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \ \mathcal{H}_2^9 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$\mathcal{F}_1(t) = \mathcal{F}_2(t) = \begin{bmatrix} 0.1\ sin(t) & 0 \\ 0 & 0.1\ sin(t) \end{bmatrix}.$$

The premise variable $\vartheta(t)$ is chosen as a state-dependent term, that is, $\vartheta(t) = z_1(t)$. Using the same procedure as in [40], the membership functions can be obtained from the property of $\chi_1(z_1(t)) + \chi_2(z_1(t)) = 1$, where $\chi_1(z_1(t)) = \frac{1}{1+e^{-z_1(t)}}$, $\chi_2(z_1(t)) = 1 - \frac{1}{1+e^{-z_1(t)}}$.
Assumption 4 is further assumed to be satisfied by

$$\overline{\mathcal{U}}_1^1 = \overline{\mathcal{U}}_2^1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{V}}_1^1 = \overline{\mathcal{V}}_2^1 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{M}}_1^1 = \overline{\mathcal{M}}_2^1 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.001 \end{bmatrix},$$
$$\overline{\mathcal{N}}_1^1 = \overline{\mathcal{N}}_2^1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{U}}_1^2 = \overline{\mathcal{U}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{V}}_1^2 = \overline{\mathcal{V}}_2^2 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.003 \end{bmatrix},$$
$$\overline{\mathcal{M}}_1^2 = \overline{\mathcal{M}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}, \overline{\mathcal{N}}_1^2 = \overline{\mathcal{N}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{U}}_1^3 = \overline{\mathcal{U}}_2^3 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix},$$
$$\overline{\mathcal{V}}_1^3 = \overline{\mathcal{V}}_2^3 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{M}}_1^3 = \overline{\mathcal{M}}_2^3 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{N}}_1^3 = \overline{\mathcal{N}}_2^3 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.004 \end{bmatrix},$$
$$\overline{\mathcal{U}}_1^4 = \overline{\mathcal{U}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.004 \end{bmatrix}, \overline{\mathcal{V}}_1^4 = \overline{\mathcal{V}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{M}}_1^4 = \overline{\mathcal{M}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.002 \end{bmatrix},$$
$$\overline{\mathcal{N}}_1^4 = \overline{\mathcal{N}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{U}}_1^5 = \overline{\mathcal{U}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.005 \end{bmatrix}, \overline{\mathcal{V}}_1^5 = \overline{\mathcal{V}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.004 \end{bmatrix},$$
$$\overline{\mathcal{M}}_1^5 = \overline{\mathcal{M}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{N}}_1^5 = \overline{\mathcal{N}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{U}}_1^6 = \overline{\mathcal{U}}_2^6 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.001 \end{bmatrix},$$
$$\overline{\mathcal{V}}_1^6 = \overline{\mathcal{V}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.005 \end{bmatrix}, \overline{\mathcal{M}}_1^6 = \overline{\mathcal{M}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.004 \end{bmatrix}, \overline{\mathcal{N}}_1^6 = \overline{\mathcal{N}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.003 \end{bmatrix},$$
$$\overline{\mathcal{U}}_1^7 = \overline{\mathcal{U}}_2^7 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{V}}_1^7 = \overline{\mathcal{V}}_2^7 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{M}}_1^7 = \overline{\mathcal{M}}_2^7 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$
$$\overline{\mathcal{N}}_1^7 = \overline{\mathcal{N}}_2^7 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.004 \end{bmatrix}, \overline{\mathcal{U}}_1^8 = \overline{\mathcal{U}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{V}}_1^8 = \overline{\mathcal{V}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix},$$
$$\overline{\mathcal{M}}_1^8 = \overline{\mathcal{M}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{N}}_1^8 = \overline{\mathcal{N}}_2^8 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.002 \end{bmatrix}.$$

Using simple calculations, we can find out $\overline{\mathcal{A}}_1^R$, $\overline{\mathcal{A}}_2^R$, $\overline{\mathcal{A}}_1^I$, $\overline{\mathcal{A}}_2^I$, $\overline{\mathcal{A}}_1^J$, $\overline{\mathcal{A}}_2^J$, $\overline{\mathcal{A}}_1^K$, $\overline{\mathcal{A}}_2^K$, $\overline{\mathcal{B}}_1^R$, $\overline{\mathcal{B}}_2^R$, $\overline{\mathcal{B}}_1^I$, $\overline{\mathcal{B}}_2^I$, $\overline{\mathcal{B}}_1^J$, $\overline{\mathcal{B}}_2^J$, $\overline{\mathcal{B}}_1^K$, $\overline{\mathcal{B}}_2^K$, $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{A}}_2$, $\widehat{\mathcal{B}}_1$, $\widehat{\mathcal{B}}_2$, $\widehat{\mathcal{G}}_1$, $\widehat{\mathcal{G}}_2$, $\widehat{\mathcal{F}}_1(t)$, $\widehat{\mathcal{F}}_2(t)$, $\widehat{\mathcal{H}}_1^1$, $\widehat{\mathcal{H}}_2^1$, $\widehat{\mathcal{H}}_1^2$, $\widehat{\mathcal{H}}_2^2$, $\widehat{\mathcal{H}}_1^3$ and $\widehat{\mathcal{H}}_2^3$.

Moreover, the activation functions $g_\alpha(z_\alpha(t))$, $g_\alpha(z_\alpha(t-\ell(t)))$ in (4.1) can be chosen as $g_\alpha(z_\alpha(t)) = 0.5\ tanh(z_\alpha(t)) + 0.5\ tanh(z_\alpha(t))i + 0.5\ tanh(z_\alpha(t))j + 0.5\ tanh(z_\alpha(t))k$, $g_\alpha(z_\alpha(t-\ell(t))) = 0.5\ tanh(z_\alpha(t-\ell(t))) + 0.5\ tanh(z_\alpha(t-\ell(t)))i + 0.5\ tanh(z_\alpha(t-\ell(t)))j + 0.5\ tanh(z_\alpha(t-\ell(t)))k$ for all $\alpha = 1, 2$. Obviously, the activation functions $g_\alpha(z_\alpha(t))$ and $g_\alpha(z_\alpha(t-\ell(t)))$ are satisfies Assumption 2 with $\widehat{\mathcal{L}}_g = diag\{0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25\}$. The delay $\ell(t)$ is defined as $\ell(t) = 0.1 + 0.2\ sin(t)$, which implies that the maximum permissible upper bound is $\ell = 0.3$. It is observable that $0 \leq \dot{\ell}(t) \leq \mu = 0 \leq 0.2\ cos(t) \leq 0.2$. By applying MATLAB LIM toolbox, the LMI conditions of Theorem (3.1) are verified. Under initial values of $\varphi_1(t) = -1 + 0.9i - 1.5j + 0.8k$, $\varphi_2(t) = 0.8 - 0.8i + j - 1.2k$, the time responses of states $z_1^R(t)$, $z_1^I(t)$, $z_1^J(t)$, $z_1^K(t)$, $z_2^R(t)$, $z_2^I(t)$, $z_2^J(t)$, $z_2^K(t)$ are illustrated in Figures (1)–(4).

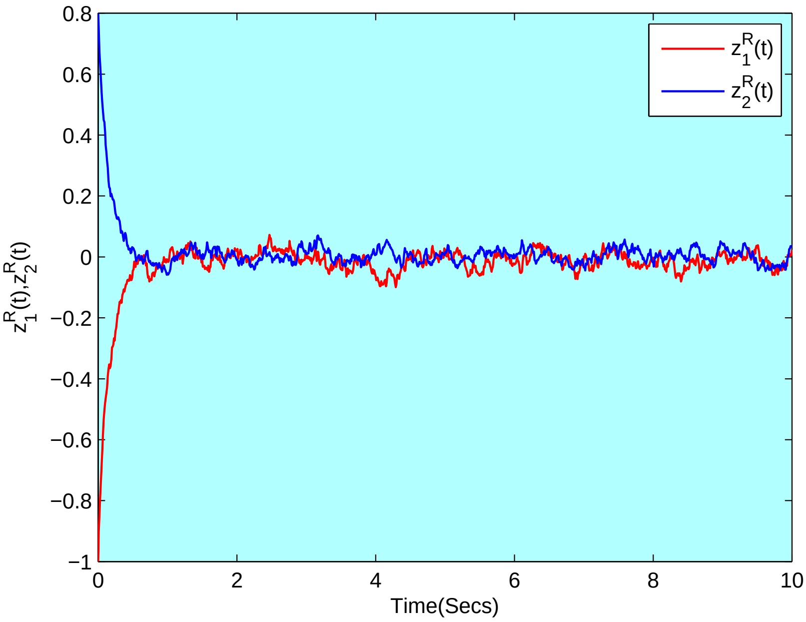

Figure 1. Time representation of the states $z_1^R(t), z_2^R(t)$ of the NN (4.1) with $\sigma^R(t, z^R(t), z^R(t - \ell(t))) = 0.1(z^R(t) + z^R(t - \ell(t))$.

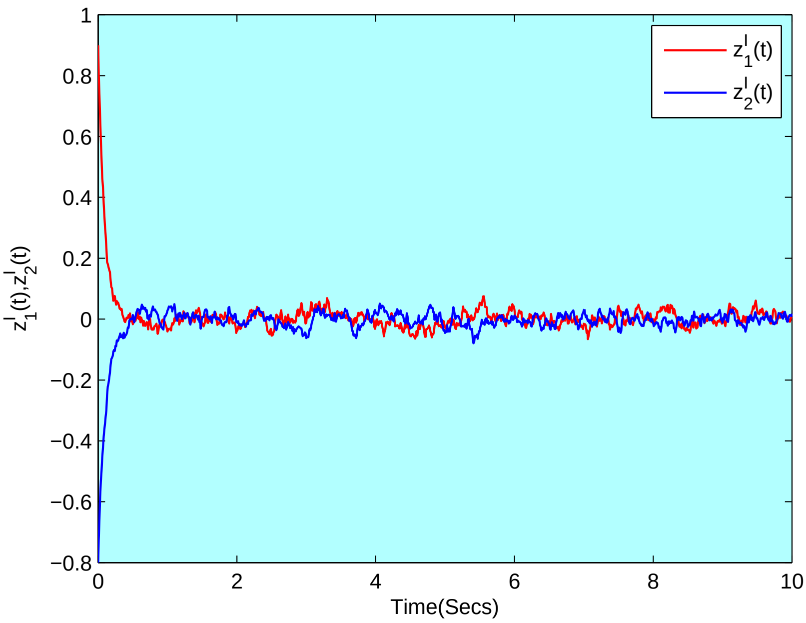

Figure 2. Time representation of the states $z_1^I(t), z_2^I(t)$ of the NN (4.1) with $\sigma^I(t, z^I(t), z^I(t - \ell(t))) = 0.1(z^I(t) + z^I(t - \ell(t))$.

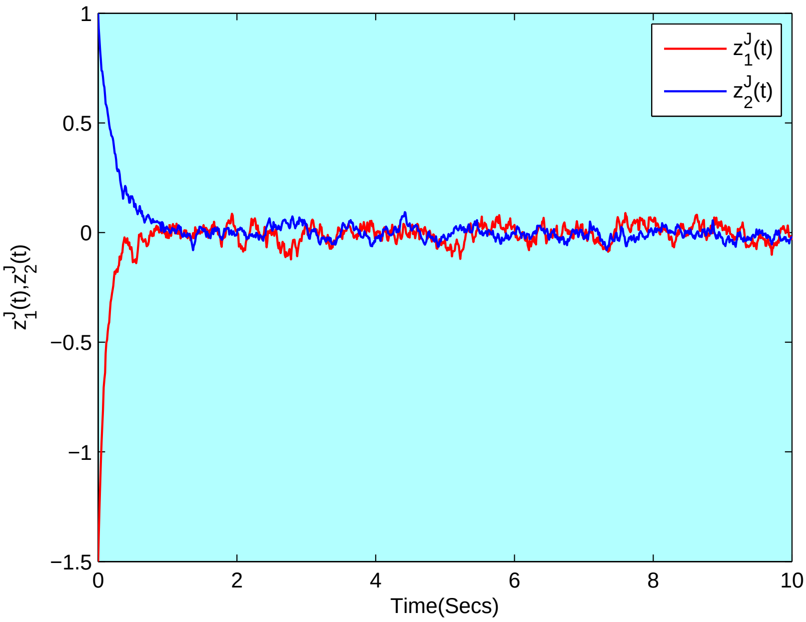

Figure 3. Time representation of the states $z_1^J(t), z_2^J(t)$ of the NN (4.1) with $\sigma^J(t, z^J(t), z^J(t - \ell(t))) = 0.1(z^J(t) + z^J(t - \ell(t)))$.

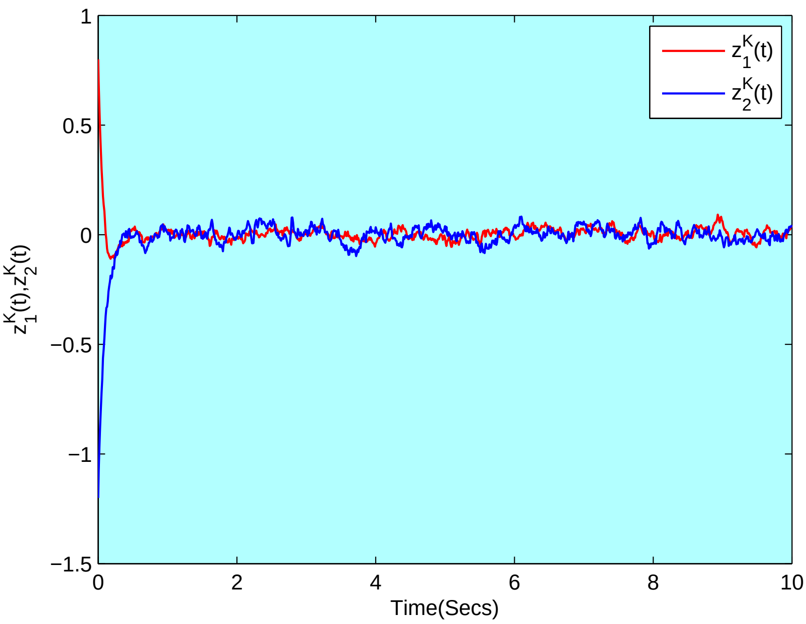

Figure 4. Time representation of the states $z_1^K(t), z_2^K(t)$ of the NN (4.1) with $\sigma^K(t, z^K(t), z^K(t - \ell(t))) = 0.1(z^K(t) + z^K(t - \ell(t)))$.

This example confirms all of the conditions associated with Theorem (3.1), then the equilibrium point of NNs (2.11) is robustly asymptotically stable in the mean square.

Example 2: Determine the T-S fuzzy stochastic QVNNs with $a = 1, 2$ as given below:

$$dz(t) = \sum_{a=1}^{2} \chi_a(\vartheta(t))\Big\{[-\overline{\mathcal{D}}_a z(t) + \overline{\mathcal{A}}_a g(z(t)) + \overline{\mathcal{B}}_a g(z(t-\ell(t)))]dt + \sigma_a(t, z(t), z(t-\ell(t)))d\omega(t)\Big\}. \quad (4.2)$$

Plant Rule 1: IF $\vartheta_1(t)$ is η_1^1, THEN

$$dz(t) = [-\overline{\mathcal{D}}_1 z(t) + \overline{\mathcal{A}}_1 g(z(t)) + \overline{\mathcal{B}}_1 g(z(t-\ell(t)))]dt + \sigma_1(t, z(t), z(t-\ell(t)))d\omega(t)\Big\}.$$

Plant Rule 2: IF $\vartheta_1(t)$ is η_1^2, THEN

$$dz(t) = [-\overline{\mathcal{D}}_2 z(t) + \overline{\mathcal{A}}_2 g(z(t)) + \overline{\mathcal{B}}_2 g(z(t-\ell(t)))]dt + \sigma_2(t, z(t), z(t-\ell(t)))d\omega(t),$$

where η_1^1 is $z_1(t)$, η_1^2 is $z_1(t)$, and let $z_1(t) \in [-s, s]$, where $0 < s = 3$, and

$$\overline{\mathcal{D}}_1 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, \ \overline{\mathcal{D}}_2 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix},$$
$$\overline{\mathcal{A}}_1 = \begin{bmatrix} 0.6 - 0.6i + 0.4j - 0.5k & 0.6 + 0.6i - 0.5j + 0.4k \\ 0.5 + 0.6i - 0.5j - 0.6k & 0.6 - 0.6i + 0.5j + 0.5k \end{bmatrix},$$
$$\overline{\mathcal{A}}_2 = \begin{bmatrix} 0.5 - 0.6i + 0.4j - 0.5k & 0.5 + 0.4i - 0.5j + 0.4k \\ 0.3 + 0.5i - 0.5j - 0.5k & 0.4 - 0.4i + 0.3j + 0.3k \end{bmatrix},$$
$$\overline{\mathcal{B}}_1 = \begin{bmatrix} 0.7 - 0.6i + 0.5j - 0.5k & 0.7 + 0.6i - 0.5j + 0.4k \\ 0.6 + 0.6i - 0.5j - 0.5k & 0.7 - 0.6i + 0.6j + 0.4k \end{bmatrix},$$
$$\overline{\mathcal{B}}_2 = \begin{bmatrix} 0.4 - 0.4i + 0.6j - 0.5k & 0.4 + 0.3i - 0.6j + 0.2k \\ 0.5 + 0.3i - 0.4j - 0.2k & 0.6 - 0.4i + 0.4j + 0.2k \end{bmatrix}.$$

The premise variable $\vartheta(t)$ is chosen as a state-dependent term, that is, $\vartheta(t) = z_1(t)$. Using the same procedure as in [40], the membership functions can be obtained from the property of $\chi_1(z_1(t)) + \chi_2(z_1(t)) = 1$, where $\chi_1(z_1(t)) = \frac{1}{1+e^{-3z_1(t)}}$, $\chi_2(z_1(t)) = 1 - \frac{1}{1+e^{-3z_1(t)}}$.
Assumption 4 is further assumed to be satisfied by

$$\overline{\mathcal{U}}_1^1 = \overline{\mathcal{U}}_2^1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \overline{\mathcal{V}}_1^1 = \overline{\mathcal{V}}_2^1 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \overline{\mathcal{M}}_1^1 = \overline{\mathcal{M}}_2^1 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.001 \end{bmatrix},$$
$$\overline{\mathcal{N}}_1^1 = \overline{\mathcal{N}}_2^1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix}, \ \overline{\mathcal{U}}_1^2 = \overline{\mathcal{U}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.002 \end{bmatrix}, \ \overline{\mathcal{V}}_1^2 = \overline{\mathcal{V}}_2^2 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.003 \end{bmatrix},$$
$$\overline{\mathcal{M}}_1^2 = \overline{\mathcal{M}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}, \ \overline{\mathcal{N}}_1^2 = \overline{\mathcal{N}}_2^2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \overline{\mathcal{U}}_1^3 = \overline{\mathcal{U}}_2^3 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix},$$
$$\overline{\mathcal{V}}_1^3 = \overline{\mathcal{V}}_2^3 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \ \overline{\mathcal{M}}_1^3 = \overline{\mathcal{M}}_2^3 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.003 \end{bmatrix}, \ \overline{\mathcal{N}}_1^3 = \overline{\mathcal{N}}_2^3 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.004 \end{bmatrix},$$
$$\overline{\mathcal{U}}_1^4 = \overline{\mathcal{U}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.004 \end{bmatrix}, \ \overline{\mathcal{V}}_1^4 = \overline{\mathcal{V}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \overline{\mathcal{M}}_1^4 = \overline{\mathcal{M}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.002 \end{bmatrix},$$
$$\overline{\mathcal{N}}_1^4 = \overline{\mathcal{N}}_2^4 = \begin{bmatrix} 0.004 & 0 \\ 0 & 0.003 \end{bmatrix}, \ \overline{\mathcal{U}}_1^5 = \overline{\mathcal{U}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.005 \end{bmatrix}, \ \overline{\mathcal{V}}_1^5 = \overline{\mathcal{V}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.004 \end{bmatrix},$$

$$\overline{\mathcal{M}}_1^5 = \overline{\mathcal{M}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{N}}_1^5 = \overline{\mathcal{N}}_2^5 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{U}}_1^6 = \overline{\mathcal{U}}_2^6 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.001 \end{bmatrix},$$

$$\overline{\mathcal{V}}_1^6 = \overline{\mathcal{V}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.005 \end{bmatrix}, \overline{\mathcal{M}}_1^6 = \overline{\mathcal{M}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.004 \end{bmatrix}, \overline{\mathcal{N}}_1^6 = \overline{\mathcal{N}}_2^6 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.003 \end{bmatrix},$$

$$\overline{\mathcal{U}}_1^7 = \overline{\mathcal{U}}_2^7 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.002 \end{bmatrix}, \overline{\mathcal{V}}_1^7 = \overline{\mathcal{V}}_2^7 = \begin{bmatrix} 0.002 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{M}}_1^7 = \overline{\mathcal{M}}_2^7 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$\overline{\mathcal{N}}_1^7 = \overline{\mathcal{N}}_2^7 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.004 \end{bmatrix}, \overline{\mathcal{U}}_1^8 = \overline{\mathcal{U}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.003 \end{bmatrix}, \overline{\mathcal{V}}_1^8 = \overline{\mathcal{V}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix},$$

$$\overline{\mathcal{M}}_1^8 = \overline{\mathcal{M}}_2^8 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \overline{\mathcal{N}}_1^8 = \overline{\mathcal{N}}_2^8 = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.002 \end{bmatrix}.$$

The following can be obtained by simple calculations $\overline{\mathcal{A}}_1^R$, $\overline{\mathcal{A}}_2^R$, $\overline{\mathcal{A}}_1^I$, $\overline{\mathcal{A}}_2^I$, $\overline{\mathcal{A}}_1^J$, $\overline{\mathcal{A}}_2^J$, $\overline{\mathcal{A}}_1^K$, $\overline{\mathcal{A}}_2^K$, $\overline{\mathcal{B}}_1^R$, $\overline{\mathcal{B}}_2^R$, $\overline{\mathcal{B}}_1^I$, $\overline{\mathcal{B}}_2^I$, $\overline{\mathcal{B}}_1^J$, $\overline{\mathcal{B}}_2^J$, $\overline{\mathcal{B}}_1^K$, $\overline{\mathcal{B}}_2^K$, $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{A}}_2$, $\widehat{\mathcal{B}}_1$ and $\widehat{\mathcal{B}}_2$. Further, the activation functions $g_\alpha(z_\alpha(t))$, $g_\alpha(z_\alpha(t-\ell(t)))$ in (4.2) can be selected as $g_\alpha(z_\alpha(t)) = 0.5\ tanh(z_\alpha(t)) + 0.5\ tanh(z_\alpha(t))i + 0.5\ tanh(z_\alpha(t))j + 0.5\ tanh(z_\alpha(t))k$, $g_\alpha(z_\alpha(t-\ell(t))) = 0.5\ tanh(z_\alpha(t-\ell(t))) + 0.5\ tanh(z_\alpha(t-\ell(t)))i + 0.5\ tanh(z_\alpha(t-\ell(t)))j + 0.5\ tanh(z_\alpha(t-\ell(t)))k$ for all $\alpha = 1, 2$. Clearly, the activation functions $g_\alpha(z_\alpha(t))$, $g_\alpha(z_\alpha(t-\ell(t)))$ are satisfies Assumption 2 with $\widehat{\mathcal{L}}_g = diag\{0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25\}$.

The delay $\ell(t)$ is regarded as $\ell(t) = 0.1 + 0.2\ sin(t)$, implying that the maximum permissible upper bound is $\ell = 0.3$. It is observable that $0 \le \dot{\ell}(t) \le \mu = 0 \le 0.2\ cos(t) \le 0.2$.

The LMI conditions of Theorem (3.5) are verified by applying MATLAB LIM toolbox. Under randomly selected 15 initial values, the time responses of states $z_1^R(t)$, $z_1^I(t)$, $z_1^J(t)$, $z_1^K(t)$, $z_2^R(t)$, $z_2^I(t)$, $z_2^J(t)$, $z_2^K(t)$ are illustrated in Figures (5)–(7).

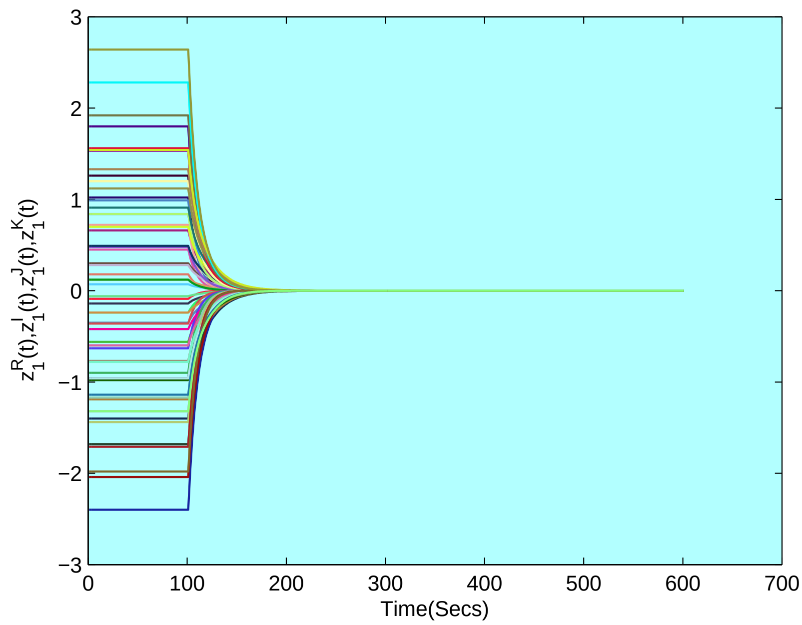

Figure 5. Time representation of the states $z_1^R(t)$, $z_1^I(t)$, $z_1^J(t)$, $z_1^K(t)$ of the NN (4.2) without stochastic disturbance.

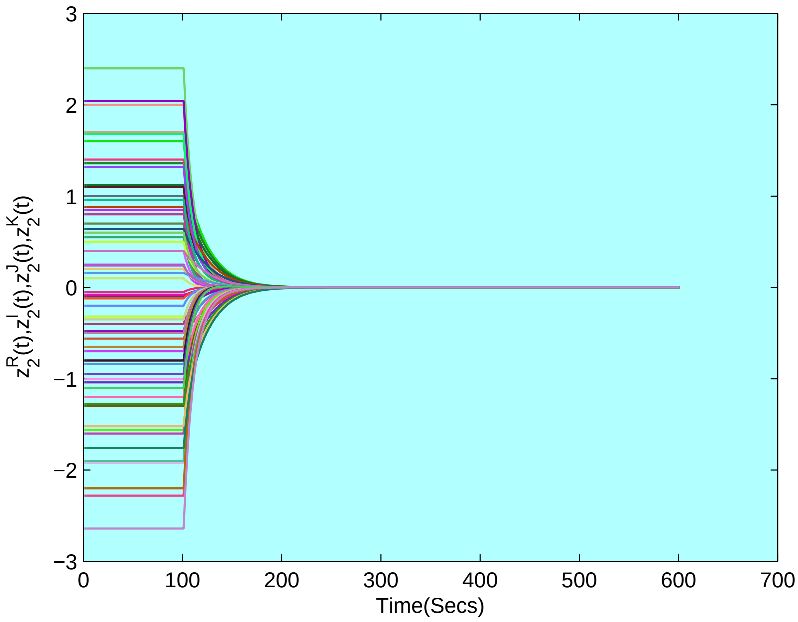

Figure 6. Time representation of the states $z_2^R(t)$, $z_2^I(t)$, $z_2^J(t)$, $z_2^K(t)$ of the NN (4.2) without stochastic disturbance.

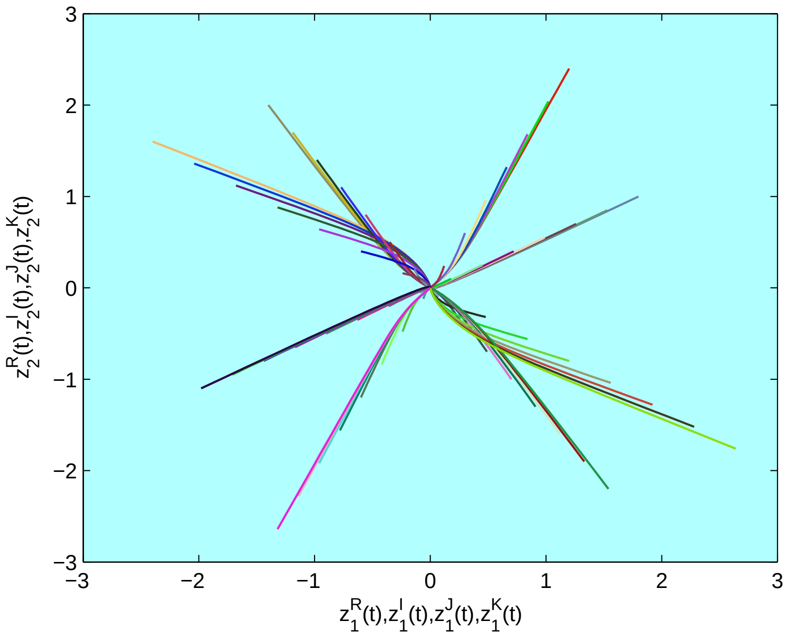

Figure 7. Phase representation of the states $z_1(t)$, $z_2(t)$ of the NNs (4.2) without stochastic disturbance.

This example confirms all of the conditions associated with Theorem (3.5), then the equilibrium point of NNs (3.19) is globally asymptotically stable in the mean square.

5. Conclusions

This paper investigated the robust and global asymptotic stability problem for a class of T-S FUQVSNNs with discrete time-varying delays using the system decomposition method. By applying T-S fuzzy models and stochastic analysis, we first considered a general form of T-S FUQVSNNs with time delays. Then, we presented some delay-dependent stability conditions for the considered NNs using LKFs to ensure the robust and global asymptotic stability. Furthermore, we established our results in terms of real-valued LMIs that can be solved in MATLAB LMI toolbox. Finally, two numerical examples are presented with their simulations to demonstrate the validity of the theoretical analysis. By using the results of this paper, we can analyze various dynamic behaviours of T-S FUQVSNNs including finite-time stability, passivity, state estimation, synchronization, and others. There are certain advancements worth investigating further in this proposed area of research. Soon, we will attempt to investigate the stability of delayed impulsive T-S FUQVSNNs in the finite-time case.

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Conflict of interest

The authors declare no conflict of interest.

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