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Research article

Fixed point results of fuzzy mappings with applications

Amer Hassan Albargi[1*] **and Jamshaid Ahmad**[2]

[1] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: aalbarqi@kau.edu.sa.

Abstract: Jleli and Samet introduced the notion of $\mathcal{F}$-metric space as a generalization of traditional metric space and proved Banach contraction principle in the setting of these generalized metric spaces. The aim of this article is to utilize $\mathcal{F}$-metric space and establish some common α-fuzzy fixed point theorems for rational (β-ϕ)-contractive conditions. Our results extend, generalize and unify some well-known results in the literature. As application of our main result, we discuss the solution of fuzzy integrodifferential equations in the setting of a generalized Hukuhara derivative.

Keywords: $\mathcal{F}$-metric space; α-fuzzy fixed point; $\beta_{\mathcal{F}}$-admissible; Hukuhara derivative
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1. Introduction

Fixed point theory is considered to be the most fascinating and vital field of research in the growth of nonlinear analysis. In this extent, Banach fixed point theorem [1] is pioneer result for investigators in last few decades. This theorem plays a significant and essential role in solving the existence and uniqueness of solution to different problems in mathematics, physics, engineering, medicines, and social sciences which guides to mathematical models design by system of nonlinear integral equations, functional equations, and differential equations. In 1960, Zadeh [2] presented the theory of fuzzy set to handle the capricious which generated the imprecision or non-recognition in the first choice to negligence. Heilpern [3] gave the notion of fuzzy mappings and established fixed point theorems in metric linear space. Estruch et al. [4] obtained fuzzy fixed point results for fuzzy mappings in the background of complete metric space. Subsequently, many researchers extended and generalized the result of Estruch et al. [4] in different generalized metric spaces with different contractions.

Fuzzy differential equations and fuzzy integral equations play a significant role in modeling dynamic systems in which doubts or ambiguities conceptss flourish. These concepts have been built up in specific theoretical directions, and countless use in constructive applications have been examined. Many foundations for analyzing fuzzy differential equations are given. The fundamental and the utmost charming accession is employing the Hukuhara differntiability (H-differentiability) for fuzzy valued functions (see [5, 6]). Later on, Kaleva [7] investigated the solution of Fuzzy differential equations. Seikkala [8] solved an initial value problem by considering fuzzy initial value and deterministic or fuzzy function. The investigations regarding the existence and uniqueness of solutions of fuzzy differential and integral equations, large number of redears have used definite fixed point theorems. Although, Subrahmanyam et al. [9] discussed the solutions of integral equations respecting fuzzy multivalued mappings by adopting the well-known Banach contraction principle. Illamizar-Roa et al. [10] discussed the existence and uniqueness of solution of fuzzy initial value problem in the backgroun of a generalized Hukuhara derivatives. These fuzzy differential and integral equations are applied in digital images, specially to restore or separates the images into segments. The researchers can see [11–16] for more details in this direction.

On the other hand, Jleli et al. [17] introduced a new metric space named as $\mathcal{F}$-metric space to generalize the classical metric space in 2018. Later on, Alnaser et al. [18] utilized $\mathcal{F}$-metric space and investigated some fixed point theorems for rational contraction. Al-Mezel et al. [19] introduced $(\alpha\beta, \phi)$-contractions in $\mathcal{F}$-metric space and obtained some generalized results. Recently, Alansari et al. [20] studied some common fuzzy fixed point results in this $\mathcal{F}$-metric space.

In this paper, we establish some common α-fuzzy fixed point theorems for rational $(\beta$-$\phi)$-contractive conditions in the setting of $\mathcal{F}$-metric space to generalize certain results of literature. We also supply a nontrivial example to support our leading result. As an application, we discuss the solution of fuzzy integrodifferential equations in the setting of the generalized Hukuhara derivative which are used in digital images to the better reconstruction in less time.

2. Preliminaries

Definition 2.1. *[2, 3] Let $\mathcal{W} \neq \emptyset$. A fuzzy set in $\mathcal{W}$ is a function with $\mathcal{W}$ as domain and $[0, 1]$ as co-domain. If Ξ_1 is a fuzzy set and $\kappa \in \mathcal{W}$, then $\Xi_1(\kappa)$ is professed to be the grade of membership of κ in Ξ_1. An α -level set of Ξ_1 is represented by $[\Xi_1]_\alpha$ and is defined in this way:*

$$[\Xi_1]_\alpha = \{\kappa : \Xi_1(\kappa) \geq \alpha\} \text{ if } \alpha \in (0, 1],$$

$$[\Xi_1]_0 = \overline{\{\kappa : \Xi_1(\kappa) > 0\}},$$

where $\overline{\Xi_2}$ is the closure of the set Ξ_2. If $\mathcal{W}$ is a metric space, then $I^{\mathcal{W}}$ is the collection of all fuzzy sets in $\mathcal{W}$. For $\Xi_1, \Xi_2 \in I^{\mathcal{W}}$, $\Xi_1 \subset \Xi_2$ means $\Xi_1(\kappa) \leq \Xi_2(\kappa)$ for all $\kappa \in \mathcal{W}$. We symbolize the fuzzy set $\chi_{\{\kappa\}}$ by $\{\kappa\}$ before it is expressed, where $\chi_{\{\kappa\}}$ is the characteristic function of the crisp set Ξ_1. Let $\mathcal{W}_1$ be an arbitrary set, $\mathcal{W}_2$ be a metric space. A mapping O is called fuzzy mapping if O is a mapping from $\mathcal{W}_1$ into $I^{\mathcal{W}_2}$. A fuzzy mapping O is a fuzzy subset on $\mathcal{W}_1 \times \mathcal{W}_2$ with membership function $O(\kappa)(\omega)$. The function $O(\kappa)(\omega)$ is the grade of membership of ω in $O(\kappa)$.

Definition 2.2. *[14] Let $O_1, O_2 : \mathcal{W} \to I^{\mathcal{W}}$. A point $\kappa \in \mathcal{W}$ is called a common α-fuzzy fixed point of O_1 and O_2 if there exists $\alpha \in [0, 1]$ such that $\kappa \in [O_1\kappa]_\alpha \cap [O_2\kappa]_\alpha$.*

In 2018, Jleli and Samet [17] introduced a fascinating metric space named as $\mathcal{F}$-metric space as follows:

Let $f : (0, +\infty) \to \mathbb{R}$ and $\mathcal{F}$ denotes the set of functions f satisfying:

($\mathcal{F}_1$) $0 < \kappa < t$ implies $f(\kappa) \leq f(t)$,
($\mathcal{F}_2$) For $\{\kappa_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \kappa_n = 0$ if and only if $\lim_{n\to\infty} f(\kappa_n) = -\infty$.

Definition 2.3. *[17] Let $\mathcal{W} \neq \emptyset$, and let $d_{\mathcal{F}} : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$. Assume that there exists $(f, h) \in \mathcal{F} \times [0, +\infty)$ such that*

(D_1) $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$, $d_{\mathcal{F}}(\kappa, \omega) = 0$ if and only if $\kappa = \omega$,
(D_2) $d_{\mathcal{F}}(\kappa, \omega) = d_{\mathcal{F}}(\omega, \kappa)$, for all $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$,
(D_3) For every $(\kappa, \omega) \in \mathcal{W} \times \mathcal{W}$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(u_i)_{i=1}^N \subset \mathcal{W}$, with $(u_1, u_N) = (\kappa, \omega)$, we have

$$d_{\mathcal{F}}(\kappa, \omega) > 0 \Rightarrow f(d_{\mathcal{F}}(\kappa, \omega)) \leq f(\sum_{i=1}^{N-1} d_{\mathcal{F}}(\kappa_i, \kappa_{i+1})) + h.$$

Then $d_{\mathcal{F}}$ is called a $\mathcal{F}$-metric on $\mathcal{W}$ and $(\mathcal{W}, d_{\mathcal{F}})$ is called an $\mathcal{F}$-metric space.

Example 2.1. [17] The function $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$

$$d_{\mathcal{F}}(\kappa, \omega) = \begin{cases} (\kappa - \omega)^2 & \text{if } (\kappa, \omega) \in [0, 3] \times [0, 3], \\ |\kappa - \omega| & \text{if } (\kappa, \omega) \notin [0, 3] \times [0, 3], \end{cases}$$

with $f(t) = \ln(t)$ and $h = \ln(3)$, is a $\mathcal{F}$-metric.

Definition 2.4. *[17] Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space.*

(i) Let $\{\kappa_n\} \subseteq \mathcal{W}$. The sequence $\{\kappa_n\}$ is said to be $\mathcal{F}$-convergent to $\kappa \in \mathcal{W}$ if $\{\kappa_n\}$ is convergent to κ with respect to the $\mathcal{F}$-metric $d_{\mathcal{F}}$.

(ii) The sequence $\{\kappa_n\}$ is said to be $\mathcal{F}$-Cauchy, if and only if

$$\lim_{n,m\to\infty} d_{\mathcal{F}}(\kappa_n, \kappa_m) = 0.$$

(iii) If every $\mathcal{F}$-Cauchy sequence in $\mathcal{F}$-metric space $(\mathcal{W}, d_{\mathcal{F}})$ is $\mathcal{F}$-convergent to an element of $\mathcal{W}$, then $(\mathcal{W}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete.

Theorem 2.1. *[17] Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$. Assume that these assertions hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) There exists $\lambda \in (0, 1)$ such that

$$d_{\mathcal{F}}(\mathcal{O}(\kappa), \mathcal{O}(\omega)) \leq \lambda d_{\mathcal{F}}(\kappa, \omega).$$

Then there exists $\kappa^* \in \mathcal{W}$ such that $\mathcal{O}\kappa^* = \kappa^*$. Furthermore, for any $\kappa_0 \in \mathcal{W}$, the sequence $\{\kappa_n\} \subset \mathcal{W}$ defined by

$$\kappa_{n+1} = \mathcal{O}(\kappa_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to κ^*.

Definition 2.5. *[19, 20] Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $C(2^{\mathcal{W}})$ be the set of all nonempty compact subsets of $\mathcal{W}$ and $\Xi_1, \Xi_2 \in C(2^{\mathcal{W}})$. Then,*

$$d_{\mathcal{F}}(\kappa, \Xi_1) = \inf \{d_{\mathcal{F}}(\kappa, \omega) : \omega \in \Xi_1\},$$

$$d_{\mathcal{F}}(\Xi_1, \Xi_2) = \inf \{d_{\mathcal{F}}(\kappa, \omega) : \kappa \in \Xi_1, \omega \in \Xi_2\}.$$

A Hausdorff metric $H_{\mathcal{F}}$ on $C(2^{\mathcal{W}})$ induced by $\mathcal{F}$-metric $d_{\mathcal{F}}$ is given as

$$H_{\mathcal{F}}(\Xi_1, \Xi_2) = \begin{cases} \max\left\{\sup_{\kappa\in\Xi_1} d_{\mathcal{F}}(\kappa, \Xi_2), \sup_{\omega\in\Xi_2} d_{\mathcal{F}}(\omega, \Xi_1)\right\}, & \textit{if it exists,} \\ \infty, & \textit{otherwise.} \end{cases}$$

In 2012, Samet et al. [21] began the notions of β-admissible mapping in this way.

Definition 2.6. *[21] Let $O : \mathcal{W} \to \mathcal{W}$ and $\alpha : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$. Then O is called a β-admissible mapping if*

$$\kappa, \omega \in \mathcal{W}, \quad \beta(\kappa, \omega) \geq 1 \quad \Longrightarrow \quad \beta(O\kappa, O\omega) \geq 1.$$

Definition 2.7. *[22, 23] A nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ is called a comparison function, if $\phi^n(t)_{n\in\mathbb{N}} \to 0$ as $n \to \infty$, for all $t \in (0, +\infty)$, where ϕ^n represents the nth iterate of ϕ.*

We represent the set of these comparison functions by Ψ.

Lemma 2.1. *[22, 23] If $\phi \in \Psi$, then these conditions hold:*
(i) Each iterate ϕ^i of ϕ, for $i \geq 1$ is a comparison function;
(ii) $\phi(t) < t$, for all $t > 0$,
(iii) ϕ is continuous at 0.

Lemma 2.2. *[20] Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be nonempty closed and compact subsets of a $\mathcal{F}$-metric space $(\mathcal{W}, d_{\mathcal{F}})$. If $\kappa \in \mathcal{W}_1$, then $d_{\mathcal{F}}(\kappa, \mathcal{W}_2) \leq H_{\mathcal{F}}(\mathcal{W}_1, \mathcal{W}_2)$.*

3. Main results

Motivated with the notion of β-admissible mapping, we define the concept of $\beta_{\mathcal{F}}$-admissible mapping in $\mathcal{F}$-metric space as follows:

Definition 3.1. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $\beta : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ and let O_1, O_2 be fuzzy mapping from $\mathcal{W}$ into $\mathcal{F}(\mathcal{W})$. The pair (O_1, O_2) is called $\beta_{\mathcal{F}}$-admissible if these assertions hold:*

(i) For each $\kappa \in \mathcal{W}$ and $\omega \in [O_1\kappa]_{\alpha_{O_1}(\kappa)}$, where $\alpha_{O_1}(\kappa) \in (0, 1]$, with $\beta(\kappa, \omega) \geq 1$, we have $\beta(\omega, z) \geq 1$ for all $z \in [O_2\omega]_{\alpha_{O_2}(\omega)} \neq \emptyset$, where $\alpha_{O_2}(\omega) \in (0, 1]$,

(ii) For each $\kappa \in \mathcal{W}$ and $\omega \in [O_2\kappa]_{\alpha_{O_2}(\kappa)}$, where $\alpha_{O_2}(\kappa) \in (0, 1]$, with $\beta(\kappa, \omega) \geq 1$, we have $\beta(\omega, z) \geq 1$ for all $z \in [O_1\omega]_{\alpha_{O_1}(\omega)} \neq \emptyset$, where $\alpha_{O_1}(\omega) \in (0, 1]$.

Theorem 3.1. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $\beta : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ and let $O_1, O_2 : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ be fuzzy mappings. Assume that for each $\kappa \in \mathcal{W}$, there exist $\alpha_{O_1}(\kappa), \alpha_{O_2}(\kappa) \in (0, 1]$ such that $[O_1\kappa]_{\alpha_{O_1}(\kappa)}, [O_2\kappa]_{\alpha_{O_2}(\kappa)} \in C(2^{\mathcal{W}})$. Assume that these assertions also hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete,

(ii) For $\kappa_0 \in \mathcal{W}$, there exists $\alpha_{O_1}(\kappa_0)$ or $\alpha_{O_2}(\kappa_0) \in (0, 1]$ such that $\kappa_1 \in [O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}$ or $\kappa_1 \in [O_2\kappa_0]_{\alpha_{O_2}(\kappa_0)}$ with $\beta(\kappa_0, \kappa_1) \geq 1$,

(iii) There exists $\phi \in \Psi$ such that

$$\max\{\beta(\kappa,\omega),\beta(\omega,\kappa)\}\, H_{\mathcal{F}}\left([O_1\kappa]_{\alpha_{O_1}(\kappa)},[O_2\omega]_{\alpha_{O_2}(\omega)}\right) \leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega), d_{\mathcal{F}}(\kappa,[O_1\kappa]_{\alpha_{O_1}(\kappa)}), d_{\mathcal{F}}(\omega,[O_2\omega]_{\alpha_{O_2}(\omega)}), \\ \frac{d_{\mathcal{F}}(\kappa,[O_1\kappa]_{\alpha_{O_1}(\kappa)})d_{\mathcal{F}}(\omega,[O_2\omega]_{\alpha_{O_2}(\omega)})}{1+d_{\mathcal{F}}(\kappa,\omega)} \end{array}\right)\right) \tag{3.1}$$

for all $\kappa,\omega\in\mathcal{W}$,

(iii) (O_1,O_2) *is* $\beta_{\mathcal{F}}$*-admissible,*

(iv) If $\{\kappa_n\}$ *is a sequence in* $\mathcal{W}$ *such that* $\beta(\kappa_n,\kappa_{n+1})\geq 1$ *and* $\kappa_n\to\kappa$ *as* $n\to\infty$*, then* $\beta(\kappa_n,\kappa)\geq 1$*, for all n.*

Then there exists some $\kappa^*\in[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\cap[O_2\kappa^*]_{\alpha_{O_2}(\kappa^*)}$.

Proof. For $\kappa_0\in\mathcal{W}$, there exists $\alpha_{O_1}(\kappa_0)\in(0,1]$ such that $[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}\in C(2^{\mathcal{W}})$. Since $[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}$ is a nonempty compact subset of $\mathcal{W}$, so there exists $\kappa_1\in[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}$ such that $d_{\mathcal{F}}(\kappa_0,\kappa_1)=d_{\mathcal{F}}(\kappa_0,[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)})$. Now for κ_1, there exists $\alpha_{O_2}(\kappa_1)\in(0,1]$ such that $[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\in C(2^{\mathcal{W}})$. Since$[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}$ is a nonempty compact subset of $\mathcal{W}$, so there exists $\kappa_2\in[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}$ such that $d_{\mathcal{F}}(\kappa_1,\kappa_2)=d_{\mathcal{F}}(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)})$. By hypothesis (ii), Lemma 2.2 and inequality 3.1, we have

$$\begin{aligned}
d_{\mathcal{F}}(\kappa_1,\kappa_2) &= d_{\mathcal{F}}\left(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\leq H_{\mathcal{F}}\left([O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \beta(\kappa_0,\kappa_1)H_{\mathcal{F}}\left([O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \max\{\beta(\kappa_0,\kappa_1),\beta(\kappa_1,\kappa_0)\}\, H_{\mathcal{F}}\left([O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}\left(\kappa_0,[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}\right), d_{\mathcal{F}}\left(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right), \\ \frac{d_{\mathcal{F}}\left(\kappa_0,[O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}\right)d_{\mathcal{F}}\left(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)}{1+d_{\mathcal{F}}(\kappa_0,\kappa_1)}\end{array}\right)\right)\\
&\leq \phi\left(\max\left(d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_1,\kappa_2), \frac{d_{\mathcal{F}}(\kappa_0,\kappa_1)\, d_{\mathcal{F}}(\kappa_1,\kappa_2)}{1+d_{\mathcal{F}}(\kappa_0,\kappa_1)}\right)\right)\\
&\leq \phi\left(\max\left(d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_1,\kappa_2)\right)\right)\\
&= \phi\left(\max\left(d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_1,\kappa_2)\right)\right).
\end{aligned} \tag{3.2}$$

If $\max\left(d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_1,\kappa_2)\right)=d_{\mathcal{F}}(\kappa_1,\kappa_2)$, then (3.2) becomes

$$d_{\mathcal{F}}(\kappa_1,\kappa_2)\leq\phi\left(d_{\mathcal{F}}(\kappa_1,\kappa_2)\right)<d_{\mathcal{F}}(\kappa_1,\kappa_2),$$

which is a contradiction. It follows that $\max\left(d_{\mathcal{F}}(\kappa_0,\kappa_1), d_{\mathcal{F}}(\kappa_1,\kappa_2)\right)=d_{\mathcal{F}}(\kappa_0,\kappa_1)$. Therefore, we have

$$d_{\mathcal{F}}(\kappa_1,\kappa_2)\leq\phi\left(d_{\mathcal{F}}(\kappa_0,\kappa_1)\right). \tag{3.3}$$

Now for $\kappa_2\in\mathcal{W}$, there exists $\alpha_{O_1}(\kappa_2)\in(0,1]$ such that $[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}\in C(2^{\mathcal{W}})$. Since $[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}$ is a nonempty compact subset of $\mathcal{W}$, so there exists $\kappa_3\in[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}$ such that $d_{\mathcal{F}}(\kappa_2,\kappa_3)=d_{\mathcal{F}}(\kappa_2,[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)})$. As $\beta(\kappa_0,\kappa_1)\geq 1$ and the pair (O_1,O_2) is $\beta_{\mathcal{F}}$-admissible, so $\beta(\kappa_1,\kappa_2)\geq 1$. Again by hypothesis (ii), Lemma 2.2 and inequality 3.1, we have

$$
\begin{aligned}
d_{\mathcal{F}}(\kappa_2,\kappa_3) &= d_{\mathcal{F}}\left(\kappa_2,[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}\right) \leq H_{\mathcal{F}}\left([O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)},[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}\right)\\
&= H_{\mathcal{F}}\left([O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \beta(\kappa_2,\kappa_1)H_{\mathcal{F}}\left([O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \max\{\beta(\kappa_2,\kappa_1),\beta(\kappa_1,\kappa_2)\}H_{\mathcal{F}}\left([O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)},[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)\\
&\leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa_2,\kappa_1), d_{\mathcal{F}}\left(\kappa_2,[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}\right), d_{\mathcal{F}}\left(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right),\\ \frac{d_{\mathcal{F}}\left(\kappa_2,[O_1\kappa_2]_{\alpha_{O_1}(\kappa_2)}\right)d_{\mathcal{F}}\left(\kappa_1,[O_2\kappa_1]_{\alpha_{O_2}(\kappa_1)}\right)}{1+d_{\mathcal{F}}(\kappa_2,\kappa_1)}\end{array}\right)\right)\\
&\leq \phi\left(\max\left(d_{\mathcal{F}}(\kappa_2,\kappa_1), d_{\mathcal{F}}(\kappa_2,\kappa_3), d_{\mathcal{F}}(\kappa_1,\kappa_2), \frac{d_{\mathcal{F}}(\kappa_2,\kappa_3)\, d_{\mathcal{F}}(\kappa_1,\kappa_2)}{1+d_{\mathcal{F}}(\kappa_2,\kappa_1)}\right)\right)\\
&\leq \phi\left(\max\left(d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_2,\kappa_3), d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_2,\kappa_3)\right)\right)\\
&= \phi\left(\max\left(d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_2,\kappa_3)\right)\right). \qquad (3.4)
\end{aligned}
$$

If $\max\left(d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_2,\kappa_3)\right) = d_{\mathcal{F}}(\kappa_2,\kappa_3)$, then (3.4) becomes

$$
d_{\mathcal{F}}(\kappa_2,\kappa_3) \leq \phi\left(d_{\mathcal{F}}(\kappa_2,\kappa_3)\right) < d_{\mathcal{F}}(\kappa_2,\kappa_3),
$$

which is a contradiction. It follows that $\max\left(d_{\mathcal{F}}(\kappa_1,\kappa_2), d_{\mathcal{F}}(\kappa_2,\kappa_3)\right) = d_{\mathcal{F}}(\kappa_1,\kappa_2)$. Therefore, we have

$$
d_{\mathcal{F}}(\kappa_2,\kappa_3) \leq \phi\left(d_{\mathcal{F}}(\kappa_1,\kappa_2)\right). \qquad (3.5)
$$

Pursuing in this way by induction, we can construct a sequence $\{\kappa_n\}$ in $\mathcal{W}$ such that $\kappa_{2n+1} \in [O_1\kappa_{2n}]_{\alpha_{O_1}(\kappa_{2n})}$, $\kappa_{2n+2} \in [O_2\kappa_{2n+1}]_{\alpha_{O_2}(\kappa_{2n+1})}$ and $\beta(\kappa_{n-1},\kappa_n) \geq 1$,

$$
d_{\mathcal{F}}(\kappa_{2n+1},\kappa_{2n+2}) \leq \phi(d_{\mathcal{F}}(\kappa_{2n},\kappa_{2n+1})) \qquad (3.6)
$$

and

$$
d_{\mathcal{F}}(\kappa_{2n+2},\kappa_{2n+3}) \leq \phi(d_{\mathcal{F}}(\kappa_{2n+1},\kappa_{2n+2})) \qquad (3.7)
$$

for all n. It follows from (3.6) and (3.7), we get

$$
d_{\mathcal{F}}(\kappa_n,\kappa_{n+1}) \leq \phi(d_{\mathcal{F}}(\kappa_{n-1},\kappa_n)) \leq \cdots \leq \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)). \qquad (3.8)
$$

Let $\epsilon > 0$ be a given positive number and $(f,h) \in \mathcal{F} \times [0,+\infty)$ be such that (D_3) holds. By $(\mathcal{F}_2)$, there exists $\eta > 0$ such that

$$
0 < t < \eta \Longrightarrow f(t) < f(\epsilon) - h. \qquad (3.9)
$$

Let $n(\epsilon) \in \mathbb{N}$ such that $\sum_{n\geq n(\epsilon)} \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)) < \eta$. Hence by (3.9) and $(\mathcal{F}_1)$, we have

$$
f\left(\sum_{i=n}^{m-1} \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)\right) \leq f\left(\sum_{n\geq n(\epsilon)} \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)\right) \leq f(\epsilon) - h.
$$

Now for $m > n \geq n(\epsilon)$, we have

$$\begin{aligned} f\left(d_{\mathcal{F}}(\kappa_n,\kappa_m)\right) &\le f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\kappa_i,\kappa_{i+1})\right)+h \\ &\le f\left(\sum_{i=n}^{m-1} \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)\right)+h \\ &\le f\left(\sum_{n\ge n(\epsilon)} \phi^n(d_{\mathcal{F}}(\kappa_0,\kappa_1)\right)+h \\ &\le f(\epsilon). \end{aligned}$$

It follows by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\kappa_n,\kappa_m)<\epsilon$, $m>n\ge n(\epsilon)$. It shows that $\{\kappa_n\}$ is $\mathcal{F}$-Cauchy. As $(\mathcal{W},d_{\mathcal{F}})$ is $\mathcal{F}$-complete, so there exists $\kappa^*\in\mathcal{W}$ such that $\{\kappa_n\}$ is $\mathcal{F}$-convergent to κ^*, i.e.,

$$\lim_{n\to\infty} d_{\mathcal{F}}(\kappa_n,\kappa^*)=0. \tag{3.10}$$

Now we prove that $\kappa^*\in[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}$, so we assume that $d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)>0$. By condition (iv), we have $\beta(\kappa_{2n-1},\kappa^*)\ge 1$, for all $n\in\mathbb{N}$.

Thus by the definition of f and (D_3), we get

$$\begin{aligned} & f\left(d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \\ \le\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+d_{\mathcal{F}}\left(\kappa_{2n},[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)+h \\ \le\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+H_{\mathcal{F}}\left([O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})},[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)+h \\ \le\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+\beta(\kappa_{2n-1},\kappa^*)H_{\mathcal{F}}\left([O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})},[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)+h \\ \le\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+\max\{\beta(\kappa_{2n-1},\kappa^*),\beta(\kappa^*,\kappa_{2n-1})\}H_{\mathcal{F}}\left([O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})},[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)+h \\ =\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+\max\{\beta(\kappa_{2n-1},\kappa^*),\beta(\kappa^*,\kappa_{2n-1})\}H_{\mathcal{F}}\left([O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)},[O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})}\right)\right)+h \\ \le\ & f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1}),d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}\left(\kappa_{2n-1},[O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})}\right), \\ \frac{d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)d_{\mathcal{F}}\left(\kappa_{2n-1},[O_2\kappa_{2n-1}]_{\alpha_{O_2}(\kappa_{2n-1})}\right)}{1+d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1})} \end{array}\right)\right)\right)+h \\ \le\ & f\left(\begin{array}{c} d_{\mathcal{F}}(\kappa^*,\kappa_{2n}) \\ +\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1}),d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}(\kappa_{2n-1},\kappa_{2n}),\frac{d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)d_{\mathcal{F}}(\kappa_{2n-1},\kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1})} \end{array}\right)\right) \end{array}\right)+h. \end{aligned} \tag{3.11}$$

Now, we analyze (3.11) under the following cases:

Case 1. If $\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1}),d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}(\kappa_{2n-1},\kappa_{2n}),\frac{d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)d_{\mathcal{F}}(\kappa_{2n-1},\kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1})} \end{array}\right)=d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1})$. Then (3.11) becomes

$$f\left(d_{\mathcal{F}}\left(\kappa^*,[O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)\le f\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n})+\phi\left(d_{\mathcal{F}}(\kappa^*,\kappa_{2n-1})\right)\right)+h.$$

Now since $\{\kappa_n\}$ is $\mathcal{F}$-convergent to κ^*, so by $(\mathcal{F}_2)$ and the properties of $\phi \in \Psi$ and taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) = \lim_{n\to\infty} f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})\right) + h = -\infty,$$

which is a contradiction.

Case 2. If $\max\begin{pmatrix} d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1}), d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n}), \frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})} \end{pmatrix} = d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)$. Then (3.11) becomes

$$f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \leq f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \phi\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)\right) + h.$$

Taking the limit as $n \to \infty$ and using the continuity of f, we have

$$f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \leq f\left(\phi\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right)\right) + h. \tag{3.12}$$

For $h = 0$, from (3.12) by $(\mathcal{F}_1)$, we have

$$d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) < \phi\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right),$$

which is a contradiction to the fact that $\phi \in \Psi$ and $\phi(t) < t$, for all $t > 0$. Hence $d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) = 0$, that is, $\kappa^* \in [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}$.

Case 3. If $\max\begin{pmatrix} d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1}), d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n}), \frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})} \end{pmatrix} = d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})$. Then (3.11) becomes

$$f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \leq f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \phi\left(d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})\right)\right) + h.$$

Now since $\{\kappa_n\}$ is $\mathcal{F}$-convergent to κ^*, so by $(\mathcal{F}_2)$ and the properties of $\phi \in \Psi$ and taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) = \lim_{n\to\infty} f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})\right) + h = -\infty,$$

which is a contradiction. Hence $d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) = 0$, that is, $\kappa^* \in [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}$.

Case 4. If $\max\begin{pmatrix} d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1}), d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right), \\ d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n}), \frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})} \end{pmatrix} = \frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1+d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})}$. Then (3.11) becomes

$$f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \leq f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \phi\left(\frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1 + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})}\right)\right) + h.$$

Now since $\{\kappa_n\}$ is $\mathcal{F}$-convergent to κ^*, so by $(\mathcal{F}_2)$ and the properties of $\phi \in \Psi$ and taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} f\left(d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right)\right) \leq \lim_{n\to\infty} f\left(d_{\mathcal{F}}(\kappa^*, \kappa_{2n}) + \frac{d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) d_{\mathcal{F}}(\kappa_{2n-1}, \kappa_{2n})}{1 + d_{\mathcal{F}}(\kappa^*, \kappa_{2n-1})}\right) + h = -\infty,$$

which is a contradiction. Therefore, we have $d_{\mathcal{F}}\left(\kappa^*, [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}\right) = 0$, that is, $\kappa^* \in [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)}$. Doing the same, we can prove that $\kappa^* \in [O_2\kappa^*]_{\alpha_{O_2}(\kappa^*)}$. Thus $\kappa^* \in [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)} \cap [O_2\kappa^*]_{\alpha_{O_2}(\kappa^*)}$. □

Example 3.1. Let $\mathcal{W} = [0, +\infty)$, define $\mathcal{F}$- metric $d_{\mathcal{F}} : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ by

$$d_{\mathcal{F}}(\kappa, \omega) = \begin{cases} (\kappa - \omega)^2 & \text{if } (\kappa, \omega) \in [0, 4] \times [0, 4], \\ |\kappa - \omega| & \text{if } (\kappa, \omega) \notin [0, 4] \times [0, 4], \end{cases}$$

whenever $\kappa, \omega \in \mathcal{W}$ and $f(t) = \ln(t)$ for $t > 0$ and $h = \ln(4)$. Then $(\mathcal{W}, d_{\mathcal{F}})$ is a $\mathcal{F}$-complete $\mathcal{F}$- metric space but it is not a metric space because $d_{\mathcal{F}}$ does not satisfy the triangle inequality as

$$d_{\mathcal{F}}(1, 4) = 9 > 5 = d_{\mathcal{F}}(1, 3) + d_{\mathcal{F}}(3, 4).$$

Furthermore, let $\alpha \in (0, 1]$ and define fuzzy mappings $O_1, O_2 : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ in this way:

(i) If $\kappa = 0$,

$$O_1(\kappa)(\iota) = \begin{cases} 1 \text{ if } \iota = 0, \\ 0 \text{ if } \iota \neq 0. \end{cases}$$

(ii) If $0 < \kappa < \infty$,

$$O_1(\kappa)(\iota) = \begin{cases} \alpha \text{ if } 0 \leq \iota < \frac{\kappa^2}{60}, \\ \frac{\alpha}{3} \text{ if } \frac{\kappa^2}{60} \leq \iota < \frac{\kappa^2}{30}, \\ \frac{\alpha}{6} \text{ if } \frac{\kappa^2}{30} \leq \iota < \kappa^2, \\ 0 \text{ if } \kappa^2 \leq \iota < \infty. \end{cases}$$

$$O_2(\omega)(\iota) = \begin{cases} \alpha \text{ if } 0 \leq \iota < \frac{\omega^2}{40}, \\ \frac{\alpha}{3} \text{ if } \frac{\omega^2}{40} \leq \iota < \frac{\omega^2}{30}, \\ \frac{\alpha}{24} \text{ if } \frac{\omega^2}{30} \leq \iota < \omega^2, \\ 0 \text{ if } \omega^2 \leq \iota < \infty. \end{cases}$$

Now we define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{12}t$ for $t \geq 0$. Then $\phi \in \Psi$. Now for $\kappa \in \mathcal{W}$, there exist $\alpha_{O_1}(\kappa) = (\frac{\alpha}{3}) \in (0, 1]$ and $\alpha_{O_2}(\kappa) = (\frac{\alpha}{6}) \in (0, 1]$ such that $[O_1\kappa]_{(\frac{\alpha}{3})}, [O_2\kappa]_{(\frac{\alpha}{6})} \in C(2^{\mathcal{W}})$. Define $\beta : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ by

$$\beta(\kappa, \omega) = \begin{cases} 1 \text{ if } \kappa \neq \omega, \\ 0 \text{ if } \kappa = \omega. \end{cases}$$

Now if $\kappa = \omega = 0$, then $[O_1\kappa]_{(\frac{\alpha}{3})} = [O_2\omega]_{(\frac{\alpha}{6})} = \{0\}$. Thus,

$$\begin{aligned} &\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}\left([O_1\kappa]_{(\frac{\alpha}{3})}, [O_2\omega]_{(\frac{\alpha}{6})}\right) = 0 \\ \leq \quad &\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa, \omega), d_{\mathcal{F}}(\kappa, [O_1\kappa]_{(\frac{\alpha}{3})}), d_{\mathcal{F}}(\omega, [O_2\omega]_{(\frac{\alpha}{6})}), \\ \frac{d_{\mathcal{F}}(\kappa, [O_1\kappa]_{(\frac{\alpha}{3})}) d_{\mathcal{F}}(\omega, [O_2\omega]_{(\frac{\alpha}{6})})}{1 + d_{\mathcal{F}}(\kappa, \omega)} \end{array}\right)\right). \end{aligned}$$

Now if $\kappa, \omega \in (0, \infty)$, then

$$[O_1\kappa]_{(\frac{\alpha}{3})} = \left\{\iota \in \mathcal{W} : O_1\kappa(\iota) \geq \frac{\alpha}{3}\right\} = \left[0, \frac{\kappa^2}{30}\right].$$

Similarly,

$$[O_2\omega]_{(\frac{\alpha}{6})} = \left[0, \frac{\omega^2}{30}\right].$$

Thus for $\kappa \neq \omega$ and by the definition of $d_{\mathcal{F}}$, we have

$$\begin{aligned} & \max\{\beta(\kappa,\omega),\beta(\omega,\kappa)\} H_{\mathcal{F}}\left([O_1\kappa]_{(\frac{\alpha}{3})},[O_2\omega]_{(\frac{\alpha}{6})}\right) \\ = & \left(\frac{\kappa^2}{30}-\frac{\omega^2}{30}\right)^2 \leq \left|\left(\frac{\kappa+\omega}{30}(\kappa-\omega)\right)\right|^2 \\ \leq & \frac{1}{12}|\kappa-\omega|^2 = \frac{1}{12}d_{\mathcal{F}}(\kappa,\omega) \\ \leq & \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega), d_{\mathcal{F}}(\kappa,[O_1\kappa]_{(\frac{\alpha}{3})}), d_{\mathcal{F}}(\omega,[O_2\omega]_{(\frac{\alpha}{6})}), \\ \frac{d_{\mathcal{F}}(\kappa,[O_1\kappa]_{(\frac{\alpha}{3})})d_{\mathcal{F}}(\omega,[O_2\omega]_{(\frac{\alpha}{6})})}{1+d_{\mathcal{F}}(\kappa,\omega)} \end{array}\right)\right). \end{aligned}$$

Thus all assertions of Theorem 3.1 are satisfied. Thus there exists $0 \in [0,+\infty)$ such that $0 \in [O_1 0]_{(\frac{\alpha}{3})} \cap [O_2 0]_{(\frac{\alpha}{6})}$.

Corollary 3.1. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $\beta : \mathcal{W}\times\mathcal{W} \to [0,\infty)$ and let $O : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ be a fuzzy mapping. Assume that for each $\kappa \in \mathcal{W}$, there exist $\alpha_O(\kappa) \in (0,1]$ such that $[O\kappa]_{\alpha_O(\kappa)} \in C(2^{\mathcal{W}})$. Assume that these assertions also hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete,

(ii) For $\kappa_0 \in \mathcal{W}$,there exists $\alpha_O(\kappa_0) \in (0,1]$ such that $\kappa_1 \in [O\kappa_0]_{\alpha_O(\kappa_0)}$ with $\beta(\kappa_0,\kappa_1) \geq 1$,

(iii) There exists $\phi \in \Psi$ such that

$$\begin{aligned} & \max\{\beta(\kappa,\omega),\beta(\omega,\kappa)\} H_{\mathcal{F}}\left([O\kappa]_{\alpha_O(\kappa)},[O\omega]_{\alpha_O(\omega)}\right) \\ \leq & \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega), d_{\mathcal{F}}(\kappa,[O\kappa]_{\alpha_O(\kappa)}), d_{\mathcal{F}}(\omega,[O\omega]_{\alpha_O(\omega)}), \\ \frac{d_{\mathcal{F}}(\kappa,[O\kappa]_{\alpha_O(\kappa)})d_{\mathcal{F}}(\omega,[O\omega]_{\alpha_O(\omega)})}{1+d_{\mathcal{F}}(\kappa,\omega)} \end{array}\right)\right) \end{aligned}$$

for all $\kappa,\omega \in \mathcal{W}$,

(iii) O is $\beta_{\mathcal{F}}$-admissible.

(iv) If $\{\kappa_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(\kappa_n,\kappa_{n+1}) \geq 1$ and $\kappa_n \to \kappa$ as $n \to \infty$, then $\beta(\kappa_n,\kappa) \geq 1$, for all n.

Then there exists some $\kappa^* \in [O\kappa^*]_{\alpha_O(\kappa^*)}$.

Proof. Taking one fuzzy mapping from $\mathcal{W}$ into $\mathcal{I}^{\mathcal{W}}$ in Theorem 3.1. □

Corollary 3.2. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and let $O_1, O_2 : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ be fuzzy mappings. Assume that for each $\kappa \in \mathcal{W}$, there exist $\alpha_{O_1}(\kappa), \alpha_{O_2}(\kappa) \in (0,1]$ such that $[O_1\kappa]_{\alpha_{O_1}(\kappa)}, [O_2\kappa]_{\alpha_{O_2}(\kappa)} \in C(2^{\mathcal{W}})$. Assume that these assertions also hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-complete,

(ii) For $\kappa_0 \in \mathcal{W}$, there exists $\alpha_{O_1}(\kappa_0)$ or $\alpha_{O_2}(\kappa_0) \in (0,1]$ such that $\kappa_1 \in [O_1\kappa_0]_{\alpha_{O_1}(\kappa_0)}$ or $\kappa_1 \in [O_2\kappa_0]_{\alpha_{O_2}(\kappa_0)}$,

(iii) There exists $\phi \in \Psi$ such that

$$H_{\mathcal{F}}\left([O_1\kappa]_{\alpha_{O_1}(\kappa)},[O_2\omega]_{\alpha_{O_2}(\omega)}\right) \leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega), d_{\mathcal{F}}(\kappa,[O_1\kappa]_{\alpha_{O_1}(\kappa)}), d_{\mathcal{F}}(\omega,[O_2\omega]_{\alpha_{O_2}(\omega)}), \\ \frac{d_{\mathcal{F}}(\kappa,[O_1\kappa]_{\alpha_{O_1}(\kappa)})d_{\mathcal{F}}(\omega,[O_2\omega]_{\alpha_{O_2}(\omega)})}{1+d_{\mathcal{F}}(\kappa,\omega)} \end{array}\right)\right)$$

for all $\kappa,\omega \in \mathcal{W}$, then there exists some $\kappa^* \in [O_1\kappa^*]_{\alpha_{O_1}(\kappa^*)} \cap [O_2\kappa^*]_{\alpha_{O_2}(\kappa^*)}$.

Proof. Taking $\beta : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ by $\beta(\kappa, \omega) = 1$, for all $\kappa, \omega \in \mathcal{W}$. □

Corollary 3.3. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and let $O : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ be fuzzy mapping. Assume that for each $\kappa \in \mathcal{W}$, there exist $\alpha_O(\kappa) \in (0, 1]$ such that $[O\kappa]_{\alpha_O(\kappa)} \in C(2^{\mathcal{W}})$. Assume that these assertions also hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ be an $\mathcal{F}$-complete,

(ii) For $\kappa_0 \in \mathcal{W}$, there exists $\alpha_O(\kappa_0) \in (0, 1]$ such that $\kappa_1 \in [O\kappa_0]_{\alpha_O(\kappa_0)}$,

(iii) There exists $\phi \in \Psi$ such that

$$H_{\mathcal{F}}\left([O\kappa]_{\alpha_O(\kappa)}, [O\omega]_{\alpha_O(\omega)}\right) \leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa, \omega), d_{\mathcal{F}}(\kappa, [O\kappa]_{\alpha_O(\kappa)}), d_{\mathcal{F}}(\omega, [O\omega]_{\alpha_O(\omega)}), \\ \frac{d_{\mathcal{F}}(\kappa, [O\kappa]_{\alpha_O(\kappa)}) d_{\mathcal{F}}(\omega, [O\omega]_{\alpha_O(\omega)})}{1 + d_{\mathcal{F}}(\kappa, \omega)} \end{array}\right)\right)$$

for all $\kappa, \omega \in \mathcal{W}$.

Then there exists some $\kappa^* \in [O\kappa^*]_{\alpha_O(\kappa^*)}$.

Proof. Taking one fuzzy mapping from $\mathcal{W}$ into $\mathcal{I}^{\mathcal{W}}$ in Corollary 3.2. □

Corollary 3.4. *Let $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $\beta : \mathcal{W} \times \mathcal{W} \to [0, \infty)$ and let $\mathcal{R}_1, \mathcal{R}_2 : \mathcal{W} \to CB(\mathcal{W})$. Assume that these conditions hold:*

(i) $(\mathcal{W}, d_{\mathcal{F}})$ be a $\mathcal{F}$-complete,

(ii) For each $\kappa_0 \in \mathcal{W}$, there exists $\kappa_1 \in \mathcal{R}_1\kappa_0$ with $\beta(\kappa_0, \kappa_1) \geq 1$.

(iii) There exists $\phi \in \Psi$ such that

$$\max\{\beta(\kappa, \omega), \beta(\omega, \kappa)\} H_{\mathcal{F}}(\mathcal{R}_1\kappa, \mathcal{R}_2\omega) \leq \phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa, \omega), d_{\mathcal{F}}(\kappa, \mathcal{R}_1\kappa), d_{\mathcal{F}}(\omega, \mathcal{R}_2\omega), \\ \frac{d_{\mathcal{F}}(\kappa, \mathcal{R}_1\kappa) d_{\mathcal{F}}(\omega, \mathcal{R}_2\omega)}{1 + d_{\mathcal{F}}(\kappa, \omega)} \end{array}\right)\right)$$

for all $\kappa, \omega \in \mathcal{W}$,

(iii) $(\mathcal{R}_1, \mathcal{R}_2)$ is β-admissible,

(iv) If $\{\kappa_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(\kappa_n, \kappa_{n+1}) \geq 1$ and $\kappa_n \to \kappa$ as $n \to \infty$, then $\beta(\kappa_n, \kappa) \geq 1$, for all n.

Then there exists some $\kappa^* \in \mathcal{R}_1\kappa^* \cap \mathcal{R}_2\kappa^*$.

Proof. Let $\alpha_{O_1}, \alpha_{O_2} : \mathcal{W} \to (0, 1]$ be any two arbitrary mappings and $O_1, O_2 : \mathcal{W} \to \mathcal{I}^{\mathcal{W}}$ be defined in this way:

$$O_1(\kappa)(t) = \begin{cases} \alpha_{O_1}(\kappa), & \text{if } t \in \mathcal{R}_1\kappa, \\ 0, & \text{if } t \notin \mathcal{R}_1\kappa, \end{cases}$$

and

$$O_2(\kappa)(t) = \begin{cases} \alpha_{O_2}(\kappa), & \text{if } t \in \mathcal{R}_2\kappa, \\ 0, & \text{if } t \notin \mathcal{R}_2\kappa. \end{cases}$$

Then for all $\kappa \in \mathcal{W}$, we get

$$[O_1\kappa]_{O_1(\kappa)} = \{t \in \mathcal{W} : O_1(\kappa)(t) \geq \alpha_{O_1}(\kappa)\} = \mathcal{R}_1\kappa.$$

Similarly,

$$[O_2\kappa]_{O_2(\kappa)} = \{t \in \mathcal{W} : O_2(\kappa)(t) \geq \alpha_{O_2}(\kappa)\} = \mathcal{R}_2\kappa.$$

Hence,

$$H_{\mathcal{F}}\left([O_1\kappa]_{O_1(\kappa)},[O_2\omega]_{O_2(\omega)}\right)=H_{\mathcal{F}}(\mathcal{R}_1\kappa,\mathcal{R}_2\omega)$$

for all $\kappa,\omega\in\mathcal{W}$ and by Theorem 3.1, there exists $\kappa^*\in\mathcal{W}$ such that

$$\kappa^*\in[O_1\kappa^*]_{O_1(\kappa^*)}\cap[O_2\kappa^*]_{O_2(\kappa^*)}=\mathcal{R}_1\kappa^*\cap\mathcal{R}_2\kappa^*.$$

□

Corollary 3.5. *Let $(\mathcal{W},d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space, $\beta:\mathcal{W}\times\mathcal{W}\to[0,\infty)$ and let $\mathcal{R}:\mathcal{W}\to CB(\mathcal{W})$. Assume that these conditions hold:*

(i) $(\mathcal{W},d_{\mathcal{F}})$ be an $\mathcal{F}$-complete,

(ii) For each $\kappa_0\in\mathcal{W}$, there exists $\kappa_1\in\mathcal{R}_1\kappa_0$ with $\beta(\kappa_0,\kappa_1)\geq 1$,

(iii) There exists $\phi\in\Psi$ such that

$$\max\{\beta(\kappa,\omega),\beta(\omega,\kappa)\}H_{\mathcal{F}}(\mathcal{R}\kappa,\mathcal{R}\omega)\leq\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega),d_{\mathcal{F}}(\kappa,\mathcal{R}\kappa),d_{\mathcal{F}}(\omega,\mathcal{R}\omega),\\ \frac{d_{\mathcal{F}}(\kappa,\mathcal{R}\kappa)d_{\mathcal{F}}(\omega,\mathcal{R}\omega)}{1+d_{\mathcal{F}}(\kappa,\omega)}\end{array}\right)\right)$$

for all $\kappa,\omega\in\mathcal{W}$,

(iii) $\mathcal{R}$ is β-admissible,

(iv) If $\{\kappa_n\}$ is a sequence in $\mathcal{W}$ such that $\beta(\kappa_n,\kappa_{n+1})\geq 1$ and $\kappa_n\to\kappa$ as $n\to\infty$, then $\beta(\kappa_n,\kappa)\geq 1$ for all n.

Then there exists some $\kappa^*\in\mathcal{R}\kappa^*$.

Proof. Taking one multivalued mapping from $\mathcal{W}$ into $CB(\mathcal{W})$ in Corollary 3.4. □

Corollary 3.6. *Let $(\mathcal{W},d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and let $\mathcal{R}_1,\mathcal{R}_2:\mathcal{W}\to CB(\mathcal{W})$. Assume that these conditions hold:*

(i) $(\mathcal{W},d_{\mathcal{F}})$ be a $\mathcal{F}$-complete,

(ii) For each $\kappa_0\in\mathcal{W}$, there exists $\kappa_1\in\mathcal{R}_1\kappa_0$,

(iii) There exists $\phi\in\Psi$ such that

$$H_{\mathcal{F}}(\mathcal{R}_1\kappa,\mathcal{R}_2\omega)\leq\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega),d_{\mathcal{F}}(\kappa,\mathcal{R}_1\kappa),d_{\mathcal{F}}(\omega,\mathcal{R}_2\omega),\\ \frac{d_{\mathcal{F}}(\kappa,\mathcal{R}_1\kappa)d_{\mathcal{F}}(\omega,\mathcal{R}_2\omega)}{1+d_{\mathcal{F}}(\kappa,\omega)}\end{array}\right)\right)$$

for all $\kappa,\omega\in\mathcal{W}$.

Then there exists some $\kappa^*\in\mathcal{R}_1\kappa^*\cap\mathcal{R}_2\kappa^*$.

Proof. Taking $\beta:\mathcal{W}\times\mathcal{W}\to[0,\infty)$ by $\beta(\kappa,\omega)=1$, for all $\kappa,\omega\in\mathcal{W}$ in Corollary 3.4. □

Corollary 3.7. *Let $(\mathcal{W},d_{\mathcal{F}})$ be a $\mathcal{F}$-metric space and let $\mathcal{R}:\mathcal{W}\to CB(\mathcal{W})$. Assume that these conditions hold:*

(i) $(\mathcal{W},d_{\mathcal{F}})$ be an $\mathcal{F}$-complete,

(ii) For each $\kappa_0\in\mathcal{W}$, there exists $\kappa_1\in\mathcal{R}\kappa_0$,

(iii) There exists $\phi\in\Psi$ such that

$$H_{\mathcal{F}}(\mathcal{R}\kappa,\mathcal{R}\omega)\leq\phi\left(\max\left(\begin{array}{c} d_{\mathcal{F}}(\kappa,\omega),d_{\mathcal{F}}(\kappa,\mathcal{R}\kappa),d_{\mathcal{F}}(\omega,\mathcal{R}\omega),\\ \frac{d_{\mathcal{F}}(\kappa,\mathcal{R}\kappa)d_{\mathcal{F}}(\omega,\mathcal{R}\omega)}{1+d_{\mathcal{F}}(\kappa,\omega)}\end{array}\right)\right)$$

for all $\kappa,\omega\in\mathcal{W}$.

Then there exists some $\kappa^*\in\mathcal{R}\kappa^*$.

Proof. Taking one multivalued mapping in Corollary 3.6. □

4. Applications

In the present section, we discuss the solution of fuzzy integrodifferential equations in the context of generalized Hukuhara derivative.

We denote $K_c(\mathbb{R})$ the family of all non-empty convex and compact subsets of real numbers $\mathbb{R}$. The notion of Hausdorff metric H in $K_c(\mathbb{R})$ is given in this way:

$$H(\aleph_1, \aleph_2) = \max\left\{ \sup_{a\in\aleph_1} \inf_{b\in\aleph_2} \|a-b\|_{\mathbb{R}}, \sup_{b\in\aleph_2} \inf_{a\in\aleph_1} \|a-b\|_{\mathbb{R}} \right\},$$

for $\aleph_1, \aleph_2 \in K_c(\mathbb{R})$. Then the pair $(K_c(\mathbb{R}), H)$ is considered as complete metric space (see [12]).

Definition 4.1. *A function* $\wp : (-\infty, +\infty) \to [0,1]$ *is professed to be a fuzzy number if these assertions hold:*

(i) There exists $t_0 \in \mathbb{R}$ *such that* $\wp(t_0) = 1$,

(ii) For $0 \leq \lambda \leq 1$,

$$\wp\left(\lambda t_1 + (1-\lambda)t_2\right) \geq \min\left\{\wp(t_1), \wp(t_2)\right\}$$

for all $t_1, t_2 \in \mathbb{R}$.

(iii) $\wp$ *is upper semicontinuous,*

(iv) $[\wp]^0 = cl\{t \in \mathbb{R} : \wp(t) > 0\}$ *is compact.*

As a consequence, E^1 denotes the set of fuzzy numbers in $\mathbb{R}$ with the following property.

For $\alpha \in (0,1]$, $[\wp]^\alpha = \{t \in \mathbb{R} : \wp(t) > \alpha\} = \left[\wp_l^\alpha, \wp_r^\alpha\right]$ represents α - cut of the fuzzy set $\wp$. For $\wp \in E^1$, one has that $[\wp]^\alpha \in K_c(\mathbb{R})$ for each $\alpha \in [0,1]$. The supremum on E^1 is defined by

$$d_\infty(\wp_1, \wp_2) = \sup_{\alpha\in[0,1]} \max\left\{\left|\wp_{1,l}^\alpha - \wp_{2,l}^\alpha\right|, \left|\wp_{1,r}^\alpha - \wp_{2,r}^\alpha\right|\right\}$$

for every $\wp_1, \wp_2 \in E^1$, where $\wp_r^\alpha - \wp_l^\alpha = diam([\wp])$ is called the diameter of $[\wp]$. We designate the class of all continuous fuzzy functions given on $[a,b]$, for $\rho > 0$ as $C([a,b], E^1)$.

From [13], it is famous that the space $C([a,b], E^1)$ is a complete metric space regarding

$$d(\wp_1, \wp_2) = \sup_{t\in J} d_\infty(\wp_1(t), \wp_2(t)))$$

for $\wp_1, \wp_2 \in C([a,b])$.

Lemma 4.1. *[7] Let* $\wp_1, \wp_2 : [a,b] \to E^1$ *and* $\eta \in \mathbb{R}$. *Then,*

(i) $\int_a^b (\wp_1 + \wp_2)(t)dt = \int_a^b \wp_1(t)dt + \int_a^b \wp_2(t)dt$,

(ii) $\int_a^b \eta\wp_1(t)dt = \eta \int_a^b \wp_1(t)dt$,

(iii) $d_\infty(\wp_1(t), \wp_2(t))$ *is integrable,*

(iv) $d_\infty(\int_a^b \wp_1(t)dt, \int_a^b \wp_2(t)dt) \leq \int_a^b d_\infty(\wp_1(t), \wp_2(t))dt$,

for $t \in [a,b]$.

Definition 4.2. *[10] Suppose that E^n denotes the family of all fuzzy numbers in $\mathbb{R}^n$ and $\wp, \omega, \ell \in E^n$. A point ℓ is called the Hukuhara difference of $\wp$ and ω , if $\wp = \omega + \ell$ is satisfied. If this Hukuhara difference exists, then it is described by $\wp\ \Theta_H\ \omega$ (or $\wp - \omega$). Evidently, if $\wp\ \Theta_H\ \wp = \{0\}$, and if $\wp\ \Theta_H\ \omega$ exists, then it is unique.*

Definition 4.3. *[10] Let $g : (a, b) \to E^n$. The function g is called a strongly generalized differentiable (or GH-differentiable) at $t_0 \in (a, b)$, if $\exists g^{/}_{G}(t_0) \in E^n$ such that*

$$g(t_0 + \delta)\Theta_H g(t_0), g(t_0)\Theta_H g(t_0 - \delta)$$

and

$$\lim_{\delta \to 0^+} \frac{g(t_0 + \delta)\Theta_H g(t_0)}{\delta} = \lim_{\delta \to 0^+} \frac{g(t_0)\Theta_H g(t_0 - \delta)}{\delta} = g^{/}_{G}(t_0).$$

Now Considering

$$\begin{cases} \wp^{/}(t) = g(t, \wp(t)), \quad t \in J = [a, \rho] \\ \wp(0) = \wp_0, \end{cases} \tag{4.1}$$

where $\wp^{/}$ is appropriated as GH-differentiable and $g : J \times E^1 \to E^1$ is continuous. The initial data $\wp_0$ is supposed in E^1. We show the family of all $g : J \to E^1$ with continuous derivative as $C^1(J, E^1)$.

Lemma 4.2. *A function $\wp \in C^1(J, E^1)$ is a solution of (4.1) if and only if it satisfies the following:*

$$\wp(t) = \wp_0 \Theta_H (-1) \int_a^t g(s, \wp(s))ds, \quad t \in J = [a, \rho].$$

Theorem 4.1. *Let $g : J \times E^1 \to E^1$ be continuous such that:*
(i) For $\wp < \omega$ and $t \in J$, we have $g(t, \wp) < g(t, \omega)$;
(ii) There exist some constants $\tau > 0$ such that $\lambda \in (0, \frac{1}{2(\rho - a)})$, such that

$$\|g(t, \wp(t)) - g(t, \omega(t))\|_{\mathbb{R}} \le \tau \max_{t \in J} \left\{ d_\infty(\wp, \omega) e^{-\tau(t-a)} \right\}$$

if $\wp < \omega$ for each $t \in J$ and $\wp, \omega \in E^1$, where $d_\infty(\wp, \omega)$ is the supremum on E^1. Then (4.1) has a fuzzy solution in $C^1(J, E^1)$.

Proof. Let $\tau > 0$ and $C^1(J, E^1)$ equipped with

$$d_\tau(\wp, \omega) = \sup_{t \in J} \left\{ d_\infty(\wp(t), \omega(t)) e^{-\tau(t-a)} \right\},$$

$\wp, \omega \in C^1(J, E^1)$. Then with $g(\wp) = \ln(\wp), \wp > 0$ and $h = 0$, $\left(C^1(J, E^1), d_\tau\right)$ is complete $\mathcal{F}$-complete metric space.

Let $M, Q : \mathcal{W} \to (0, 1]$. For $\wp \in \mathcal{W}$, take

$$L_\wp(t) = \wp_0 \Theta_H (-1) \int_a^t g(s, \wp(s))ds.$$

Assume $\wp < \omega$. Then it follows by assumption (i) that

$$\begin{aligned} L_{\wp}(t) &= \wp_0 \Theta_H(-1) \int_a^t g(s, \wp(s)) ds \\ &< \wp_0 \Theta_H(-1) \int_a^t g(s, \omega(s)) ds \\ &= R_{\omega}(t). \end{aligned}$$

Thus $L_{\wp}(t) \neq R_{\omega}(t)$. Consider $O_1, O_2 : \mathcal{W} \to E^{\mathcal{W}}$ defined by

$$\mu_{O_1\wp}(r) = \begin{cases} M(\wp), & \text{if } r(t) = L_{\wp}(t), \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{O_2\omega}(r) = \begin{cases} Q(\omega), & \text{if } r(t) = R_{\omega}(t), \\ 0, & \text{otherwise.} \end{cases}$$

Take $\alpha_{O_1}(\wp) = M(\wp)$ and $\alpha_{O_2}(\omega) = Q(\omega)$, we get

$$[O_1\wp]_{\alpha_{O_1}(\wp)} = \{r \in \mathcal{W} : (O_1\wp)(t) \geq M(\wp)\} = \{L_{\wp}(t)\},$$

and likewise $[O_2\omega]_{\alpha_{O_1}(\omega)} = \{R_{\omega}(t)\}$, so

$$\begin{aligned} H\left([O_1\wp]_{\alpha_{O_1}(\wp)}, [O_2\omega]_{\alpha_{O_1}(\omega)}\right) &= \max \left\{ \begin{array}{c} \sup_{\wp \in [O_1\wp]_{\alpha_{O_1}(\wp)}, \omega \in [O_2\omega]_{\alpha_{O_1}(\omega)}} \inf \|\wp - \omega\|_{\mathbb{R}}, \\ \sup_{\omega \in [O_2\omega]_{\alpha_{O_1}(\omega)}, \wp \in [O_1\wp]_{\alpha_{O_1}(\wp)}} \inf \|\wp - \omega\|_{\mathbb{R}} \end{array} \right\} \\ &\leq \max \left\{ \sup_{t \in J} \left\| L_{\wp}(t) - R_{\omega}(t) \right\|_{\mathbb{R}} \right\} = \sup_{t \in J} \left\| L_{\wp}(t) - R_{\omega}(t) \right\|_{\mathbb{R}} \\ &= \sup_{t \in J} \left\| \int_a^t g(s, \wp(s)) ds - \int_0^t g(s, \omega(s)) ds \right\|_{\mathbb{R}} \\ &\leq \sup_{t \in J} \left\{ \int_a^t \| g(s, \wp(s)) - g(s, \omega(s)) \| \, ds \right\} \\ &\leq \sup_{t \in J} \left\{ \int_a^t du \lambda \max \left\{ D_{\infty}(\wp, \omega) e^{-\tau(t-a)} \right\} ds \right\} \\ &\leq \lambda \sup_{t \in J} \left\{ (t-a) \max \left\{ D_{\infty}(\wp, \omega) e^{-\tau(t-a)} \right\} \right\} \\ &\leq \lambda(\rho - a) d_{\tau}(\wp, \omega) \leq \frac{1}{2} d_{\tau}(\wp, \omega) \\ &= \phi\left(d_{\tau}(\wp, \omega)\right) \\ &\leq \phi\left(\max \left(\begin{array}{c} d_{\tau}(\wp, \omega), d_{\tau}(\wp, [O_1\wp]_{\alpha_{O_1}(\wp)}), d_{\tau}(\omega, [O_2\omega]_{\alpha_{O_1}(\omega)}), \\ \frac{d_{\tau}(\wp, [O_1\wp]_{\alpha_{O_1}(\wp)}) d_{\tau}(\omega, [O_2\omega]_{\alpha_{O_1}(\omega)})}{1 + d_{\tau}(\wp, \omega)} \end{array} \right) \right). \end{aligned}$$

Hence, all the hypotheses of Theorem 3.1 are satisfied with $\phi(t) = \frac{1}{2}t$, for $t > 0$. Thus $\wp^*$ is a solution of (4.1). □

5. Conclusions

In this article, we have proved some significant common α-fuzzy fixed point theorems for rational (β, ϕ)-contractive conditions in the context of complete $\mathcal{F}$-metric spaces. The established theorems improved and generalized different conventional theorems in fuzzy fixed point theory. We also discussed the solution of fuzzy integrodifferential equations in the background of a generalized Hukuhara derivative as application of our leading result which deals with uncertainties in decision making. The established results are important contribution and generalization of the existing results in fuzzy fixed point theory. Our results can be extended and improved for intuitionistic fuzzy mappings as a future work.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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