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Research article

Application of subordination and superordination for multivalent analytic functions associated with differintegral operator

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Abstract: The results from this paper are related to the geometric function theory. In order to obtain them, we use the technique based on the properties of the differential subordination and superordination one of the newest techniques used in this field, we obtain some differential subordination and superordination results for multivalent functions defined by differintegral operator with j-derivatives $\Im_p(\nu,\rho;\ell)f(z)$ for $\ell > 0$, $\nu,\rho \in \mathbb{R}$, such that $(\rho - j) \geq 0, \nu > -\ell p$, $(p \in \mathbb{N})$ in the open unit disk U. Differential sandwich result is also obtained. Also, the results are followed by some special cases and counter examples.

Keywords: analytic function; multivalent functions; higher order derivatives; subordination; superordination
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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,\upsilon]$ be the subclass of $H(U)$ including form-specific functions

$$f(z) = a + a_\upsilon z^\upsilon + a_{\upsilon+1} z^{\upsilon+1} + \dots \quad (a \in \mathbb{C}),$$

we denote by $H = H[1,1]$.

Also, $A(p)$ should denote the class of multivalent analytic functions in U, with the power series expansion of the type:

$$f(z) = z^p + \sum_{\upsilon=p+1}^{\infty} a_\upsilon z^\upsilon \quad (p \in \mathbb{N} = \{1,2,3,..\}). \tag{1.1}$$

Upon differentiating j-times for each one of the (1.1) we obtain:

$$\begin{aligned} f^{(j)}(z) &= \delta(p,j)z^{p-j} + \sum_{\upsilon=p+1}^{\infty} \delta(\upsilon,j)a_\upsilon z^{\upsilon-j} \qquad z \in U, \\ \delta(p,j) &= \frac{p!}{(p-j)!} \qquad (p \in \mathbb{N},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ p \geq j). \end{aligned} \tag{1.2}$$

Numerous mathematicians, for instance, have looked at higher order derivatives of multivalent functions (see [1,3,6,9,16,27,28,31]).

For $f, \hbar \in H$, the function f is subordinate to $\hbar$ or the function $\hbar$ is said to be superordinate to f in U and we write $f(z) \prec \hbar(z)$, if there exists a Schwarz function ω in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = \hbar(\omega(z))$, $z \in U$. If $\hbar$ is univalent in U, then $f(z) \prec \hbar(z)$ iff $f(0) = \hbar(0)$ and $f(U) \subset \hbar(U)$. (see [7,21]).

In the concepts and common uses of fractional calculus (see, for example, [14,15] see also [2]; the Riemann-Liouville fractional integral operator of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) is one of the most widely used operators (see [29]) given by:

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\mu)^{\alpha-1} f(\mu)\, \mathrm{d}\mu \qquad (x > 0;\ \Re(\alpha) > 0) \tag{1.3}$$

applying the well-known (Euler's) Gamma function $\Gamma(\alpha)$. The Erdélyi-Kober fractional integral operator of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) is an interesting alternative to the Riemann-Liouville operator I_{0+}^{α}, defined by:

$$(I_{0+;\sigma,\eta}^{\alpha} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \mu^{\sigma(\eta+1)-1} (x^\sigma - \mu^\sigma)^{\alpha-1} f(\mu)\, \mathrm{d}\mu \tag{1.4}$$

$$(x > 0;\ \Re(\alpha) > 0),$$

which corresponds essentially to (1.3) when $\sigma - 1 = \eta = 0$, since

$$(I_{0+;1,0}^{\alpha} f)(x) = x^{-\alpha} (I_{0+}^{\alpha} f)(x) \qquad (x > 0;\ \Re(\alpha) > 0).$$

Mainly motivated by the special case of the definition (1.4) when $x = \sigma = 1$, $\eta = \nu - 1$ and $\alpha = \rho - \nu$, here, we take a look at the integral operator $\Im_p(\nu, \rho, \mu)$ with $f \in A(p)$ by (see [11])

$$\Im_p(\nu,\rho;\ell)f(z) = \frac{\Gamma(\rho+\ell p)}{\Gamma(\nu+\ell p)\Gamma(\rho-\nu)} \int_0^1 \mu^{\nu-1} (1-\mu)^{\rho-\nu-1} f(z\mu^\ell)\, \mathrm{d}\mu$$

$$(\ell > 0;\ \nu, \rho \in \mathbb{R};\ \rho > \nu > -\ell p;\ p \in \mathbb{N}).$$

Evaluating (Euler's) Gamma function by using the Eulerian Beta-function integral as following:

$$\mathrm{B}(\alpha,\beta) := \begin{cases} \displaystyle\int_0^1 \mu^{\alpha-1} (1-\mu)^{\beta-1}\, \mathrm{d}\mu & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0'), \end{cases}$$

we readily find that

$$\Im_p(\nu,\rho;\ell)f(z) = \begin{cases} z^p + \frac{\Gamma(\rho+p\ell)}{\Gamma(\nu+p\ell)} \sum\limits_{\upsilon=p+1}^{\infty} \frac{\Gamma(\nu+\upsilon\ell)}{\Gamma(\rho+\upsilon\ell)} a_\upsilon z^\upsilon & (\rho > \nu) \\ \\ f(z) & (\rho = \nu). \end{cases} \tag{1.5}$$

It is readily to obtain from (1.5) that

$$z(\Im_p(\nu,\rho;\ell)f(z))' = (\frac{\nu}{\ell} + p)(\Im_p(\nu+1,\rho;\ell)f(z)) - \frac{\nu}{\ell}(\Im_p(\nu,\rho;\ell)f(z)). \tag{1.6}$$

The integral operator $\Im_p(\nu,\rho;\ell)f(z)$ should be noted as a generalization of several other integral operators previously discussed for example,

(i) If we set $p = 1$, we get $\tilde{I}(\nu,\rho;\ell)f(z)$ defined by Ŕaina and Sharma ([22] with $m = 0$);

(ii) If we set $\nu = \beta, \rho = \beta + 1$ and $\ell = 1$, we obtain $\Im_p^\beta f(z)(\beta > -p)$ it was presented by Saitoh et al. [24];

(iii) If we set $\nu = \beta, \rho = \alpha + \beta - \delta + 1,\ \ell = 1$, we obtain $\Re_{\beta,p}^{\alpha,\delta} f(z)(\delta > 0; \alpha \geq \delta - 1; \beta > -p)$ it was presented by Aouf et al. [4];

(iv) If we put $\nu = \beta, \rho = \alpha + \beta,\ \ell = 1$, we get $Q_{\beta,p}^{\alpha} f(z)(\alpha \geq 0; \beta > -p)$ it was investigated by Liu and Owa [18];

(v) If we put $p = 1, \nu = \beta, \rho = \alpha + \beta,\ \ell = 1$, we obtain $\Re_\beta^\alpha f(z)(\alpha \geq 0; \beta > -1)$ it was introduced by Jung et al. [13];

(vi) If we put $p = 1,\ \nu = \alpha - 1,\ \rho = \beta - 1,\ \ell = 1$, we obtain $L(\alpha,\beta)f(z)(\alpha,\beta \in \mathbb{C}\backslash\mathbb{Z}_0,\ \mathbb{Z}_0 = \{0, -1, -2, ...\})$ which was defined by Carlson and Shaffer [8];

(vii) If we put $p = 1,\ \nu = \nu - 1,\ \rho = j,\ \ell = 1$ we obtain $I_{\nu,j} f(z)(\nu > 0; j \geq -1)$ it was investigated by Choi et al. [10];

(viii) If we put $p = 1,\ \nu = \alpha, \rho = 0,\ \ell = 1$, we obtain $D^\alpha f(z)(\alpha > -1)$ which was defined by Ruscheweyh [23];

(ix) If we put $p = 1,\ \nu = 1,\ \rho = m,\ \ell = 1$, we obtain $I_m f(z)(m \in \mathbb{N}_0)$ which was introduced by Noor [21];

(x) If we set $p = 1, \nu = \beta, \rho = \beta + 1,\ \ell = 1$ we obtain $\Im_\beta f(z)$ which was studied by Bernadi [5];

(xi) If we set $p = 1, \nu = 1,\ \rho = 2,\ \ell = 1$ we get $\Im f(z)$ which was defined by Libera [17].

2. Key lemmas

We state various definition and lemmas which are essential to obtain our results.

Definition 1. *([20], Definition 2, p.817) We denote by Q the set of the functions f that are holomorphic and univalent on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta : \zeta \in \partial U \ \ \textit{and} \ \ \lim_{z\to\zeta} f(z) = \infty\},$$

and satisfy $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1. *([12]; see also ([19], Theorem 3.1.6, p.71)) Assume that $h(z)$ is convex (univalent) function in U with $h(0) = 1$, and let $\varphi(z) \in H$, is analytic in U. If*

$$\varphi(z) + \frac{1}{\gamma} z\varphi'(z) \prec h(z) \qquad (z \in U),$$

where $\gamma \neq 0$ and $Re(\gamma) \geq 0$. Then

$$\varphi(z) \prec \Psi(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)dt \prec h(z) \qquad (z \in U),$$

and $\Psi(z)$ is the best dominant.

Lemma 2. *([26]; Lemma 2.2, p.3) Suppose that q is convex function in U and let $\psi \in C$ with $\varkappa \in C^* = C\backslash\{0\}$ with*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\frac{\psi}{\varkappa}\right\}, \; z \in U.$$

If $\lambda(z)$ is analytic in U, and

$$\psi\lambda(z) + \varkappa z\lambda'(z) \prec \psi q(z) + \varkappa z q'(z),$$

therefore $\lambda(z) \prec q(z)$, and q is the best dominant .

Lemma 3. *([20]; Theorem 8, p.822) Assume that q is convex univalent in U and suppose $\delta \in C$, with $Re(\delta) > 0$. If $\lambda \in H[q(0), 1] \cap Q$ and $\lambda(z) + \delta z\lambda'(z)$ is univalent in U, then*

$$q(z) + \delta z q'(z) \prec \lambda(z) + \delta z\lambda'(z),$$

implies

$$q(z) \prec \lambda(z) \qquad (z \in U)$$

and q is the best subordinant.

For a, ϱ, c and $c(c \notin \mathbb{Z}_0^-)$ real or complex number the Gaussian hypergeometric function is given by

$${}_2F_1(a, \varrho; c; z) = 1 + \frac{a\varrho}{c} \cdot \frac{z}{1!} + \frac{a(a+1)\varrho(\varrho+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots .$$

The previous series totally converges for $z \in U$ to a function analytical in U (see, for details, ([30], Chapter 14)) see also [19].

Lemma 4. *For a, ϱ and c $(c \notin \mathbb{Z}_0^-)$, real or complex parameters,*

$$\int_0^1 t^{\varrho-1}(1-t)^{c-\varrho-1}(1-zt)^{-x}dt = \frac{\Gamma(\varrho)\Gamma(c-a)}{\Gamma(c)} {}_2F_1(a, \varrho; c; z) \quad (\mathrm{Re}(c) > \mathrm{Re}(\varrho) > 0); \tag{2.1}$$

$${}_2F_1(a, \varrho; c; z) = {}_2F_1(\varrho, a; c; z); \tag{2.2}$$

$${}_2F_1(a, \varrho; c; z) = (1-z)^{-a} {}_2F_1(a, c-\varrho; c; \frac{z}{z-1}); \tag{2.3}$$

$${}_2F_1(1, 1; 2; \frac{az}{az+1}) = \frac{(1+az)\ln(1+az)}{az}; \tag{2.4}$$

$${}_2F_1(1, 1; 3; \frac{az}{az+1}) = \frac{2(1+az)}{az}\left(1 - \frac{\ln(1+az)}{az}\right). \tag{2.5}$$

3. Main results

Throughout the sequel, we assume unless otherwise indicated $-1 \leq D < C \leq 1$, $\delta > 0$, $\ell > 0$, $\nu, \rho \in \mathbb{R}$, $\nu > -\ell p$, $p \in \mathbb{N}$ and $(\rho - j) \geq 0$. We shall now prove the subordination results stated below:

Theorem 1. *Let $0 \leq j < p$, $0 < r \leq 1$ and for $f \in A(p)$ assume that*

$$\frac{(\mathfrak{I}_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U, \tag{3.1}$$

whenever $\delta \in (0,+\infty)\backslash\mathbb{N}$. Let define the function Φ_j by

$$\Phi_j(z) = (1-\alpha)\left(\frac{\left(\mathfrak{I}_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} + \alpha\frac{\left(\mathfrak{I}_p(\nu+1,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\left(\frac{\left(\mathfrak{I}_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta-1},$$

such that the powers are all the principal ones, i.e., log1=0. Whether

$$\Phi_j(z) \prec \left[\frac{p!}{(p-j)!}\right]^{\delta}\left(\frac{1+Cz}{1+Dz}\right)^{r}, \tag{3.2}$$

then

$$\left(\frac{\left(\mathfrak{I}_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \prec \left[\frac{p!}{(p-j)!}\right]^{\delta} p(z), \tag{3.3}$$

where

$$p(z) = \begin{cases} \left(\frac{C}{D}\right)^r \sum\limits_{i\geq 0} \frac{(-r)_i}{i!}\left(\frac{C-D}{C}\right)^i (1+Dz)^{-i}\, {}_2F_1\left(i,1;1+\frac{\delta(\nu+\ell p)}{\alpha\ell};\frac{Dz}{1+Dz}\right) & (D \neq 0); \\ {}_2F_1\left(-r,\frac{\delta(\nu+\ell p)}{\alpha\ell};1+\frac{\delta(\nu+\ell p)}{\alpha\ell};-Cz\right) & (D = 0), \end{cases}$$

and $\left[\frac{p!}{(p-j)!}\right]^{\delta} p(z)$ is the best dominant of (3.3). Moreover, there are

$$\Re\left(\frac{\left(\mathfrak{I}_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} > \left[\frac{p!}{(p-j)!}\right]^{\delta}\zeta, \quad z \in U, \tag{3.4}$$

where ζ is given by:

$$\zeta = \begin{cases} \left(\frac{C}{D}\right)^r \sum\limits_{i\geq 0} \frac{(-r)_i}{i!}\left(\frac{C-D}{C}\right)^i (1-D)^{-i}\, {}_2F_1\left(i,1;1+\frac{\delta(\nu+\ell p)}{\alpha\ell};\frac{D}{D-1}\right) & (D \neq 0); \\ {}_2F_1\left(-r,\frac{\delta(\nu+\ell p)}{\alpha\ell};1+\frac{\delta(\nu+\ell p)}{\alpha\ell};C\right) & (D = 0), \end{cases}$$

then (3.4) is the best possible.

Proof. Let

$$\phi(z) = \left(\frac{(p-j)!}{p!}\frac{\left(\mathfrak{I}_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}, \quad (z \in U). \tag{3.5}$$

It is observed that the function $\phi(z) \in H$, which is analytic in U and $\phi(0) = 1$. Differentiating (3.5) with respect to z, applying the given equation, the hypothesis (3.2), and the knowing that

$$\begin{aligned} z(\mathfrak{I}_p(\nu,\rho;\ell)f(z))^{(j+1)} &= (\frac{\nu}{\ell}+p)(\mathfrak{I}_p(\nu+1,\rho;\ell)f(z))^{(j)} - (\frac{\nu}{\ell}+j)(\mathfrak{I}_p(\nu,\rho;\ell)f(z))^{(j)} \\ (0 &\leq j < p), \end{aligned} \tag{3.6}$$

we get

$$\phi(z) + \frac{z\phi'(z)}{\frac{\delta(\nu+\ell p)}{\alpha\ell}} \prec \left(\frac{1+Cz}{1+Dz}\right)^r = q(z) \quad (z \in U).$$

We can verify that the above equation $q(z)$ is analytic and convex in U as following

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) &= -1 + (1-r)\Re\left(\frac{1}{1+Cz}\right) + (1+r)\Re\left(\frac{1}{1+Dz}\right) \\ &> -1 + \frac{1-r}{1+|C|} + \frac{1+r}{1+|D|} \geq 0 \quad (z \in U). \end{aligned}$$

Using Lemma 1, there will be

$$\phi(z) \prec p(z) = \frac{\delta(\nu+\ell p)}{\alpha\ell} z^{-\frac{\delta(\nu+\ell p)}{\alpha\ell}} \int_0^z t^{\frac{\delta(\nu+\ell p)}{\alpha\ell}-1}\left(\frac{1+Ct}{1+Dt}\right)^r dt.$$

In order to calculate the integral, we define the integrand in the type

$$t^{\frac{\delta(\nu+\ell p)}{\alpha\ell}-1}\left(\frac{1+Ct}{1+Dt}\right)^r = t^{\frac{\delta(\nu+\ell p)}{\alpha\ell}-1}\left(\frac{C}{D}\right)^r\left(1 - \frac{C-D}{C+CDt}\right)^r,$$

using Lemma 4 we obtain

$$p(z) = \left(\frac{C}{D}\right)^r \sum_{i\geq 0} \frac{(-r)_i}{i!}(\frac{C-D}{C})^i(1+Dz)^{-i}\ {}_2F_1(i,1;1+\frac{\delta(\nu+\ell p)}{\alpha\ell};\frac{Dz}{1+Dz})(D \neq 0).$$

On the other hand if $D = 0$ we have

$$p(z) =_2 F_1(-r,\frac{\delta(\nu+\ell p)}{\alpha\ell};1+\frac{\delta(\nu+\ell p)}{\alpha\ell};-Cz),$$

where the identities (2.1)–(2.3), were used after changing the variable, respectively. This proof the inequality (3.3).

Now, we'll verify it

$$\inf\{\Re p(z) : |z| < 1\} = p(-1). \tag{3.7}$$

Indeed, we have

$$\Re\left(\frac{1+Cz}{1+Dz}\right)^r \geq \left(\frac{1-C\sigma}{1-D\sigma}\right)^r \quad (|z| < \sigma < 1).$$

Setting

$$\hbar(s,z) = \left(\frac{1+Csz}{1+Dsz}\right)^r \quad (0 \leq s \leq 1;\ z \in U)$$

and

$$dv(s) = \frac{\delta(\nu + \ell p)}{\alpha\ell} s^{\frac{\delta(\nu+\ell p)}{\alpha\ell}-1} ds$$

where $dv(s)$ is a positive measure on the closed interval [0,1], we get that

$$p(z) = \int_0^1 \hbar(s,z)dv(s),$$

so that

$$\Re p(z) \geq \int_0^1 \left(\frac{1 - Cs\sigma}{1 - Ds\sigma}\right)^r dv(s) = p(-\sigma) \quad (|z| < \sigma < 1).$$

Now, taking $\sigma \to 1^-$ we get the result (3.7). The inequality (3.4) is the best possible since $\left[\frac{p!}{(p-j)!}\right]^\delta p(z)$ is the best dominant of (3.3). □

If we choose $j = 1$ and $\alpha = \delta = 1$ in Theorem 1, we get:

Corollary 1. *Let* $0 < r \leq 1$. *If*

$$\frac{\left(\Im_p(\nu + 1, \rho; \ell) f(z)\right)'}{z^{p-1}} \prec p\left(\frac{1 + Cz}{1 + Dz}\right)^r,$$

then

$$\Re\left(\frac{\left(\Im_p(\nu, \rho; \ell) f(z)\right)'}{z^{p-1}}\right) > p\zeta_1, \quad z \in U, \tag{3.8}$$

where ζ_1 *is given by:*

$$\zeta_1 = \begin{cases} \left(\frac{C}{D}\right)^r \sum\limits_{i\geq 0} \frac{(-r)_i}{i!} (\frac{C-D}{C})^i (1-D)^{-i} \, {}_2F_1(i, 1; 1 + \frac{(\nu+\ell p)}{\ell}; \frac{D}{D-1}) & (D \neq 0); \\ {}_2F_1(-r, \frac{(\nu+\ell p)}{\ell}; 1 + \frac{(\nu+\ell p)}{\ell}; C) & (D = 0), \end{cases}$$

then (3.8) is the best possible.

If we choose $\nu = \rho = 0$ and $\ell = 1$ in Theorem 1, we get:

Corollary 2. *Let* $0 \leq j < p$, $0 < r \leq 1$ *and as* $f \in A(p)$ *assume that*

$$\frac{f^{(j)}(z)}{z^{p-j}} \neq 0, \quad z \in U,$$

whenever $\delta \in (0, +\infty)\backslash\mathbb{N}$. *Let define the function* Φ_j *by*

$$\Phi_j(z) = [1 - \alpha(1 - \frac{j}{p})]\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)^\delta + \alpha\left(\frac{zf^{(j+1)}(z)}{pf^{(j)}(z)}\right)\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)^\delta, \tag{3.9}$$

such that the powers are all the principal ones, i.e., log1=0. If

$$\Phi_j(z) \prec \left[\frac{p!}{(p-j)!}\right]^{\delta}\left(\frac{1+Cz}{1+Dz}\right)^{r},$$

then

$$\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)^{\delta} \prec \left[\frac{p!}{(p-j)!}\right]^{\delta} p_1(z), \tag{3.10}$$

where

$$p_1(z) = \begin{cases} \left(\frac{C}{D}\right)^{r} \sum\limits_{i\geq 0} \frac{(-r)_i}{i!}\left(\frac{C-D}{C}\right)^{i}(1+Dz)^{-i}\ {}_2F_1(i,1;1+\frac{\delta p}{\alpha};\frac{Dz}{1+Dz}) & (D\neq 0); \\ {}_2F_1(-r,\frac{\delta p}{\alpha};1+\frac{\delta p}{\alpha};-Cz) & (D=0), \end{cases}$$

and $\left[\frac{p!}{(p-j)!}\right]^{\delta} p_1(z)$ *is the best dominant of (3.10). Morover, there are*

$$\Re\left(\frac{f^{(j)}(z)}{z^{p-j}}\right)^{\delta} > \left[\frac{p!}{(p-j)!}\right]^{\delta}\zeta_2, \quad z\in U, \tag{3.11}$$

where ζ_2 *is given by*

$$\zeta_2 = \begin{cases} \left(\frac{C}{D}\right)^{r} \sum\limits_{i\geq 0} \frac{(-r)_i}{i!}\left(\frac{C-D}{C}\right)^{i}(1-D)^{-i}\ {}_2F_1(i,1;1+\frac{\delta p}{\alpha};\frac{D}{D-1}) & (D\neq 0); \\ {}_2F_1(-r,\frac{\delta p}{\alpha};1+\frac{\delta p}{\alpha};C) & (D=0), \end{cases}$$

then (3.11) is the best possible.

If we put $\delta = 1$ and $r = 1$ in Corollary 2, we get:

Corollary 3. *Let* $0 \leq j < p$, *and for* $f \in A(p)$ *say it*

$$\frac{f^{(j)}(z)}{z^{p-j}} \neq 0, \quad z \in U.$$

Let define the function Φ_j *by*

$$\Phi_j(z) = [(1-\alpha(1-\frac{j}{p})]\frac{f^{(j)}(z)}{z^{p-j}} + \alpha\frac{f^{(j+1)}(z)}{pz^{p-j-1}}.$$

If

$$\Phi_j(z) \prec \frac{p!}{(p-j)!}\frac{1+Cz}{1+Dz},$$

then

$$\frac{f^{(j)}(z)}{z^{p-j}} \prec \frac{p!}{(p-j)!}p_2(z), \tag{3.12}$$

where

$$p_2(z) = \begin{cases} \frac{C}{D} + (1-\frac{C}{D})(1+Dz)^{-1}\ {}_2F_1(1,1;1+\frac{p}{\alpha};\frac{Dz}{1+Dz}) & (D\neq 0); \\ 1+\frac{p}{p+\alpha}Cz, & (D=0), \end{cases}$$

and $\frac{p!}{(p-j)!}p_2(z)$ *is the best dominant of (3.12). Morover there will be*

$$\Re\left(\frac{f^{(j)}(z)}{z^{p-j}}\right) > \frac{p!}{(p-j)!}\zeta_3, \quad z\in U, \tag{3.13}$$

where ζ_3 is given by:

$$\zeta_3 = \begin{cases} \frac{C}{D} + (1 - \frac{C}{D})(1 - D)^{-1}\ {}_2F_1(1, 1; 1 + \frac{p}{\alpha}; \frac{D}{D-1}) & (D \neq 0); \\ 1 - \frac{p}{p+\alpha}C, & (D = 0), \end{cases}$$

then (3.13) is the best possible.

For $C = 1, D = -1$ and $j = 1$ Corollary 3, leads to the next example:

Example 1. *(i) For $f \in A(p)$ suppose that*

$$\frac{f'(z)}{z^{p-1}} \neq 0, \quad z \in U.$$

Let define the function Φ_j by

$$\Phi_j(z) = [1 - (\alpha - \frac{\alpha}{p})]\frac{f'(z)}{z^{p-1}} + \alpha\frac{f''(z)}{pz^{p-2}} \prec p\frac{1+z}{1-z},$$

then

$$\frac{f'(z)}{z^{p-1}} \prec p\frac{1+z}{1-z}, \tag{3.14}$$

and

$$\Re\left(\frac{f'(z)}{z^{p-1}}\right) > p\zeta_4, \quad z \in U, \tag{3.15}$$

where ζ_4 is given by:

$$\zeta_4 = -1 + {}_2F_1(1, 1; \frac{p+\alpha}{\alpha}; \frac{1}{2}),$$

then (3.15) is the best possible.

(ii) For $p = \alpha = 1$, (i) leads to:

For $f \in A$ suppose that

$$f'(z) \neq 0, \quad z \in U.$$

Let define the function Φ_j by

$$\Phi_j(z) = f'(z) + zf''(z) \prec \frac{1+z}{1-z},$$

then

$$\Re\left(f'(z)\right) > -1 + 2\ln 2, \quad z \in U.$$

So the estimate is best possible.

Theorem 2. *Let $0 \leq j < p$, $0 < r \leq 1$ as for $f \in A(p)$. Assume that $\mathcal{F}_\alpha$ is defined by*

$$\mathcal{F}_\alpha(z) = \alpha(\frac{\nu}{\ell} + p)(\Im_p(\nu + 1, \rho; \ell)f(z)) + (1 - \alpha - \alpha(\frac{\nu}{\ell}))(\Im_p(\nu, \rho; \ell)f(z)). \tag{3.16}$$

If

$$\frac{\mathcal{F}_{\alpha}^{(j)}(z)}{z^{p-j}} \prec (1-\alpha+\alpha p)\frac{p!}{(p-j)!}\left(\frac{1+Cz}{1+Dz}\right)^{r}, \tag{3.17}$$

then

$$\frac{\left(\mathfrak{I}_{p}(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!}p(z), \tag{3.18}$$

where

$$p(z)=\begin{cases} \left(\frac{C}{D}\right)^{r}\sum\limits_{i\geq 0}\frac{(-r)_i}{i!}(\frac{C-D}{C})^{i}(1+Dz)^{-i}\,{}_2F_1(i,1;1+\frac{(1-\alpha+\alpha p)}{\alpha};\frac{Dz}{1+Dz}) & (D\neq 0);\\ {}_2F_1(-r,\frac{(1-\alpha+\alpha p)}{\alpha};1+\frac{(1-\alpha+\alpha p)}{\alpha};-Cz) & (D=0),\end{cases}$$

and $\frac{p!}{(p-j)!}p(z)$ *is the best dominant of (3.18). Moreover, there will be*

$$\Re\left(\frac{\left(\mathfrak{I}_{p}(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right) > \frac{p!}{(p-j)!}\eta,\ \ z\in U, \tag{3.19}$$

where η *is given by:*

$$\eta=\begin{cases} \left(\frac{C}{D}\right)^{r}\sum\limits_{i\geq 0}\frac{(-r)_i}{i!}(\frac{C-D}{C})^{i}(1+D)^{-i}\,{}_2F_1(i,1;1+\frac{(1-\alpha+\alpha p)}{\alpha};\frac{D}{D-1}) & (D\neq 0);\\ {}_2F_1(-r,\frac{(1-\alpha+\alpha p)}{\alpha};1+\frac{(1-\alpha+\alpha p)}{\alpha};C) & (D=0),\end{cases}$$

then (3.19) is the best possible.

Proof. By using the definition (3.16) and the inequality (3.6), we have

$$\mathcal{F}_{\alpha}^{(j)}(z)=\alpha z(\mathfrak{I}_{p}(\nu,\rho;\ell)f(z))^{(j+1)}+(1-\alpha+\alpha j)(\mathfrak{I}_{p}(\nu,\rho;\ell)f(z))^{(j)}, \tag{3.20}$$

for $0\leq j<p$. *Putting*

$$\phi(z)=\frac{(p-j)!}{p!}\frac{\left(\mathfrak{I}_{p}(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}},\quad (z\in U), \tag{3.21}$$

we have that $\phi\in H$. Differentiating (3.21), and using (3.17), (3.20), we get

$$\phi(z)+\frac{z\phi'(z)}{\frac{(1-\alpha+\alpha p)}{\alpha}} \prec \left(\frac{1+Cz}{1+Dz}\right)^{r} \qquad (z\in U).$$

Following the techniques of Theorem 1, we can obtain the remaining part of the proof. □

If we choose $j=1$ and $r=1$ in Theorem 2, we get:

Corollary 4. *For* $f\in A(p)$ *let the function* $\mathcal{F}_{\alpha}$ *define by 3.16. If*

$$\frac{\mathcal{F}_{\alpha}'(z)}{z^{p-1}} \prec p(1-\alpha+\alpha p)\frac{1+Cz}{1+Dz},$$

then

$$\Re\left(\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)'}{z^{p-1}}\right) > p\eta_1, \quad z\in U, \tag{3.22}$$

where η_1 is given by:

$$\eta_1 = \begin{cases} \frac{C}{D} + (1-\frac{C}{D})(1-D)^{-1}\ {}_2F_1(1,1;1+\frac{1-\alpha+\alpha p}{\alpha};\frac{D}{D-1}) & (D\neq 0);\\ 1-\frac{1-\alpha+\alpha p}{1+\alpha p}C & (D=0),\end{cases}$$

then (3.22) is the best possible.

Example 2. *If we choose $p = C = \alpha = 1$ and $D = -1$ in Corollary 4, we obtain:*

For

$$\mathcal{F}(z) = (\frac{\nu}{\ell}+1)(\Im(\nu+1,\rho;\ell)f(z)) - (\frac{\nu}{\ell})(\Im(\nu,\rho;\ell)f(z)).$$

If

$$\mathcal{F}'(z) \prec \frac{1+z}{1-z},$$

then

$$\Re\left(\left(\Im(\nu,\rho;\ell)f(z)\right)'\right) > -1+2\ln 2, \quad z\in U,$$

the result is the best possible.

Theorem 3. *Let $0\le j<p$, $0<r\le 1$ as for $\theta>-p$ assume that $J_{p,\theta}: A(p)\to A(p)$ defined by*

$$J_{p,\theta}(f)(z) = \frac{p+\theta}{z^\theta}\int_0^z t^{\theta-1}f(t)dt, \quad z\in U. \tag{3.23}$$

If

$$\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!}\left(\frac{1+Cz}{1+Dz}\right)^r, \tag{3.24}$$

then

$$\frac{\left(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z)\right)^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!}p(z), \tag{3.25}$$

where

$$p(z) = \begin{cases} \left(\frac{C}{D}\right)^r \sum\limits_{i\ge 0}\frac{(-r)_i}{i!}(\frac{C-D}{C})^i(1+Dz)^{-i}\ {}_2F_1(i,1;1+\theta+p;\frac{Dz}{1+Dz}) & (D\neq 0);\\ {}_2F_1(-r,\theta+p;1+\theta+p;Cz) & (D=0),\end{cases}$$

and $\frac{p!}{(p-j)!}p(z)$ is the best dominant of (3.25). Moreover, there will be

$$\Re\left(\frac{\left(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z)\right)^{(j)}}{z^{p-j}}\right) > \frac{p!}{(p-j)!}\beta, \quad z\in U, \tag{3.26}$$

where β *is given by:*

$$\beta = \begin{cases} \left(\frac{C}{D}\right)^r \sum\limits_{i\geq 0} \frac{(-r)_i}{i!}\left(\frac{C-D}{C}\right)^i (1+D)^{-i}\, {}_2F_1(i,1;1+\theta+p;\frac{D}{D-1}) & (D \neq 0); \\ {}_2F_1(-r,\theta+p;1+\theta+p;-C) & (D = 0), \end{cases}$$

then (3.26) is the best possible.

Proof. Suppose

$$\phi(z) = \frac{(p-j)!}{p!} \frac{\left(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z)\right)^{(j)}}{z^{p-j}}, \quad (z \in U),$$

we have that $\phi \in H$. Differentiating the above definition, by using (3.24) and

$$z\left(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z)\right)^{(j+1)} = (\theta+p)(\Im_p(\nu,\rho;\ell)f(z))^{(j)}$$
$$-(\theta+j)(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z))^{(j)} \quad (0 \leq j < p),$$

we get

$$\phi(z) + \frac{z\phi'(z)}{\theta+p} \prec \left(\frac{1+Cz}{1+Dz}\right)^r.$$

Now, we obtain (3.25) and the inequality (3.26) follow by using the same techniques in Theorem 1. □

If we set $j = 1$ and $r = 1$ in Theorem 3, we get:

Corollary 5. *For* $\theta > -p$, *let the operator* $J_{p,\theta} : A(p) \to A(p)$ *defined by (3.25). If*

$$\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)'}{z^{p-1}} \prec p\frac{1+Cz}{1+Dz},$$

then

$$\Re\left(\frac{\left(\Im_p(\nu,\rho;\ell)J_{p,\theta}(f)(z)\right)'}{z^{p-1}}\right) > p\beta_1, \quad z \in U, \tag{3.27}$$

where β_1 *is given by:*

$$\beta_1 = \begin{cases} \frac{C}{D} + (1-\frac{C}{D})(1-D)^{-1}\, {}_2F_1(1,1;1+\theta+p;\frac{D}{D-1}) & (D \neq 0); \\ 1 - \frac{\theta+p}{1+\theta+p}C & (D = 0), \end{cases}$$

then (3.27) is the best possible.

Example 3. *If we choose* $p = C = \theta = 1$ *and* $D = -1$ *in Corollary 5, we get:*

If

$$(\Im(\nu,\rho;\ell)f(z))' \prec \frac{1+z}{1-z},$$

then

$$\Re\left((\Im(\nu,\rho;\ell)J_{1,1}(f)(z))'\right) > -1 + 4(1-\ln 2),$$

the result is the best possible.

Theorem 4. *Let q is univalent function in U, such that q satisfies*

$$Re\left(1+\frac{zq^{''}(z)}{q'(z)}\right)>\max\left\{0;-\frac{\delta(\nu+\ell p)}{\alpha\ell}\right\},\quad z\in U. \tag{3.28}$$

Let $0\leq j<p$, $0<r\leq 1$ *and for* $f\in A(p)$ *assume* that

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}}\neq 0,\quad z\in U,$$

whenever $\delta\in(0,+\infty)\backslash\mathbb{N}$. *Let the function* Φ_j *defined by (3.1), and assume that it satisfies:*

$$\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)\prec q(z)+\frac{\alpha\ell}{\delta(\nu+\ell p)}zq^{'}(z). \tag{3.29}$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}\prec q(z), \tag{3.30}$$

and $q(z)$ *is the best dominant of (3.30).*

Proof. Let $\phi(z)$ is defined by (3.5), from Theorem 1 we get

$$\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)=\phi(z)+\frac{\alpha\ell}{\delta(\nu+\ell p)}z\phi^{'}(z). \tag{3.31}$$

Combining (3.29) and (3.31) we find that

$$\phi(z)+\frac{\alpha\ell}{\delta(\nu+\ell p)}z\phi^{'}(z)\prec q(z)+\frac{\alpha\ell}{\delta(\nu+\ell p)}zq^{'}(z). \tag{3.32}$$

The proof of Theorem 4 follows by using Lemma 2 and (3.32). □

Taking $q(z)=\left(\frac{1+Cz}{1+Dz}\right)^r$ in Theorem 4, we obtain:

Corollary 6. *Suppose that*

$$Re\left(\frac{1-Dz}{1+Dz}+\frac{(r-1)(C-D)z}{(1+Dz)(1+Cz)}\right)>\max\left\{0;-\frac{\delta(\nu+\ell p)}{\alpha\ell}\right\},\quad z\in U.$$

Let $0\leq j<p$, $0<r\leq 1$ *and for* $f\in A(p)$ *satisfies*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}}\neq 0,\quad z\in U,$$

whenever $\delta\in(0,+\infty)\backslash\mathbb{N}$. *Let the function* Φ_j *defined by (3.1), satisfies:*

$$\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)\prec\left(\frac{1+Cz}{1+Dz}\right)^r+\frac{\alpha\ell}{\delta(\nu+\ell p)}\left(\frac{1+Cz}{1+Dz}\right)^r\frac{r(C-D)z}{(1+Dz)(1+Cz)}.$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}\prec\left(\frac{1+Cz}{1+Dz}\right)^{r},\tag{3.33}$$

so $\left(\frac{1+Cz}{1+Dz}\right)^r$ *is the best dominant of (3.33).*

Taking $q(z)=\frac{1+Cz}{1+Dz}$ in Theorem 4, we get:

Corollary 7. *Suppose that*

$$Re\left(\frac{1-Dz}{1+Dz}\right)>\max\left\{0;-\frac{\delta(\nu+\ell p)}{\alpha\ell}\right\},\ \ z\in U.$$

Let $0\le j<p$, $0<r\le 1$ *and for* $f\in A(p)$ *satisfies*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}}\neq 0,\quad z\in U,$$

whenever $\delta\in(0,+\infty)\backslash\mathbb{N}$. *Let the function* Φ_j *defined by (3.1), satisfies:*

$$\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)\prec\frac{1+Cz}{1+Dz}+\frac{\alpha\ell}{\delta(\nu+\ell p)}\frac{(C-D)z}{(1+Dz)^2}.$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}\prec\frac{1+Cz}{1+Dz},\tag{3.34}$$

so $\frac{1+Cz}{1+Dz}$ *is the best dominant of (3.34).*

If we put $\nu=\rho=0$ and $\ell=1$ in Theorem 4, we get:

Corollary 8. *Let q is univalent function in U, such that q satisfies*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\frac{\delta p}{\alpha}\right\},\ \ z\in U.$$

For $f\in A(p)$ *satisfies*

$$\frac{f^{(j)}(z)}{z^{p-j}}\neq 0,\quad z\in U.$$

Let the function Φ_j *defined by (3.9), satisfies:*

$$\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)\prec q(z)+\frac{\alpha}{\delta p}zq'(z).\tag{3.35}$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{f^{(j)}(z)}{z^{p-j}}\right)^{\delta}\prec q(z),\tag{3.36}$$

so $q(z)$ *is the best dominant of (3.36).*

Taking $C=1$ and $D=-1$ in Corollaries 6 and 7 we get:

Example 4. *(i) For $f \in A(p)$ assume that*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U.$$

Let the function Φ_j defined by (3.1), and assume that it satisfies:

$$\left[\frac{(p-j)!}{p!}\right]^{\delta} \Phi_j(z) \prec \left(\frac{1+z}{1-z}\right)^{r} + \frac{\alpha\ell}{\delta(\nu+\ell p)}\left(\frac{1+z}{1-z}\right)^{r}\frac{2rz}{1-z^2}.$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{r}, \tag{3.37}$$

so $\left(\frac{1+z}{1-z}\right)^{r}$ is the best dominant of (3.37).

(ii) For $f \in A(p)$ say it

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U.$$

Let the function Φ_j defined by (3.1), and assume that it satisfies:

$$\left[\frac{(p-j)!}{p!}\right]^{\delta} \Phi_j(z) \prec \frac{1+z}{1-z} + \frac{\alpha\ell}{\delta(\nu+\ell p)}\frac{2z}{1-z^2}.$$

Then,

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \prec \frac{1+z}{1-z}, \tag{3.38}$$

so $\frac{1+z}{1-z}$ is the best dominant of (3.38).

If we put $p = C = \alpha = \delta = 1,\ D = -1$ and $j = 0$ in Corollary 8 we get:

Example 5. *For $f \in A$ suppose that*

$$\frac{f(z)}{z} \neq 0, \quad z \in U,$$

and

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{r} + \left(\frac{1+z}{1-z}\right)^{r}\frac{2rz}{1-z^2}.$$

Then,

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{r}, \tag{3.39}$$

and $\left(\frac{1+z}{1-z}\right)^{r}$ is the best dominant of (3.39).

Remark 1. *For $\nu = \rho = 0,\ \ell = p = r = 1$ and $j = 0$ in Theorem 4, we get the results investigated by Shanmugam et al. ([25], Theorem 3.1).*

Theorem 5. *Let $0 \leq j < p$, and for $f \in A(p)$ assume that*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U,$$

whenever $\delta \in (0,+\infty)\backslash\mathbb{N}$. Suppose that

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \in H \cap Q$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)$ is univalent in U, where the function Φ_j is defined by (3.1). If q is convex (univalent) function in U, and

$$q(z) + \frac{\alpha\ell}{\delta(\nu+\ell p)}zq^{'}(z) \prec \left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z),$$

then

$$q(z) \prec \left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}, \tag{3.40}$$

so $q(z)$ is the best subordinate of (3.40).

Proof. Let ϕ is defined by (3.5), from (3.31) we get

$$q(z) + \frac{\alpha\ell}{\delta(\nu+\ell p)}zq^{'}(z) \prec \left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z) = \phi(z) + \frac{\alpha\ell}{\delta(\nu+\ell p)}z\phi^{'}(z).$$

The proof of Theorem 5 followes by an application of Lemma 3. □

Taking $q(z) = \left(\frac{1+Cz}{1+Dz}\right)^r$ in Theorem 5, we get:

Corollary 9. *Let $0 \leq j < p$, $0 < r \leq 1$ and for $f \in A(p)$ assume that*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U.$$

Suppose that

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \in H \cap Q$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)$ is univalent in U, where the function Φ_j is defined by (3.1). If

$$\left(\frac{1+Cz}{1+Dz}\right)^r + \frac{\alpha\ell}{\delta(\nu+\ell p)}\left(\frac{1+Cz}{1+Dz}\right)^r\frac{r(C-D)z}{(1+Dz)(1+Cz)} \prec \left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z),$$

then

$$\left(\frac{1+Cz}{1+Dz}\right)^{r} \prec \left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}, \tag{3.41}$$

so $\left(\frac{1+Cz}{1+Dz}\right)^{r}$ *is the best dominant of (3.41).*

Taking $q(z) = \frac{1+Cz}{1+Dz}$ and $r = 1$ in Theorem 5, we get:

Corollary 10. *Let* $0 \le j < p$, *and for* $f \in A(p)$ *assume that*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U,$$

whenever $\delta \in (0,+\infty)\backslash\mathbb{N}$. *Assume that*

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \in H \cap Q$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)$ *is univalent in* U, *where the function* Φ_j *is defined by (3.1). If*

$$\frac{1+Cz}{1+Dz} + \frac{\alpha\ell}{\delta(\nu+\ell p)}\frac{(C-D)z}{(1+Dz)^2} \prec \left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z),$$

then

$$\frac{1+Cz}{1+Dz} \prec \left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta}, \tag{3.42}$$

so $\frac{1+Cz}{1+Dz}$ *is the best dominant of (3.42).*

Combining results of Theorems 4 and 5, we have

Theorem 6. *Let* $0 \le j < p$, *and for* $f \in A(p)$ *assume that*

$$\frac{(\Im_p(\nu,\rho;\ell)f(z))^{(j)}}{z^{p-j}} \neq 0, \quad z \in U.$$

Suppose that

$$\left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \in H[q(0),1] \cap Q$$

such that $\left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z)$ *is univalent in* U, *where the function* Φ_j *is defined by (3.1). Let* q_1 *is convex (univalent) function in* U, *and assume that* q_2 *is convex in* U, *that* q_2 *satisfies (3.28). If*

$$q_1(z) + \frac{\alpha\ell}{\delta(\nu+\ell p)}zq_1'(z) \prec \left[\frac{(p-j)!}{p!}\right]^{\delta}\Phi_j(z) \prec q_2(z) + \frac{\alpha\ell}{\delta(\nu+\ell p)}zq_2'(z),$$

then

$$q_1(z) \prec \left(\frac{(p-j)!}{p!}\frac{\left(\Im_p(\nu,\rho;\ell)f(z)\right)^{(j)}}{z^{p-j}}\right)^{\delta} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinate and best dominant of the above subordination.*

4. Conclusions

We used the application of higher order derivatives to obtained a number of interesting results concerning differential subordination and superordination relations for the operator $\Im_p(\nu, \rho; \ell)f(z)$ of multivalent functions analytic in U, the differential subordination outcomes are followed by some special cases and counters examples. Differential sandwich-type results have been obtained. Our results we obtained are new and could help the mathematicians in the field of Geometric Function Theory to solve other special results in this field.

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Conflict of interest

The authors declare no conflict of interest.

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