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Research article

Large time behavior of the Euler-Poisson system coupled to a magnetic field

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Abstract: In this paper, the large time behavior of globally smooth solutions of the Cauchy problem for the three dimensional Euler-Poisson system of compressible fluids coupled to a magnetic field is studied. We prove that the smooth solutions (near a given constant equilibrium state) of the problem converge asymptotically to a stationary solution exponentially fast as t goes to ∞.

Keywords: stability; asymptotic behavior of solutions; Euler-Poisson system; magnetic field
Mathematics Subject Classification: 35B35, 35B40, 35M10

1. Introduction and main results

In this paper, we are interested in the large time behavior of solutions to a three dimensional Euler-Poisson system of compressible fluids coupled to a magnetic field:

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.1}$$

$$\partial_t(nu) + \operatorname{div}(nu \otimes u) + \nabla p(n) = qn\nabla\phi + \operatorname{curl} B \times B - nu, \tag{1.2}$$

$$\partial_t B - \operatorname{curl}(u \times B) = \Delta B, \tag{1.3}$$

$$\operatorname{div} B = 0, \tag{1.4}$$

$$\Delta\phi = n - 1, \tag{1.5}$$

for $x \in \mathbb{T}^3$ and $t > 0$. In the above equations, $\mathbb{T}^3$ is 3-dimensional torus. The unknown functions (n, u, B, ϕ) represent the density, the velocity, the magnetic field, and the gravitational potential or electrical potential, respectively. Throughout this paper, we assume that the pressure function $p(n)$ satisfies the usual γ-law,

$$p(n) = \frac{n^\gamma}{\gamma}, \quad n > 0, \tag{1.6}$$

for some constant $\gamma > 1$. When $q = -1$, the system (1.1)–(1.5) is used to model the evolution of a magnetic stars [1, 2] And when $q = 1$, the system (1.1)–(1.5) is used to describe the dynamics of a plasma or semiconductor. It is obvious that equation (1.4) is redundant with equation (1.3), as soon as they are satisfied by the initial conditions $\operatorname{div} B_0 = 0$. In this paper, we consider the case of $q = 1$. Without taking magnetic effects into account, system (1.1)–(1.5) is reduced to the Euler-Poisson equations. The system (1.1)–(1.5) is supplemented with the initial conditions

$$n(x,0) = n_0(x), u(x,0) = u_0, B(x,0) = B_0, \quad x \in \mathbb{T}^3, \tag{1.7}$$

which satisfies the compatibility condition

$$\operatorname{div} B_0 = 0, \quad \Delta\phi_0 = n_0 - 1. \tag{1.8}$$

Without taking magnetic effects into account, system (1.1)–(1.5) reduces to the Euler-Poisson equations, whose asymptotic behavior has been studied by many authors in the literature. Specifically, this model and its large-time behavior have been studied for Cauchy problem or initial-boundary value problem in the case $B = 0$. Luo-Natalini-Xin [3] established the global existence of smooth solutions to the Cauchy problem for the one-dimensional isentropic Euler-Poisson (or hydrodynamic) model for semiconductors for small initial data and proved that, as $t \to \infty$, these solutions converge to the stationary solutions of the drift-diffusion equations. Hsiao-Yang [4] investigated the asymptotic behavior of smooth solutions of the Cauchy problem and the initial-boundary value problem of Euler-Poisson system. For the case of multi-dimensional system (1.1)–(1.5) with $B = 0$, Hsiao-Wang [5] established the global existence and asymptotic behavior of the spherically symmetrical solution and proved that the solution to the problem converges to a stationary solution asymptotically exponentially fast in time. The solution without symmetry was studied by Hsiao-Markowich-Wang in [6], where the asymptotic behavior of globally smooth solutions of the Cauchy problem was proved. Hsiao-Wang-Zhao [7] also obtain the corresponding result when the Poisson equation has the following form

$$\Delta\phi = n - b(x)$$

for two-dimensional space. Hsiao-Ju-Wang [8] extended the result of [7] to the case of d-dimensional space for $d = 2$ and $d = 3$. In [9], Wu-Tan-Wang investigated the global existence and asymptotic behavior of smooth solutions near a non-flat steady state to the compressible non-isentropic Euler-Poisson system by some concise energy estimates and an interpolation trick. We also mention that many mathematicians have made contributions to the large time behavior and global existence of smooth or weak solutions to the related models. See [10–15] and the references therein. More recently, the local well-posedness and quasi-neutral limit for the system (1.1)–(1.5) without magnetic diffusion has been investigated by Yang [16]. Our main goal here is to establish the global existence of smooth solutions around a constant state $(1, 0, 0, 0)$, which is a equilibrium solution of system (1.1)–(1.5), and the decay rate of the global smooth solutions in time for the system (1.1)–(1.5). However, due to the adjunction of magnetic effects, the proof to the global existence of smooth solutions and the decay rate of the global smooth solutions in time becomes more complex and difficult. Because of these effects, some key estimates in Ref. [6] have to be reconsidered, and our analysis depends heavily on the special nonlinear structure of the system (1.1)–(1.5). Our main result reads as follows:

Theorem 1.1. *Let* (1.8) *hold. Assume that* $n(\cdot,0)-1\in H^3(\mathbb{T}^3)$ *and* $\int_{\mathbb{T}^3} B_0 dx = 0$. *There exists* $\delta>0$ *such that if*

$$\|(n_0-1,u_0,\nabla\phi_0,B_0)\|_{H^3(\mathbb{T}^3)}+\|(n_t,u_t.\nabla\phi_t,B_t)(\cdot,0)\|_{H^2(\mathbb{T}^3)}\le\delta,$$

then the Cauchy problem (1.1)–(1.8) *admits a unique global smooth solution* (n,u,ϕ,B) *satisfying*

$$\begin{aligned}&\|(n-1,u,\nabla\phi,B)\|^2_{H^3(\mathbb{T}^3)}+\|(n_t,u_t.\nabla\phi_t,B_t)\|^2_{H^2(\mathbb{T}^3)}\\&\le C_0\left[\|(n_0-1,u_0,\nabla\phi_0,B_0)\|^2_{H^3(\mathbb{T}^3)}+\|(n_t,u_t.\nabla\phi_t,B_t)(\cdot,0)\|^2_{H^2(\mathbb{T}^3)}\right]e^{-\alpha t}\end{aligned}$$

for some positive constants α *and* C_0.

Before ending the introduction, we give some notations and vector analysis formulas used throughout the current paper. The letter C denote various positive constants, which can be different from one line to another one. The symbol ":"means summation over both matrix indices. $|U|$ denotes some norm of a vector or matrix U. Also, we denote

$$\|\cdot\|=\|\cdot\|_{L^2(\mathbb{T}^3)},\ \ \|\cdot\|_\infty=\|\cdot\|_{L^\infty(\mathbb{T}^3)},\ \ \|\cdot\|_k=\|\cdot\|_{H^k(\mathbb{T}^3)},\ \ k\in\mathbb{N}^*.$$

The following vector analysis formulas will be repeatedly used, see [17].

$$\operatorname{curl}(a\times b)=a\operatorname{div}b-b\operatorname{div}a+(b\cdot\nabla)a-(a\cdot\nabla)b, \tag{1.9}$$

$$\operatorname{div}(a\times b)=b\cdot\operatorname{curl}a-a\cdot\operatorname{curl}b, \tag{1.10}$$

$$\operatorname{curl}\operatorname{curl}a=\Delta a+\nabla\operatorname{div}a, \tag{1.11}$$

$$\operatorname{curl}(fa)=\nabla f\times\operatorname{curl}a+f\operatorname{curl}a, \tag{1.12}$$

$$(a\cdot\nabla)b=\operatorname{curl}a\times b+\nabla(a\cdot b)+\nabla a\cdot b,\quad \nabla a\cdot b=\sum_{j=1}^{3}a_jb_j, \tag{1.13}$$

$$\operatorname{curl}a\times a=(a\cdot\nabla)a-\frac{1}{2}\nabla|a|^2, \tag{1.14}$$

where, a and b are vector functions, f is a scalar function.

2. Proof of Theorem 1.1

Assume that $(n,u,\nabla\phi,B)$ is a smooth solution to the Cauchy problem of the system (1.1)–(1.5) with initial data (1.7) which satisfies (1.8). We introduce the transformation

$$n(x,t)=1+\rho(x,t). \tag{2.1}$$

Then, the system (1.1)–(1.5) becomes

$$\partial_t\rho+\operatorname{div}((1+\rho)u)=0, \tag{2.2}$$

$$\partial_t u+(u\cdot\nabla)u+\nabla h(1+\rho)=\nabla\phi+\frac{1}{1+\rho}(\operatorname{curl}B\times B)-u, \tag{2.3}$$

$$\partial_t B-\operatorname{curl}(u\times B)=\Delta B, \tag{2.4}$$

$$\mathrm{div}B = 0, \tag{2.5}$$

$$\Delta\phi = \rho, \tag{2.6}$$

with the initial conditions

$$\rho(x,0) = \rho_0 = n_0 - 1, u(x,0) = u_0, B(x,0) = B_0 \tag{2.7}$$

and the compatible condition

$$\mathrm{div}B_0 = 0, \quad \Delta\phi_0 = \rho_0, \tag{2.8}$$

where $h(s)$ is defined by

$$h'(s) = p'(s)/s.$$

To prove Theorem 1.1, we need the existence of the local solution and the a priori estimates. The former is given by Lemma 2.1 whose proof can be found in [16], and the latter is proved in Lemma 2.2.

Lemma 2.1. *Assume that the initial conditions*

$$\rho_0 \in H^3(\mathbb{T}^3), \quad u_0 \in H^3(\mathbb{T}^3), \quad B_0 \in H^3(\mathbb{T}^3)$$

be satisfied. Then there exists a unique smooth solution $(\rho, u, \nabla\phi, B)$ *of* (2.2)–(2.8) *satisfying*

$$\rho, u, \nabla\phi, B \in C^1([0, T_{\max}), H^2(\mathbb{T}^3)) \cap C([0, T_{\max}), H^3(\mathbb{T}^3))$$

defined on a maximal interval of existence $[0, T_{\max})$*. Furthermore, if* $T_{\max} < +\infty$*, then*

$$\begin{aligned}&\|(\rho, u, \nabla\phi, B)\|_3^2 + \|(\rho_t, u_t, \nabla\phi_t, B_t)\|_2^2 \\ &+ \int_0^t (\|(\rho, u, \nabla\phi, B)\|_3^2 + \|(\rho_s, u_s, \nabla\phi_s, B_s)\|_2^2) ds \to +\infty\end{aligned}$$

as $t \to T_{\max}$.

Lemma 2.2. *There exist positive constants* δ_1, α_1 *and* C_1 *such that, for any* $T > 0$*, if*

$$\sup_{t\in[0,T]} (\|(\rho, u, \nabla\phi, B)\|_3 + \|(\rho_t, u_t, \nabla\phi_t, B_t)\|_2) \le \delta_1, \tag{2.9}$$

then for any $T > 0$*,*

$$\begin{aligned}&\|(\rho, u, \nabla\phi, B)\|_3^2 + \|(\rho_t, u_t, \nabla\phi_t, B_t)\|_2^2 \\ &\le C_1(\|(\rho_0, u_0, \nabla\phi_0, B_0)\|_3^2 + \|(\rho_t, u_t, \nabla\phi, B_t)(\cdot, 0)\|_2^2) e^{-\alpha_1 t}.\end{aligned} \tag{2.10}$$

Proof. From the assumption (2.9), we get

$$\begin{aligned}&\sup_{x\in\mathbb{T}^3} |(\rho, u, \nabla\phi, B, \partial_x\rho, \partial_x u, \partial_x\nabla\phi, \partial_x B, \rho_t, u_t, \nabla\phi_t, B_t)| \\ &\le C(\|(\rho, u, \nabla\phi, B)\|_3 + \|(\rho_t, u_t, \nabla\phi_t, B_t)\|_2) \le C\delta_1.\end{aligned} \tag{2.11}$$

Using (2.2) and (2.11), we have

$$\int_{\mathbb{T}^3} |\rho_t|^2 dx = -\int_{\mathbb{T}^3} \rho_t \mathrm{div}((1+\rho)u) dx$$

$$
\begin{aligned}
&= -\int_{\mathbb{T}^3} \rho_t(1+\rho)\mathrm{div} u dx - \int_{\mathbb{T}^3} \rho_t u \cdot \nabla\rho dx \\
&\leq \frac{1}{2}\int_{\mathbb{T}^3} |\rho_t|^2 + C\int_{\mathbb{T}^3} (|\nabla u|^2 + |\nabla\rho|^2) dx.
\end{aligned}
$$

Then, we get

$$
\|\rho_t\| \leq C(\|\nabla u\| + \|\nabla\rho\|). \tag{2.12}
$$

Similarly, we have

$$
\|\partial_x^j \rho_t\| \leq C \sum_{k=1}^{j+1} (\|\partial_x^k \rho\| + \|\partial_x^k u\|), \quad j = 1, 2. \tag{2.13}
$$

Take ∂_x^j, $j = 1, 2, 3$ of (2.6) and multiply the resulting equation by $\partial_x^j \phi$, then integrate it over $\mathbb{T}^3$ to get

$$
\begin{aligned}
\int_{\mathbb{T}^3} |\partial_x^j \nabla\phi|^2 &= -\int_{\mathbb{T}^3} \partial_x^j \Delta\phi \partial_x^j \phi dx \\
&= \int_{\mathbb{T}^3} \partial_x^{j-1} N \partial_x^j \nabla\phi dx \\
&\leq \frac{1}{2}\int_{\mathbb{T}^3} |\partial_x^j N|^2 dx + \frac{1}{2}\int_{\mathbb{T}^3} |\partial_x^{j-1}\nabla\phi|^2 dx,
\end{aligned}
$$

which implies

$$
\|\partial_x^j \nabla\phi\| \leq C\|\partial_x^{j-1} N\|, \quad j = 1, 2, 3. \tag{2.14}
$$

Similarly, we get

$$
\|\partial_x^j \nabla\phi_t\| \leq C \sum_{k=0}^{j} (\|\partial_x^k \rho\| + \|\partial_x^k u\|), \quad j = 0, 1, 2, 3. \tag{2.15}
$$

Multiplying (2.3) by $(1+\rho)u$ and integrating it over $\mathbb{T}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} (1+\rho)|u|^2 dx - \frac{1}{2}\int_{\mathbb{T}^3} \rho_t |u|^2 dx + \int_{\mathbb{T}^3} (1+\rho)|u|^2 dx \\
&\quad + \int_{\mathbb{T}^3} \nabla(h(1+\rho) - h(1))((1+\rho)u) dx + \int_{\mathbb{T}^3} (u\cdot\nabla)u \cdot (1+\rho)u dx \\
&\quad - \int_{\mathbb{T}^3} \nabla\phi \cdot (1+\rho)u dx - \int_{\mathbb{T}^3} (\mathrm{curl}\, B \times B) \cdot u dx = 0.
\end{aligned} \tag{2.16}
$$

We assume that δ_1 is chosen so small that

$$
\frac{1}{2} \leq 1+\rho \leq 2. \tag{2.17}
$$

Then, from (1.6), we have

$$
0 < D_1 \leq h'(1+\rho) \leq D_2 < +\infty, \quad |h^{(k)}(1+\rho)| \leq D_3 < +\infty \tag{2.18}
$$

for some positive constants D_1, D_2, D_3 and an arbitrary positive integer k.

From (2.2), by integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{T}^3} \nabla(h(1+\rho)-h(1))((1+\rho)u)dx \\
=&-\int_{\mathbb{T}^3}(h(1+\rho)-h(1))\mathrm{div}((1+\rho)u)dx \\
=&\frac{d}{dt}\int_{\mathbb{T}^3}\int_0^{\rho}(h(1+s)-h(1))dsdx \\
=&\frac{d}{dt}\int_{\mathbb{T}^3}\int_0^{\rho}(h(1+s)-h(1))dsdx. \qquad (2.19)
\end{aligned}$$

Similarly, one gets

$$-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|u|^2dx+\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot(1+\rho)udx=0. \qquad (2.20)$$

Using (2.2), (2.6) and (2.11), we have

$$\int_{\mathbb{T}^3}\nabla\phi\cdot(1+\rho)udx=-\int_{\mathbb{T}^3}\phi\partial_t\rho dx=\int_{\mathbb{T}^3}\phi\Delta\phi_t dx=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla\phi|^2dx. \qquad (2.21)$$

Multiplying (2.4) by B and integrating it over $\mathbb{T}^3$, with the help of (2.11), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|B|^2dx-\int_{\mathbb{T}^3}\mathrm{curl}\,(u\times B)\cdot Bdx+\int_{\mathbb{T}^3}|\nabla B|^2dx \\
=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|B|^2dx-\int_{\mathbb{T}^3}(u\times B)\cdot\mathrm{curl}\,Bdx+\int_{\mathbb{T}^3}|\nabla B|^2dx \\
=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|B|^2dx+\int_{\mathbb{T}^3}(\mathrm{curl}\,B\times B)\cdot udx+\int_{\mathbb{T}^3}|\nabla B|^2dx \\
=&0. \qquad (2.22)
\end{aligned}$$

So, from (2.16) and (2.19)–(2.22), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\left[\frac{1+\rho}{2}|u|^2+\int_0^{\rho}(h(1+s)-h(1))ds+\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}|B|^2\right]dx \\
&\quad+\int_{\mathbb{T}^3}(1+\rho)|u|^2dx+\int_{\mathbb{T}^3}|\nabla B|^2dx=0. \qquad (2.23)
\end{aligned}$$

Differentiating (2.3) with respect to t and then multiplying the resulting equation by $(1+\rho)u_t$, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|u_t|^2dx-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|u_t|^2dx \\
&\quad+\int_{\mathbb{T}^3}(1+\rho)|u_t|^2dx-\int_{\mathbb{T}^3}\partial_t h(1+\rho)\mathrm{div}((1+\rho)u_t)dx \\
&\quad+\int_{\mathbb{T}^3}\partial_t[(u\cdot\nabla)u]\cdot(1+\rho)u_tdx-\int_{\mathbb{T}^3}\nabla\phi_t\cdot(1+\rho)u_tdx \\
&\quad-\int_{\mathbb{T}^3}\partial_t\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right)\cdot(1+\rho)u_tdx=0. \qquad (2.24)
\end{aligned}$$

Differentiating (2.2) with respect to t, we get

$$\operatorname{div}((1+\rho)u_t) = -\rho_{tt} - \operatorname{div}(\rho_t u). \tag{2.25}$$

Using (2.11), (2.17), (2.18) and (2.25), by integration by parts, we have

$$\begin{aligned}
&-\int_{\mathbb{T}^3} \partial_t h(1+\rho)\operatorname{div}((1+\rho)u_t)dx\\
=&\int_{\mathbb{T}^3} h'(1+\rho)\rho_t\rho_{tt}dx + \int_{\mathbb{T}^3} h'(1+\rho)\rho_t\operatorname{div}(\rho_t u)dx\\
=&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} h'(1+\rho)|\rho_t|^2dx - \frac{1}{2}\int_{\mathbb{T}^3} h''(1+\rho)\rho_t|\rho_t|^2dx\\
&+\int_{\mathbb{T}^3} h'(1+\rho)|\rho_t|^2\operatorname{div}u dx - \frac{1}{2}\int_{\mathbb{T}^3}\operatorname{div}(h'(1+\rho)u)|\rho_t|^2dx\\
\geq&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} h'(1+\rho)|\rho_t|^2dx - C\delta_1\|\rho_t\|^2.
\end{aligned} \tag{2.26}$$

Similarly, we have

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|u_t|^2dx + \int_{\mathbb{T}^3}\partial_t[(u\cdot\nabla)u]\cdot(1+\rho)u_tdx\\
=&-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|u_t|^2dx + \int_{\mathbb{T}^3}(u\cdot\nabla)u_t\cdot(1+\rho)u_tdx + \int_{\mathbb{T}^3}(u_t\cdot\nabla)u\cdot(1+\rho)u_tdx\\
=&\int_{\mathbb{T}^3}(u_t\cdot\nabla)u\cdot(1+\rho)u_tdx\\
\geq&-C\delta_1\|u_t\|^2.
\end{aligned} \tag{2.27}$$

Due to (2.11), (2.16), (2.19) and (2.25), we get

$$\begin{aligned}
&-\int_{\mathbb{T}^3}\nabla\phi_t\cdot(1+\rho)u_tdx\\
=&-\int_{\mathbb{T}^3}\phi_t\rho_{tt}dx - \int_{\mathbb{T}^3}\phi_t\operatorname{div}(\rho u)_tdx\\
=&-\int_{\mathbb{T}^3}\phi_t\Delta\phi_{tt}dx + \int_{\mathbb{T}^3}\nabla\phi_t(\rho u)_tdx\\
\geq&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla\phi_t|^2dx - C\delta_1\|(u_t,\nabla\phi_t,\rho_t)\|^2.
\end{aligned} \tag{2.28}$$

From (2.25), we get

$$\begin{aligned}
&-\int_{\mathbb{T}^3}\partial_t\left(\frac{1}{1+\rho}(\operatorname{curl}B\times B)\right)\cdot(1+\rho)u_tdx\\
=&\int_{\mathbb{T}^3}\frac{\rho_t}{(1+\rho)^2}(\operatorname{curl}B\times B)\cdot(1+\rho)u_tdx\\
&-\int_{\mathbb{T}^3}(\operatorname{curl}B_t\times B)\cdot u_tdx - \int_{\mathbb{T}^3}(\operatorname{curl}B\times B_t)\cdot u_tdx
\end{aligned}$$

$$\geq -\int_{\mathbb{T}^3}(\mathrm{curl}\, B_t\times B)\cdot u_t dx - C\delta_1\|(u_t,\rho_t,B_t)\|^2. \tag{2.29}$$

Differentiating (2.4) with respect to t and then multiplying the resulting equation by B_t, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|B_t|^2dx-\int_{\mathbb{T}^3}\mathrm{curl}\,(u_t\times B)\cdot B_t dx\\&\quad-\int_{\mathbb{T}^3}\mathrm{curl}\,(u\times B_t)\cdot B_t dx+\int_{\mathbb{T}^3}|\nabla B_t|^2dx=0.\end{aligned} \tag{2.30}$$

it is easy to get

$$-\int_{\mathbb{T}^3}\mathrm{curl}\,(u_t\times B)\cdot B_t dx=\int_{\mathbb{T}^3}(\mathrm{curl}\, B_t\times B)\cdot u_t dx. \tag{2.31}$$

Noting $\mathrm{div}B = 0$ and using the vector analysis formula (1.9), we get

$$\begin{aligned}&-\int_{\mathbb{T}^3}\mathrm{curl}\,(u\times B_t)\cdot B_t dx\\&=\int_{\mathbb{T}^3}\mathrm{div}u|B_t|^2dx-\int_{\mathbb{T}^3}(B_t\cdot\nabla)u\cdot B_t dx+\int_{\mathbb{T}^3}(u\cdot\nabla)B_t\cdot B_t dx\\&=\frac{1}{2}\int_{\mathbb{T}^3}\mathrm{div}u|B_t|^2dx-\int_{\mathbb{T}^3}(B_t\cdot\nabla)u\cdot B_t dx\\&\geq -C\delta_1\|B_t\|^2.\end{aligned} \tag{2.32}$$

In view of (2.30)–(2.32), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|B_t|^2dx+\int_{\mathbb{T}^3}|\nabla B_t|^2dx+\int_{\mathbb{T}^3}(\mathrm{curl}\, B_t\times B)\cdot u_t dx\leq C\delta_1\|B_t\|^2. \tag{2.33}$$

Combining (2.24) with (2.26)–(2.29) and (2.33) together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\left[h'(1+\rho)|\rho_t|^2+(1+\rho)|u_t|^2+|\nabla\phi_t|^2+|B_t|^2\right]dx\\&\quad+\int_{\mathbb{T}^3}(1+\rho)|u_t|^2dx+\int_{\mathbb{T}^3}|\nabla B_t|^2dx\\&\leq C\delta_1\|(\rho_t,u_t,\nabla\phi_t,B_t)\|^2.\end{aligned} \tag{2.34}$$

From (2.23) and (2.34), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^3}\Bigg[\frac{1+\rho}{2}|u|^2+\int_0^\rho(h(1+\rho)-h(1)ds+\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}|B|^2\\&\qquad+\frac{1}{2}h'(1+\rho)|\rho_t|^2+\frac{1}{2}(1+\rho)|u_t|^2+\frac{1}{2}|\nabla\phi_t|^2+\frac{1}{2}|B_t|^2\Bigg]dx\\&\qquad+\int_{\mathbb{T}^3}(1+\rho)(|u|^2+|u_t|^2)dx+\int_{\mathbb{T}^3}(|\nabla B|^2+|\nabla B_t|^2)dx\\&\leq C\delta_1\|(\rho_t,u_t,\nabla\phi_t,B_t)\|^2.\end{aligned} \tag{2.35}$$

By (1.9), (2.4), (2.5) and (2.11), we have

$$\begin{aligned}\int_{\mathbb{T}^3}|B_t|^2dx=&\int_{\mathbb{T}^3}B_t\cdot\mathrm{curl}\,(u\times B)dx-\int_{\mathbb{T}^3}\nabla B_t\cdot\nabla Bdx\\=&-\int_{\mathbb{T}^3}B_t\cdot(B\mathrm{div}u-(B\cdot\nabla)u+(u\cdot\nabla)B)dx-\int_{\mathbb{T}^3}\nabla B_t\cdot\nabla Bdx\\\leq&\frac{1}{2}\int_{\mathbb{T}^3}|B_t|^2dx+\frac{1}{2}\int_{\mathbb{T}^3}(|\nabla B|^2+|\nabla B_t|^2)dx+C\delta_1\int_{\mathbb{T}^3}(|\nabla u|^2+|u|^2)dx.\end{aligned}$$

Then, we get

$$\|B_t\|^2\leq C\|(\nabla B,\nabla B_t,\nabla u,u)\|^2. \tag{2.36}$$

Now we need to estimate $\int_{\mathbb{T}^3}|\nabla u|^2dx$. To do this, we use the formulation

$$\int_{\mathbb{T}^3}|\nabla u|^2dx=\int_{\mathbb{T}^3}|\mathrm{div}u|^2dx+\int_{\mathbb{T}^3}|\mathrm{curl}\,u|^2dx$$

to control $\int_{\mathbb{T}^3}|\nabla u|^2dx$. Taking div of the Eq (2.3) and multiplying the resulting equation by $(1+\rho)\mathrm{div}u$, by integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|\mathrm{div}u|^2dx+\int_{\mathbb{T}^3}(1+\rho)|\mathrm{div}u|^2dx\\=&-\int_{\mathbb{T}^3}\Delta h(1+\rho)(1+\rho)\mathrm{div}udx-\int_{\mathbb{T}^3}\mathrm{div}(u\cdot\nabla u)(1+\rho)\mathrm{div}udx\\&+\int_{\mathbb{T}^3}\rho(1+\rho)\mathrm{div}udx+\int_{\mathbb{T}^3}\mathrm{div}\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right)(1+\rho)\mathrm{div}udx,\end{aligned} \tag{2.37}$$

where we used $\Delta\phi=\rho$.

First, we deal with the first term in the right hand side of the Eq (2.37), which is more difficult to control. To do this, we rewrite the Eq (2.2) into the following formulation:

$$(1+\rho)\mathrm{div}u=-\rho_t+u\cdot\nabla\rho. \tag{2.38}$$

With the help of (2.38) and (2.11), we have

$$\begin{aligned}&-\int_{\mathbb{T}^3}\Delta h(1+\rho)((1+\rho)\mathrm{div}u)dx\\=&\int_{\mathbb{T}^3}\mathrm{div}(h'(1+\rho)\nabla\rho)\rho_tdx+\int_{\mathbb{T}^3}\mathrm{div}(h'(1+\rho)\nabla\rho)u\cdot\nabla\rho dx\\=&-\int_{\mathbb{T}^3}h'(1+\rho)\nabla\rho\nabla\rho_tdx-\int_{\mathbb{T}^3}h'(1+\rho)\nabla\rho\cdot\nabla(u\cdot\nabla\rho)dx\\=&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}h'(1+\rho)|\nabla\rho|^2dx+\frac{1}{2}\int_{\mathbb{T}^3}h''(1+\rho)\rho_t|\nabla\rho|^2dx\\&-\int_{\mathbb{T}^3}h'(1+\rho)\nabla u:(\nabla\rho\otimes\nabla\rho)dx+\frac{1}{2}\int_{\mathbb{T}^3}\mathrm{div}(h'(1+\rho)u)|\nabla\rho|^2dx\\\leq&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}h'(1+\rho)|\nabla\rho|^2dx+C\delta_1\|\nabla\rho\|^2.\end{aligned} \tag{2.39}$$

Using (2.2) and (2.11), we have

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}\operatorname{div}(u\cdot\nabla u)(1+\rho)\operatorname{div}u dx\\
=&-\int_{\mathbb{T}^3}\partial_{x_i}(u_j\partial_{x_j}u_i)(1+\rho)\operatorname{div}u dx\\
=&-\int_{\mathbb{T}^3}\partial_{x_i}u_j\partial_{x_j}u_i(1+\rho)\operatorname{div}u dx-\int_{\mathbb{T}^3}u_j\partial_{x_j}(\partial_{x_i}u_i)(1+\rho)\operatorname{div}u dx\\
=&-\int_{\mathbb{T}^3}\partial_{x_i}u_j\partial_{x_j}u_i(1+\rho)\operatorname{div}u dx-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\operatorname{div}u|^2dx\\
\le& C\delta_1\|\nabla u\|^2.
\end{aligned}\tag{2.40}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\rho(1+\rho)\operatorname{div}u dx\\
=&-\int_{\mathbb{T}^3}\rho\rho_t dx-\int_{\mathbb{T}^3}\rho u\cdot\nabla\rho dx\\
\le&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\rho|^2dx+C\|(u,\rho)\|^2.
\end{aligned}\tag{2.41}
$$

Now, we begin to deal with the last term in the right hand side of the Eq (2.37), which is most difficult to control. Using (2.4), (2.5), (2.36) and the vector analysis formulas (1.9)–(1.14), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\operatorname{div}\left(\frac{1}{1+\rho}(\operatorname{curl}B\times B)\right)(1+\rho)\operatorname{div}u dx\\
=&\int_{\mathbb{T}^3}\operatorname{div}(\operatorname{curl}B\times B)\operatorname{div}u dx-\int_{\mathbb{T}^3}\frac{1}{1+\rho}(\operatorname{curl}B\times B)\nabla\rho\operatorname{div}u dx\\
\le&\int_{\mathbb{T}^3}B\cdot\Delta B\operatorname{div}u dx-\int_{\mathbb{T}^3}|\operatorname{curl}B|^2\operatorname{div}u dx+C\delta_1\int_{\mathbb{T}^3}(|\nabla u|^2+|\nabla\rho|^2)dx\\
=&\int_{\mathbb{T}^3}B\cdot B_t\operatorname{div}u dx-\int_{\mathbb{T}^3}B\cdot B_t\operatorname{div}u dx+C\delta_1\int_{\mathbb{T}^3}(|\nabla u|^2+|\nabla\rho|^2+|\nabla B|^2)dx\\
\le& C\|(\nabla u,\nabla\rho,\nabla B,u)\|^2.
\end{aligned}\tag{2.42}
$$

Combining (2.37) with (2.39)–(2.42), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{T}^3}(1+\rho)|\operatorname{div}u|^2dx+h'(1+\rho)|\nabla\rho|^2dx+|\rho|^2dx\right]+\int_{\mathbb{T}^3}(1+\rho)|\operatorname{div}u|^2dx\\
\le& C\delta_1\|(\nabla u,\nabla\rho,\nabla B,u,\rho)\|^2.
\end{aligned}\tag{2.43}
$$

Similarly we take curl of (2.3) and multiply the resulting equation by $(1+\rho)\operatorname{curl}u$ in $L^2(\mathbb{T}^3)$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|\operatorname{curl}u|^2dx+\int_{\mathbb{T}^3}(1+\rho)|\operatorname{curl}u|^2dx\\
=&-\int_{\mathbb{T}^3}\operatorname{curl}(u\cdot\nabla u)(1+\rho)\operatorname{curl}u dx
\end{aligned}
$$

$$+\int_{\mathbb{T}^3}\operatorname{curl}\left(\frac{1}{1+\rho}(\operatorname{curl}B\times B)\right)(1+\rho)\operatorname{curl}u\,dx, \tag{2.44}$$

where we used $\operatorname{curl}(\nabla a)=0$.

Direct calculation and integration by parts give

$$\begin{aligned}
&-\int_{\mathbb{T}^3}\operatorname{curl}(u\cdot\nabla u)(1+\rho)\operatorname{curl}u\,dx\\
=&-\int_{\mathbb{T}^3}(1+\rho)(\partial_{x_k}(u_j\partial_{x_j}u_i)-\partial_{x_i}(u_j\partial_{x_j}u_k))(\partial_{x_k}u_i-\partial_{x_i}u_k)dx\\
=&-\int_{\mathbb{T}^3}(1+\rho)(\partial_{x_k}u_j\partial_{x_j}u_i-\partial_{x_i}u_j\partial_{x_j}u_k)(\partial_{x_k}u_i-\partial_{x_i}u_k)dx\\
&-\int_{\mathbb{T}^3}(1+\rho)(u_j\partial_{x_k}(\partial_{x_j}u_i)-u_j\partial_{x_i}(\partial_{x_j}u_k))(\partial_{x_k}u_i-\partial_{x_i}u_k)dx\\
=&-\int_{\mathbb{T}^3}(1+\rho)(\partial_{x_k}u_j\partial_{x_j}u_i-\partial_{x_i}u_j\partial_{x_j}u_k)(\partial_{x_k}u_i-\partial_{x_i}u_k)udx\\
&-\int_{\mathbb{T}^3}(1+\rho)u_j(\partial_{x_j}(\partial_{x_k}u_i-\partial_{x_i}u_k)(\partial_{x_k}u_i-\partial_{x_i}u_k)dx\\
=&-\int_{\mathbb{T}^3}(1+\rho)(\partial_{x_k}u_j\partial_{x_j}u_i-\partial_{x_i}u_j\partial_{x_j}u_k)(\partial_{x_k}u_i-\partial_{x_i}u_k)udx\\
&-\frac{1}{2}\int_{\mathbb{T}^3}(1+\rho)u\cdot\nabla|\operatorname{curl}u|^2dx\\
=&-\int_{\mathbb{T}^3}(1+\rho)(\partial_{x_k}u_j\partial_{x_j}u_i-\partial_{x_i}u_j\partial_{x_j}u_k)(\partial_{x_k}u_i-\partial_{x_i}u_k)udx\\
&-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\operatorname{curl}u|^2dx\\
\leq& C\delta_1\|\nabla u\|^2.
\end{aligned}\tag{2.45}$$

Using the vector analysis formulas (1.9), (1.12), (1.13), (2.5), we have

$$\begin{aligned}
&\int_{\mathbb{T}^3}\operatorname{curl}\left(\frac{1}{1+\rho}(\operatorname{curl}B\times B)\right)(1+\rho)\operatorname{curl}u\,dx\\
=&\int_{\mathbb{T}^3}\nabla\left(\frac{1}{1+\rho}\right)\times(\operatorname{curl}B\times B)(1+\rho)\operatorname{curl}u\,dx\\
&+\int_{\mathbb{T}^3}\operatorname{curl}(\operatorname{curl}B\times B)\cdot\operatorname{curl}u\,dx\\
=&-\int_{\mathbb{T}^3}\frac{\nabla\rho}{1+\rho}\times(\operatorname{curl}B\times B)\operatorname{curl}u\,dx\\
&+\int_{\mathbb{T}^3}(B\cdot\nabla)\operatorname{curl}B\cdot\operatorname{curl}u\,dx-\int_{\mathbb{T}^3}(\operatorname{curl}B\cdot\nabla)B\operatorname{curl}u\,dx\\
\leq&\int_{\mathbb{T}^3}(B\cdot\nabla)\,B\cdot\operatorname{curl}u\,dx+C\delta_1\int_{\mathbb{T}^3}(|\nabla B|^2+|\nabla u|^2)dx\\
=&\int_{\mathbb{T}^3}(\operatorname{curl}B\times\operatorname{curl}B)\operatorname{curl}u\,dx+\int_{\mathbb{T}^3}\nabla(B\cdot\operatorname{curl}B)\operatorname{curl}u\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}(\nabla B\cdot \operatorname{curl} B)\cdot \operatorname{curl} u dx + C\delta_1\int_{\mathbb{T}^3}(|\nabla B|^2+|\nabla u|^2)dx\\
\leq& C\delta_1\|(\nabla B,\nabla u)\|^2,
\end{aligned}
\tag{2.46}
$$

where, we used $\operatorname{div}(\operatorname{curl} u)=0$. Thus, (2.44) together with (2.45)–(2.46), implies

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|\operatorname{curl} u|^2dx+\int_{\mathbb{T}^3}(1+\rho)|\operatorname{curl} u|^2dx\leq C\delta_1\|(\nabla B,\nabla u)\|^2. \tag{2.47}
$$

Combining (2.43) and (2.47), we have, with the help of the smallness of δ_1, that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\left[(1+\rho)|\nabla u|^2+h'(1+\rho)|\nabla\rho|^2+|\rho|^2\right]dx+\int_{\mathbb{T}^3}(1+\rho)|\nabla u|^2dx\\
\leq& C\delta_1\|(\nabla u,\nabla\rho,\nabla B,u,\rho)\|^2.
\end{aligned}
\tag{2.48}
$$

On the other hand, multiplying (2.2) by $\nabla\rho$, and integrating it over $\mathbb{T}^3$ to get

$$
\begin{aligned}
\int_{\mathbb{T}^3}h'(1+\rho)|\nabla\rho|^2dx=&-\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot\nabla\rho dx+\int_{\mathbb{T}^3}\nabla\phi\cdot\nabla\rho dx\\
&+\int_{\mathbb{T}^3}\frac{1}{1+\rho}(\operatorname{curl} B\times B)\cdot\nabla\rho dx-\int_{\mathbb{T}^3}(u_t+u)\cdot\nabla\rho dx\\
\leq& C\delta_1\int_{\mathbb{T}^3}(|\nabla\rho|^2+|\nabla B|^2)dx-\int_{\mathbb{T}^3}\Delta\phi\rho dx\\
&+\epsilon\int_{\mathbb{T}^3}|\nabla\rho|^2dx+\int_{\mathbb{T}^3}(|u|^2+|u_t|^2)dx\\
=& C\delta_1\int_{\mathbb{T}^3}(|\nabla\rho|^2+|\nabla B|^2)dx-\int_{\mathbb{T}^3}|\rho|^2dx\\
&+\epsilon\int_{\mathbb{T}^3}|\nabla\rho|^2dx+\int_{\mathbb{T}^3}(|u|^2+|u_t|^2)dx.
\end{aligned}
\tag{2.49}
$$

With the help of (2.18) and the smallness of δ_1 and ϵ, we have

$$
\int_{\mathbb{T}^3}(|\rho|^2+|\nabla\rho|^2)dx\leq C\int_{\mathbb{T}^3}(|u|^2+|u_t|^2+|\nabla B|^2)dx. \tag{2.50}
$$

Combining (2.35) with (2.12) and (2.48)–(2.50), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\Big[(1+\rho)|u|^2+2\int_0^\rho(h(1+\rho)-h(1)ds+|\nabla\phi|^2+|B|^2+h'(1+\rho)|\rho_t|^2\\
&+(1+\rho)|u_t|^2+|\nabla\phi_t|^2+|B_t|^2+(1+\rho)|\nabla u|^2+h'(1+\rho)|\nabla\rho|^2+|\rho|^2\Big]dx\\
&+C\|(\rho,\nabla\rho,\rho_t,u,\nabla u,u_t,\nabla\phi,\nabla\phi_t,B,B_t)\|^2\leq 0.
\end{aligned}
\tag{2.51}
$$

Take ∇ of (2.4) and multiply the resulting equation by ∇B, then integrate over $\mathbb{T}^3$, using integration by parts, to get

$$
0=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla B|^2dx+\int_{\mathbb{T}^3}\operatorname{curl}(u\times B)\Delta Bdx+\int_{\mathbb{T}^3}|\Delta B|^2dx \tag{2.52}
$$

$$\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla B|^2dx - C\delta_1\int_{\mathbb{T}^3}(|\nabla u|^2+|\nabla B|^2)dx + \frac{1}{2}\int_{\mathbb{T}^3}|\Delta B|^2dx, \tag{2.53}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla B|^2dx \leq C\delta_1\|(\nabla u,\nabla B)\|^2. \tag{2.54}$$

Thus, (2.51) together with (2.52), (2.51), (2.14) with $j=1$, implies

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^3}\Big[(1+\rho)(|u|^2+|u_t|^2+|\nabla u|^2)+\int_0^\rho(h(1+s)-h(1)ds+|\nabla\phi|^2+|\nabla\phi_t|^2\\&\quad+\lambda|\partial_x\nabla\phi|^2+|B|^2+|\nabla B|^2+|B_t|^2+h'(1+\rho)(|\rho_t|^2+|\nabla\rho|^2)+|\rho|^2\Big]dx\\&\quad+C\|(\rho,\nabla\rho,\rho_t,u,\nabla u,u_t,\nabla\phi,\partial_x\nabla\phi,\nabla\phi_t,B,\nabla B,B_t)\|^2\leq 0\end{aligned} \tag{2.55}$$

for some positive constant $\lambda>0$.

The next step is to estimate the higher-order derivatives. Taking ∂_x^2 of (2.3) and multiplying the resulting equation by $(1+\rho)\partial_x^2u$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|\partial_x^2u|^2dx+\int_{\mathbb{T}^3}(1+\rho)|\partial_x^2u|^2dx-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_x^2u|^2dx\\&\quad-\int_{\mathbb{T}^3}\partial_x^2(h(1+\rho)-h(1))\partial_x^2\mathrm{div}u\,dx+\int_{\mathbb{T}^3}\partial_x^2\phi\,\mathrm{div}((1+\rho)\partial_x^2u)dx\\&\quad+\int_{\mathbb{T}^3}(1+\rho)\partial_x^2(u\cdot\nabla u)\partial_x^2u\,dx-\int_{\mathbb{T}^3}\partial_x^2(\mathrm{curl}\,B\times B)\partial_x^2u\,dx\\&\quad-\int_{\mathbb{T}^3}\left[\partial_x^2\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right)-\frac{1}{1+\rho}\partial_x^2(\mathrm{curl}\,B\times B)\right](1+\rho)\partial_x^2u\,dx=0.\end{aligned} \tag{2.56}$$

Using (2.2), (2.6), (2.11), (2.13) and (2.18), one gets

$$\begin{aligned}&-\int_{\mathbb{T}^3}\partial_x^2(h(1+\rho))\partial_x^2\mathrm{div}u\,dx\\&=\int_{\mathbb{T}^3}\partial_x^2(h(1+\rho))\partial_x^2\left(\frac{\rho_t+u\cdot\nabla\rho}{1+\rho}\right)dx\\&\geq\int_{\mathbb{T}^3}\frac{h'(1+\rho)}{1+\rho}\partial_x^2\rho\partial_x^2\rho_t dx+\int_{\mathbb{T}^3}\frac{h'(1+\rho)}{1+\rho}\partial_x^2\rho(u\cdot\nabla\partial_x^2\rho)dx\\&\quad-C\delta_1\|\partial_x\rho,\partial_x^2\rho,\partial_xu,\partial_x^2u)\|^2\\&=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\frac{h'(1+\rho)}{1+\rho}|\partial_x^2\rho|^2dx\\&\quad-\frac{1}{2}\int_{\mathbb{T}^3}\left[\left(\frac{h'(1+\rho)}{1+\rho}\right)_t+\mathrm{div}\left(\frac{h'(1+\rho)}{1+\rho}u\right)\right]|\partial_x^2\rho|^2dx\\&\quad-C\delta_1\|(\partial_x\rho,\partial_x^2\rho,\partial_xu,\partial_x^2u)\|^2\\&\geq\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\frac{h'(1+\rho)}{1+\rho}|\partial_x^2\rho|^2dx\end{aligned}$$

$$- C\delta_1\|(\partial_x\rho, \partial_x^2\rho, \partial_x u, \partial_x^2 u)\|^2, \tag{2.57}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^3} \partial_x^2\phi \operatorname{div}((1+\rho)\partial_x^2 u)dx \\
=&\int_{\mathbb{T}^3} \partial_x^2\phi\partial_x^2 \operatorname{div} u dx + \int_{\mathbb{T}^3} \partial_x^2\phi \operatorname{div}(\rho\partial_x^2 u)dx \\
=&-\int_{\mathbb{T}^3} \partial_x^2\phi\partial_x^2(\rho_t + \operatorname{div}(\rho u))dx + \int_{\mathbb{T}^3} \partial_x^2\phi \operatorname{div}(\rho\partial_x^2 u)dx \\
=&\int_{\mathbb{T}^3} \partial_x^2\nabla\phi\partial_x^2\nabla\phi_t dx + \int_{\mathbb{T}^3} \partial_x^2\nabla\phi\partial_x^2(\rho u)dx - \int_{\mathbb{T}^3} \partial_x^2\nabla\phi(\rho\partial_x^2 u)dx \\
\geq&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |\partial_x^2\nabla\phi|^2 dx - C\delta_1\|(\partial_x\rho, \partial_x^2\rho, \partial_x u, \partial_x^2 u, \partial_x^2\nabla\phi)\|^2.
\end{aligned} \tag{2.58}$$

From (2.2), by integration by parts, we have

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{T}^3} \rho_t|\partial_x^2 u|^2 dx + \int_{\mathbb{T}^3} (1+\rho)\partial_x^2(u\cdot\nabla u)\partial_x^2 u dx \\
=&-\frac{1}{2}\int_{\mathbb{T}^3} \rho_t|\partial_x^2 u|^2 dx + \int_{\mathbb{T}^3} (1+\rho)(u\cdot\nabla\partial_x^2 u)\partial_x^2 u dx \\
&+\int_{\mathbb{T}^3} (1+\rho)(\partial_x^2(u\cdot\nabla u) - (u\cdot\nabla\partial_x^2 u))\partial_x^2 u dx \\
=&\int_{\mathbb{T}^3} (1+\rho)(\partial_x^2(u\cdot\nabla u) - (u\cdot\nabla\partial_x^2 u))\partial_x^2 u dx \\
\geq&-C\delta_1\|\partial_x^2 u\|^2.
\end{aligned} \tag{2.59}$$

Direct calculations give

$$\begin{aligned}
&-\int_{\mathbb{T}^3} \partial_x^2(\operatorname{curl} B\times B)\partial_x^2 u dx \\
=&-\int_{\mathbb{T}^3} (\partial_x^2\operatorname{curl} B\times B)\partial_x^2 u dx - \int_{\mathbb{T}^3} (\partial_x^2(\operatorname{curl} B\times B) - (\partial_x^2\operatorname{curl} B\times B))\partial_x^2 u dx \\
=&-\int_{\mathbb{T}^3} \partial_x^2(\operatorname{curl} B\times B)\partial_x^2 u dx \\
\geq&-\int_{\mathbb{T}^3} (\partial_x^2\operatorname{curl} B\times B)\partial_x^2 u dx - C\delta_1\|(\partial_x^2, \partial_x^2 B)\|^2.
\end{aligned} \tag{2.60}$$

and

$$\begin{aligned}
&-\int_{\mathbb{T}^3} \left[\partial_x^2\left(\frac{1}{1+\rho}(\operatorname{curl} B\times B)\right) - \frac{1}{1+\rho}\partial_x^2(\operatorname{curl} B\times B)\right](1+\rho)\partial_x^2 u dx \\
\geq&-C\delta_1\|(\partial_x^2 u, \partial_x^2\rho, \partial_x\rho, \partial_x^2 B, \partial_x B)\|^2.
\end{aligned} \tag{2.61}$$

Taking ∂_x^2 of the Eq (2.4) and then taking the L^2 inner product of the resulting equation with $\partial_x^2 B$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3} |\partial_x^2 B|^2 dx + \int_{\mathbb{T}^3} |\partial_x^2\nabla B|^2 dx - \int_{\mathbb{T}^3} \partial_x^2(\operatorname{curl}(u\times B))\cdot\partial_x^2 B dx$$

$$\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\partial_x^2 B|^2dx+\int_{\mathbb{T}^3}|\partial_x^2\nabla B|^2dx+\int_{\mathbb{T}^3}(\partial_x^2\mathrm{curl}\,B\times B)\partial_x^2udx$$
$$-C\delta_1\|(\partial_x^2u,\partial_xu,\partial_x^2B,\partial_xB)\|^2. \tag{2.62}$$

Combining (2.56) with (2.56)–(2.61), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\left[(1+\rho)|\partial_x^2u|^2+\frac{h'(1+\rho)}{1+\rho}|\partial_x^2\rho|^2+|\partial_x^2\nabla\phi|^2+|\partial_x^2B|^2\right]dx$$
$$+\int_{\mathbb{T}^3}((1+\rho)|\partial_x^2u|^2+|\partial_x^2\nabla B|^2dx$$
$$\leq C\delta_1\|(\partial_x^2u,\partial_xu,\partial_x^2\rho,\partial_x\rho,\partial_x^2B,\partial_xB,\partial_x^2\nabla\phi)\|^2. \tag{2.63}$$

As (2.50), differentiating (2.3) with respect to x and multiplying the resulting equation by $\partial_x\nabla\rho$, we get

$$\|(\partial_x^2\rho,\partial_x\rho)\|^2\leq C\|(\partial_x^2u,\partial_xu,\partial_x^2B)\|^2. \tag{2.64}$$

Taking ∂_{xt} of (2.3) and then taking the L^2 inner product of the resulting equation with $(1+\rho)\partial_xu_t$ in $\mathbb{T}^3$, one gets

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(1+\rho)|\partial_xu_t|^2dx+\int_{\mathbb{T}^3}(1+\rho)|\partial_xu_t|^2dx-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_xu_t|^2dx$$
$$-\int_{\mathbb{T}^3}\partial_x(h(1+\rho)_t-\partial_x\phi_t)\mathrm{div}((1+\rho)\partial_xu_t)dx+\int_{\mathbb{T}^3}(1+\rho)\partial_x(u\cdot\nabla u)_t\cdot\partial_xu_tdx$$
$$-\int_{\mathbb{T}^3}(1+\rho)\partial_x\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right)_t\partial_xu_tdx=0. \tag{2.65}$$

In view of (2.2), (2.13) and (2.14), we have

$$-\int_{\mathbb{T}^3}(\partial_x(h(1+\rho))_t-\partial_x\phi_t)\mathrm{div}((1+\rho)\partial_xu_t)dx$$
$$=-\int_{\mathbb{T}^3}(1+\rho)\partial_x(h'(1+\rho)\rho_t)\partial_x\mathrm{div}u_tdx-\int_{\mathbb{T}^3}\partial_x(h'(1+\rho)\rho_t)\partial_xu_t\cdot\nabla\rho dx$$
$$-\int_{\mathbb{T}^3}\partial_x\phi_t\partial_x\mathrm{div}u_tdx-\int_{\mathbb{T}^3}\partial_x\phi_t\partial_x\mathrm{div}(\rho u)_tdx$$
$$\geq-\int_{\mathbb{T}^3}(1+\rho)\partial_x(h'(1+\rho)\rho_t)\partial_x\left(\frac{\rho_t+u\cdot\nabla\rho}{1+\rho}\right)_tdx$$
$$-\int_{\mathbb{T}^3}(1+\rho)\partial_x\phi_t\partial_x(\rho_t+\mathrm{div}(\rho u))_tdx$$
$$-C\delta_1\int_{\mathbb{T}^3}(|\partial_x\rho|^2+|\partial_x\rho_t|^2+|\partial_xu_t|^2)dx$$
$$\geq\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}(h'(1+\rho)|\partial_x\rho_t|^2+(1+\rho)|\partial_x\nabla\phi_t|^2)dx$$
$$-C\delta_1\|(\partial_x\rho,\partial_x\rho_t,\partial_xu_t,\partial_xu,\partial_x^2\rho,\partial_x^2u,\rho_{tt},\partial_x\nabla\phi_t)\|^2 \tag{2.66}$$

and

$$-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_xu_t|^2dx+\int_{\mathbb{T}^3}(1+\rho)\partial_x(u\cdot\nabla u)_t\cdot\partial_xu_tdx$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_x u_t|^2dx + \int_{\mathbb{T}^3}(1+\rho)(u\cdot\nabla\partial_x u_t)\cdot\partial_x u_t dx \\
&\quad + \int_{\mathbb{T}^3}(1+\rho)(\partial_x(u\cdot\nabla u)_t - u\cdot\nabla\partial_x u_t)\cdot\partial_x u_t dx \\
&= \int_{\mathbb{T}^3}(1+\rho)(\partial_x(u\cdot\nabla u)_t - u\cdot\nabla\partial_x u_t)\cdot\partial_x u_t dx \\
&\geq -C\delta_1\|(\partial_x u_t, \partial_x^2 u)\|^2.
\end{aligned}
\tag{2.67}
$$

For the third term in (2.65), we have

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}(1+\rho)\partial_x\left(\frac{1}{1+\rho}(\mathrm{curl}\, B\times B)\right)_t\partial_x u_t dx \\
\geq& -\int_{\mathbb{T}^3}(\mathrm{curl}\,\partial_x B_t\times B)\partial_x u_t dx \\
&- C\delta_1\|(\partial_x\rho_t, \partial_x u_t, \partial_x B_t)\|^2.
\end{aligned}
\tag{2.68}
$$

Taking ∂_{xt} of (2.4) and then taking the L^2 inner product of the resulting equation with $\partial_x B_t$ in $\mathbb{T}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}|\partial_x B_t|^2dx + \int_{\mathbb{T}^3}|\partial_x\nabla B_t|^2dx \\
=&\int_{\mathbb{T}^3}\mathrm{curl}\,\partial_x(u\times B)_t\partial_x B_t dx \\
=&\int_{\mathbb{T}^3}\mathrm{curl}\,(\partial_x u_t\times B)\partial_x B_t dx + \int_{\mathbb{T}^3}(\mathrm{curl}\,\partial_x(u\times B)_t - \mathrm{curl}\,(\partial_x u_t\times B))\partial_x B_t dx \\
\leq& -\int_{\mathbb{T}^3}(\mathrm{curl}\,\partial_x B_t\times B)\partial_x u_t dx + C\delta_1\|(\partial_x u_t, \partial_x^2 u, \partial_x B_t)\|^2.
\end{aligned}
\tag{2.69}
$$

Thus, (2.65) together with (2.66)–(2.70) and (2.15) with $j=1$, implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}[(1+\rho)|\partial_x u_t|^2 + h'(1+\rho)|\partial_x\rho_t|^2 + |\partial_x\nabla\phi_t|^2 + |\partial_x B_t|^2]dx \\
&+C\int_{\mathbb{T}^3}((1+\rho)|\partial_x u_t|^2 + |\partial_x\nabla B_t|^2 + |\partial_x\nabla\phi_t|^2)dx \\
\leq& C\delta_1\|(\partial_x\rho, \partial_x\rho_t, \partial_x u_t, \partial_x u, \partial_x^2\rho, \partial_x^2 u, \rho_{tt}, \partial_x\nabla\phi_t, \partial_x B_t)\|^2 + C\|(\partial_x u, \partial_x\rho)\|^2.
\end{aligned}
\tag{2.70}
$$

Combining (2.63), (2.64), (2.70), (2.13) with $j=1$ and (2.14) with $j=2$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\Bigg[(1+\rho)|\partial_x^2 u|^2 + \frac{h'(1+\rho)}{1+\rho}|\partial_x^2\rho|^2 + |\partial_x^2\nabla\phi|^2 + |\partial_x^2 B|^2 \\
&\qquad + (1+\rho)|\partial_x u_t|^2 + h'(1+\rho)|\partial_x\rho_t|^2 + (1+\rho)|\partial_x\nabla\phi_t|^2 + |\partial_x B_t|^2\Bigg]dx \\
\leq& C\delta_1\|(\partial_x\rho, \partial_x\rho_t, \partial_x u_t, \partial_x u, \partial_x^2\rho, \partial_x^2 u, \rho_{tt}, \partial_x\nabla\phi_t, \partial_x B_t)\|^2 + C\|(\partial_x u, \partial_x\rho)\|^2.
\end{aligned}
\tag{2.71}
$$

Notice that in obtaining the estimates on the first and the second derivatives, we have used the smallness of $|(\rho, \partial_x\rho, u, \partial_x u, \nabla\phi, \partial_x\nabla\phi, B, \partial_x B)|$ and $|(\rho_t, u_t, \nabla\phi_t, B_t)|$. However, the above arguments do not work for the third derivatives because we cannot obtain the smallness of $\partial_x^2\rho, \partial_x^2 u, \partial_x^2\phi, \partial_x^2 B$ and $\partial_x\rho_t, \partial_x u_t, \partial_x\nabla\phi_t, \partial_x B_t$. Then we need to obtain the estimates of the third derivatives. Taking ∂_x^3 of (2.3) and multiplying the resulting equation by $(1+\rho)\partial_x^3 u$, and taking∂_x^3 of (2.3) and multiplying the resulting equation by $\partial_x^3 B$, and summing up them, by integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}[(1+\rho)|\partial_x^3 u|^2 + |\partial_x^3 B|^2]dx + \int_{\mathbb{T}^3}[(1+\rho)|\partial_x^3 u|^2+)|\partial_x^3\nabla B|^2]dx\\
&-\int_{\mathbb{T}^3}\partial_x^3(h(1+\rho))\partial_x^3\mathrm{div}u dx - \frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_x^3 u|^2 dx + \int_{\mathbb{T}^3}\partial_x^3\phi\,\mathrm{div}((1+\rho)\partial_x^3 u)dx\\
&+\int_{\mathbb{T}^3}(1+\rho)\partial_x^3(u\cdot\nabla u)\partial_x^3 u dx - \int_{\mathbb{T}^3}\partial_x^3(\mathrm{curl}\,B\times B)\partial_x^3 u dx\\
&-\int_{\mathbb{T}^3}\left[\partial_x^3\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right) - \frac{1}{1+\rho}\partial_x^3(\mathrm{curl}\,B\times B)\right](1+\rho)\partial_x^3 u dx\\
&-\int_{\mathbb{T}^3}\mathrm{curl}\,(\partial_x^3 u\times B)\partial_x^3 B dx - \int_{\mathbb{T}^3}(\mathrm{curl}\,\partial_x^3(u\times B) - \mathrm{curl}\,(\partial_x^3 u\times B))\partial_x^3 B dx\\
=&0. \qquad (2.72)
\end{aligned}$$

Similar to (2.57)–(2.64), by some tedious but straightforward calculations, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\left[(1+\rho)|\partial_x^3 u|^2 + \frac{h'(1+\rho)}{1+\rho}|\partial_x^3\rho|^2 + |\partial_x^3\nabla\phi|^2 + |\partial_x^3 B|^2\right]dx\\
&+\int_{\mathbb{T}^3}((1+\rho)|\partial_x^3 u|^2 + |\partial_x^3\nabla B|^2)dx\\
\le& C\delta_1\|(\partial_x^3\rho, \partial_x^2\rho, \partial_x\rho, \partial_x^2 u, \partial_x u, \partial_x^3\nabla\phi, \partial_x^3 B)\|^2 + \epsilon\|\partial_x^3\rho\|^2 \qquad (2.73)
\end{aligned}$$

with the help of the smallness of ϵ and δ_1.

As (2.50), we easily get

$$\|(\partial_x^3\rho, \partial_x^2\rho)\|^2 \le C\|(\partial_x^3 u, \partial_x^2 u, \partial_x u)\|^2. \qquad (2.74)$$

Taking $\partial_x^2\partial_t$ of (2.3) and taking the L^2 inner product of the resulting equation with$(1+\rho)\partial_x^2 u_t$, and taking $\partial_x^2\partial_t$ of (2.4) and taking the L^2 inner product of the resulting equation with $\partial_x^2 B_t$, and summing up them, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\left[(1+\rho)|\partial_x^2 u_t|^2 + |\partial_x^2 B_t|^2\right]dx + \int_{\mathbb{T}^3}\left[(1+\rho)|\partial_x^2 u_t|^2 + |\partial_x^2\nabla B_t|^2\right]dx\\
&-\int_{\mathbb{T}^3}\partial_x(h(1+\rho)_t - \partial_x\phi_t)\mathrm{div}((1+\rho)\partial_x u_t)dx + \int_{\mathbb{T}^3}(1+\rho)\partial_x(u\cdot\nabla u)_t\cdot\partial_x u_t dx\\
&-\frac{1}{2}\int_{\mathbb{T}^3}\rho_t|\partial_x u_t|^2 dx - \int_{\mathbb{T}^3}(1+\rho)\partial_x\left(\frac{1}{1+\rho}(\mathrm{curl}\,B\times B)\right)_t\partial_x u_t dx\\
&-\int_{\mathbb{T}^3}\mathrm{curl}\,\partial_x^2(u\times B)_t\partial_x^2 B_t dx. \qquad (2.75)
\end{aligned}$$

Similarly, the terms in (2.75) can be estimated as the previous calculations to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}[(1+\rho)|\partial_x^2 u_t|^2 + h'(1+\rho)|\partial_x^2\rho_t|^2 + (1+\rho)|\partial_x^2\nabla\phi_t|^2 + |\partial_x^2 B_t|^2]dx$$

$$
\begin{aligned}
&+C\int_{\mathbb{T}^3}((1+\rho)|\partial_x^2 u_t|^2+|\partial_x^2\nabla B_t|^2)dx\\
\le& C\delta_1\|(\partial_x^3\rho,\partial_x^3 u,\partial_x^2\nabla\phi_t)\|^2\\
&+C\|(\partial_x u,\partial_x u_t,\partial_x^2 u,\partial_x\rho,\partial_x^2\rho,\partial_x B,\partial_x^2 B)\|^2+C\|(\partial_x^2 u,\partial_x^2\rho)\|^2.
\end{aligned}\tag{2.76}
$$

Combining (2.73), (2.74), (2.76), (2.13) with $j=2$, (2.14) with $j=3$, and (2.15) with $j=2$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^3}\Big[(1+\rho)|\partial_x^3 u|^2+\frac{h'(1+\rho)}{1+\rho}|\partial_x^3\rho|^2+|\partial_x^3\nabla\phi|^2+|\partial_x^3 B|^2\\
&+(1+\rho)|\partial_x^2 u_t|^2+h'(1+\rho)|\partial_x^2\rho_t|^2+|\partial_x^2\nabla\phi_t|^2+|\partial_x^2 B_t|^2\Big]dx\\
&+C\|(\partial_x^2\rho_t,\partial_x^3\rho,\partial_x^2 u_t,\partial_x^3 u,\partial_x^3\nabla\phi,\partial_x^2\nabla\phi_t,\partial_x^2 B_t,\partial_x^3 B)\|^2\\
\le& C\|(\partial_x^2\rho,\partial_x\rho,\partial_x^2 u,\partial_x u)\|^2.
\end{aligned}\tag{2.77}
$$

with the help of the smallness of ϵ and δ_1.

Finally, combining (2.55), (2.63) and (2.77), with the help of the smallness of δ_1, we have

$$
\frac{d}{dt}\int_{\mathbb{T}^3}Gdx+C(\|(\rho,u,\nabla\phi,B)\|_3^2+\|(\rho_t,u_t,\nabla\phi_t,B_t)\|_2^2)\le 0,\tag{2.78}
$$

where

$$
\begin{aligned}
G=&A\Big\{\Big[(1+\rho)(|u|^2+|u_t|^2+|\nabla u|^2)+\int_0^\rho(h(1+\rho)-h(1)ds+|\nabla\phi|^2+|\nabla\phi_t|^2\\
&+\lambda|\nabla^2\phi|^2+|B|^2+|\nabla B|^2+|B_t|^2+h'(1+\rho)(|\rho_t|^2+|\nabla\rho|^2)+|\rho|^2\Big]\\
&+\Big[(1+\rho)|\partial_x^2 u|^2+\frac{h'(1+\rho)}{1+\rho}|\partial_x^2\rho|^2+|\partial_x^2\nabla\phi|^2+|\partial_x^2 B|^2\\
&+(1+\rho)|\partial_x u_t|^2+h'(1+\rho)|\partial_x\rho_t|^2+(1+\rho)|\partial_x\nabla\phi_t|^2+|\partial_x B_t|^2\Big]\\
&+\Big[(1+\rho)|\partial_x^3 u|^2+\frac{h'(1+\rho)}{1+\rho}|\partial_x^3\rho|^2+|\partial_x^3\nabla\phi|^2+|\partial_x^3 B|^2\\
&+(1+\rho)|\partial_x^2 u_t|^2+h'(1+\rho)|\partial_x^2\rho_t|^2+|\partial_x^2\nabla\phi_t|^2+|\partial_x^2 B_t|^2\Big]\Big\}
\end{aligned}
$$

for some positive constant A. It is easy to see that G satisfies

$$
\begin{aligned}
&c\left(\|(\rho,u,\nabla\phi,B)\|_3^2+\|(\rho_t,u_t,\nabla\phi_t,B_t)\|_2^2\right)\\
&\le\int_{\mathbb{T}^3}Gdx\\
&\le C\left(\|(\rho,u,\nabla\phi,B)\|_3^2+\|(\rho_t,u_t,\nabla\phi_t,B_t)\|_2^2\right)
\end{aligned}\tag{2.79}
$$

for some positive constants $C>c$.

So, (2.78) and (2.79) implies (2.10). And Lemma 2.2 is proved. □

Theorem 1.1 follows from the standard argument by using the local existence theorem (Lemma 2.1) and the A priori estimates given in Lemma 2.2. In fact, from the A priori estimation in Lemma 2.2, we can obtain that $(n-1,u,\nabla\phi,B)$ is uniformly bounded in space $C^1(([0,T),H^2(\mathbb{T}^3))\bigcap C([0,T),H^3(\mathbb{T}^3))$ for any $T>0$, which implies that the system (1.1)–(1.5) has a unique global smooth solution satisfying the exponential decay estimate, for detail, see [18].

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Conflict of interest

The authors declare no competing interests.

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