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Research article

An efficient two-level factored method for advection-dispersion problem with spatio-temporal coefficients and source terms

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Abstract: A two-level factored implicit scheme is considered for solving a two-dimensional unsteady advection-dispersion equation with spatio-temporal coefficients and source terms subjected to suitable initial and boundary conditions. The approach reduces multi-dimensional problems into pieces of one-dimensional subproblems and then solves tridiagonal systems of linear equations. The computational cost of the algorithm becomes cheaper and makes the method more attractive. Furthermore, the two-level approach is unconditionally stable, temporal second-order accurate and spatial fourth-order convergent. The developed numerical scheme is faster and more efficient than a broad range of methods widely studied in the literature for the considered initial-boundary value problem. The stability of the proposed procedure is analyzed in the $L^{\infty}(t_0, T_f; L^2)$-norm whereas the convergence rate of the algorithm is numerically analyzed using the $L^2(t_0, T_f; L^2)$-norm. Numerical examples are provided to verify the theoretical result.

Keywords: two-dimensional advection-dispersion equation; spatio-temporal coefficients; Crank-Nicolson approach; a two-level factored Crank-Nicolson method; stability and convergence rate
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1. Introduction and motivation

The two-dimensional nonstationary advection-dispersion equation is one of the popular and important models describing the contaminant transport in aquifers. The solute migration is subject to physical, chemical and biological activities such as: contaminant density, absorption and desorption, retardation, degradation and chemical-biological reactions. A general theory of dispersion of pollutants was developed in unsteady flow in heterogeneous aquifers [1]. Both temporal and spatial variations in groundwater velocity have been analyzed in [2, 3]. Advection-dispersion model is often used in several fields such as environmental sciences, groundwater hydrology, petroleum engineering, chemical engineering and biological sciences for predicting solute concentration. Furthermore, advection-dispersion can serve as a model equation for heat conduction, Burgers' equations, Shallow water problems, mixed Stokes-Darcy models and Navier-Stokes equations [4–15]. The development of efficient and accurate numerical approaches in approximate solutions for these equations is of great importance in the computational fluid dynamic community and has been analyzed by many authors [16–30]. Because of a wide set of applications of solute transport problems, a large class of numerical schemes have been discussed in approximate solutions. Concrete models are often approximated by advection-dispersion equation in a simple geometry (different geological formation, i.e., aquifer, aquitard, and etc.). In [31] the authors provided efficient solutions to transient advection-dispersion with spatio-temporal approximation. The obtained solutions lead to transient computed ones which are free of spurious oscillations and numerical diffusions for any values of Peclet number. The authors [32] discussed a broad range of finite element schemes in an efficient solution of the advection-dispersion. The numerical model of two-dimensional flow and transport equation was developed in simulating transient water flow and nonreactive solute transport in heterogeneous, unsaturated porous media containing air and water in [33]. It is established in [27] a numerical technique for solving the variable saturated solute transport equation that is free of oscillations and limits numerical dispersions. The authors [19, 34, 35] have developed an explicit scheme, implicit method and predictor-corrector procedure to solve the two-dimensional solute transport through a clay membrane barrier. In the analysis, the first order spatial derivatives are approximated by the fourth-order accurate finite difference representation. Efficient computed solutions of contaminant transport in heterogeneous aquifers which arises from the numerical treatment of both convective and cross-dispersive terms of the advection-dispersion equation have been deeply studied in [18, 36–38]. Although some methods mentioned above are fast, temporal second-order convergent and spatial fourth-order accurate, the theoretical analysis has not been considered. Explicit and predictor-corrector finite difference formulations require a suitable time-step restriction to maintain the stability of the algorithm while fully implicit approaches provide a substantial amount of computations at each time level. To overcome this drawback, a two-level factored Crank-Nicolson technique is proposed to solve the two-dimensional advection-dispersion equation with spatio-temporal coefficients and source terms in an efficient manner. The method consists of reducing a multidimensional problem into a set of one-dimensional subproblems which are easily solvable. Solving each subproblem is equivalent to finding the solution of a tridiagonal linear system of equations, which can be easily obtained by applying the Thomas technique. This considerably reduces the computational cost of the algorithm at each calculating time. Furthermore, the constructed approach is unconditionally stable, second-order convergent in time, spatial fourth-order accurate and it is easy to implement than a broad range of numerical methods applied to the considered initial-boundary value problems (2.1)–(2.3). For more details, we refer the readers to [19, 27, 31–34, 36, 39–41].

We recall that the aim of this study is to analyze an efficient computed solution of the initial-boundary value problems (2.1)–(2.3). Specifically, the analysis considers the following three items:

i) Mathematical formulation and full description of the two-level factored technique for solving the unsteady advection-diffusion equation with spatio-temporal coefficients and source term (2.1) subjects to initial-boundary conditions (2.2) and (2.3).

ii) Stability analysis of the numerical approach.

iii) A wide set of numerical examples which confirm the theoretical result.

In the remainder of this paper, we proceed as follows: Section 2 deals with the mathematical formulation of the considered model together with a detailed description of the new method for solving the system of Eqs (2.1)–(2.3). The unconditional stability of the two-level factored Crank-Nicolson formulation is established in Section 3, using the Von Neumann stability approach. We present and discuss in Section 4 a broad range of numerical evidence to confirm the theory (stability and convergence rate). Section 5 considers the general conclusion and presents our future works.

2. Mathematical formulation and description of the three-level factored Crank-Nicolson method

This section considers the mathematical formulation of the two-dimensional unsteady advection-dispersion equation with spatio-temporal dispersion coefficients with source term together with a detailed description of the two-level factored Crank-Nicolson formulation for solving the proposed model.

Consider the solute invades the groundwater level from the point source. The contaminant being of a significantly higher density than the groundwater moves towards the bottom of the shallow aquifer along vertical downward from each point, the pollutant is bound to spread in the horizontal plane along the unsteady porous media flow. For describing the two-dimensional hydrodynamic dispersion in homogeneous, isotropic porous media can be expressed as

$$\frac{\partial c}{\partial t} - \frac{\partial}{\partial x}\left(\widehat{D}_1\frac{\partial c}{\partial x} - \widehat{u}c\right) - \frac{\partial}{\partial y}\left(\widehat{D}_2\frac{\partial c}{\partial y} - \widehat{v}c\right) + \widehat{\mu}c - q = 0, \quad \text{on} \quad \Omega\times(t_0, T_f], \tag{2.1}$$

with initial condition

$$c(x, y, t_0) = \varphi_1(x, y), \quad \text{on} \quad \overline{\Omega}, \tag{2.2}$$

and boundary condition

$$c(x, y, t) = \varphi_2(x, y, t), \quad \text{on} \quad \Gamma\times(t_0, T_f], \tag{2.3}$$

where

- $c = c(x, y, t)$, is the solute concentration of the dispersing contaminant mass,
- $\widehat{u} = \widehat{u}(x, t)$ and $\widehat{v} = \widehat{v}(y, t)$, are called velocity components along the longitudinal direction (x-axis) and the lateral direction (y-axis), respectively,
- $\widehat{D}_1 = \widehat{D}_1(x, t)$ and $\widehat{D}_2 = \widehat{D}_2(y, t)$, denote the dispersion coefficients along the longitudinal direction and the lateral direction, respectively,
- $q = q(x, y, t)$ and $\widehat{\mu} = \widehat{\mu}(x, y, t)$, are the source of pollutant mass injected at a point of the infinite horizontal groundwater flow domain and first-order reaction rate, respectively,
- $\varphi_1 = \varphi_1(x, y)$, represents the initial condition. This indicates that the region is not solute free before the pollutant's source is injected into it,
- $\varphi_2 = \varphi_2(x, y, t)$, designates the boundary condition which suggests that $\Omega = (a_1, b_1)\times(a_2, b_2)$, where a_i and b_i ($i = 1, 2$) are real numbers,

- Γ denotes the boundary of Ω,
- t_0 and T_f, are the initial and final times, respectively.

In the literature [42–44], it is shown that: (a) All the coefficients may be reduced to constants, (b) dispersion coefficients are expressed in a homogeneous quadratic spatial form while velocity components consider the homogeneous linear spatial expression and (c) the dispersion coefficient may be time-dependent and velocity components temporally dependent or constants. In this work, we focus on the case where homogeneous quadratic and linear spatial expressions are considered along the longitudinal and lateral directions. Thus, the dispersion and velocity coefficients are defined as

$$\widehat{D}_1(x,t) = D_{x_0}(\alpha_2 + \alpha_1 x)^2 f_1(mt) \text{ and } \widehat{u}(x,t) = u_0(\alpha_2 + \alpha_1 x) f_2(mt), \tag{2.4}$$

and

$$\widehat{D}_2(y,t) = D_{y_0}(\beta_2 + \beta_1 y)^2 f_1(mt) \text{ and } \widehat{v}(y,t) = v_0(\beta_2 + \beta_1 y) f_2(mt), \tag{2.5}$$

where (D_{x_0}, D_{y_0}) and (u_0, v_0) are constant dispersion coefficients and velocity components (which are assumed to be nonnegative), respectively, in the corresponding directions in a steady flow domain through a homogeneous porous medium, $\alpha_1 > 0$ and $\beta_1 > 0$ denote the spatial dependent parameters along the x-axis and y-axis, respectively. Their significant or insignificant values represent the medium as homogeneous or heterogeneous. $m > 0$ is called the temporal dependence parameter which is chosen such that, the functions $f_i(mt)$, for $i = 1, 2, \widehat{\mu}(x, y, t)$ and $\frac{f_2(mt)}{f_1(mt)}$ are nonnegative, increasing in time and tend to 1 when m goes to zero, α_2 and β_2 are two positive constants. To ensure the nonnegativity of the convective terms, we assume that both functions $\frac{u_0}{2\alpha_1 D_{x_0}}\frac{f_2(mt)}{f_1(mt)}$ and $\frac{v_0}{2\beta_1 D_{y_0}}\frac{f_2(mt)}{f_1(mt)}$ are greater than or equal one. Armed with the above tools, we are ready to provide a detailed description of the two-level factored Crank-Nicolson approach for solving the initial-boundary value problems (2.1)–(2.3).

Let K, M and N be three positive integers. Set $k := \Delta t = \frac{T_f - t_0}{K}$; $h_x := \Delta x = \frac{b_1 - a_1}{M}$ and $h_y := \Delta y = \frac{b_2 - a_2}{N}$, be the time step and grid spacings, respectively. Set $t^n = t_0 + kn$, $n = 0, 1, 2, ..., K$; $x_i = a_1 + ih_x$, $i = 0, 1, \cdots, M$; and $y_j = a_2 + jh_y$, $0, 1, \cdots, N$. In addition, suppose $\Omega_k = \{t^n, 0 \le n \le K\}$; $\overline{\Omega}_h = \{(x_i, y_j), 0 \le i \le M,\ 0 \le j \le N\}$; $\Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$.

Let $C_h = \{c_{ij}^n,\ n = 0, 1, ..., K,\ 0 \le i \le M,\ 0 \le j \le N\}$, where $c_{ij}^n = c(x_i, y_j, t^n)$, be the space of grid functions defined on $\Omega_h \times \Omega_k$. We introduce the following operators

$$\delta_t c_{ij}^{n+1} = \frac{c_{ij}^{n+1} - c_{ij}^n}{k};\ \Delta_x c_{ij}^n = \frac{c_{i+1,j}^n - c_{ij}^n}{h_x};\ \nabla_x c_{ij}^n = \frac{c_{ij}^n - c_{i-1,j}^n}{h_x};\ \delta^x c_{ij}^n = \frac{c_{i+1,j}^n - c_{i-1,j}^n}{2h_x};\ \Delta_y c_{ij}^n = \frac{c_{i,j+1}^n - c_{ij}^n}{h_y};$$

$$\nabla_y c_{ij}^n = \frac{c_{ij}^n - c_{i,j-1}^n}{h_y};\ \delta^y c_{ij}^n = \frac{c_{i,j+1}^n - c_{i,j-1}^n}{2h_y};\ \delta_x^2 c_{ij}^n = \frac{\Delta_x c_{ij}^n - \nabla_x c_{ij}^n}{h_x} \text{ and } \delta_y^2 c_{ij}^n = \frac{\Delta_y c_{ij}^n - \nabla_y c_{ij}^n}{h_y}. \tag{2.6}$$

Using Eq (2.6), it is easy to see that $\delta^x c_{ij}^n = \frac{1}{2}\left(\Delta_x c_{ij}^n + \nabla_x c_{ij}^n\right)$, $\delta^y c_{ij}^n = \frac{1}{2}\left(\Delta_y c_{ij}^n + \nabla_y c_{ij}^n\right)$, $\delta_x^2 c_{ij}^n = \frac{c_{i+1,j}^n - 2c_{ij}^n + c_{i-1,j}^n}{h_x^2}$ and $\delta_y^2 c_{ij}^n = \frac{c_{i,j+1}^n - 2c_{ij}^n + c_{i,j-1}^n}{h_y^2}$. We define the following discrete norms

$$\|c^n\|_{L^2(\Omega)} = \left(h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N} |c_{ij}^n|^2\right)^{\frac{1}{2}} \text{ and } |||c|||_{L^\infty(t_0,T_f;L^2)} = \max_{1\le n\le K} \|c^n\|_{L^2(\Omega)}, \tag{2.7}$$

where $|\cdot|$ denotes the $\mathbb{C}$-norm. The spaces $L^2(\Omega)$ and $L^\infty(t_0, T_f; L^2(\Omega))$ are equipped with the norms $\|\cdot\|_{L^2}$ and $|||\cdot|||_{L^\infty(t_0,T_f;L^2)}$, respectively. We recall that a two-level factored Crank-Nicolson procedure consists to reducing problems in many space variables into a sequence of one-dimensional subproblems and then find the solution of linear systems with associated tridiagonal matrix. This considerably reduces the computational cost of the scheme.

For the convenience of writing, we should provide a simple expression of Eq (2.1) which will be considered in the following. By direct computations and rearranging terms, Eq (2.1) can be rewritten as

$$\frac{\partial c}{\partial t} = D_1\frac{\partial^2 c}{\partial x^2} + D_2\frac{\partial^2 c}{\partial y^2} - u\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y} - \mu c + q, \tag{2.8}$$

where

$$D_1 = \widehat{D}_1,\ D_2 = \widehat{D}_2,\ u = \widehat{u} - \frac{\partial \widehat{D}_1}{\partial x},\ v = \widehat{v} - \frac{\partial \widehat{D}_2}{\partial y} \text{ and } \mu = \widehat{\mu} + \frac{\partial \widehat{u}}{\partial x} + \frac{\partial \widehat{v}}{\partial y}. \tag{2.9}$$

The application of the Taylor series expansion for c about (x_i, y_j, t^n) with time step k using backward and forward differences gives

$$c_{ij}^n = c_{ij}^{n+1} - kc_{t,ij}^{n+1} + \frac{k^2}{2}c_{2t,ij}^{n+1} + O(k^3) \text{ and } c_{ij}^{n+1} = c_{ij}^n + kc_{t,ij}^n + \frac{k^2}{2}c_{2t,ij}^n + O(k^3), \tag{2.10}$$

where $c_t = \frac{\partial c}{\partial t}$ and $c_{2t} = \frac{\partial^2 c}{\partial t^2}$. Combining both equations in (2.10) and performing direct calculations, it is not hard to observe that

$$\frac{c_{ij}^{n+1} - c_{ij}^n}{k} = \frac{1}{2}\left(c_{t,ij}^{n+1} + c_{t,ij}^n\right) + O(k^2). \tag{2.11}$$

Utilizing Eq (2.8), direct computations result in

$$c_{t,ij}^n = D_{1,i}^n c_{2x,ij}^n + D_{2,j}^n c_{2y,ij}^n - u_i^n c_{x,ij}^n - v_j^n c_{y,ij}^n - \mu_{ij}^n c_{ij}^{n+1} + q_{ij}^n, \tag{2.12}$$

and

$$c_{t,ij}^{n+1} = D_{1,i}^{n+1} c_{2x,ij}^{n+1} + D_{2,j}^{n+1} c_{2y,ij}^{n+1} - u_i^{n+1} c_{x,ij}^{n+1} - v_j^{n+1} c_{y,ij}^{n+1} - \mu_{ij}^{n+1} c_{ij}^{n+1} + q_{ij}^{n+1}. \tag{2.13}$$

Expanding the Taylor series for c about (x_i, y_j, t^n) and (x_i, y_j, t^{n+1}) with space steps h_x and h_y, using central difference representations, this yields

$$c_{x,ij}^{n+1} = \delta^x c_{ij}^{n+1} + O(h_x^2);\ c_{x,ij}^n = \delta^x c_{ij}^n + O(h_x^2),\ c_{y,ij}^{n+1} = \delta^y c_{ij}^{n+1} + O(h_y^2),\ c_{y,ij}^n = \delta^y c_{ij}^n + O(h_y^2), \tag{2.14}$$

$$c_{2x,ij}^{n+1} = \delta_x^2 c_{ij}^{n+1} + O(h_x^2);\ c_{2x,ij}^n = \delta_x^2 c_{ij}^n + O(h_x^2),\ c_{y,ij}^{n+1} = \delta_y^2 c_{ij}^{n+1} + O(h_y^2),\ c_{2y,ij}^n = \delta_y^2 c_{ij}^n + O(h_y^2). \tag{2.15}$$

Substituting the second and fourth equations of (2.14) and (2.15) into relation (2.13) and the first and third equations of (2.14) and (2.15) into relation (2.12), it is easy to see that

$$c_{t,ij}^n = D_{1,i}^n \delta_x^2 c_{ij}^n + D_{2,j}^n \delta_y^2 c_{ij}^n - u_i^n \delta^x c_{ij}^n - v_j^n \delta^y c_{ij}^n - \mu_{ij}^n c_{ij}^n + q_{ij}^n + O(h_x^2 + h_y^2), \tag{2.16}$$

and

$$c_{t,ij}^{n+1} = D_{1,i}^{n+1} \delta_x^2 c_{ij}^{n+1} + D_{2,j}^{n+1} \delta_y^2 c_{ij}^{n+1} - u_i^{n+1} \delta^x c_{ij}^{n+1} - v_j^{n+1} \delta^y c_{ij}^{n+1} - \mu_{ij}^{n+1} c_{ij}^{n+1} + q_{ij}^{n+1} + O(h_x^2 + h_y^2). \tag{2.17}$$

Plugging Eqs (2.11), (2.16), (2.17) and rearranging terms, we obtain

$$\frac{c_{ij}^{n+1}-c_{ij}^{n}}{k}=\frac{1}{2}\Big\{D_{1,i}^{n+1}\delta_x^2c_{ij}^{n+1}+D_{1,i}^{n}\delta_x^2c_{ij}^{n}+D_{2,j}^{n+1}\delta_y^2c_{ij}^{n+1}+D_{2,j}^{n}\delta_y^2c_{ij}^{n}-u_i^{n+1}\delta^xc_{ij}^{n+1}-u_i^{n}\delta^xc_{ij}^{n}$$
$$-v_j^{n+1}\delta^yc_{ij}^{n+1}-v_j^{n}\delta^yc_{ij}^{n}-\mu_{ij}^{n+1}c_{ij}^{n+1}-\mu_{ij}^{n}c_{ij}^{n}+q_{ij}^{n+1}+q_{ij}^{n}\Big\}+O(k^2+h_x^2+h_y^2).$$

Solving this equation for c_{ij}^{n+1} provides

$$\Big\{\mathcal{J}-\frac{k}{2}\Big[D_{1,i}^{n+1}\delta_x^2+D_{2,j}^{n+1}\delta_y^2-u_i^{n+1}\delta^x-v_j^{n+1}\delta^y-\mu_{ij}^{n+1}\mathcal{J}\Big]\Big\}c_{ij}^{n+1}$$
$$=\Big\{\mathcal{J}+\frac{k}{2}\Big[D_{1,i}^{n}\delta_x^2+D_{2,j}^{n}\delta_y^2-u_i^{n}\delta^x-v_j^{n}\delta^y-\mu_{ij}^{n}\mathcal{J}\Big]\Big\}c_{ij}^{n}+\frac{k}{2}(q_{ij}^{n+1}+q_{ij}^{n})+O(k^3+kh_x^2+kh_y^2), \quad (2.18)$$

where $\mathcal{J}$ denotes the identity operator. Since $(1-a)(1-b)=1-a-b+ab$, for any real numbers a and b, a factored expression is obtained by adding the following term

$$\frac{k^2}{4}\Big[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]\Big[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]c_{ij}^{n+1},$$

to both sides of (2.18) and by manipulating the right hand side of the new equation. This fact allows to write

$$\Big\{\mathcal{J}-\frac{k}{2}\Big[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]\Big\}\Big\{\mathcal{J}-\frac{k}{2}\Big[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]\Big\}c_{ij}^{n+1}$$
$$=\Big\{\mathcal{J}+\frac{k}{2}\Big[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\Big]\Big\}\Big\{\mathcal{J}+\frac{k}{2}\Big[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\Big]\Big\}c_{ij}^{n}+\frac{k}{2}(q_{ij}^{n+1}+q_{ij}^{n})+\xi_{ij}^{n}, \quad (2.19)$$

where ξ_{ij}^{n} is the error term which is given by

$$\xi_{ij}^{n}=\frac{k^2}{4}\Big\{\Big[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]\Big[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\Big]c_{ij}^{n+1}$$
$$-\Big[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\Big]\Big[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\Big]c_{ij}^{n}\Big\}+O(k^3+kh_x^2+kh_y^2). \quad (2.20)$$

Tracking the truncation error $O(k^3+kh_x^2+kh_y^2)$ in Eq (2.18) and replacing the exact solution c_{ij}^{n} with the computed one C_{ij}^{n}, it follows a one-step linearized implicit scheme defined as

$$\Big\{\mathcal{J}-\frac{k}{2}\Big[D_{1,i}^{n+1}\delta_x^2+D_{2,j}^{n+1}\delta_y^2-u_i^{n+1}\delta^x-v_j^{n+1}\delta^y-\mu_{ij}^{n+1}\mathcal{J}\Big]\Big\}C_{ij}^{n+1}$$
$$=\Big\{\mathcal{J}+\frac{k}{2}\Big[D_{1,i}^{n}\delta_x^2+D_{2,j}^{n}\delta_y^2-u_i^{n}\delta^x-v_j^{n}\delta^y-\mu_{ij}^{n}\mathcal{J}\Big]\Big\}C_{ij}^{n}+\frac{k}{2}(q_{ij}^{n+1}+q_{ij}^{n}). \quad (2.21)$$

In addition, using relation (2.19) a two-step linearized equation can be constructed as follows

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}c_{ij}^*$$

$$=\left\{\mathcal{J}+\frac{k}{2}\left[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}\left\{\mathcal{J}+\frac{k}{2}\left[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}c_{ij}^n+\frac{k}{2}(q_{ij}^{n+1}+q_{ij}^n)+\xi_{ij}^n, \quad (2.22)$$

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}c_{ij}^{n+1}=c_{ij}^*, \quad (2.23)$$

where the superscript asterisk denotes an intermediate value and ξ_{ij}^n is defined by Eq (2.20).

Many splitting methods in a numerical solution of the transport equations have been developed to advance the solution in time. The most popular of these techniques is the compact ADI methods and the three-level time-split MacCormack deeply studied in [15, 45]. Fully implicit schemes may be constructed in many different ways (see, for example, Eq (2.21)). The most common of these techniques is the Euler implicit formulation or Crank-Nicolson method. Although these approaches do not require a time step restriction for stability (unconditionally stable), they produce a large system of linear equations to be solved as efficiently as possible. For two-dimensional problems, this becomes a big challenge when calculating a numerical solution utilizing one-step implicit models. To overcome this difficulty, this work develops a two-level factored Crank-Nicolson procedure.

Omitting the error term ξ_{ij}^n in Eq (2.22) and combining the new equation with (2.23), we obtain the desired numerical algorithm. For $n = 0, 1, 2, \cdots, K-1$, $i = 1, 2, \cdots, M-1$, and $j = 1, 2, \cdots, N-1$,

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}C_{ij}^*$$

$$=\left\{\mathcal{J}+\frac{k}{2}\left[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}\left\{\mathcal{J}+\frac{k}{2}\left[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}C_{ij}^n+\frac{k}{2}(q_{ij}^{n+1}+q_{ij}^n), \quad (2.24)$$

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}C_{ij}^{n+1}=C_{ij}^*, \quad (2.25)$$

subjects to initial and boundary conditions,

$$C_{ij}^0=\varphi_{1,ij},\ C_{0j}^*=C_{0j}^{n+1}=\varphi_{2,0j}^{n+1},\ C_{Mj}^*=C_{Mj}^{n+1}=\varphi_{2,Mj}^{n+1},\ C_{i0}^*=C_{i0}^{n+1}=\varphi_{2,i0}^{n+1}, \text{and}\ C_{iN}^*=C_{iN}^{n+1}=\varphi_{2,iN}^{n+1}, \quad (2.26)$$

for $i = 0, 1, 2, \cdots, M$ and $j = 0, 1, 2, \cdots, N$. Relations (2.24)–(2.26) represent a two-level factored Crank-Nicolson approach.

Now, we introduce the following operators

$$\mathcal{P}_x^+=\mathcal{J}-\frac{k}{2}\left[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right],\quad \mathcal{P}_y^+=\mathcal{J}-\frac{k}{2}\left[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right],$$

$$\mathcal{P}_x^-=\mathcal{J}+\frac{k}{2}\left[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\ \text{and}\ \mathcal{P}_y^-=\mathcal{J}+\frac{k}{2}\left[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right], \quad (2.27)$$

which play a crucial role in the stability analysis of the proposed models (2.24)–(2.26).

It is worth mentioning that the two-level factored Crank-Nicolson algorithm deals with two stages, as specified in the difference equations (2.24) and (2.25). In each phase, both operators $\mathcal{P}_x^{\pm}$ and $\mathcal{P}_y^{\pm}$

calculate implicitly. Thus, the growth of the error cannot cause any instability in the algorithm. Finally, it comes from Eq (2.20) that the truncation error ψ_{ij}^n satisfies: $\psi_{ij}^n = O(k^2+h_x^4+h_y^4)$ (indeed, $kh_x^2 \leq k^2+h_x^4$ and $kh_y^2 \leq k^2+h_y^4$). Thus, the new approach is second order accurate in time and fourth order convergent in space.

In the following, we assume that the exact solution $c \in L^\infty(t_0, T_f; H^2(\Omega)) \cap H^1(t_0, T_f; L^2(\Omega))$, that is, there is a positive constant ϱ, independent of the time step k and the space steps h_x and h_y such that,

$$|||c|||_{L^\infty(t_0,T_f;H^2)} + |||c|||_{H^1(t_0,T_f;L^2)} \leq \varrho. \tag{2.28}$$

3. Unconditional stability of the two-level factored Crank-Nicolson procedure

We analyze the unconditional stability of the proposed approach (2.24)–(2.26) in an approximate solution of the two-dimensional nonstationary advection-dispersion equation with spatio-temporal coefficients and source terms (2.1). We assume that the boundary condition given by Eq (2.3) is accurate so that an algebraic criterion for the stability analysis of the proposed technique is satisfied by the amplification factor can be determined by applying the Fourier method to the difference equations (2.24) and (2.25). Following the Von Neumann criterion for the necessary condition of stability, we suppose that both analytical and numerical solutions c_{ij}^n and C_{ij}^n together with the error $e_{ij}^n = c_{ij}^n - C_{ij}^n$ can be expressed in the form of Fourier series

$$c_{ij}^n = \widetilde{c}^n \exp\widehat{i}(i\phi_x h_x + j\phi_y h_y), \quad C_{ij}^n = \widetilde{C}^n \exp\widehat{i}(i\phi_x h_x + j\phi_y h_y) \text{ and } e_{ij}^n = \widetilde{e}^n \exp\widehat{i}(i\phi_x h_x + j\phi_y h_y), \tag{3.1}$$

where $c_{ij}^n = c(x_i, y_j, t^n)$ and $C_{ij}^n = C(x_i, y_j, t^n)$ are the exact solutions of Eqs (2.22), (2.23), (2.24) and (2.25), respectively. Furthermore, $\widetilde{c}^n$, $\widetilde{C}^n$ and $\widetilde{e}^n = \widetilde{c}^n - \widetilde{C}^n$, are the amplitudes at time level n, $\widehat{i}$ denotes the imaginary unit, ϕ_x and ϕ_y are called the wave numbers in the x and y directions, respectively. The products $\phi_x h_x$ and $\phi_y h_y$ represent the phase angles.

Theorem 3.1. *(Unconditional stability of the proposed approach). Suppose c_{ij}^n and C_{ij}^n be the solutions provided by Eqs* (2.22), (2.23), (2.24) *and* (2.25), *respectively. Under the assumptions stated in page* 3, *the paragraph below Eq* (2.5) *(that is, the physical parameters: m, u_0, v_0, D_{x_0}, D_{y_0}, α_i and β_i ($i = 1, 2$) given in Eqs* (2.4) *and* (2.5) *are nonnegative, the functions $f_i(mt)$ ($i = 1, 2$) and $\widehat{\mu}(x, y, t)$ given in relations* (2.4) *and* (2.1), *respectively, and $\frac{f_2(mt)}{f_1(mt)}$ are nonnegative and time variable increasing and both functions $\frac{u_0}{2\alpha_1 D_{x_0}}\frac{f_2(mt)}{f_1(mt)}$ and $\frac{v_0}{2\beta_1 D_{y_0}}\frac{f_2(mt)}{f_1(mt)}$ are greater than or equal one), the two-level factored Crank-Nicolson approach* (2.24)–(2.26) *applied to the initial-boundary value problems* (2.1)–(2.3) *is unconditionally stable. That is,*

$$|||C|||_{L^\infty(t_0,T_f;L^2)} \leq C_\varrho, \tag{3.2}$$

where C_ϱ is a positive parameter which depends on ϱ but is independent of the time step k and grid sizes h_x and h_y.

The following result (namely Lemma 3.1) plays a crucial role in the proof of the stability analysis of the two-level factored Crank-Nicolson formulation given by Eqs (2.24)–(2.26).

Lemma 3.1. *Under the hypotheses of Theorem* 3.1*, the operators* $\mathcal{P}_x^{\pm}$ *and* $\mathcal{P}_y^{\pm}$ *defined in Eq* (2.27) *satisfy*

$$\frac{\left|\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))\right|^2}{\left|\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))\right|^2} \leq \frac{\left[1-\frac{k}{2}\left(4D_{1,i}^n\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^n\right)\right]^2+\frac{k^2}{4}\left[u_i^n\frac{\sin(\phi_x h_x)}{h_x}\right]^2}{\left[1+\frac{k}{2}\left(4D_{1,i}^n\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^n\right)\right]^2+\frac{k^2}{4}\left[u_i^n\frac{\sin(\phi_x h_x)}{h_x}\right]^2} \leq 1, \tag{3.3}$$

and

$$\frac{\left|\mathcal{P}_y^-(\exp(\widehat{ii}\phi_y h_y))\right|^2}{\left|\mathcal{P}_y^+(\exp(\widehat{ii}\phi_y h_y))\right|^2} \leq \frac{\left[1-\frac{k}{2}\left(4D_{2,j}^n\frac{\sin^2(\phi_y h_y/2)}{h_y^2}+\frac{1}{2}\mu_{ij}^n\right)\right]^2+\frac{k^2}{4}\left[v_j^n\frac{\sin(\phi_y h_y)}{h_y}\right]^2}{\left[1+\frac{k}{2}\left(4D_{2,y}^n\frac{\sin^2(\phi_y h_y/2)}{h_y^2}+\frac{1}{2}\mu_{ij}^n\right)\right]^2+\frac{k^2}{4}\left[v_j^n\frac{\sin(\phi_y h_y)}{h_y}\right]^2} \leq 1. \tag{3.4}$$

Proof. (Of Lemma 3.1). For the sake of convenience, we should prove only estimate (3.3). The proof of inequality (3.4) is similar.

Utilizing both operators δ^x and δ_x^2 (respectively, $\mathcal{P}_x^{\pm}$) defined in relation (2.6) (respectively, Eq (2.27)), it holds

$$\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x)) = \Bigg\{\exp(\widehat{ii}\phi_x h_x)+\frac{k}{2}\Bigg[D_{1,i}^n\frac{\exp(\widehat{i}(i+1)\phi_x h_x)-2\exp(\widehat{ii}\phi_x h_x)+\exp(\widehat{i}(i-1)\phi_x h_x)}{h_x^2}$$

$$-u_i^n\frac{\exp(\widehat{i}(i+1)\phi_x h_x)-\exp(\widehat{i}(i-1)\phi_x h_x)}{2h_x}-\frac{1}{2}\mu_{ij}^n\exp(\widehat{ii}\phi_x h_x)\Bigg]\Bigg\}$$

$$=\left\{1+\frac{k}{2}\left[D_{1,i}^n\frac{\exp(\widehat{i}\phi_x h_x)-2+\exp(-\widehat{i}\phi_x h_x)}{h_x^2}-u_i^n\frac{\exp(\widehat{i}\phi_x h_x)-\exp(-\widehat{i}\phi_x h_x)}{2h_x}-\frac{1}{2}\mu_{ij}^n\right]\right\}\exp(\widehat{ii}\phi_x h_x). \tag{3.5}$$

But, it is easy to see that $\exp(\widehat{i}\phi_x h_x)-2+\exp(-\widehat{i}\phi_x h_x) = 2\cos(\phi_x h_x)-2 = -4\sin^2(\phi_x h_x/2)$ and $\exp(\widehat{i}\phi_x h_x)-\exp(-\widehat{i}\phi_x h_x) = 2\widehat{i}\sin(\phi_x h_x)$. A combination of this together with Eq (3.5) provides

$$\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x)) = \left\{1+\frac{k}{2}\left[-4D_{1,i}^n\frac{\sin^2(\phi_x h_x/2)}{h_x^2}-\widehat{i}u_i^n\frac{\sin(\phi_x h_x)}{h_x}-\frac{1}{2}\mu_{ij}^n\right]\right\}\exp(\widehat{ii}\phi_x h_x).$$

Squared modulus of both sides results in

$$\left|\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))\right|^2 = \left[1-\frac{k}{2}\left(4D_{1,i}^n\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^n\right)\right]^2+\frac{k^2}{4}\left[u_i^n\frac{\sin(\phi_x h_x)}{h_x}\right]^2. \tag{3.6}$$

In addition, it is not hard to see that

$$\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x)) = \Bigg\{\exp(\widehat{ii}\phi_x h_x)-\frac{k}{2}\Bigg[D_{1,i}^{n+1}\frac{\exp(\widehat{i}(i+1)\phi_x h_x)-2\exp(\widehat{ii}\phi_x h_x)+\exp(\widehat{i}(i-1)\phi_x h_x)}{h_x^2}$$

$$-u_i^{n+1}\frac{\exp(\widehat{i}(i+1)\phi_x h_x)-\exp(\widehat{i}(i-1)\phi_x h_x)}{2h_x}-\frac{1}{2}\mu_{ij}^{n+1}\exp(\widehat{ii}\phi_x h_x)\Bigg]\Bigg\}$$

$$=\left\{1+\frac{k}{2}\left[4D_{1,i}^{n+1}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\widehat{i}u_i^{n+1}\frac{\sin(\phi_x h_x)}{h_x}+\frac{1}{2}\mu_{ij}^{n+1}\right]\right\}\exp(\widehat{ii}\phi_x h_x).$$

The squared modulus gives

$$\left|\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))\right|^2 = \left[1 + \frac{k}{2}\left(4D_{1,i}^{n+1}\frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2}\mu_{ij}^{n+1}\right)\right]^2 + \frac{k^2}{4}\left[u_i^{n+1}\frac{\sin(\phi_x h_x)}{h_x}\right]^2. \quad (3.7)$$

To establish estimate (3.3), we should prove the following inequalities

$$\frac{k^2}{4}\left[u_i^n\frac{\sin(\phi_x h_x)}{h_x}\right]^2 \leq \frac{k^2}{4}\left[u_i^{n+1}\frac{\sin(\phi_x h_x)}{h_x}\right]^2, \quad (3.8)$$

and

$$\left[1 - \frac{k}{2}\left(4D_{1,i}^{n}\frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2}\mu_{ij}^{n}\right)\right]^2 \leq \left[1 + \frac{k}{2}\left(4D_{1,i}^{n+1}\frac{\sin^2(\phi_x h_x/2)}{h_x^2} + \frac{1}{2}\mu_{ij}^{n+1}\right)\right]^2. \quad (3.9)$$

Using Eqs (2.4) and (2.9), simple computations yield

$$D_{1,x}^n = D_{x_0}(\alpha_2+\alpha_1 x_i)^2 f_1(mt^n),\ D_{1,x}^{n+1} = D_{x_0}(\alpha_2+\alpha_1 x_i)^2 f_1(mt^{n+1}),\ \widehat{u}_i^n = u_0(\alpha_2+\alpha_1 x_i)f_2(mt^n),$$

$$\widehat{u}_i^{n+1} = u_0(\alpha_2+\alpha_1 x_i)f_2(mt^{n+1}),\ u_i^n = u_0(\alpha_2+\alpha_1 x_i)f_2(mt^n) - 2D_{x_0}\alpha_1(\alpha_2+\alpha_1 x_i)f_1(mt^n),$$

$$u_i^{n+1} = u_0(\alpha_2+\alpha_1 x_i)f_2(mt^{n+1}) - 2D_{x_0}\alpha_1(\alpha_2+\alpha_1 x_i)f_1(mt^{n+1}),\ \mu_{ij}^n = \widehat{\mu}_{ij}^n + (\alpha_1 u_0 + \beta_1 v_0)f_2(mt^n),$$

$$\mu_{ij}^{n+1} = \widehat{\mu}_{ij}^{n+1} + (\alpha_1 u_0 + \beta_1 v_0)f_2(mt^{n+1}). \quad (3.10)$$

Now, since the parameters α_i, are positive and the functions $f_i(mt) \geq 0$, and $\frac{u_0}{2\alpha_1 D_{x_0}}\frac{f_2(mt)}{f_1(mt)} \geq 1$, are increasing in time variable, using Eq (3.10) it is easy to see that

$$\frac{\left[u_i^n\frac{\sin(\phi_x h_x)}{h_x}\right]^2}{\left[u_i^{n+1}\frac{\sin(\phi_x h_x)}{h_x}\right]^2} = \frac{(u_i^n)^2}{(u_i^{n+1})^2} = \frac{[u_0(\alpha_2+\alpha_1 x_i)f_2(mt^n) - 2D_{x_0}\alpha_1(\alpha_2+\alpha_1 x_i)f_1(mt^n)]^2}{[u_0(\alpha_2+\alpha_1 x_i)f_2(mt^{n+1}) - 2D_{x_0}\alpha_1(\alpha_2+\alpha_1 x_i)f_1(mt^{n+1})]^2}$$

$$= \frac{(f_1(mt^n))^2}{(f_1(mt^{n+1}))^2}\frac{\left[\frac{u_0 f_2(mt^n)}{2D_{x_0}\alpha_1 f_1(mt^n)} - 1\right]^2}{\left[\frac{u_0 f_2(mt^{n+1})}{2D_{x_0}\alpha_1 f_1(mt^{n+1})} - 1\right]^2} \leq 1.$$

The last estimate comes from both inequalities: $0 \leq f_1(mt^n) \leq f_1(mt^{n+1})$ and $1 \leq \frac{u_0 f_2(mt^n)}{2D_{x_0}\alpha_1 f_1(mt^n)} \leq \frac{u_0 f_2(mt^{n+1})}{2D_{x_0}\alpha_1 f_1(mt^{n+1})}$. This completes the proof of estimate (3.8).

Now, let's prove estimate (3.9). Since the functions $f_1(mt)$, $f_2(mt)$, $\widehat{\mu}(x,y,t)$ are increasing in time variable and the parameters D_{x_0}, D_{y_0}, u_0, v_0, α_i, $i = 1,2$, and β_1 are nonnegative, utilizing relation (3.10), it is not hard to observe that

$$D_{1,i}^n = D_{x_0}(\alpha_2+\alpha_1 x_i)^2 f_1(mt^n) \leq D_{x_0}(\alpha_2+\alpha_1 x_i)^2 f_1(mt^{n+1}) = D_{1,i}^{n+1}, \quad (3.11)$$

and

$$\mu_{ij}^n = \widehat{\mu}_{ij}^n + \alpha_1 u_0 f_2(mt^n) + \beta_1 v_0 f_2(mt^n) \leq \widehat{\mu}_{ij}^{n+1} + \alpha_1 u_0 f_2(mt^{n+1}) + \beta_1 v_0 f_2(mt^{n+1}) = \mu_{ij}^{n+1}. \quad (3.12)$$

Plugging estimates (3.11) and (3.12), direct calculations provide

$$\left[1+\frac{k}{2}\left(4D_{1,i}^{n}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n}\right)\right]^2 \leq \left[1+\frac{k}{2}\left(4D_{1,i}^{n+1}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n+1}\right)\right]^2, \tag{3.13}$$

and

$$\left[1-\frac{k}{2}\left(4D_{1,i}^{n}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n}\right)\right]^2 \leq \left[1+\frac{k}{2}\left(4D_{1,i}^{n}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n}\right)\right]^2. \tag{3.14}$$

A combination of (3.13) and (3.14) results in

$$\left[1-\frac{k}{2}\left(4D_{1,i}^{n}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n}\right)\right]^2 \leq \left[1+\frac{k}{2}\left(4D_{1,i}^{n+1}\frac{\sin^2(\phi_x h_x/2)}{h_x^2}+\frac{1}{2}\mu_{ij}^{n+1}\right)\right]^2.$$

This ends the proof of inequality (3.9). Summing side by side estimates (3.8) and (3.9) and dividing the obtained inequality by the right-hand side to get estimate (3.3). □

Proof. (Of Theorem 3.1).

Subtracting the difference equation (2.24) from Eq (2.22) and approximation (2.25) from (2.23) provide

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}e_{ij}^{*}=\left\{\mathcal{J}+\frac{k}{2}\left[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}\left\{\mathcal{J}+\frac{k}{2}\left[D_{2,j}^{n}\delta_y^2\right.\right.$$

$$\left.\left.-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}e_{ij}^{n}+\xi_{ij}^{n}, \tag{3.15}$$

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}e_{ij}^{n+1}=e_{ij}^{*}, \tag{3.16}$$

where the predicted error term e_{ij}^{*} is defined as $e_{ij}^{*}=c_{ij}^{*}-C_{ij}^{*}$. Now, substituting Eq (3.16) into (3.15), it is not difficult to observe that

$$\left\{\mathcal{J}-\frac{k}{2}\left[D_{1,i}^{n+1}\delta_x^2-u_i^{n+1}\delta^x-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}\left\{\mathcal{J}-\frac{k}{2}\left[D_{2,j}^{n+1}\delta_y^2-v_j^{n+1}\delta^y-\frac{1}{2}\mu_{ij}^{n+1}\mathcal{J}\right]\right\}e_{ij}^{n+1}$$

$$=\left\{\mathcal{J}+\frac{k}{2}\left[D_{1,i}^{n}\delta_x^2-u_i^{n}\delta^x-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}\left\{\mathcal{J}+\frac{k}{2}\left[D_{2,j}^{n}\delta_y^2-v_j^{n}\delta^y-\frac{1}{2}\mu_{ij}^{n}\mathcal{J}\right]\right\}e_{ij}^{n}+\xi_{ij}^{n}. \tag{3.17}$$

Using relation (2.27), Eq (3.17) becomes

$$\mathcal{P}_x^{+}\mathcal{P}_y^{+}(e_{ij}^{n+1})=\mathcal{P}_x^{-}\mathcal{P}_y^{-}(e_{ij}^{n})+\xi_{ij}^{n}, \tag{3.18}$$

where ξ_{ij}^{n} is defined by (2.20). Utilizing the last equation in (3.1), it is not hard to see that

$$e_{ij}^{n}=\widetilde{e}^{n}\exp\widehat{i}(i\phi_x h_x+j\phi_y h_y)=\widetilde{e}^{n}\exp(\widehat{i}i\phi_x h_x)\exp(\widehat{i}j\phi_y h_y),$$

and

$$e_{ij}^{n+1}=\widetilde{e}^{n+1}\exp\widehat{i}(i\phi_x h_x+j\phi_y h_y)=\widetilde{e}^{n+1}\exp(\widehat{i}i\phi_x h_x)\exp(\widehat{i}j\phi_y h_y).$$

This fact, together with Eq (3.18) give

$$\widetilde{e}^{n+1}\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))\mathcal{P}_y^+(\exp(\widehat{i}j\phi_y h_y)) = \widetilde{e}^n\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))\mathcal{P}_y^-(\exp(\widehat{i}j\phi_y h_y)) + \xi_{ij}^n.$$

Omitting the error term ξ_{ij}^n, this can be approximated as

$$\widetilde{e}^{n+1}\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))\mathcal{P}_y^+(\exp(\widehat{i}j\phi_y h_y)) = \widetilde{e}^n\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))\mathcal{P}_y^-(\exp(\widehat{i}j\phi_y h_y)),$$

which is equivalent to

$$\frac{\widetilde{e}^{n+1}}{\widetilde{e}^n} = \frac{\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))}{\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))}\frac{\mathcal{P}_y^-(\exp(\widehat{i}j\phi_y h_y))}{\mathcal{P}_y^+(\exp(\widehat{ii}\phi_x h_x))}. \tag{3.19}$$

We remind that relation (3.19) defines the amplification factor provided by the numerical method (2.24)–(2.26). To show the unconditional stability of the proposed approach, we must prove that the squared modulus of the amplification factor given by (3.19) is less than or equal 1.

Taking the squared modulus in both sides of Eq (3.19), we get

$$\left|\frac{\widetilde{e}^{n+1}}{\widetilde{e}^n}\right|^2 = \left|\frac{\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))}{\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))}\right|^2\left|\frac{\mathcal{P}_y^-(\exp(\widehat{i}j\phi_y h_y))}{\mathcal{P}_y^+(\exp(\widehat{ii}\phi_x h_x))}\right|^2. \tag{3.20}$$

But, it comes from estimates (3.3) and (3.4) of Lemma 3.1 that

$$\frac{\left|\mathcal{P}_x^-(\exp(\widehat{ii}\phi_x h_x))\right|^2}{\left|\mathcal{P}_x^+(\exp(\widehat{ii}\phi_x h_x))\right|^2} \le 1 \;\text{ and }\; \frac{\left|\mathcal{P}_y^-(\exp(\widehat{ii}\phi_y h_y))\right|^2}{\left|\mathcal{P}_y^+(\exp(\widehat{ii}\phi_y h_y))\right|^2} \le 1. \tag{3.21}$$

Now, a combination of Eq (3.20) and estimates (3.21) results in

$$\frac{\left|\widetilde{e}^{n+1}\right|^2}{\left|\widetilde{e}^n\right|^2} \le 1,$$

which can be rewritten as

$$\left|\widetilde{e}^{n+1}\right| \le \left|\widetilde{e}^n\right|, \;\text{ for }\; n = 1, 2, \cdots, K-1.$$

By mathematical induction, it is not hard to see that, for $n = 1, 2, \cdots, K$

$$\left|\widetilde{e}^n\right| \le \left|\widetilde{e}^1\right|. \tag{3.22}$$

Utilizing the definition of L^2-norm given by (2.7) and Eq (3.1), simple computations give

$$\|e^n\|_{L^2(\Omega)} = \left(h_x h_y \sum_{i=1}^{M}\sum_{j=1}^{N}|e_{ij}^n|^2\right)^{\frac{1}{2}} = ((b_1-a_1)(b_2-a_2))^{\frac{1}{2}}\left|\widetilde{e}^n\right|. \tag{3.23}$$

Furthermore, it is easy to observe that $\|C^n\|_{L^2(\Omega)} - \|c^n\|_{L^2(\Omega)} \le \|c^n - C^n\|_{L^2(\Omega)} = \|e^n\|_{L^2(\Omega)}$. This fact, together with estimate (3.22), Eq (3.23) and inequality (2.28) yield

$$\|C^n\|_{L^2(\Omega)} \le \varrho + ((b_1-a_1)(b_2-a_2))^{\frac{1}{2}}\left|\widetilde{e}^1\right|, \;\text{ for }\; n = 1, 2, \cdots, K.$$

In fact, c is the analytical solution of the initial-boundary value problems (2.1)–(2.3), which satisfies estimate (2.28). Taking the maximum over n, this completes the proof of Theorem 3.1. □

4. Numerical experiments and convergence rate

In this section, we carry out numerical experiments to demonstrate the efficiency and effectiveness of the proposed two-level factored Crank-Nicolson scheme (2.24)–(2.26) applied to the initial-boundary value problems (2.1)–(2.3). Two examples are taken in [44] to confirm our theoretical statements. In each test, the results show that the considered approach provides satisfactory performances. The predicted convergence rate and unconditional stability from the theory are confirmed (Section 3, Theorem 3.1 and Section 2, Page 7, first paragraph, line 5) and Section 3, Theorem 3.1). Furthermore, both tables and graphs corresponding to the approximate solution (see Figures 1–8 and Tables 1 and 2) suggest that the proposed technique is unconditionally stable and convergent with order $O(k^2 + h^4)$. Specifically, the convergence rate of the two-level factored Crank-Nicolson method is obtained by listing the errors between the numerical solution and the analytical ones with different values of the mesh size $h = h_x = h_y$ and time step k. As indicated in [44], the dispersion and velocity coefficients are given by

$$\widehat{D}_1(x,t) = D_{x_0}(\alpha_2 + \alpha_1 x)^2 f_1(mt),\ \ \widehat{D}_2(y,t) = D_{y_0}(\beta_2 + \beta_1 y)^2 f_1(mt)\ \ , \widehat{u}(x,t) = u_0(\alpha_2 + \alpha_1 x) f_2(mt),$$

and

$$\widehat{v}(y,t) = v_0(\beta_2 + \beta_1 y) f_2(mt).$$

The source of pollutant mass injected $q(x,y,t)$ is given by

$$q(x,y,t) = C_0 \widehat{u}(x_0,t)\widehat{u}(y_0,t)\delta(x - x_0)\delta(y - y_0).$$

In this study, we take

$$\alpha_1 = \beta_1 \in \{3 \times 10^{-1}, 1\},\ \ \alpha_2 = \beta_2 \in \{1,2\},\ \ D_{x_0} = 2 \times 10^{-1},\ \ D_{y_0} = 2 \times 10^{-2},\ \ u_0 = 5 \times 10^{-1},$$

$$v_0 = 5 \times 10^{-2},\ \ x_0 = y_0 = 3,\ \ C_0 = 1,\ \ \Omega = (0,6) \times (0,6).$$

$f_1(mt)$, $f_2(mt)$ and $\widehat{\mu}(x,y,t)$ are functions defined as: $f_1(mt) = f_2(mt) \in \{1, \frac{t}{1+t}\}$ and $\widehat{\mu}(x,y,t) = 0$. The initial condition $\varphi_1(x,y)$ and the boundary one $\varphi_2(x,y,t)$, are directly obtained from the exact solution.

To verify the theoretical results, we take the space step and time step in the range $h \in \{2^{-r},\ r = 1,2,..,5\}$ and $k = 2^{-l}$, $l = 2,4,\cdots,10$, respectively. We calculate the norms of analytical solution $\|c\|_{L^2}$, approximate ones $\|C\|_{L^2}$ and error estimates, $\|E\|_{L^2}$ related to the proposed approach to see that the approach is unconditionally stable, spatial fourth-order accurate and temporal second-order convergent. Furthermore, for different values of k and h, we plot the exact and computed solutions together with the error versus n. This analysis indicates that the considered technique is faster and more efficient than a wide set of numerical schemes widely studied in the literature for solving the initial-boundary value problems (2.1)–(2.3). Finally, it follows from Tables 1 and 2 that the "$CR = \log_2(E(2h)/E(h))$" obtained from the approximation errors in two adjacent space-levels can be used to estimate the corresponding convergence rate with respect to h.

• Test 1 (Case: $f_1(mt) = f_2(mt) = 1$). In [44] the dispersion coefficients are defined as

$$\widehat{D}_1(x,t) = D_{x_0}(\alpha_2 + \alpha_1 x)^2 f_1(mt)\ \text{ and }\ \widehat{D}_2(y,t) = D_{y_0}(\beta_2 + \beta_1 y)^2 f_1(mt),$$

whereas the velocity coefficients are given by

$$\widehat{u}(x,t) = u_0(\alpha_2 + \alpha_1 x) f_2(mt) \text{ and } \widehat{v}(y,t) = v_0(\beta_2 + \beta_1 y) f_2(mt).$$

In this example, we take $\alpha_1 = \beta_1 = 3\times10^{-1}$, $\alpha_2 = \beta_2 = 1$, $(t_0, T_f) = (1,5)$ and $\Omega = (0,6)^2$. Furthermore, $\widehat{\mu}(x,y,t) = 0$ and the function $q(x,y,t)$ is given

$$q(x,y,t) = C_0 \widehat{u}(x_0,t)\widehat{u}(y_0,t)\delta(x-x_0)\delta(y-y_0),$$

where $\delta(\cdot)$ denotes the dirac function. The analytical solution c taken in [44] is given by

$$c(x,y,t) = \frac{\exp[-(\alpha_1 u_0 + \beta_1 v_0 + \widehat{\mu})t]}{4\pi t\sqrt{D_{x_0}D_{y_0}}(\alpha_1 x_0 + \alpha_2)(\beta_1 y_0 + \beta_2)} \exp\left[-\frac{1}{4D_{x_0}t}\left(\frac{1}{\alpha_1}\log\left(\frac{\alpha_1 x + \alpha_2}{\alpha_1 x_0 + \alpha_2}\right) + (\alpha_1 D_{x_0} - u_0)t\right)^2\right]$$

$$\times \exp\left[-\frac{1}{4D_{y_0}t}\left(\frac{1}{\beta_1}\log\left(\frac{\beta_1 y + \beta_2}{\beta_1 y_0 + \beta_2}\right) + (\beta_1 D_{y_0} - v_0)t\right)^2\right].$$

The initial condition φ_1 and the boundary one φ_2 are obtained from the exact solution c. We assume that the grid spacing h and time step k satisfy $k = h^2$.

Table 1. Analytical solution "c", numerical one "C", error "E" and convergence rates "$CR = \log_2(E(2h)/E(h))$" of the proposed algorithm with different mesh size h.

h	$\vert\vert\vert c\vert\vert\vert_{L^2}$	$\vert\vert\vert C\vert\vert\vert_{L^2}$	$\vert\vert\vert E(h)\vert\vert\vert_{L^2}$	RC
2^{-1}	5.3686×10^{-1}	6.0311×10^{-1}	1.0839×10^{0}	–
2^{-2}	3.1571×10^{-1}	2.6278×10^{-1}	1.2311×10^{-1}	3.1382
2^{-3}	2.0893×10^{-1}	1.9175×10^{-1}	9.5015×10^{-3}	3.6957
2^{-4}	1.2512×10^{-1}	1.1945×10^{-1}	5.6779×10^{-4}	4.0648
2^{-5}	7.0305×10^{-2}	6.7212×10^{-2}	3.0865×10^{-5}	4.2013

• Test 2 (Case: $f_1(mt) = f_2(mt) = \frac{t}{1+t}$). In this case, the dispersion and velocity coefficients are given by

$$\widehat{D}_1(x,t) = D_{x_0}(\alpha_2 + \alpha_1 x)^2 f_1(mt),\ \ \widehat{D}_2(y,t) = D_{y_0}(\beta_2 + \beta_1 y)^2 f_1(mt),\ \ \widehat{u}(x,t) = u_0(\alpha_2 + \alpha_1 x) f_2(mt),$$

and

$$\widehat{v}(y,t) = v_0(\beta_2 + \beta_1 y) f_2(mt).$$

As in Test 1, we set $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 2$, $(t_0, T_f) = (1,5)$ and $\Omega = (0,6)^2$. Furthermore, $\widehat{\mu}(x,y,t) = 0$ and the function $q(x,y,t)$ is defined as

$$q(x,y,t) = C_0 \widehat{u}(x_0,t)\widehat{u}(y_0,t)\delta(x-x_0)\delta(y-y_0),$$

where $\delta(\cdot)$ denotes the dirac function. The analytical solution c taken in [44] is given by

$$c(x,y,t) = \frac{\exp[-(\alpha_1 u_0 + \beta_1 v_0 + \widehat{\mu})t]}{4\pi\tau\sqrt{D_{x_0}D_{y_0}}(\alpha_1 x_0 + \alpha_2)(\beta_1 y_0 + \beta_2)} \exp\left[-\frac{1}{4D_{x_0}\tau}\left(\frac{1}{\alpha_1}\log\left(\frac{\alpha_1 x + \alpha_2}{\alpha_1 x_0 + \alpha_2}\right) + \alpha_1 D_{x_0}\tau - u_0 t\right)^2\right]$$

$$\times \exp\left[-\frac{1}{4D_{y_0}\tau}\left(\frac{1}{\beta_1}\log\left(\frac{\beta_1 y + \beta_2}{\beta_1 y_0 + \beta_2}\right) + \beta_1 D_{y_0}\tau - v_0 t\right)^2\right],$$

where $\tau = t - \ln(1+t)$. The initial and boundary conditions φ_1 and φ_2, respectively, are determined by the analytical solution c. Similar to Test 1, the time step k and space step h satisfy $k = h^2$.

Like in Test 1, the time step and mesh grid are chosen such that: $k = 2^{-l}$, $l = 2, 4, ..., 10$ and $h \in \{2^{-l}, l = 1, 2, ..., 5\}$. We list in Table 2 the approximate solution "C", the exact one "c" and error "E" related to a two-level factored Crank-Nicolson formulation to see that the proposed approach is convergent with accuracy $O(k^2 + h^4)$. Furthermore, we plot the exact solution and computed one together with the error versus n to see the efficiency of the developed method.

Table 2. Approximate solution "C", exact solution "c", error "E" and convergence rate "$CR = \log_2(E(2h)/E(h))$" for the proposed technique with $k = h^2$.

h	$\vert\vert\vert c\vert\vert\vert_{L^2}$	$\vert\vert\vert C\vert\vert\vert_{L^2}$	$\vert\vert\vert E(h)\vert\vert\vert_{L^2}$	RC
2^{-1}	6.8685×10^{-1}	7.3217×10^{-1}	6.6773×10^{-2}	–
2^{-2}	1.1299×10^{-2}	1.1623×10^{-2}	4.6232×10^{-3}	3.8523
2^{-3}	2.7992×10^{-3}	2.7987×10^{-3}	3.4042×10^{-4}	3.7635
2^{-4}	9.8520×10^{-4}	8.0479×10^{-4}	2.2246×10^{-5}	3.9357
2^{-5}	4.3567×10^{-4}	4.2106×10^{-4}	1.3807×10^{-6}	4.0101

The theoretical analysis provided in Section 3 (Theorem 3.1) and Section 2, has suggested that the proposed numerical scheme is unconditionally stable, temporal second-order accurate and spatial fourth-order convergent. We observe from Tests 1 and 2 that the expected results from the theory are confirmed. More precisely, Tables 1 and 2 show that the proposed two-level factored method is convergent with accuracy $O(k^2 + h^4)$.

Figures 1–8 indicate that the constructed two-level factored approach is unconditionally stable and convergent. This numerical result confirms the theoretical one discussed in Section 3 (Theorem 3.1).

Stability analysis and convergence of the new two-level factored scheme.

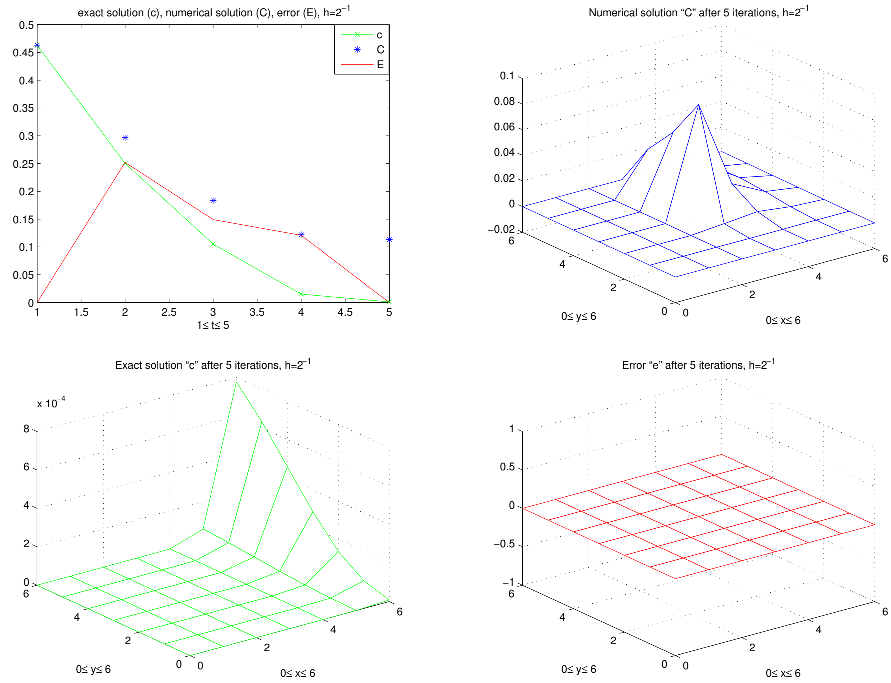

Figure 1. Test 1: decay constant $\mu = 0$, $f_1(mt) = f_2(mt) = 1$, $\alpha_1 = 3 \times 10^{-1}$ and $\alpha_2 = 1$.

Analysis of stability and convergence of the proposed two-level factored approach.

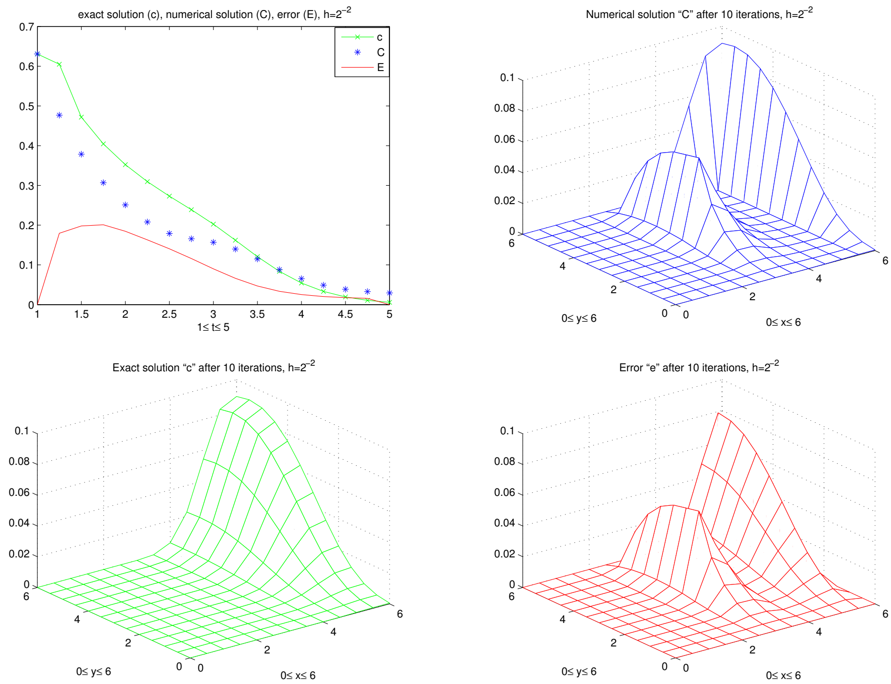

Figure 2. Test 1: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = 1$.

Stability analysis and convergence of the developed two-level factored numerical method.

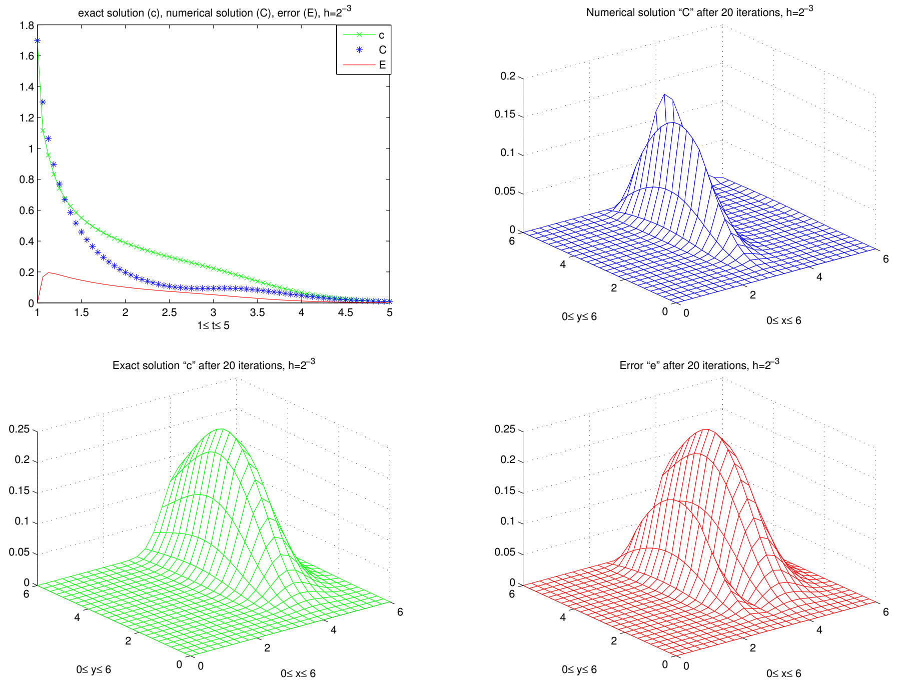

Figure 3. Test 1: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = 1$.

Analysis of stability and convergence of the constructed two-level factored scheme.

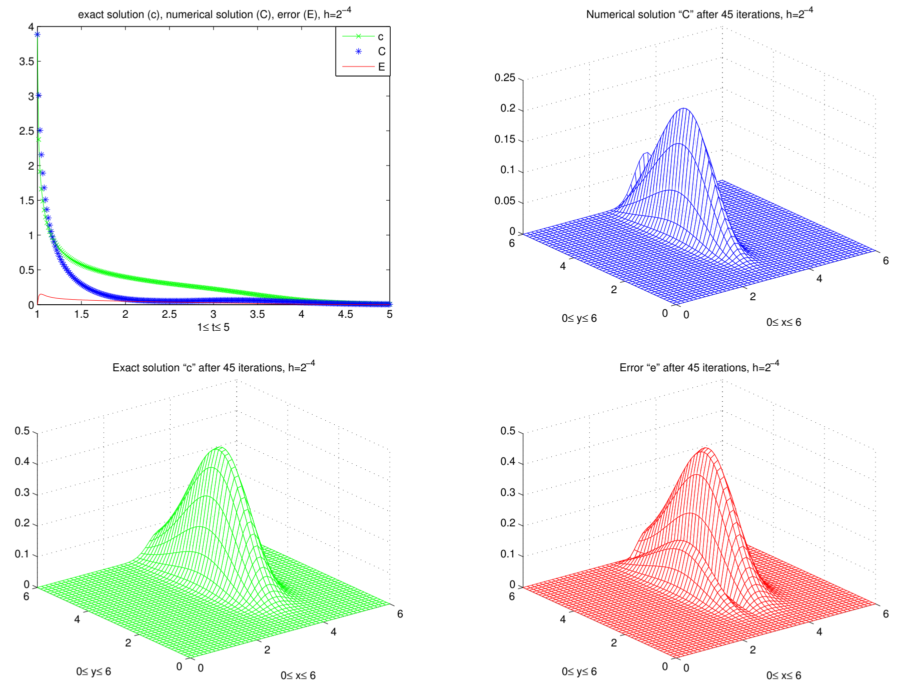

Figure 4. Test 1: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = 1$.

Stability analysis and convergence of the new algorithm.

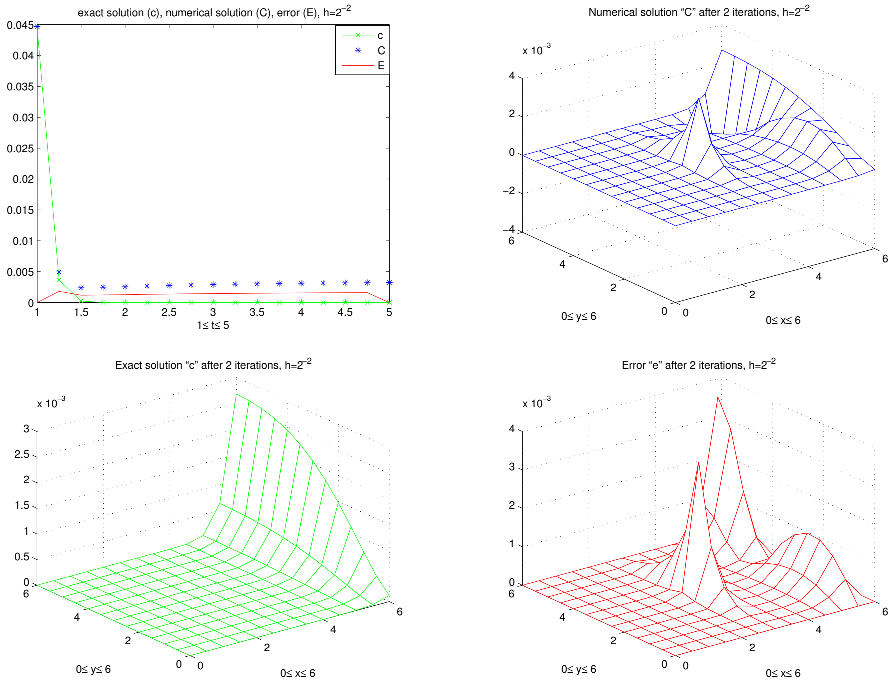

Figure 5. Test 2: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = \frac{t}{1+t}$.

Stability and convergence of a two-level factored Crank-Nicolson method.

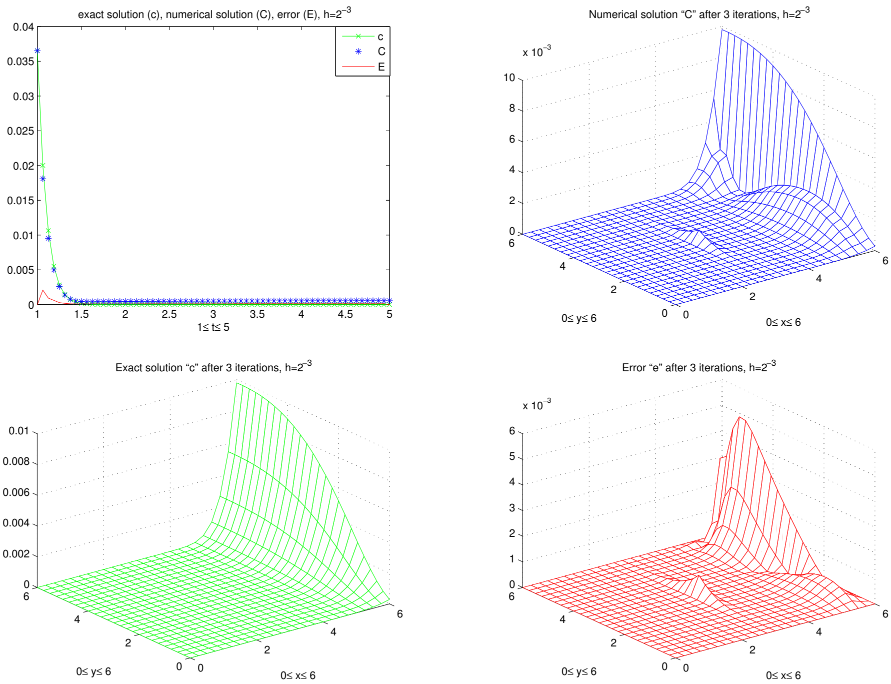

Figure 6. Test 2: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = \frac{t}{1+t}$.

Analysis of stability and convergence of the proposed two-level factored technique.

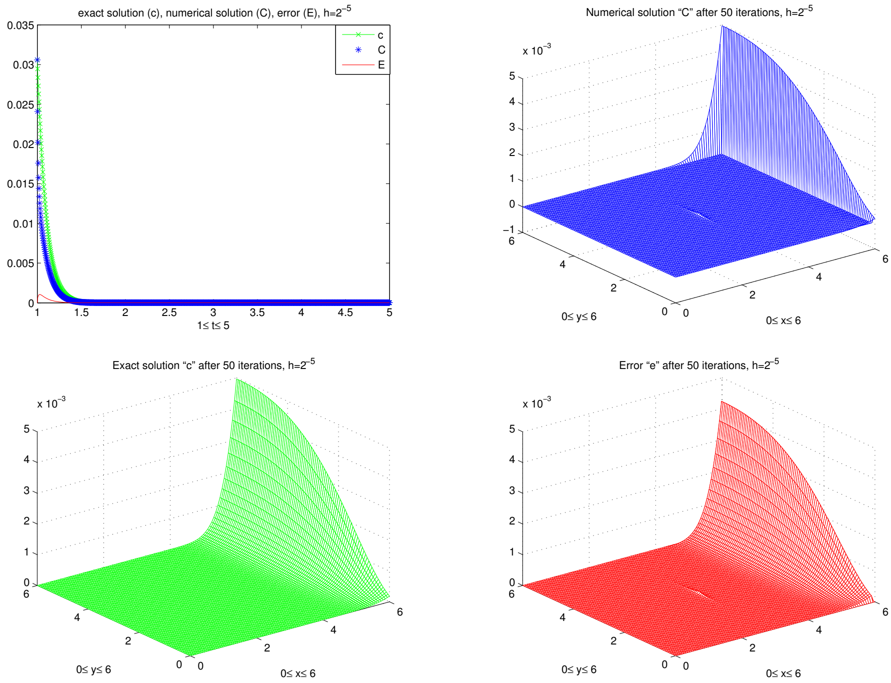

Figure 7. Test 2: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = \frac{t}{1+t}$.

Stability and convergence of the developed two-level factored approach.

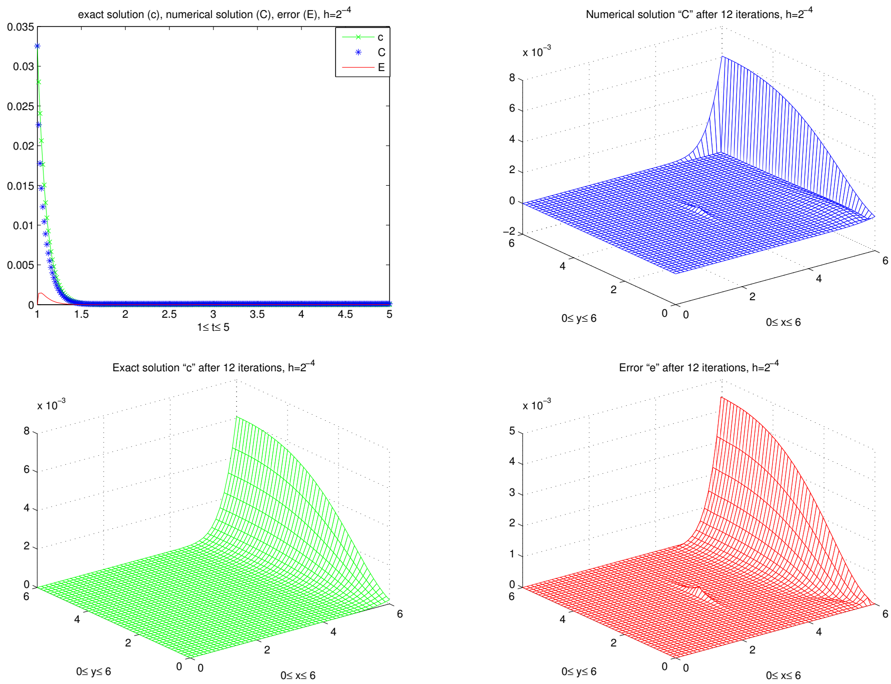

Figure 8. Test 2: decay constant $\mu = 0$ and $f_1(mt) = f_2(mt) = \frac{t}{1+t}$.

5. Conclusions and future works

In this paper, we have proposed a two-level factored numerical scheme to solve the two-dimensional evolutionary advection-dispersion equation with spatio-temporal dispersion coefficients and source terms (2.1) subjects to suitable initial and boundary conditions (2.2) and (2.3) and we have analyzed in detail the stability together with the convergence rate of the method. The theoretical study has shown that the proposed approach is unconditionally stable, temporal second-order accurate and spatial fourth-order convergent (Section 3, Theorem 3.1 and Section 2. This theory is confirmed by two numerical tests (see both Figures 1–8 and Tables 1 and 2). Numerical evidence also indicated that the new algorithm is: (a) More efficient and effective than a large set of numerical techniques [6, 9, 12, 19, 27, 41, 45, 46] applied to the initial-boundary value problems (2.1)–(2.3); (b) Fast and robust tools for the integration of general systems of PDEs. Moreover, the two-level factored formulation is an efficient scheme for solving from low to high Reynolds number flows where the viscous region is too thin by providing fewer computations at each calculation step. This substantially reduces the computational cost of the method. In addition, for multi-dimensional problems, the procedure reduces to solve a tridiagonal system of equations which should be easily obtained by the application of the Thomas algorithm. The future works will apply the two-level factored approach to two-dimensional time-fractional convection-diffusion equation with source terms.

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Conflict of interest

The author declares that he has no conflict of interest.

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