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Research article

A piecewise homotopy Padé technique to approximate an arbitrary function

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Abstract: The Padé approximation and its enhancements provide a more accurate approximation of functions than the Taylor series truncation. A new technique for approximating functions into rational functions is proposed in this paper. This technique is based on the homotopy Padé technique and introduces new parameters known as merging parameters. These parameters are added to the Tayler series before the Padé process is computed. To control error, the merging parameters and dividing the interval into subintervals are used. Two illustrative examples are used to demonstrate the validity and reliability of the proposed novel approximation. The robustness and efficiency of the proposed approximation were demonstrated by computing the absolute error and comparing the results to those of the standard Padé technique and the generalized restrictive Padé technique. Also, Hard-core scattering problem and Debye-Hukel function are tested by the proposed technique. The piecewise homotopy Padé method is an excellent path to approximate any function. The proposed new approximation's efficacy and accuracy have been validated using Mathematica 12.

Keywords: rational approximation; homotopy Padé technique; Padé approximation; the generalized restrictive Padé technique
Mathematics Subject Classification: 65C30, 65L12

1. Introduction

Due to its numerous applications in physical sciences, engineering, and other applied sciences [1–9], rational approximation of an arbitrary function is an important topic in numerical analysis. The Padé approximation is a form of rational function approximation that is unique and well-known. Henri Padé invented the technique in 1890. To approximate more functions, this sort of approximation was modified in several ways, including the N-point Padé approximant, restricted Padé, the generalized restrictive Padé and a homotopy Padé [10–12].

Padé and two-point Padé approximations have also been adapted to approximate multivariable functions [13]. Some systems have been discussed by approaches based on Padé approximations. The system of Markov function and Nikishin system are discussed by Hermit-Padé approximations [14,15]. A global Padé approximation of the generalized Mittag-Leffler function is introduced [16,17]. Some of the approaches are developed based on Padé approximations to handle the differential systems.

In 2009, Yan-Ming and Yong Chen handle the nonlinear differential-difference equations by Padé approximation and the Adomian decomposition method [18]. Also, Adomian method and Padé approximation were used to explore the power system oscillations [19]. In 2013, the Variational iteration method is adopted by multivariate Padé approximation to handle linear and nonlinear fractional order partial differential equations [20]. Also, the Padé evolutionary cooperative multi-simplex algorithm is introduced to the treatment of nonlinear partial differential equations [7]. In 2022, Ibrahim discussed blood flow of the Carreau-Yasuda Nano fluid flooded in gyrotactic microorganisms by a differential transform method combined by Padé approximation [21]. To speed up the solutions of nonlinear differential equations originating from the homotopy analysis method (HAM), Liao and Cheung proposed a homotopy Padé methodology as follows [12]:

Let $f(x,q)=f_0(x)+f_1(x)q+f_2(x)q^2+f_3(x)q^3+\ldots$, such that

$$f(x)=f(x,q)\big|_{q=1}, \tag{1.1}$$

where q is the embedding parameter and belongs to the interval $[0,1]$. First, we employ the traditional (m,n) Padé technique concerning the embedding parameter q to obtain (m,n) Padé approximation:

$$H_{f(x,q)}(m,n)=\frac{\sum_{i=0}^{m} b_i(x)q^i}{1+\sum_{i=1}^{n} c_i(x)q^i}, \tag{1.2}$$

such that

$$H_{f(x,q)}(m,n)-f(x,q)=O\left(q^{m+n+1}\right). \tag{1.3}$$

The development of homotopy methods based on the embedding parameter q belongs to the interval $[0,1]$. If $q=0$ occurs the equation initial condition and the exact solution occurs when $q=1$. Then, setting $q=1$ in (1.2), and using (1.1), we have the so-called (m,n) homotopy Padé approximant:

$$H_{f(x)}(m,n)=\frac{\sum_{i=0}^{m} b_i(x)}{1+\sum_{i=1}^{n} c_i(x)}. \tag{1.4}$$

Semi-analytic methods have recently been used to solve a wide range of differential equations [22–28]. Many problems were also solved using the Padé and homotopy Padé techniques [29–36].

In Section 2, we introduce a technique based on the homotopy Padé technique for giving rational approximations for an arbitrary function defined on the interval by dividing the interval into short subintervals, and using the Taylor series to extend the function at a point inside each subinterval. As a result, we use the homotopy Padé technique for each sub-interval. This technique has the advantage of providing a better approximation of the function than the truncation of its Taylor series. As a simple way to control the absolute error, we introduce new parameters, called merging parameters, for terms of a power series. Section 3 summarizes some numerical examples to demonstrate that the proposed method outperforms the traditional Padé technique and the generalized restrictive Padé approximation [36]. Section 4 introduces two practical problems and the analysis.

2. A piecewise homotopy Padé approximant

Let $f(x)$ be given by the formal power series at $x = x_k$:

$$f(x) = \sum_{i=0}^{\infty} a_{i,k} (x - x_k)^i, \quad 2.1)$$

where $k = 1, 2, 3, \ldots, \lambda$ and λ sub-intervals of the interval $x \in [0, L]$. The power series (2.1) can be written in the form:

$$f(x,q) = \sum_{i=0}^{\infty} A_{i,k}(x) q^i = A_{0,k}(x) + A_{1,k}(x) q + A_{2,k}(x) q^2 + \ldots, \quad (2.2)$$

where

$$A_{0.k}(x) = \sum_{i=0}^{k_{0,k}} a_{i,k} (x - x_k)^i, \; A_{1.k}(x) = \sum_{i=k_{0,k}+1}^{k_{0,k}+k_{1,k+1}} a_{i,k} (x - x_k)^i,$$

$$A_{2,k}(x) = \sum_{i=k_{0,k}+k_{1,k}+2}^{k_{0,k}+k_{1,k}+k_{2,k}+2} a_{i,k} (x - x_k)^i,$$

and so on, where $k_{0,k}, k_{1,k}, \ldots$ are called the merger parameters for terms of the power series (2.1) and $q \in [0,1]$, such that

$$f(x,1) = f(x). \quad (2.3)$$

Definition 2.1. *Let $f(x)$ analytic and defined on an interval $[0, L]$, the points x_i form a sequence $0 = x_0 < x_1 < x_2 \ldots\ldots < x_\lambda = L$, dividing the interval $[0, L]$ into λ sub-intervals. Then, piecewise homotopy Padé approximant (2.2) for $f(x,q)$ in the interval $[0, L]$ is defined by*

$$PH_{f(x,q)}\left[(m_1,n_1),(m_2,n_2),\ldots,(m_\lambda,n_\lambda)\right]=\begin{cases} H_{f(x,q)}(m_1,n_1), x\in[x_0,x_1],\\ H_{f(x,q)}(m_2,n_2), x\in[x_1,x_2],\\ \vdots \\ H_{f(x,q)}(m_\lambda,n_\lambda), x\in[x_{\lambda-1},x_\lambda]. \end{cases} \tag{2.4}$$

Note that

$$PH_{f(x,q)}\left[(m_1,n_1),(m_2,n_2),\ldots,(m_\lambda,n_\lambda)\right]-f(x,q)=\begin{cases} O\left(q^{m_1+n_1+1}\right),\ x\in[x_0,x_1],\\ O\left(q^{m_2+n_2+1}\right),\ x\in[x_1,x_2],\\ \vdots \\ O\left(q^{m_\lambda+n_\lambda+1}\right),\ x\in[x_{\lambda-1},x_\lambda]. \end{cases} \tag{2.5}$$

Then, setting $q=1$ in (2.4) and using (2.3), we have the so-called piecewise homotopy Padé approximant in the interval $[0,L]$ for $f(x)$ in (2.1) of the form:

$$PH_{f(x)}=PH_{f(x,1)}\left[(m_1,n_1),(m_2,n_2),\ldots,(m_\lambda,n_\lambda)\right]. \tag{2.6}$$

Remark 2.1. *If* $(m_k,n_k)=(m,n)$ *for all k, the following notation may be used:*

$$PH_{f(x)}=PH_{f(x,1)}(m,n), \tag{2.7}$$

By controlling the values of merger parameters, we find the relation between (m,n) *homotopy Padé approximant and traditional* (m,n) *Padé approximant in the following theorem:*

Theorem 2.1. *If all merger parameters for terms of the power series (2.2) are zero, then the* (m,n) *piecewise homotopy Padé approximant* $PH_{f(x,1)}$ *is the same as the traditional* $(m,n)_k$ *Padé approximant for* $(x-x_k)$ *at* $k=1,2,...,\lambda.$

Proof. Because all merger parameters for terms of the power series (2.2) are equal to zero, $f(x,q)$ can be written in the following form:

$$f(x,q)=\sum_{i=0}^{\infty}a_{i,k}v_k^{\ i}, \tag{2.8}$$

where $v_k=(x-x_k)q$. To obtain the (m,n) homotopy Padé approximant, first employ the traditional (m,n) Padé technique with respect to v_k for the series (2.8), as follows:

$$\sum_{i=0}^{\infty}a_{i,k}v_k^{\ i}=\frac{p_{0,k}+p_{1,k}v_k+p_{2,k}v_k^{\ 2}+\ldots+p_{m,k}v_k^{\ m}}{1+s_{1,k}v_k+s_{2,k}v_k^{\ 2}++\ldots+s_{n,k}v_k^{\ n}}+O\left(v_k^{\ m+n+1}\right). \tag{2.9}$$

By cross-multiplying equation (2.9), we get

$$\left(1+s_{1,k}v_k+s_{2,k}v_k{}^2+\ldots+s_{n,k}v_k{}^n\right)\left(a_{0,k}+a_{1,k}v_k+a_{2,k}v_k{}^2+\ldots\right)=p_{0,k}+p_{1,k}v_k+p_{2,k}v_k{}^2+\ldots+p_{m,k}v_k{}^m. \quad (2.10)$$

From (2.10), one can obtain the set of equations

$$\begin{cases} a_{0,k}=p_{0,k}, \\ a_{1,k}+a_{0,k}s_{1,k}=p_{1,k}, \\ a_{2,k}+a_{1,k}s_{1,k}+a_{0,k}s_{2,k}=p_{2,k}, \\ \vdots \\ a_{m,k}+a_{m-1,k}s_{1,k}+\ldots+a_{0,k}s_{m,k}=p_{m,k} \end{cases} \quad (2.11)$$

and

$$\begin{cases} a_{m+1,k}+a_{m,k}s_{1,k}+\ldots+a_{m-n+1,k}s_{n,k}=0, \\ a_{m+2,k}+a_{m+1,k}s_{1,k}+\ldots+a_{m-n+2,k}s_{n,k}=0, \\ a_{m+n,k}+a_{m+n,k}s_{1,k}+\ldots+a_{m,k}s_{n,k}=0, \end{cases} \quad (2.12)$$

where $a_{i,k}=0$ for $i<0$ and $s_{j,k}=0$ for $j>n$.

By solving (2.11) and (2.12) directly, we get

$$PH_{f(x,q)}=\frac{N_1}{D_1}, \quad (2.13)$$

where

$$D_1=\begin{vmatrix} a_{m-n+1,k} & a_{m-n+2,k} & \cdots & a_{m+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,k} & a_{m+1,k} & \cdots & a_{m+n,k} \\ v_k{}^n & v_k{}^{n-1} & \ldots & 1 \end{vmatrix},$$

$$N_1=\begin{vmatrix} a_{m-n+1,k} & a_{m-n+2,k} & \cdots & a_{m+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,k} & a_{m+1,k} & \cdots & a_{m+n,k} \\ \sum_{j=n}^{m} a_{j-n,k}v_k{}^j & \sum_{j=n-1}^{m} a_{j-n+1,k}v_k{}^j & \ldots & \sum_{j=0}^{m} a_{j,k}v_k{}^j \end{vmatrix}.$$

If the lower index on a sum exceeds the upper (j), the sum is replaced by zero. Setting $q=1$ so that $v_k=\left(x-x_k\right)$, then the $\left(m,n\right)$ piecewise homotopy Padé approximant $PH_{f(x,1)}$ (2.13) becomes the traditional $\left(m,n\right)_k$ Padé approximant for $\left(x-x_k\right)$ at $k:1,2,...,\lambda$.

The order of truncation error of the proposed technique is given by the following theorem:

Theorem 2.2. *The piecewise homotopy Padé approximant and the analytic function* $f\left(x\right)$ *satisfy the* *following equation:*

$$PH_{f(x)}\left[(m_1,n_1),(m_2,n_2),\ldots,(m_\lambda,n_\lambda)\right]-f(x)=\Xi, \tag{2.14}$$

where

$$\Xi=\begin{cases} O(x-x_1)^{k_{0,1}+k_{1,1}+\ldots+k_{(m_1+n_1),1}+m_1+n_1+1},\ x\in[x_0,x_1], \\ O(x-x_2)^{k_{0,2}+k_{1,2}+\ldots+k_{(m_2+n_2),2}+m_2+n_2+1},\ x\in[x_1,x_2], \\ \vdots \\ O(x-x_\lambda)^{k_{0,\lambda}+k_{1,\lambda}+\ldots+k_{(m_\lambda+n_\lambda),\lambda}+m_\lambda+n_\lambda+1},\ x\in[x_{\lambda-1},x_\lambda]. \end{cases} \tag{2.15}$$

Proof. Let $k=j$, then $x\in\left[x_{j-1},x_j\right]$ and by truncated in series (2.2), then

$$\begin{aligned} f(x,q) &= \sum_{i=0}^{m_j+n_j+1} A_{i,j}(x)q^i = A_{0,j}(x)+A_{1,j}(x)q+\ldots+A_{(m_j+n_j+1),j}(x)q^{m_j+n_j+1} \\ &= \sum_{i=0}^{k_{0,j}} a_{i,j}(x-x_j)^i+\sum_{i=k_{0,j}+1}^{k_{0,j}+k_{1,j}} a_{i,j}(x-x_j)^i q+\ldots+\sum_{i=k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j-1),j}+1}^{k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j),j}+m_j+n_j} a_{i,j}(x-x_j)^i q^{m_j+n_j} \\ &\quad +\left(O(\left[x-x_j\right]^{k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j),j}+m_j+n_j+1}\right)+\ldots)q^{m_j+n_j+1}. \end{aligned}$$

From (2.3), then

$$\begin{aligned} f(x)=f(x,1) &= \sum_{i=0}^{k_{0,j}} a_{i,j}(x-x_j)^i+\sum_{i=k_{0,j}+1}^{k_{0,j}+k_{1,j}} a_{i,j}(x-x_j)^i+\ldots++\sum_{i=k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j-1),j}+1}^{k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j),j}+m_j+n_j} a_{i,j}(x-x_j)^i \\ &\quad +O(\left[x-x_j\right]^{k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j),j}+m_j+n_j+1}). \end{aligned}$$

Therefore, the order of truncation error of the proposed technique in subinterval $x\in\left[x_{j-1},x_j\right]$ is

$$O\left(x-x_j\right)^{k_{0,j}+k_{1,j}+\ldots+k_{(m_j+n_j),j}+m_j+n_j+1}. \tag{2.16}$$

By putting $j=1,2,\ldots,\lambda$ in Eq (2.16), we get that the piecewise homotopy Padé approximant and the analytic function $f(x)$ satisfy (2.14).

Remark 2.2. *If the merger parameters for terms of the power series (2.2) are equal zero, then* $PH_{f(x)}\left[(m_1,n_1),(m_2,n_2),\ldots,(m_\lambda,n_\lambda)\right]-f(x)=O(x-x_k)^{m_k+n_k+1}$ *for each sub-interval in Eq (2.14).*

3. Numerical examples

To provide some indication of the accuracy of the constructed approximation, the following examples are given.

Example 3.1. Consider the function

$$f(x)=e^{x},$$

and $x\in[0,2]$. According to Eq (2.1), then,

$$f(x)=e^{x_k}\left(1+(x-x_k)+\frac{1}{2}(x-x_k)^2+\frac{1}{3!}(x-x_k)^3+\frac{1}{4!}(x-x_k)^4+\ldots\right), \tag{3.1}$$

we choose the merger parameters for terms of the power series (3.1) as follows:

$$k_{0,k}=k_{1,k}=\ldots=k_{m+n,k}=1,$$

and from (2.2), the power series (3.1) becomes

$$\begin{aligned} f(x,q)=e^{x_k}[1+(x-x_k)+\left(\frac{1}{2}(x-x_k)^2+\frac{1}{3!}(x-x_k)^3\right)q+\left(\frac{1}{4!}(x-x_k)^4+\frac{1}{5!}(x-x_k)^5\right)q^2 \\ +\left(\frac{1}{6!}(x-x_k)^6+\frac{1}{7!}(x-x_k)^7\right)q^3+\ldots]. \end{aligned} \tag{3.2}$$

Now the piecewise homotopy Padé approximant is applied for (3.2) in the interval $x\in[0,2]$. By dividing the interval $[0,2]$ into four subintervals, the width of each is equal to 0.5. We also choose x_k at the beginning of each subinterval, as shown in Table 1. The absolute error is defined by

$$Absolute\ error\ =\left|PH_{f(x,1)}(m,n)-e^{x}\right|. \tag{3.3}$$

Table 1. The absolute error (3.3) for different (m,n) orders of piecewise homotopy Padé approximant.

(m,n)	x_k	Sub-interval	$0+x_k$	$0.1+x_k$	$0.2+x_k$	$0.3+x_k$	$0.4+x_k$	$0.5+x_k$
(2,2)	0	[0,0.5]	0	2.22×10^{-16}	3.97×10^{-14}	2.30×10^{-12}	4.11×10^{-11}	3.86×10^{-10}
	0.5	[0.5,1]	0	2.22×10^{-16}	6.61×10^{-14}	3.80×10^{-12}	6.78×10^{-11}	6.37×10^{-10}
	1	[1,1.5]	0	4.44×10^{-16}	1.08×10^{-13}	6.26×10^{-12}	1.11×10^{-10}	1.05×10^{-9}
	1.5	[1.5,2]	0	8.88×10^{-16}	1.78×10^{-13}	1.03×10^{-11}	1.84×10^{-10}	1.73×10^{-9}
(3,3)	0	[0,0.5]	0	2.22×10^{-16}	4.44×10^{-16}	2.22×10^{-16}	0	1.11×10^{-15}
	0.5	[0.5,1]	0	0	4.44×10^{-16}	8.88×10^{-16}	0	2.22×10^{-15}
	1	[1,1.5]	0	4.44×10^{-16}	8.88×10^{-16}	4.44×10^{-16}	8.88×10^{-16}	3.55×10^{-15}
	1.5	[1.5,2]	0	8.88×10^{-16}	1.77×10^{-15}	8.88×10^{-16}	8.88×10^{-16}	3.55×10^{-15}

Table 2. Some absolute error values for different methods of Example 3.1.

The method	x=0.2	x=0.65	x=0.9
Padé approximant (2,2)	5.44×10^{-7}	3.16×10^{-4}	2.11×10^{-3}
GR Padé approximant (2,2/2)	1.32×10^{-9}	1.36×10^{-7}	6.37×10^{-7}
PH Padé approximant (2,2)	3.97×10^{-14}	3.77×10^{-15}	6.78×10^{-11}

Example 3.2. Consider the function

$$f\left(x\right)=1-ln\left(1-x\right), \tag{3.4}$$

and $x\in\left[0,0.5\right]$. According to (2.1), then,

$$\begin{aligned} f\left(x\right)=&\left(1-ln(1-x_k)\right)-\frac{x-x_k}{-1+x_k}+\frac{\left(x-x_k\right)^2}{2\left(-1+x_k\right)^2}-\frac{\left(x-x_k\right)^3}{3\left(-1+x_k\right)^3}+\frac{\left(x-x_k\right)^4}{4\left(1-x_k\right)^4}\\ &+\frac{\left(x-x_k\right)^5}{5\left(1-x_k\right)^5}+\frac{\left(x-x_k\right)^6}{6\left(1-x_k\right)^6}+\frac{\left(x-x_k\right)^7}{7\left(1-x_k\right)^7}+\ldots \end{aligned} \tag{3.5}$$

By using (2.2) and choosing the merger parameters for terms of the power series (3.5): $k_0=k_1=\ldots=k_{m+n}=1$, then the power series (3.5) becomes

$$\begin{aligned} f\left(x,q\right)=&\left(1-ln\left[1-x_k\right]\right)-\frac{x-x_k}{-1+x_k}+\left(\frac{\left(x-x_k\right)^2}{2\left(-1+x_k\right)^2}-\frac{\left(x-x_k\right)^3}{3\left(-1+x_k\right)^3}\right)q\\ &+\left(\frac{\left(x-x_k\right)^4}{4\left(1-x_k\right)^4}+\frac{\left(x-x_k\right)^5}{5\left(1-x_k\right)^5}\right)q^2+\left(\frac{\left(x-x_k\right)^6}{6\left(1-x_k\right)^6}+\frac{\left(x-x_k\right)^7}{7\left(1-x_k\right)^7}\right)q^3+\ldots \end{aligned} \tag{3.6}$$

Again, when the merger parameters are $k_{0,k}=k_{1,k}=\ldots=k_{m+n,k}=2$, then the power series (3.5) becomes

$$\begin{aligned} f\left(x,q\right)=&\left(1-\ln\left(1-x_k\right)\right)-\frac{x-x_k}{-1+x_k}+\frac{\left(x-x_k\right)^2}{2\left(-1+x_k\right)^2}+\left(-\frac{\left(x-x_k\right)^3}{3\left(-1+x_k\right)^3}+\frac{\left(x-x_k\right)^4}{4\left(1-x_k\right)^4}+\frac{\left(x-x_k\right)^5}{5\left(1-x_k\right)^5}\right)q\\ &+\left(\frac{\left(x-x_k\right)^6}{6\left(1-x_k\right)^6}+\frac{\left(x-x_k\right)^7}{7\left(1-x_k\right)^7}+\frac{\left(x-x_k\right)^8}{8\left(1-x_k\right)^8}\right)q^2+\ldots \end{aligned} \tag{3.7}$$

For series in (3.6) and (3.7) at the interval $x\in\left[0,0.5\right]$, the piecewise homotopy Padé approximant order (2,2) can be used. Divide the interval $\left[0,0.5\right]$ into two sub-intervals, each with a width of 0.25. In addition, as shown in Table 3, we use x_k at the start of each sub-interval. The absolute error is defined as follows:

$$Absolute\ error\ =\left|PH_{f(x,1)}\left(2,2\right)-1+ln\left(1-x\right)\right|. \tag{3.8}$$

Table 3 displays the absolute error (3.8) for various merger parameter values. As merger parameters increase, the accuracy of the piecewise homotopy Padé approximant order (2,2) for Example 3.2 increases. This means that the current technique allows for the control of error accuracy rather than an increase in Padé order. The absolute maximum errors in the interval [0,0.5] for Example 3.2 of the generalized restrictive Padé approximant [37], and the piecewise homotopy Padé approximant are shown in Table 4.

Table 2 shows that the piecewise homotopy Padé approximant approximates Example 3.2 more accurately than the generalized restrictive Padé approximant for the same order (m,n) [37].

Table 3 shows the absolute error (3.8) for different values of the merger parameters. One can see the accuracy of the piecewise homotopy Padé approximant order (2,2) for Example 3.2 increases as merger parameters increase. This means that the present technique gives a way to control the accuracy of error instead of an increase of Padé order. Table 4 shows the absolute maximum errors in the interval [0,0.5] for Example 3.2 of the generalized restrictive Padé approximant [37] and the piecewise homotopy Padé approximant. From Table 2, it is clear that the piecewise homotopy Padé approximant has more accuracy to approximate the Example 3.2 in the interval [0,0.5] than the generalized restrictive Padé approximant for the same orders (m,n) [37]. Furthermore, as the merging parameters increase, the accuracy of the piecewise homotopy Padé approximant for different orders increases.

Table 3. The absolute error (3.8) for different values of the merger parameters for Example 3.2.

(m,n)	The merger parameters	x_k	Sub-interval	$0+x_k$	$0.1+x_k$	$0.2+x_k$
(2,2)	$k_{0,k}=k_{1,k}=\ldots=k_{4,k}=1$	0	[0,0.25]	0	3.03×10^{-13}	3.54×10^{-10}
		0.25	[0.25,0.5]	0	5.59×10^{-12}	7.07×10^{-9}
(2,2)	$k_{0,k}=k_{1,k}=\ldots=k_{4,k}=2$	0	[0,0.25]	0	0	7.17×10^{-14}
		0.25	[0.25,0.5]	0	2.22×10^{-16}	5.84×10^{-12}

Table 4. The absolute maximum errors in the interval [0,0.5].

m	α	n	$Max_{x\in[0,0.5]}\lvert G[m,\alpha/n]-f(x)\rvert$	$Max_{x\in[0,0.5]}\lvert PH_{f(x,1)}(m,n)-f(x)\rvert$, $k_{0,k}=k_{1,k}=\ldots=k_{m+n,k}=1$	$Max_{x\in[0,0.5]}\lvert PH_{f(x,1)}(m,n)-f(x)\rvert$, $k_{0,k}=k_{1,k}=\ldots=k_{m+n,k}=2$
2	1	0	4.09347×10^{-3}	3.21075×10^{-4}	8.07847×10^{-6}
2	2	1	1.79839×10^{-5}	3.23250×10^{-6}	2.47180×10^{-8}
2	2	2	1.22010×10^{-6}	7.60845×10^{-8}	1.83653×10^{-10}
3	2	2	2.88994×10^{-7}	2.68453×10^{-9}	2.09743×10^{-12}

4. Practical problems and discussion

4.1. Hard-core scattering problem

Consider the following hard-core scattering problem:

$$S(x)=\int_0^x \frac{\sin t}{t^3}-\frac{\cos t}{t^4}\,dt. \tag{4.1}$$

Baker and Gammel address this issue when calculating the scattering length of a repulsive square-well potential [38]. The integral's Maclaurin expansion (4.1) is as follows:

$$\frac{x}{3}-\frac{x^3}{90}+\frac{x^5}{4200}-\frac{x^7}{317520}+\frac{x^9}{35925120}-\frac{x^{11}}{5708102400}+\ldots \tag{4.2}$$

and Taylor expansion of the integral (4.1) at $x=3$ is

$$\frac{1}{18}\left(3\cos(3)-\sin(3)+16.6378\right)+\frac{1}{27}(x-3)\left(-3\cos(3)+\sin(3)\right)+(x-3)^2\left(\frac{\cos(3)}{18}+\frac{\sin(3)}{27}\right)+\ldots \tag{4.3}$$

We show that the proposed method can handle this problem by dividing $x\in[0,6]$ into two subintervals [0,3] and [3,6]. We set the merger parameters to one and apply piecewise homotopy Padé approximant (2,2) of Eqs (4.2) and (4.3).

Figure 1 depicts the absolute error, and the maximum errors are 3×10^{-12} and 0.000015 in the sub intervals [0,3] and [3,6], respectively. We increase the parameter values to two to demonstrate the effect of the merger parameters. As illustrated in Figure 2, the maximum absolute error becomes 3×10^{-16} and 8×10^{-7} in sub intervals [0,3] and [3,6]. The piecewise homotopy Padé approximant method can achieve very good accuracy by increasing the values of merger parameters.

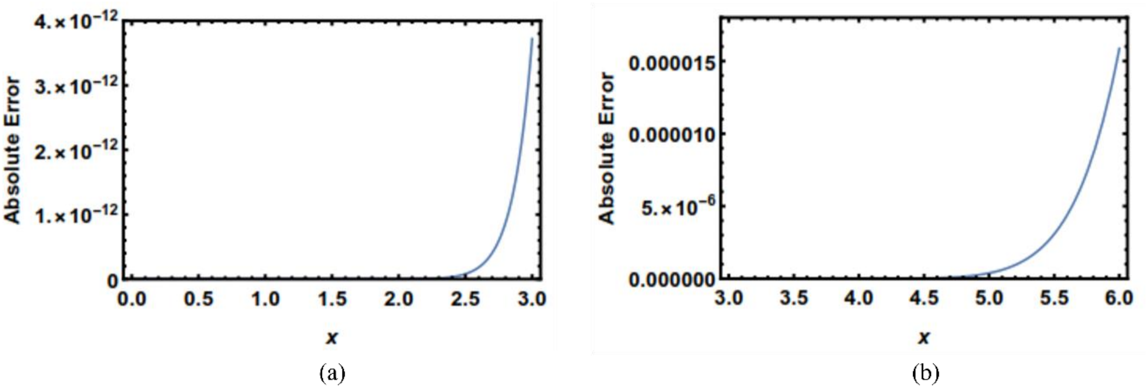
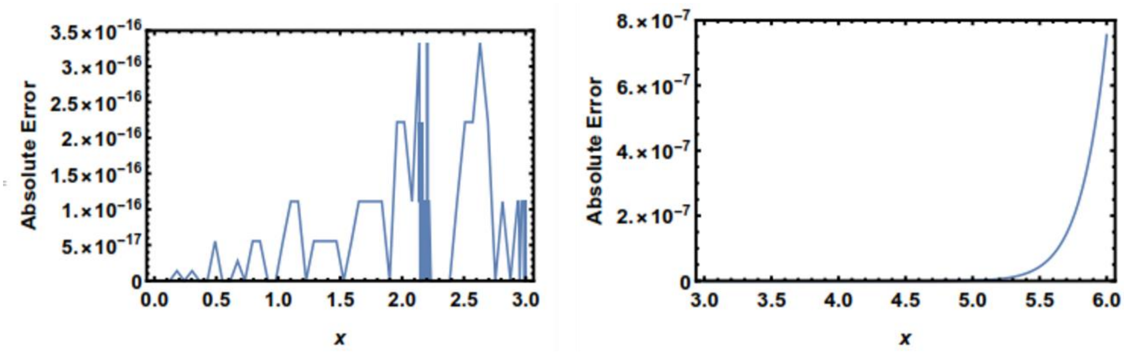


Figure 1. The absolute error of piecewise homotopy Padé approximant (2,2) when the merger parameters is one.



(a) (b)

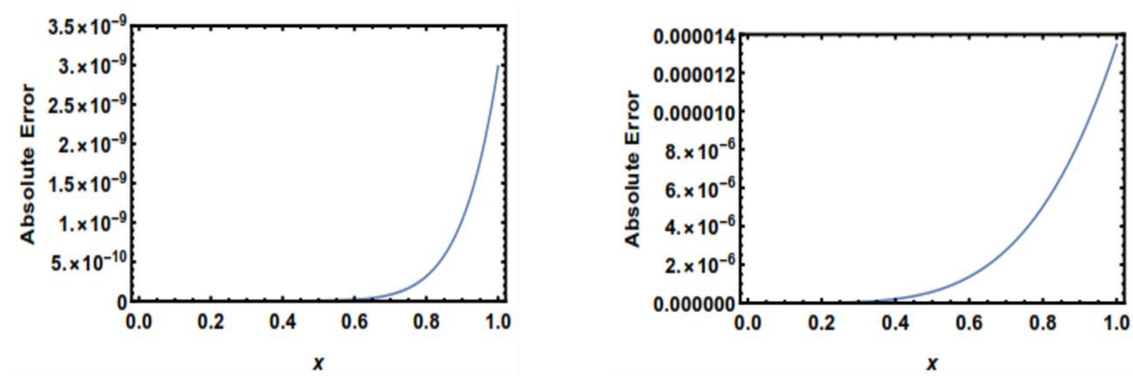
Figure 2. The absolute error of piecewise homotopy Padé approximant (2,2) when the merger parameters is two.

4.2. Debye-Hukel function

Consider the Debye-Hukel function, which is given by

$$D(x)=\frac{2}{x}-\frac{2}{x^2}\left(1-e^{-x}\right). \tag{4.4}$$

The Debye-Huckel theory of strong electrolytes introduces this function [39]. We used the piecewise homotopy Padé method (2,2) and the Padé approximant (2,2) in the interval $x\in[0,1]$. According to Figure 3, the maximum errors for the piecewise homotopy Padé method and the Padé approximant are 3×10^{-9} and 0.000014, respectively. According to this figure, the piecewise homotopy Padé approximant method outperforms the Padé approximant method in terms of accuracy. We can also improve the piecewise homotopy Padé approximant solution by dividing the interval $x\in[0,1]$ into subintervals. Figure 4 shows that, the absolute error is improved when splitting the interval $x\in[0,1]$ into two subintervals $x\in[0,0.5]$ and [0.5,1].



(a) (b)

Figure 3. Absolute error by (a) piecewise homotopy Padé approximant (2,2) and (b) Padé approximant (2,2).

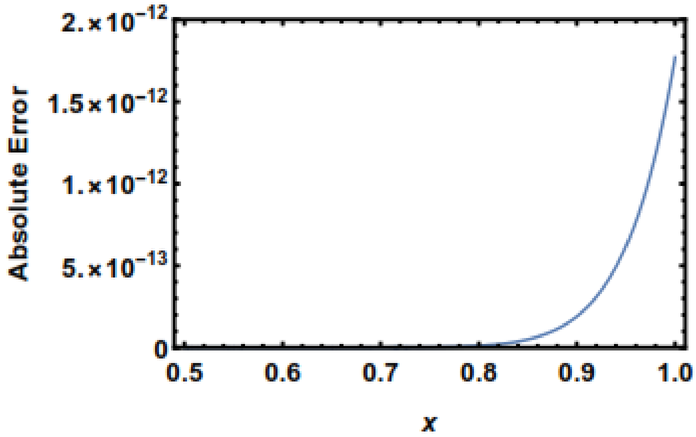


Figure 4. Absolute error by piecewise homotopy Padé approximant (2,2) in interval $x \in [0.5,1]$.

5. Conclusions

The issue of function approximation is fascinating, and the Padé technique, in particular, deserves further attention. In this paper, we propose a novel approximation methodology based on the homotopy Padé technique, and the splitting the solution interval into sub intervals. The piecewise homotopy Padé approach is a good way to approximate any function that is specified over a period of time. The most crucial aspect of our research is to directly detect and monitor absolute inaccuracy by introducing merging parameters for power series terms. The Illustrative examples and practical problems show that the proposed technique is more accurate than both the classic Padé and the generalized restrictive Padé techniques.

The authors are confident that the proposed approximation's solution can be combined with semi-analytic and iterative methods to solve differential equations in a wide range. We are currently working on using the piecewise homotopy Padé technique to find an approximated solution to the well-known inverse Langevin function that outperforms the commonly used methods.

Conflict of interest

We declare that there are no conflicts of interest regarding the publication of this paper.

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