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Research article

Existence of periodic wave for a perturbed MEW equation

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Abstract: A perturbed MEW equation including small backward diffusion, dissipation and nonlinear term is considered by the geometric singular perturbation theory. Based on the monotonicity of the ratio of Abelian integrals, we prove the existence of periodic wave on a manifold for perturbed MEW equation. By Chebyshev system criterion, the uniqueness of the periodic wave is obtained. Furthermore, the monotonicity of the wave speed is proved and the range of the wave speed is obtained. Additionally, the monotonicity of period is given by Picard-Fuchs equation.

Keywords: perturbed MEW equation; geometric singular perturbation theory; periodic wave; Abelian integral; Chebyshev system; Picard-Fuchs equation
Mathematics Subject Classification: 34C25, 34C60, 37C27

1. Introduction

Nonlinear wave equations have been widely used to describe natural phenomena of science, engineering, geology, economics, meteorology, chemistry, and physics. The phenomena of dispersion, dissipation, diffusion, reaction and convection play a major role in nonlinear wave equations. Therefore, it is very important to find exact solutions of nonlinear evolution equations. Recently, there are many useful techniques to obtain the exact traveling wave solutions, such as the inverse scattering transform method [1], modified simple equation method [2], the Bäcklund transformation [3], the Lie symmetry method [4], the Darboux transformation [5], the multiple exp-function method [6], Hirota's bilinear method [7], the sine-cosine method [8], the bifurcation theory of dynamic system [9]. A well-known nonlinear wave equations is equal width (EW) equation

$$u_t + uu_x - bu_{xxt} = 0 \tag{1.1}$$

with the boundary conditions $u \to 0$ as $x \to \pm\infty$. Since it describes many physical phenomena such as shallow water waves and ion acoustic plasma waves, it is a model for the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes [10]. The regularized long wave (RLW) equation is given by

$$u_t + u_x + \frac{1}{2}(u_x^2) - u_{xxt} = 0 \tag{1.2}$$

with $-\infty < x < +\infty$, $t > 0$. The RLW equation appears in many physical applications, for example, the nonlinear transverse waves in shallow water, ion-acoustic waves in plasma, elastic media, optical fibres, hydromagnetic wave in cold plasma [11]. The development of an equal width undular bore is investigated in [12] and compared with that of the RLW bore. The modified equal width (MEW) equation

$$u_t + a(u^3)_x - bu_{xxt} = 0, \tag{1.3}$$

is based upon the EW (1.2). (1.3) is a nonlinear wave equation with cubic nonlinearity which admits solitary wave solutions with the same width, which is considered to explain many physical phenomena [13–15].

Based on many applications of MEW equation, there is a variety of investigations on MEW equation. In 2000, the author studied the solitary wave motion and interaction for MEW equation by finite element methods [16]. The modified equal width equation and its variants were investigated by Wazwaz [8] by a sine-cosine ansatz and the tanh method, compactons, solitons, solitary patterns, and periodic solution are obtained. In [17], the propagation of the solitary wave for time split MEW equation and space split MEW equation was investigated by quintic B-spline collocation method. After that, the numerical solution of the MEW equation was proposed by the collocation method using the radial basis functions with first order accurate forward difference approximation [18]. Lu [19] solved the modified equal width equation by variational iteration method which provides remarkable accuracy in comparison with the analytical solution. Cheng and Liew [20] derived formulae for an improved element-free Galerkin method for MEW equation by numerical examples. In [21], the motion of a single solitary wave and interaction of two solitary waves for the MEW equation were studied. More recently, Shi and Zhang [22] obtained the periodic solutions, dark solutions, soliton solutions and soliton-like solutions of the space-time fractional MEW equation by ansatz method. Additionally, there are lots of investigations on finding the traveling wave solutions for combined systems based on MEW equation, such as, ZK-MEW equation [23, 24], KP-MEW equation [25, 26], KP-MEW-Burgers [27]. However, there is few investigation on the traveling wave solution especially periodic wave solution for the perturbed MEW equation.

In this paper, we consider the following perturbed MEW equation

$$u_t + (u^2)_x + u_{xxt} + \varepsilon((uu_x)_x - u_{xx} + u_{xxxx}) = 0, \tag{1.4}$$

where ε is a small nonnegative parameter. In Eq (1.4), u_t is the evolution term, the second term represents the nonlinear term, u_{xx} is the backward diffusion, u_{xxxx} is the dissipative term, $(uu_x)_x$ is the nonlinear term when the Marangoni effect is considered which describing the opposite to the Bénard convection [28, 29]. The Eq (1.4) describes the water motion in a wide range of weak dissipative circumstances. We focus on studying the existence of the periodic wave solution for perturbed MEW equation (1.4) and finding the number of periodic wave solution by using geometric singular perturbation theory. Furthermore, for the persisted periodic wave solution, we give the monotonicity of the period.

On the topic about finding the existence of traveling wave solution for perturbed nonlinear wave equations by using geometric singular perturbation theory, there are also lots of excellent productions. Ogama [30] established the existence of solitary waves and periodic waves for the perturbed KdV equation. Fan and Tian [31] studied the existence of solitary wave solution of perturbed mKdV-KS equation. Tang [32] gave the condition of solitary wave solution persisted. Mansour [33] constructed solitary waves for a generalized nonlinear dispersive-dissipative equation. Yan et al. [34] proved the persistence of solitary waves and periodic waves to a perturbed generalized KdV equation. By investigating the ratio of Abelian integrals, Du and his cooperators proved the existence of traveling wave solutions for some delayed nonlinear wave equations [35–38]. Chen [39, 40] studied the existence of traveling wave solutions for perturbed KdV equation. In references [41,42], two different generalized perturbed BBM equations were considered. Motivated by the references, we present the existence of uniqueness isolated periodic wave solution of (1.4) by using geometric singular perturbation theory. Combing with the Chebyshev system criterion and symbolic computation, the monotonicity of the ratio of Abelian integrals is given. Moreover, for the periodic wave solution, the property of the wave speed, the monotonicity and the range of period are obtained.

The rest of this paper is organized as follows. In Section 2, the geometric singular perturbation theory is introduced and our main result is stated. In Section 3, the existence of unique periodic wave for the perturbed MEW equation is proved by using geometric singular perturbation theory. Chebyshev system and symbolic computation are used to verify the monotonicity of the ratio of Abelian integrals, which is more effective than the method used in the references [39–41]. In Section 4, we investigate the monotonicity of period by Picard-Fuchs equation, which is not been considered in the references [42, 43].

2. Preliminaries and main results

We aim to prove the traveling wave persists for sufficiently small $\varepsilon > 0$ by using geometric singular perturbation theory, so we introduce the geometric singular perturbation theory which is due to [44,45], firstly.

Consider the system

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{2.1}$$

where $\prime = \frac{d}{dt}$, $0 < \varepsilon \ll 1$ is a real and small parameter, $x = (x_1, x_2, \ldots, x_k)^T \in R^k$, $y = (y_1, y_2, \ldots, y_l)^T \in R^l$. The following hypothesis about the system (2.1) is needed.

(H1) The functions f and g are both C^r on a set $U \times I$, where $0 < r < +\infty$ $U \subset R^{k+l}$ is open and I is an open interval containing 0.

With a change of time scaling $\tau = \varepsilon t$, system (2.1) can be written as

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = g(x, y, \varepsilon), \end{cases} \tag{2.2}$$

where $\dot{} = \frac{d}{d\tau}$. The time scale τ is slow and t is fast. When $\varepsilon \neq 0$, systems (2.1) and (2.2) are equivalent. Thus system (2.1) is called *the fast system*, while system (2.2) is called *the slow system*. In (2.1), letting $\varepsilon \to 0$, we obtain the *layer* system

$$\begin{cases} x'(t) = f(x, y, 0), \\ y'(t) = 0. \end{cases} \tag{2.3}$$

Here, x is called the fast variable, whereas y is called the slow variable. Let $\varepsilon \to 0$ in (2.2), the limit only makes sense if $f(x, y, 0) = 0$ and is given by

$$\begin{cases} f(x, y, 0) = 0, \\ \dot{y} = g(x, y, 0). \end{cases} \tag{2.4}$$

We shall assume that we are given an l-dimensional manifold, possibly with boundary, M_0 which is contained in the set $\{f(x, y, 0) = 0\}$. The manifold M_0 is *normally hyperbolic* if the layer system (2.3) at each point in M_0 has exactly l eigenvalues on the imaginary axis. Moreover, we give the following hypothesis.

(H2) The set M_0 is a compact manifold, possibly with boundary, and is normally hyperbolic relative to (2.3).

Definition 2.1. *A set M is locally invariant under the flow from* (2.1) *if it has neighborhood V so that no trajectory can leave M without also leaving V. In other words, it is locally invariant if for all $x \in M$, $x \cdot [0, t] \subset V$ implies that $x \cdot [0, t] \subset M$, similarly with $[0, t]$ replaced by $[t, 0]$, when $t < 0$, where the notation $x \cdot t$ is used to denote the application of a flow after time t to the initial condition x.*

Assume that there is a C^r function $h^0(y)$, $0 < r < +\infty$, with K being a compact domain in $\mathbb{R}^l$, such that $M_0 = \{(x, y) : x = h^0(y)\}$. Consequently, the following geometric theory of singular perturbation is established in [44].

Lemma 2.1. *For $\varepsilon > 0$ is sufficiently small, there exists a manifold M_ε lying within $O(\varepsilon)$ of M_0. M_ε is diffeomorphic to M_0 and locally invariant under the flow of* (2.1)*, and C^r in x, y and ε, for any $0 < r < +\infty$.*

Lemma 2.2. *Under the hypotheses (H1) and (H2), for $\varepsilon > 0$ is sufficiently small, there exists a function $x = h^\varepsilon(y)$ defining on K, such that the graph*

$$M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\},$$

is locally invariant under (2.1)*. Moreover, $h^\varepsilon(y)$ is C^r, for any $0 < r < +\infty$, jointly in y and ε. M_ε possesses locally invariant stable and unstable manifold $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ lying within $O(\varepsilon)$ and being C^r diffeomorphic to the stable and unstable manifold $W^s(M_0)$ and $W^u(M_0)$ of the critical manifold M_0.*

For Eq (1.4), making the traveling wave transformation $u(x, t) = u(x + ct) = u(\xi)$, where c is the wave speed. Integrating it once and neglecting the integral constant, then corresponding traveling wave system is

$$cu + u^2 + cu'' + \varepsilon(uu' - u' + u''') = 0, \tag{2.5}$$

where $'$ is the derivative respect to ξ. Taking a time scale transformation $u = c\phi$ and $\xi = z$ to (2.5), it obtains

$$\phi + \phi^2 + \frac{d^2\phi}{dz^2} + \varepsilon\left(\phi\phi' - \frac{1}{c}\frac{d\phi}{dz} + \frac{1}{c}\frac{d^3\phi}{dz^3}\right) = 0, \tag{2.6}$$

which is equivalent to the following three-dimensional system

$$\begin{cases} \dfrac{d\phi}{dz} = y, \\ \dfrac{dy}{dz} = \omega, \\ \varepsilon\dfrac{1}{c}\dfrac{d\omega}{dz} = -\phi - \phi^2 - \omega - \varepsilon\left(-\dfrac{y}{c} + \phi y\right). \end{cases} \tag{2.7}$$

When $\varepsilon = 0$, system (2.7) corresponds to a unperturbed system

$$\begin{cases} \dfrac{d\phi}{dz} = y, \\ \dfrac{dy}{dz} = -\phi - \phi^2, \end{cases} \tag{2.8}$$

which is a Hamiltonian system with the energy function

$$H(\phi, y) = \frac{y^2}{2} + \frac{\phi^2}{2} + \frac{\phi^3}{3}. \tag{2.9}$$

Clearly, (2.8) has two equilibrium points $(-1, 0)$ and $(0, 0)$. The origin $(0, 0)$ is a center and $(-1, 0)$ is a saddle. It is well known that (2.8) is determined by its potential energy function and its equilibrium points. $H(-1, 0) = H(\frac{1}{2}, 0) = \frac{1}{6}$, $H(0, 0) = 0$. Figure 1 shows a family of closed orbits surrounded by a homoclinic loop.

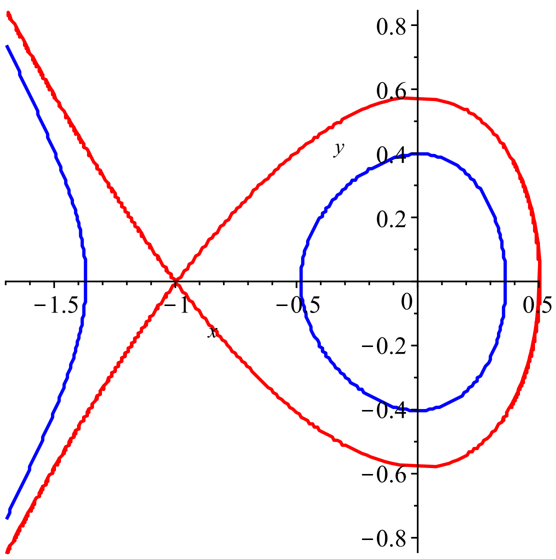

Figure 1. The portrait of system (2.8).

Let Γ_h be the closed orbits defined by $H(\phi, y) = h$ which surrounds a center. Suppose that $\phi(z, h)$ is a point lying the corresponding closed orbit Γ_h, $\phi(z, \varepsilon, h, c(\varepsilon, h))$ is the periodic wave solution of (2.5) near Γ_h on $c = c(\varepsilon, h)$. Denote that $T(h)$ is the period of $\phi(z, \varepsilon, h, c(\varepsilon, h))$, $\phi_0(z)$ is the solution corresponds to homoclinic loop, then for the traveling wave system of perturbed MEW equation (1.4), we obtain the following statements.

Theorem 2.1. *For any given traveling wave speed* $c \in (-12, +\infty)$*, there exists* $\varepsilon_0(c) > 0$ *such that when* $0 < \varepsilon < \varepsilon_0(c)$*, then Eq* (1.4) *has a unique isolated periodic wave solution* $u(x + ct)$*, which is given by* $u(x + ct) = c\phi(x + ct, c, \varepsilon, h(c, \varepsilon))$*, satisfying*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \phi(x + ct, c, \varepsilon, h) = \phi(x + ct, h),\\
&\lim_{(c,\varepsilon)\to(+\infty,0),0<\varepsilon<\varepsilon_0(c)} \phi(x + ct, c, \varepsilon, h) \to 0,\\
&\lim_{(c,\varepsilon)\to(-12,0),0<\varepsilon<\varepsilon_0(c)} \phi(x + ct, c, \varepsilon, h) \to \phi_0(x + ct).
\end{aligned}$$

Furthermore, $c(\varepsilon, h)$ *satisfies*

$$\lim_{\varepsilon\to 0} c(\varepsilon, h) = c(h), \ \frac{\partial c(\tau, h)}{\partial h} < 0,$$

where $c(h)$ *is a strictly decreasing function in* h *satisfying* $-12 < c(h) < +\infty$.

Theorem 2.2. *The period of the isolated periodic wave solution shown in Theorem 2.1 is strictly increasing for* $h \in (0, \frac{1}{6})$ *and satisfies*

$$\lim_{h\to 1/6} T(h) = +\infty, \lim_{h\to 0} T(h) = 2\pi.$$

3. Perturbation analysis

By the geometric singular perturbation theory, when $\varepsilon \neq 0$, the transformation $z = \varepsilon\tau$ is introduced to change the system (2.7) to

$$\begin{cases}
\dfrac{d\phi}{d\tau} = \varepsilon y,\\
\dfrac{dy}{d\tau} = \varepsilon\omega,\\
\dfrac{1}{c}\dfrac{d\omega}{d\tau} = -\phi - \phi^2 - \omega - \varepsilon\left(-\dfrac{1}{c}y + \phi y\right).
\end{cases} \tag{3.1}$$

System (2.7) is the slow system and (3.1) is the fast system, they are equivalent when $\varepsilon > 0$. The two different time-scales corresponds to two different limiting systems. If $\varepsilon = 0$, the flow of system (3.1) is confined to the two-dimensional invariant manifold

$$M_0 = \{(\phi, y, \omega) \in \mathbb{R}^3 : \omega = -\phi - \phi^2\}$$

and its dynamics are determined only by the first two equations of (3.1). The set M_0 is the slow manifold. Since the linearized matrix of (3.1) with $\varepsilon = 0$ is

$$\begin{pmatrix}
0 & 0 & 0\\
0 & 0 & 0\\
-c(1 + 2\phi) & 0 & -c
\end{pmatrix},$$

It is not difficult to find that the eigenvalues are 0, 0, $-c$, then the slow manifold M_0 is normally hyperbolic. From Lemma 2.2, there exists a sub-manifold M_ε of the perturbed system (2.7) of $\mathbb{R}^3$ for sufficiently small $\varepsilon > 0$, which can be written as

$$M_\varepsilon = \{(\phi, y, \omega) \in \mathbb{R}^3 : \omega = -\phi - \phi^2 + g(\phi, y, \varepsilon)\},$$

where g is a smooth function defined on a compact domain and satisfies $g(\phi, y, 0) = 0$. Then assume that $g(\phi, y, \varepsilon)$ is expanded into Taylor series $g(\phi, y, \varepsilon) = \varepsilon g_1(\phi, y) + O(\varepsilon^2)$. Substituting $\omega = -\phi - \phi^2 + g(\phi, y, \varepsilon)$ into the slow system (2.7) and comparing the coefficient of ε, it obtains

$$g_1 = \frac{2}{c}y + \left(\frac{2}{c} - 1\right)\phi y.$$

Therefore, the slow system (2.7) restricted on M_ε is given by a regular perturbed system

$$\begin{cases} \dfrac{d\phi}{dz} = y, \\ \dfrac{dy}{dz} = -\phi - \phi^2 + \varepsilon\left(\dfrac{2}{c}y + \dfrac{2-c}{c}\phi y\right) + O(\varepsilon^2). \end{cases} \tag{3.2}$$

For $h \in (0, 1/6)$, suppose that there exists a closed orbit Γ_h of $(3.2)|_{\varepsilon=0}$ which surrounds the center $(0, 0)$. $T(h)$ is the period of Γ_h. $A(h) \in \Gamma_h$ is the rightmost point on the positive ϕ-axis at $z = 0$. For $\varepsilon > 0$ sufficiently small, let Γ_{h_ε} be a piece of the orbit for (3.2) starting from $A(h)$ to the next intersection point $B(h_\varepsilon)$ with the positive ϕ-axis at $z = z(\varepsilon)$ for $0 < |h_\varepsilon - h| \ll 1$. By [46], the displacement function between $B(h_\varepsilon)$ and $A(h)$ is given by

$$\begin{aligned} d(h, c, \varepsilon) &= \int_{\widehat{AB}} dH = \int_{\widehat{AB}} (\phi + \phi^2)d\phi + ydy \\ &= \int_0^{z(\varepsilon)} \left\{(\phi + \phi^2)y + \left[-\phi - \phi^2 + \varepsilon\left(\frac{2}{c}y + \frac{2-c}{c}\phi y\right) + O(\varepsilon^2)\right]y\right\} dz \\ &= \varepsilon\frac{1}{c}\int_0^{z(\varepsilon)} [2y^2 + (2-c)\phi y^2 + O(\varepsilon)]dz \\ &\triangleq \varepsilon\Phi(h, c, \varepsilon). \end{aligned}$$

By continuousness theorem, we have

$$\lim_{\varepsilon\to0} \Gamma_{h,\varepsilon} = \Gamma_h, \ \lim_{\varepsilon\to0} B(h, \varepsilon) = B(h), \ \lim_{\varepsilon\to0} z(\varepsilon) = T(h).$$

Therefore, it has

$$\Phi(h, c, \varepsilon) = \frac{1}{c}M(h, c) + O(\varepsilon),$$

where

$$\begin{aligned} M(h, c) &= \oint_{\Gamma_h} \left[2y^2 + (2-c)\phi y^2\right] dz \\ &= \oint_{\Gamma_h} [2y + (2-c)\phi y]\, d\phi = 2J_0(h) + (2-c)J_1(h), \end{aligned} \tag{3.3}$$

with $J_i(h) = \oint_{\Gamma_h} \phi^i y d\phi$, i=0,1. $M(h, c)$ is called Melnikov function. By the Poincaré bifurcation theory, the isolated zeros of $d(h, \varepsilon)$ corresponds to limit cycles of system (3.2).

In order to investigate the zero of $M(h, c)$, the following lemmas are needed.

Lemma 3.1. *For $h \in (0, \frac{1}{6})$, $J_0(h) > 0$ and $J_0'(h) > 0$.*

Proof. Denote (ϕ, y) is a point lying on Γ_h. By Green formula, it yields

$$J_0(h) = \oint_{\Gamma_h} y d\phi = \iint_{int\Gamma_h} d\phi dy > 0.$$

Since $y^2 = 2h - \phi^2 - \frac{2}{3}\phi^3$, it has $J_i'(h) = \oint_{\Gamma_h} \phi^i \frac{\partial y}{\partial h} d\phi = \oint_{\Gamma_h} \frac{\phi^i}{y} d\phi$, then

$$J_0'(h) = \oint_{\Gamma_h} \frac{1}{y} d\phi = \int_0^{T(h)} dz = T(h) > 0.$$

□

Denote that $P(h) = \frac{J_1(h)}{J_0(h)}$, then

$$M(h, c) = J_0[2 + (2 - c)P(h)]. \tag{3.4}$$

Lemma 3.2. $J_0\left(\frac{1}{6}\right) = \frac{6}{5},\ J_1\left(\frac{1}{6}\right) = -\frac{6}{35}$. *Then,* $\frac{J_1\left(\frac{1}{6}\right)}{J_0\left(\frac{1}{6}\right)} = -\frac{1}{7}$.

Proof. Denote that $\alpha(h)$ and $\beta(h)$ are the left and right intersection points of Γ_h to ϕ-axis, respectively. From (2.9), it has $y = \pm\sqrt{-\frac{2}{3}\phi^3 - \phi^2 + 2h}$ and $\alpha\left(\frac{1}{6}\right) = -1, \beta\left(\frac{1}{6}\right) = \frac{1}{2}$, then we compute the $J_0\left(\frac{1}{6}\right)$ and $J_1\left(\frac{1}{6}\right)$ directly

$$\begin{aligned} J_0\left(\frac{1}{6}\right) &= \oint_{\Gamma_h} y d\phi = 2\int_{-1}^{\frac{1}{2}} \sqrt{-\frac{2}{3}\phi^3 - \phi^2 + \frac{1}{3}} d\phi \\ &= \frac{2\sqrt{6}}{3}\int_{-1}^{\frac{1}{2}} (1 + \phi)\sqrt{\frac{1}{2} - \phi} d\phi = \frac{6}{5}, \\ J_1\left(\frac{1}{6}\right) &= \oint_{\Gamma_h} \phi y d\phi = 2\int_{-1}^{\frac{1}{2}} \phi\sqrt{-\frac{2}{3}\phi^3 - \phi^2 + \frac{1}{3}} d\phi \\ &= \frac{2\sqrt{6}}{3}\int_{-1}^{\frac{1}{2}} \phi(1 + \phi)\sqrt{\frac{1}{2} - \phi} d\phi = -\frac{6}{35}, \end{aligned}$$

then it obtains the statements. □

Lemma 3.3. $\lim_{h\to 0} \frac{J_1(h)}{J_0(h)} = 0$.

Proof. When $h \to 0$, Γ_h approaches to the center $(0, 0)$, implying that $\phi \to 0$. By applying mean value theorem for integrals, it has

$$\lim_{h\to 0} \frac{J_1(0)}{J_0(0)} = \lim_{h\to 0} \frac{\oint_{\Gamma_h} \phi y d\phi}{\oint_{\Gamma_h} y d\phi} = \lim_{\phi\to 0} \phi = 0.$$

□

Lemma 3.4. *For $h \in (0, \frac{1}{6})$, $\frac{J_1(h)}{J_0(h)}$ is decreasing strictly from 0 to $-\frac{1}{7}$.*

Proof. Monotonicity of $\frac{J_1(h)}{J_0(h)}$ on $(0, \frac{1}{6})$ is equivalent to $\{J_0(h), J_1(h)\}$ is extended complete Chebyshev system [47], i.e. any nontrivial linear combination $a_0J_0(h) + a_2J_1(h)$ has at most one zero on $(0, \frac{1}{6})$. Denote $f_0(\phi) = 1$, $f_1(\phi) = \phi$. Setting that

$$l_i(\phi) = \left(\frac{f_i}{\Psi}\right)(\phi) - \left(\frac{f_i}{\Psi}\right)(z(\phi)),$$

where $\Psi(\phi)=\frac{1}{2}\phi^2+\frac{1}{3}\phi^3$ and $z(\phi)$ is an involution function: $(-1, 0) \to (0, \frac{1}{2})$ by $\Psi(\phi) = \Psi(z(\phi))$. Since $\Psi(\phi) - \Psi(z(\phi)) = \frac{1}{6}(\phi - z)q(\phi, z)$, where $q(\phi, y) = 2\phi^2 + 2\phi z + 2z^2 + 3\phi + 3z$. We need to prove two Wronskians $W[l_0(\phi)]$, $W[l_0(\phi), l_1(\phi)]$ are non-vanishing on $(-1, 0)$. With aids of Maple, it has

$$W[l_0(\phi)] = -\frac{(\phi - z)(\phi + z + 1)}{\phi z(\phi + 1)(z + 1)},$$
$$W[l_0(\phi), l_1(\phi)] = \frac{(\phi - z)^3 w_0(\phi, z)}{\phi^2 z^2(\phi + 1)^2(z + 1)^2(2\phi + 4z + 3)},$$

where $w_0(\phi, z) = 4\phi^2 + 6\phi z + 4z^2 + 7\phi + 7z + 3$. It is not hard to verify that for $\phi \in (-1, 0), z \in (0, 1/2)$, it has $\phi + z + 1 \neq 0$, then $W[l_0(\phi)] \neq 0$. In order to check the zero of $W[l_0(\phi), l_1(\phi)]$, we calculate the resultants of $2\phi + 4z + 3$ and $q(\phi, z)$, $w_0(\phi, z)$ and $q(\phi, z)$ respect to z. It has

$$R(2\phi + 4z + 3, q, z) = 6(2\phi + 3)(2\phi - 1) \neq 0,$$
$$R(w_0(\phi, z), q, z) = 2(8\phi^4 + 16\phi^3 - 4\phi^2 - 12\phi + 9) \neq 0$$

for $\phi \in (0, 1)$, then $W[l_0(\phi), l_1(\phi)] \neq 0$. Therefore, $\frac{J_1(h)}{J_0(h)}$ is monotonic on $(0, \frac{1}{6})$. By Lemmas 3.2 and (3.3), the assertion in this lemma is proved. □

Remark 3.1. *Here, the Chebyshev system criterion is used to prove the monotonicity of the ratio of Abelian integrals, which is more effective and simpler then linear programming method used in the references [39–41].*

From (3.4), we know that for each $h \in (0, \frac{1}{6})$, when $c = c(h) = (2 + 2P(h))/P(h)$, it derives $M(h, c) = 0$. Moreover, the monotonicity of $P(h)$ implies that the zero of $P(h)$ is unique, denoted by h^*, and $c'(h) = -2/P(h)^2 < 0$, then from Lemma 3.4, we get

$$-12 < c(h) < \infty, \lim_{h\to\frac{1}{6}} c(h) = -12, \lim_{h\to 0} c(h) = +\infty.$$

Combining with implicit function theorem, there exists $c^* = c(h^*) + O(\varepsilon)$ such that $M(h, c) + O(\varepsilon)$ has a unique zero near h^*. Therefore, the conclusion in Theorem 2.1 is obtained.

4. The period of existence of periodic wave

For the sections before, our purpose is focusing on the analysis of the relationship between the speed of periodic wave solutions and the level h. Moreover, the property of the period is also significant in reality. On this purpose, it is needed to introduce some additional properties for $J_i(h)$ and $J_i'(h)$, $i = 0, 1, 2$.

From (2.9), we have

$$J_i(h) = \oint_{\Gamma_h} \phi^i y d\phi = 2\int_\alpha^\beta \phi^i \sqrt{-\frac{2}{3}\phi^3 - \phi^2 + 2h} d\phi = 2\int_\alpha^\beta \phi^i E(\phi) d\phi,$$

where $E(\phi) = \sqrt{-\frac{2}{3}\phi^3 - \phi^2 + 2h}$, $h \in (0, \frac{1}{6})$. The derivative of the Abelian integrals is

$$J_i'(h) = \oint_{\Gamma_h} \phi^i \frac{\partial y}{\partial h} d\phi = \oint_{\Gamma_h} \frac{\phi^i}{y} d\phi = 2\int_\alpha^\beta \frac{\phi^i}{E} d\phi.$$

Then $T(h)$ is a period of $\phi(z)$ satisfying $T(h) = 2\int_\alpha^\beta dz = \oint_{\Gamma_h} \frac{1}{y} d\phi = J_0'(h)$. Therefore, we have the following lemmas.

Lemma 4.1. $\begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \Lambda(h) \begin{pmatrix} J_0' \\ J_1' \end{pmatrix}$, *where* $\Lambda(h) = \frac{1}{35}\begin{pmatrix} 42h & 7 \\ -6h & 30h-6 \end{pmatrix}$.

Proof. Since $E\frac{dE}{d\phi} = -\phi^2 - \phi$, it has

$$\begin{aligned} J_0(h) &= 2\int_\alpha^\beta E d\phi = 2\int_\alpha^\beta E^2 \frac{d\phi}{E} \\ &= 2\int_\alpha^\beta \left(-\frac{2}{3}\phi^3 - \phi^2 + 2h\right)\frac{d\phi}{E} \\ &= 2\int_\alpha^\beta \frac{2}{3}\phi\left(\frac{EdE}{d\phi} + \phi\right)\frac{d\phi}{E} - 2\int_\alpha^\beta \phi^2 \frac{d\phi}{E} + 2h\oint_{\Gamma_h} \frac{d\phi}{E} \\ &= \frac{4}{3}\int_\alpha^\beta \phi dE - \frac{2}{3}\int_\alpha^\beta \phi^2 \frac{d\phi}{E} + 4h\int_\alpha^\beta \frac{d\phi}{E}. \end{aligned}$$

From integration by part, $\int_\alpha^\beta \phi dE = \phi E|_\alpha^\beta - \int_\alpha^\beta E d\phi = -\frac{1}{2}J_0$. Moreover,

$$\int_\alpha^\beta \phi^2 \frac{d\phi}{E} = \int_\alpha^\beta \left(-\frac{EdE}{d\phi} - \phi\right)\frac{d\phi}{E} = -\int_\alpha^\beta dE - \int_\alpha^\beta \phi\frac{d\phi}{E} = -\frac{1}{2}J_1'.$$

Thus,

$$J_0 = -\frac{2}{3}J_0 + \frac{1}{3}J_1' + 2hJ_0',$$

therefore, it obtains

$$J_0 = \frac{1}{5}(6hJ_0' + J_1').$$

Similarly, we get

$$\begin{aligned} J_1(h) &= 2\int_\alpha^\beta \phi E d\phi = 2\int_\alpha^\beta \phi E^2 \frac{d\phi}{E} \\ &= 2\int_\alpha^\beta \phi\left(-\frac{2}{3}\phi^3 - \phi^2 + 2h\right)\frac{d\phi}{E} \\ &= \int_\alpha^\beta \frac{4}{3}\phi^2\left(\frac{EdE}{d\phi} + \phi\right)\frac{d\phi}{E} - 2\int_\alpha^\beta \phi^3\frac{d\phi}{E} + 4h\int_\alpha^\beta \phi\frac{d\phi}{E} \\ &= \frac{4}{3}\int_\alpha^\beta \phi^2 dE - \frac{2}{3}\int_\alpha^\beta \phi^3\frac{d\phi}{E} + 4h\int_\alpha^\beta \phi\frac{d\phi}{E}. \end{aligned} \tag{4.1}$$

Moreover, since

$$\begin{aligned}
&\int_{\alpha}^{\beta}\phi^2 dE = \phi^2 E|_{\alpha}^{\beta} - 2\int_{\alpha}^{\beta}\phi E d\phi = -J_1,\\
&\int_{\alpha}^{\beta}\phi^3\frac{d\phi}{E} = \int_{\alpha}^{\beta}\phi\left(-\frac{EdE}{d\phi}-\phi\right)\frac{d\phi}{E} = -\int_{\alpha}^{\beta}\phi dE - \int_{\alpha}^{\beta}\phi^2\frac{d\phi}{E} = \frac{1}{2}J_0 + \frac{1}{2}J_1',
\end{aligned} \tag{4.2}$$

substitute (4.2) into (4.1), it obtains

$$J_1 = \frac{6}{35}\left[-hJ_0' + (5h-1)J_1'\right].$$

Therefore the proof of the lemma is completed. □

From Lemma 4.1, we obtain the following lemmas.

Lemma 4.2. *J_0 and J_1 satisfy the Picard-Fuchs equation*

$$\begin{pmatrix} J_0' \\ J_1' \end{pmatrix} = \frac{1}{h(6h-1)}\begin{pmatrix} 5h-1 & -\frac{7}{6} \\ h & 7h \end{pmatrix}\begin{pmatrix} J_0 \\ J_1 \end{pmatrix}.$$

Lemma 4.3. *J_0' and J_1' satisfy the Picard-Fuchs equation*

$$\begin{pmatrix} J_0'' \\ J_1'' \end{pmatrix} = \frac{1}{h(6h-1)}\begin{pmatrix} -h & -\frac{1}{6} \\ h & h \end{pmatrix}\begin{pmatrix} J_0' \\ J_1' \end{pmatrix}.$$

Proof. Denote that $J = (J_0, J_1)^T$. From lemmas 4.1 and 4.2, we have $J'' = \Lambda^{-1}(I - \Lambda')J'$, where I is the unit matrix, Λ' is $\Lambda(h)$ derivative with respect to h. It is not hard to verify that

$$\Lambda^{-1}(I-\Lambda)' = \frac{1}{h(6h-1)}\begin{pmatrix} 5h-1 & -\frac{7}{6} \\ h & h \end{pmatrix}\begin{pmatrix} -\frac{1}{5} & 0 \\ \frac{6}{35} & \frac{1}{7} \end{pmatrix} = \frac{1}{h(6h-1)}\begin{pmatrix} -h & -\frac{1}{6} \\ h & h \end{pmatrix}.$$

This proves the lemma. □

Lemma 4.4. $\lim_{h\to 1/6} T(h) = +\infty$, $\lim_{h\to 0} T(h) = 2\pi$. *Furthermore, For $h \in (0, 1/6)$, $T'(h) > 0$.*

Proof. Since $h = 1/6$ corresponds the homoclinic loop connecting to saddle $(-1, 0)$, it is not hard to verify the first statement. For the latter, by Lemma 4.2, it obtains

$$\begin{aligned}
\lim_{h\to 0} T(h) &= \lim_{h\to 0} J_0'(h) = \lim_{h\to 0}\frac{1}{h(6h-1)}[(5h-1)J_0 - \frac{7}{6}J_1]\\
&= \lim_{h\to 0}\frac{J_0}{h(6h-1)}[(5h-1) - \frac{7}{6}\frac{J_1}{J_0}]\\
&= \lim_{h\to 0}\frac{(5h-1)J_0}{h(6h-1)} = \frac{5}{6}\lim_{h\to 0}\frac{J_0}{h} = 2\pi.
\end{aligned}$$

From Lemmas 4.2 and 4.3, it obtains

$$\begin{aligned} T'(h) = J_0''(h) &= \frac{1}{h(6h-1)}[-hJ_0' - \frac{1}{6}J_1'] \\ &= \frac{1}{h^2(6h-1)^2}\left[-h((5h-1)J_0 - \frac{7}{6}J_1) - \frac{1}{6}(hJ_0 + 7hJ_1)\right] \\ &= \frac{-5}{h(6h-1)^2}(h - \frac{1}{6})J_0 > 0. \end{aligned}$$

Therefore we get the Theorem 2.2. □

Remark 4.1. *In this section, we show the properties of period for the uniqueness isolated periodic wave solution, which were not considered in the references [42, 43].*

5. Conclusions

This paper mainly proves the existence of unique periodic wave solution for perturbed MEW equation with weak backward diffusion, dissipation and Marangoni effect. By geometric singular perturbation theory, the local invariant submainfold is given, and then the singular perturbation is reduced into regular perturbation. We established the existence of periodic wave on for perturbed MEW equation by analyzing the monotonicity of the ratio of Abelian integrals. Chebyshev system criterion is utilized to prove the uniqueness of the periodic wave solution. Particularly, the related properties on the periodic wave are given by Picard-Fuchs equation.

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Conflict of interest

This work does not have any conflict of interest.

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