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Research article

A novel B-spline collocation method for Hyperbolic Telegraph equation

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Abstract: The present study is concerned with the construction of a new high-order technique to establish approximate solutions of the Telegraph equation (TE). In this technique, a novel optimal B-spline collocation method based on quintic B-spline (QBS) basis functions is constructed to discretize the spatial domain and fourth-order implicit method is derived for time integration. Test problems are considered to verify the theoretical results and to demonstrate the applicability of the suggested technique. The error norm L_∞ and the rate of spatial and temporal convergence are computed and compared with those of techniques available in the literature. The obtained results show the improvement and efficiency of the proposed scheme over the existing ones. Also, it is obviously observed that the experimental rate of convergence is almost compatible with the theoretical rate of convergence.

Keywords: optimal quintic B-spline; Telegraph equation; collocation method
Mathematics Subject Classification: 76B25, 65D07, 65M50, 65N35

1. Introduction

Hyperbolic partial differential equations (HPDEs) have drawn much attention in recent years. This type of equations play a key role in understanding physical phenomena such as vibrations of structures and several atomic physics fields. The hyperbolic TE, used in the modelling of signal analysis for transmission and propagation of electric signals, is one of the basic type of HPDEs. In this paper, we deal with the approximate solution of the following TE

$$u_{tt} + 2\alpha u_t + \lambda^2 u = u_{xx} + g(x,t) \tag{1.1}$$

which satisfies the following boundary conditions (BCs)

$$u(a,t) = \Gamma_0(t), \quad u(b,t) = \Gamma_1(t), \quad t \geq 0 \tag{1.2}$$

$$u_x(a,t) = \Gamma_2(t), \quad u_x(b,t) = \Gamma_3(t), \; t \geq 0 \tag{1.3}$$

and initial conditions (ICs)

$$u(x,0) = \Psi_0(x), \quad u_t(x,0) = \Psi_1(x), \qquad a \leq x \leq b \tag{1.4}$$

where α and λ are positive constants and g is adequately differentiable forcing function of x and t. Suppose further that $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ and their derivatives are continuous functions of t and similarly Ψ_0, Ψ_1 and their derivatives are continuous function of x. A variety of the numerical methods have been proposed in literature to establish the solutions of the TE, such as the Rothe-wavelet approximation [1], legendre multiwavelet Galerkin technique [2], the Rothe-wavelet Galerkin method [3], the reproducing Kernel Hilbert space method [4], the Chebyshev wavelets approach [5], the meshfree procedure with the radial basis functions [6], the computational method based on the polynomial scaling functions [7], the numerical approach associated Hermite orthogonal functions [8], the Bessel collocation method [9], the Galerkin approach [10], the implicit three-level difference scheme [11], the differential quadrature method [12–14]. In addition, in the past few years, various B-spline collocation techniques have been implemented to obtain the approximate solution of the TE. Alshomrani et al. [15] designed a new algorithm based on modified cubic trigonometric B-spline functions. Dehghand and Shokri [16] proposed a numerical scheme structured on thin plate splines. The collocation approach based on quartic, septic and cubic B-splines were presented to compute the approximate solution of the TE in the studies [17, 18] and [19–21] respectively. Nazir et al. [22] applied the cubic trigonometric B-spline approach for the approximate solution of the TE. Sharifi and Rashidinia [23] proposed a collocation approach with extended cubic B-spline functions to solve numerically the TE. Singh et al. [24] presented a method obtained by using exponential B-spline collocation procedure in space and second-order Runge-Kutta scheme in time. Later, Singh et al. [25] derived a fourth-order cubic spline technique to solve numerically the TE. A high-order new numerical scheme is introduced in the study [26] in which the spatial integration of Eq 1.1 is managed through collocation technique and fourth-order implicit scheme is used for temporal variables.

In the present study, our aim is to produce numerical results with high order accurate by enhancing the accuracy in both time and space. For this purpose, a novel optimal B-spline collocation method based on QBS basis functions is developed to discretize the spatial domain and fourth-order implicit scheme is derived for time integration. The advantage of the suggested technique over the existing ones in the literature is that the suggested technique has an accuracy of $O\left(h^6\right)$ in the spatial direction and an accuracy of $O\left(\Delta t^4\right)$ in temporal direction and produces excellent results even with less points of the temporal and spatial domains. The structure of this paper is as follows: In section 2, the time discretization is carried out and a novel optimal B-spline collocation method is constructed. In section 3, the application of the proposed method to (1.1) is explained. In section 4, the numerical experiments are provided and the results are reported in the table form. The conclusion of the study is given in the section 5 with the remarks about the main observations.

2. Derivation of the proposed method

In this section, we derive a new high-order approach to solve the problem (1.1). Firstly, by introducing an auxiliary variable $v = u_t$, the original equation (1.1) is transformed to the following system of first-order (in time) equations

$$\begin{aligned} u_t &= v, && (2.1)\\ v_t &= u_{xx} - 2\alpha v - \lambda^2 u + g(x,t). && (2.2) \end{aligned}$$

The relevant BCs and ICs are rewritten as

$$\begin{aligned} &u(a,t) = \Gamma_0(t), \qquad u(b,t) = \Gamma_1(t),\\ &v(a,t) = \frac{\partial \Gamma_0}{\partial t}(t), \qquad v(b,t) = \frac{\partial \Gamma_1}{\partial t}(t) \end{aligned} \tag{2.3}$$

$$\begin{aligned} &u_x(a,t) = \Gamma_2(t), \qquad u_x(b,t) = \Gamma_3(t),\\ &v_x(a,t) = \frac{\partial \Gamma_2}{\partial t}(t), \qquad v_x(b,t) = \frac{\partial \Gamma_3}{\partial t}(t) \end{aligned} \tag{2.4}$$

and

$$u(x,0) = \Psi_0(x), \quad v(x,0) = \Psi_1(x). \tag{2.5}$$

Consider the uniform partition of the spatial and temporal domains represented by $\Omega = [a,b] \times [0,T]$ with the grid points (x_j, t_n), where $x_j = a + jh,\ j = 0,1,2,...,N$, $t_n = n\Delta t$, $n = 0,1,2,....$

2.1. Temporal discretization

Utilizing finite difference method which is fourth-order implicit scheme, the time discretization of Eqs (2.1) and (2.2) is derived as follows

$$u^{n+1} = u^n + \theta_1 u_t^{n+1} + \theta_2 u_t^n + \theta_3 u_{tt}^{n+1} + \theta_4 u_{tt}^n \tag{2.6}$$

and

$$v^{n+1} = v^n + \theta_1 v_t^{n+1} + \theta_2 v_t^n + \theta_3 v_{tt}^{n+1} + \theta_4 v_{tt}^n \tag{2.7}$$

where

$$\begin{aligned} u_{tt} &= -2\alpha v - \lambda^2 u + u_{xx} + g(x,t), && (2.8)\\ v_{tt} &= v_{xx} - 2\alpha\left(u_{xx} - 2\alpha v - \lambda^2 u + g(x,t)\right) - \lambda^2 v + g_t(x,t), && (2.9) \end{aligned}$$

obtained by taking partial derivative of both sides Eq (2.2) with respect to t and using Eq (2.1) and θ_1, θ_2, θ_3, θ_4 are unknown parameters to be defined later.

Using Eqs (2.1) and (2.8) in Eq (2.6) yields

$$\begin{aligned} u^{n+1} = \; & u^n + \theta_1 v^{n+1} + \theta_2 v^n + \theta_3\left(u_{xx}^{n+1} - 2\alpha v^{n+1} - \lambda^2 u^{n+1} + g(x,t_{n+1})\right)\\ & + \theta_4(u_{xx}^n - 2\alpha v^n - \lambda^2 u^n + g(x,t_n)). \end{aligned} \tag{2.10}$$

Equation (2.10) can be rearranged to the form given as Eq (2.11)

$$\left(1 + \theta_3\lambda^2\right) u^{n+1} + (2\alpha\theta_3 - \theta_1)\, v^{n+1} - \theta_3 u_{xx}^{n+1} = m(x,t_n) \tag{2.11}$$

where

$$m(x,t_n) = (1 - \theta_4\lambda^2)u^n + (-2\alpha\theta_4 + \theta_2)v^n + \theta_4 u_{xx}^n + \theta_4 g(x,t_n) + \theta_3 g(x,t_{n+1}).$$

In a similar manner, using Eqs (2.2) and (2.9) in Eq (2.7) gives

$$\begin{aligned} v^{n+1} &= v^n + \theta_1\left(u_{xx}^{n+1} - 2\alpha v^{n+1} - \lambda^2 u^{n+1} + g(x,t_{n+1})\right) \\ &+\theta_2\left(u_{xx}^{n} - 2\alpha v^{n} - \lambda^2 u^{n} + g(x,t_{n})\right) \\ &+\theta_3\left(v_{xx}^{n+1} - 2\alpha\left(u_{xx}^{n+1} - 2\alpha v^{n+1} - \lambda^2 u^{n+1} + g(x,t_{n+1})\right) - \lambda^2 v^{n+1} + g_t(x,t_{n+1})\right) \\ &+\theta_4\left(v_{xx}^{n} - 2\alpha\left(u_{xx}^{n} - 2\alpha v^{n} - \lambda^2 u^{n} + g(x,t_{n})\right) - \lambda^2 v^{n} + g_t(x,t_{n})\right). \end{aligned}$$

After simplifying the above equation, we get,

$$\begin{aligned} &\left(\lambda^2\theta_1 - 2\alpha\lambda^2\theta_3\right)u^{n+1} + \left(1 + 2\alpha\theta_1 + \lambda^2\theta_3 - 4\alpha^2\theta_3\right)v^{n+1} \\ &+ (-\theta_1 + 2\alpha\theta_3)u_{xx}^{n+1} - \theta_3 v_{xx}^{n+1} \\ &= k(x,t_n) \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} k(x,t_n) &= (-\lambda^2\theta_2 + 2\alpha\lambda^2\theta_4)u^n + (1 - 2\alpha\theta_2 - \lambda^2\theta_4 + 4\alpha^2\theta_4)v^n \\ &+(\theta_2 - 2\alpha\theta_4)u_{xx}^n + \theta_4 v_{xx}^n + (-2\alpha\theta_4 + \theta_2)g(x,t_n) + \theta_4 g_t(x,t_n) \\ &+(\theta_1 - 2\alpha\theta_3)g(x,t_{n+1}) + \theta_3 g_t(x,t_{n+1}). \end{aligned}$$

Lemma 1. *Suppose that $u, v, f \in C^6(\Omega)$ Then, when*

$$\theta_1 = \theta_2 = \frac{\Delta t}{2} \text{ and } \theta_3 = -\theta_4 = -\frac{(\Delta t)^2}{12},$$

the suggested scheme is consistent and has fourth-order accurarcy in temporal direction for the norm $\|\cdot\|_\infty$.

Proof. For the proof, see [10].

It can be also easily observed that by selecting the following parameters in Eqs (2.6) and (2.7)

$$\theta_1 = \theta_2 = \frac{\Delta t}{2} \text{ and } \theta_3 = \theta_4 = 0,$$

we achieve Crank-Nicolson scheme having an accurarcy of $O(\Delta t^2)$ in time. □

2.2. Space discretization

In this subsection, we describe a fully discrete scheme by means of a novel optimal B-spline collocation (NOBSC) method based on QBS basis functions in the spatial discretization. To this end, we first provide some properties of quintic spline interpolant (QSI) which will be used later in the formulation of NOBSCM.

2.2.1. Properties of QBS interpolant

Let $\pi = \{a = x_0 < x_1 < ... < x_N = b\}$ be a uniform partition over the interval $[a, b]$, where $x_j - x_{j-1} = h$ is the step size. In order to construct the QBS functions, ten additional mesh points are required outside the interval $[a, b]$, which are positioned as $x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1}$ and $x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5}$. By making use of the results in [27, 28], we can define QBS basis functions $\phi_i(x)$ for $i = -2, -1, ..., N+1, N+2$ as follows

$$\phi_j^5(x) = \frac{1}{120h^5}\begin{cases} (f_{j-3})^5, & \left[x_{j-3}, x_{j-2}\right] \\ \left(f_{j-3}\right)^5 - 6\left(f_{j-2}\right)^5, & \left[x_{j-2}, x_{j-1}\right] \\ \left(f_{j-3}\right)^5 - 6\left(f_{j-2}\right)^5 + 15\left(f_{j-1}\right)^5, & \left[x_{j-1}, x_j\right] \\ -\left(f_{j+3}\right)^5 + 6\left(f_{j+2}\right)^5 - 15\left(f_{j+1}\right)^5, & \left[x_j, x_{j+1}\right] \\ -\left(f_{j+3}\right)^5 + 6\left(f_{j+2}\right)^5, & \left[x_{j+1}, x_{j+2}\right] \\ -(f_{j+3})^5, & \left[x_{j+2}, x_{j+3}\right] \\ 0, & \text{otherwise,} \end{cases} \tag{2.13}$$

where $f_j = x - x_j$. The set of QBS $\{\phi_{-2}, \phi_{-1}, \ldots, \phi_{N+1}, \phi_{N+2}\}$ generates a basis over the solution interval $[a, b]$. Let $S(x, t)$ and $R(x, t)$ be QBS approximations of the exact solutions $u(x, t)$ and $v(x, t)$, respectively Then, $S(x, t)$ and $R(x, t)$ are expressed by the sum of QBS basis functions as:

$$S(x, t) = \sum_{j=-2}^{N+2} c_j(t)\phi_j(x), \quad R(x, t) = \sum_{j=-2}^{N+2} d_j(t)\phi_j(x) \tag{2.14}$$

where c_j and d_j, $j = -2, -1, 0, ..., N+2$, are unknown time dependent quantities to be determined via collocation technique with the convenient BCs. Throughout this study, we denote

$$\begin{aligned} &\frac{\partial^k S(x_j, t)}{\partial x^k} = S^{(k)}(x_j, t), \qquad \frac{\partial^k R(x_j, t)}{\partial x^k} = R^{(k)}(x_j, t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, .., j = 0, 1, ..., N. \\ &\frac{\partial^k u(x_j, t)}{\partial x^k} = u^{(k)}(x_j, t), \qquad \frac{\partial^k v(x_j, t)}{\partial x^k} = v^{(k)}(x_j, t), \end{aligned}$$

Let $S(x, t)$ and $R(x, t)$ be the QSI of $u(x, t)$ and $v(x, t)$, satisfying the following interpolation conditions

$$S(x_j, t) = u(x_j, t), \qquad 0 \le j \le N \tag{2.15}$$

$$R(x_j, t) = v(x_j, t), \qquad 0 \le j \le N \tag{2.16}$$

$$S''(x_j, t) = u''(x_j, t) + \frac{h^4}{720}u^{(6)}(x_j, t), \qquad j = 0, 1, N-1, N \tag{2.17}$$

$$R''(x_j, t) = v''(x_j, t) + \frac{h^4}{720}v^{(6)}(x_j, t). \qquad j = 0, 1, N-1, N \tag{2.18}$$

Theorem 1. *Let $S(x, t)$ and $R(x, t)$ be the QSI for $u(x, t)$ and $v(x, t) \in C^8(\Omega)$, respectively, and satisfy the interpolation conditions (2.15–2.18). Then, the below given results hold at the knot points x_j, $j = 0, 1, ..., N$:*

$$S''(x_j, t) = u''(x_j, t) + \frac{h^4}{720}u^{(6)}(x_j, t) + O(h^6) \tag{2.19}$$

$$\begin{aligned} R''(x_j,t) &= v''(x_j,t) + \frac{h^4}{720}v^{(6)}(x_j,t) + O(h^6) && (2.20)\\ S'(x_j,t) &= u'(x_j,t) + O(h^6) && (2.21)\\ R'(x_j,t) &= v'(x_j,t) + O(h^6) && (2.22)\end{aligned}$$

and

$$\begin{aligned} \left\|S^{(l)} - u^{(l)}\right\|_\infty &= O(h^{6-l}), \qquad l = 0,1,2\\ \left\|R^{(l)} - v^{(l)}\right\|_\infty &= O(h^{6-l}), \qquad l = 0,1,2\end{aligned}$$

where $K^{(l)}$, $u^{(l)}$, $P^{(l)}$ and $v^{(l)}$ denote the l th derivative w.r.t. 'x'.

Proof. For the proof see [29]. □

2.2.2. Construction of NOBSC method

Here, we construct a NOBSC method by developing a new approximation for $S''(x,t)$ and $R''(x,t)$. To do this, we first define the discrete operator ξ as follows

$$\xi M(x_j,t) = M(x_{j-2},t) - 4M(x_{j-1,}t) + 6M(x_j,t) - 4M(x_{j+1,}t) + M(x_{j+2},t), \quad j = 2,...,N-2 \tag{2.23}$$

for any function M defined at points of spatial discretization.

Lemma 2. *Let $S(x,t)$ and $R(x,t)$ be the QSI for $u(x,t)$ and $v(x,t) \in C^8(\Omega)$ and satisfy the interpolation conditions (2.15–2.18). Then we get,*

$$\begin{aligned} u^{(6)}(x_j,t) &= \frac{1}{h^4}\xi S''(x_j,t) + O(h^2), \quad j = 2,...,N-2 && (2.24)\\ v^{(6)}(x_j,t) &= \frac{1}{h^4}\xi R''(x_j,t) + O(h^2). \quad j = 2,...,N-2 && (2.25)\end{aligned}$$

Proof. We prove relation (2.24). From Eq (2.19), we have the following equality

$$\frac{S''(x_j,t)}{h^4} = \frac{u''(x_j,t)}{h^4} + \frac{1}{720}u^{(6)}(x_j,t) + O(h^2). \tag{2.26}$$

The application of the operator ξ defined by (2.23) on (2.26) gives

$$\begin{aligned} \frac{\xi S''(x_j,t)}{h^4} &= \frac{\xi S''(x_j,t)}{h^4} + \frac{1}{720}\xi u^{(6)}(x_j,t) + O(h^2)\\ &= \frac{1}{h^4}\left(u''(x_{j-2},t) - 4u''(x_{j-1},t) + 6u''(x_j,t) - 4u''(x_{j+1},t) + u''(x_{j+2},t)\right) && (2.27)\\ &+\frac{1}{720}\left(u^{(6)}(x_{j-2},t) - 4u^{(6)}(x_{j-1},t) + 6u^{(6)}(x_j,t) - 4u^{(6)}(x_{j+1},t) + u^{(6)}(x_{j+2},t)\right) + O(h^2).\end{aligned}$$

By using Taylor expansion and finite differences for the u terms at x_j on the right hand side of (2.27), we obtain,

$$\frac{\xi S''(x_j,t)}{h^4} = u^{(6)}(x_j,t) + O(h^2), \qquad j = 2,...,N-2.$$

This proves relation (2.24). In a similar manner, one can prove the relation (2.25). □

Corollary 1. *If* $u(x,t)$ *and* $v(x,t) \in C^8(\Omega)$*, then for* $j = 2, 3, ..., N-2$*, the below given approximations hold.*

$$u''(x_j,t) = S''(x_j,t) - \frac{1}{720}\xi S''(x_j,t) + O(h^6)$$

$$v''(x_j,t) = R''(x_j,t) - \frac{1}{720}\xi R''(x_j,t) + O(h^6).$$

Proof. By Lemma 2 and Theorem 1, one can prove this corollary. □

Lemma 3. *Suppose that* $S(x,t)$ *and* $R(x,t)$ *are the QSI for* $u(x,t)$ *and* $v(x,t) \in C^8(\Omega)$ *and satisfy the interpolation conditions (2.15–2.18). Then, we have*

For $j = 0, 1$:

$$u^{(6)}(x_j,t) = \frac{(3-j)\xi S''(x_2,t) - (2-j)\xi S''(x_3,t)}{h^4} + O(h^2). \tag{2.28}$$

For $(j,k) = (N-1,1), (N,0)$:

$$u^{(6)}(x_j,t) = \frac{(3-k)\xi S''(x_{N-2},t) - (2-k)\xi S''(x_{N-3},t)}{h^4} + O(h^2). \tag{2.29}$$

For $j = 0, 1$:

$$v^{(6)}(x_j,t) = \frac{(3-j)\xi R''(x_2,t) - (2-j)\xi R''(x_3,t)}{h^4} + O(h^2). \tag{2.30}$$

For $(j,k) = (N-1,1), (N,0)$:

$$v^{(6)}(x_j,t) = \frac{(3-k)\xi R''(x_{N-2},t) - (2-k)\xi R''(x_{N-3},t)}{h^4} + O(h^2). \tag{2.31}$$

Proof. We first prove relation (2.28) for $j = 1$. Consider the following approximation for $u^{(6)}(x_1,t)$,

$$u^{(6)}(x_1,t) = 2u^{(6)}(x_2,t) - u^{(6)}(x_3,t) + O(h^2). \tag{2.32}$$

Making use relation (2.24) for $j = 2, 3$, from Eq (2.32), we obtain

$$u^{(6)}(x_1,t) = \frac{2\xi S''(x_2,t)}{h^4} - \frac{\xi S''(x_3,t)}{h^4} + O(h^2).$$

Hence, this proves relation (2.28) for $j = 1$. Now, we prove relation (2.28) for $j = 0$. For this purpose, we take into consideration the following approximation for $u^{(6)}(x_0,t)$,

$$u^{(6)}(x_0,t) = 2u^{(6)}(x_1,t) - u^{(6)}(x_2,t) + O(h^2). \tag{2.33}$$

Considering relation (2.28) for $j = 1$ and relation (2.24) for $j = 2$, it follows from Eq (2.33) that

$$\begin{aligned} u^{(6)}(x_0,t) &= \frac{4\xi u^{(6)}(x_2,t)}{h^4} - \frac{2\xi u^{(6)}(x_3,t)}{h^4} - \frac{\xi u^{(6)}(x_2,t)}{h^4} + O(h^2) \\ &= \frac{3\xi u^{(6)}(x_2,t)}{h^4} - \frac{2\xi u^{(6)}(x_3,t)}{h^4} + O(h^2). \end{aligned}$$

Hence, the relation (2.28) for $j = 0$ is obtained. In a similar way, the remaining relations in Lemma 3 can be proved. Thus, the proof of Lemma 3 is completed. □

Corollary 2. *If $u(x,t)$ and $v(x,t) \in C^8(\Omega)$, then for $j = 0, 1, N-1, N$, the following relations are obtained*

$$\begin{aligned}
u''(x_j,t) &= S''(x_j,t) - \left(\frac{(3-j)\,\xi S''(x_2,t) - (2-j)\,\xi S''(x_3,t)}{720}\right) + O(h^6),\ j = 0,1 \\
u''(x_j,t) &= S''(x_j,t) - \left(\frac{(3-l)\,\xi S''(x_{N-2},t) - (2-l)\,\xi S''(x_{N-3},t)}{720}\right) + O(h^6),\ (j,l) = (N-1,1), (N,0) \\
v''(x_j,t) &= R''(x_j,t) - \left(\frac{(3-j)\,\xi R''(x_2,t) - (2-j)\,\xi R''(x_3,t)}{720}\right) + O(h^6),\ j = 0,1 \\
v''(x_j,t) &= R''(x_j,t) - \left(\frac{(3-l)\,\xi R''(x_{N-2},t) - (2-l)\,\xi R''(x_{N-3},t)}{720}\right) + O(h^6).\ (j,l) = (N-1,1), (N,0)
\end{aligned}$$

Proof. With the help of Lemma 3 and Theorem 1, one can prove corollary 2.

Let $S(x,t)$ and $R(x,t)$ be optimal QBS approximate solutions of $u(x,t)$ and $v(x,t)$, respectively. Then, the values of $S(x,t)$ and $R(x,t)$ and their first two higher-order derivatives are as follows:

$$\begin{aligned}
S(x_j,t) &= \frac{1}{120}\left(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}\right) + O(h^6), j = 0,1,...,N \\
S'(x_j,t) &= \frac{1}{24h}\left(c_{j-2} + 10c_{j-1} - 10c_{j+1} - c_{j+2}\right) + O(h^6), j = 0,1,...,N \qquad (2.34) \\
S''(x_0,t) &= \frac{1}{4320h^2}\left(\begin{array}{l} 717c_{-2} + 1448c_{-1} - 4300c_0 \\ +1322c_1 + 938c_2 - 202c_3 + 92c_4 \\ -10c_5 - 7c_6 + 2c_7 \end{array}\right) + O(h^6), \\
S''(x_1,t) &= \frac{1}{4320h^2}\left(\begin{array}{l} -2c_{-2} + 725c_{-1} + 1454c_0 \\ -4396c_1 + 1574c_2 + 602c_3 \\ +50c_4 - 4c_5 - 4c_6 + c_7 \end{array}\right) + O(h^6), \\
S''(x_j,t) &= \frac{1}{4320h^2}\left(\begin{array}{l} -c_{j-4} + 2c_{j-3} + 728c_{j-2} + \\ 1406c_{j-1} - 4270c_j + 1406c_{j+1} \\ +728c_{j+2} + 2c_{j+3} - c_{j+4} \end{array}\right) + O(h^6), j = 2,..,N-2 \\
S''(x_{N-1},t) &= \frac{1}{4320h^2}\left(\begin{array}{l} c_{N-7} - 4c_{N-6} - 4c_{N-5} + 50c_{N-4} \\ +602c_{N-3} + 1574c_{N-2} - 4396c_{N-1} \\ +1454c_N + 725c_{N+1} - 2c_{N+2} \end{array}\right) + O(h^6) \\
S''(x_N,t) &= \frac{1}{4320h^2}\left(\begin{array}{l} 2c_{N-7} - 7c_{N-6} - 10c_{N-5} + 92c_{N-4} \\ -202c_{N-3} + 938c_{N-2} + 1322c_{N-1} \\ -4300c_N + 1448c_{N+1} + 717c_{N+2} \end{array}\right) + O(h^6)
\end{aligned}$$

and

$$\begin{aligned}
R(x_j,t) &= \frac{1}{120}\left(d_{j-2} + 26d_{j-1} + 66d_j + 26d_{j+1} + d_{j+2}\right) + O(h^6), \qquad j = 0,1,...,N \\
R'(x_j,t) &= \frac{1}{24h}\left(d_{j-2} + 10d_{j-1} - 10d_{j+1} - d_{j+2}\right) + O(h^6), \qquad j = 0,1,...,N \qquad (2.35) \\
R''(x_0,t) &= \frac{1}{4320h^2}\left(\begin{array}{l} 717d_{-2} + 1448d_{-1} - 4300d_0 \\ +1322d_1 + 938d_2 - 202d_3 + 92d_4 \\ -10d_5 - 7d_6 + 2d_7 \end{array}\right) + O(h^6),
\end{aligned}$$

$$R''(x_1,t) = \frac{1}{4320h^2}\begin{pmatrix} -2d_{-2}+725d_{-1}+1454d_0 \\ -4396d_1+1574d_2+602d_3 \\ +50d_4-4d_5-4d_6+d_7 \end{pmatrix} + O(h^6),$$

$$R''(x_j,t) = \frac{1}{4320h^2}\begin{pmatrix} -d_{j-4}+2d_{j-3}+728d_{j-2}+ \\ 1406d_{j-1}-4270d_j+1406d_{j+1} \\ +728d_{j+2}+2d_{j+3}-d_{j+4} \end{pmatrix} + O(h^6), \qquad j=2,..,N-2$$

$$R''(x_{N-1},t) = \frac{1}{4320h^2}\begin{pmatrix} d_{N-7}-4d_{N-6}-4d_{N-5}+50d_{N-4} \\ +602d_{N-3}+1574d_{N-2}-4396d_{N-1} \\ +1454d_N+725d_{N+1}-2d_{N+2} \end{pmatrix} + O(h^6)$$

$$R''(x_N,t) = \frac{1}{4320h^2}\begin{pmatrix} 2d_{N-7}-7d_{N-6}-10d_{N-5}+92d_{N-4} \\ -202d_{N-3}+938d_{N-2}+1322d_{N-1} \\ -4300d_N+1448d_{N+1}+717d_{N+2} \end{pmatrix} + O(h^6).$$

□

3. Implementation of the proposed method

Substituting Eq (2.14) into Eqs (2.11) and (2.12), using Eqs (2.34) and (2.35), we get the following systems:

For $j=0$,

$$\begin{aligned} & c_{-2}^{n+1}\left[\tfrac{a_1}{120}-\tfrac{717\theta_3}{4320h^2}\right]+c_{-1}^{n+1}\left[\tfrac{26a_1}{120}-\tfrac{1448\theta_3}{4320h^2}\right]+c_0^{n+1}\left[\tfrac{66a_1}{120}+\tfrac{4300\theta_3}{4320h^2}\right] \\ & +c_1^{n+1}\left[\tfrac{26a_1}{120}-\tfrac{1322\theta_3}{4320h^2}\right]+c_2^{n+1}\left[\tfrac{a_1}{120}-\tfrac{938\theta_3}{4320h^2}\right]+c_3^{n+1}\left[\tfrac{202\theta_3}{4320h^2}\right] \\ & +c_4^{n+1}\left[-\tfrac{92\theta_3}{4320h^2}\right]+c_5^{n+1}\left[\tfrac{10\theta_3}{4320h^2}\right]+c_6^{n+1}\left[\tfrac{7\theta_3}{4320h^2}\right]+c_7^{n+1}\left[-\tfrac{2\theta_3}{4320h^2}\right] \\ & +d_{-2}^{n+1}\left[\tfrac{a_2}{120}\right]+d_{-1}^{n+1}\left[\tfrac{26a_2}{120}\right]+d_0^{n+1}\left[\tfrac{66a_2}{120}\right]+d_1^{n+1}\left[\tfrac{26a_2}{120}\right]+d_2^{n+1}\left[\tfrac{a_2}{120}\right] \\ & = m\left(x_0,t_n\right). \end{aligned} \tag{3.1}$$

For $j=1$,

$$\begin{aligned} & c_{-2}^{n+1}\left[\tfrac{2\theta_3}{4320h^2}\right]+c_{-1}^{n+1}\left[\tfrac{a_1}{120}-\tfrac{725\theta_3}{4320h^2}\right]+c_0^{n+1}\left[\tfrac{26a_1}{120}-\tfrac{1454\theta_3}{4320h^2}\right] \\ & +c_1^{n+1}\left[\tfrac{66a_1}{120}+\tfrac{4396\theta_3}{4320h^2}\right]+c_2^{n+1}\left[\tfrac{26a_1}{120}-\tfrac{1574\theta_3}{4320h^2}\right]+c_3^{n+1}\left[\tfrac{a_1}{120}-\tfrac{602\theta_3}{4320h^2}\right] \\ & +c_4^{n+1}\left[-\tfrac{50\theta_3}{4320h^2}\right]+c_5^{n+1}\left[\tfrac{4\theta_3}{4320h^2}\right]+c_6^{n+1}\left[\tfrac{4\theta_3}{4320h^2}\right]+c_7^{n+1}\left[-\tfrac{\theta_3}{4320h^2}\right] \\ & +d_{-1}^{n+1}\left[\tfrac{a_2}{120}\right]+d_0^{n+1}\left[\tfrac{26a_2}{120}\right]+d_1^{n+1}\left[\tfrac{66a_2}{120}\right]+d_2^{n+1}\left[\tfrac{26a_2}{120}\right]+d_3^{n+1}\left[\tfrac{a_2}{120}\right] \\ & = m\left(x_1,t_n\right). \end{aligned} \tag{3.2}$$

For $2 \leq j \leq N-2$

$$
\begin{aligned}
& c_{j-4}^{n+1}\left[\frac{\theta_3}{4320h^2}\right] + c_{j-3}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right] + c_{j-2}^{n+1}\left[\frac{a_1}{120} - \frac{728\theta_3}{4320h^2}\right] + c_{j-1}^{n+1}\left[\frac{26a_1}{120} - \frac{1406\theta_3}{4320h^2}\right] \\
& + c_{j}^{n+1}\left[\frac{66a_1}{120} + \frac{4270\theta_3}{4320h^2}\right] + c_{j+1}^{n+1}\left[\frac{26a_1}{120} - \frac{1406\theta_3}{4320h^2}\right] + c_{j+2}^{n+1}\left[\frac{a_1}{120} - \frac{728\theta_3}{4320h^2}\right] \\
& + c_{j+3}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right] + c_{j+4}^{n+1}\left[\frac{\theta_3}{4320h^2}\right] + d_{j-2}^{n+1}\left[\frac{a_2}{120}\right] + d_{j-1}^{n+1}\left[\frac{26a_2}{120}\right] + \\
& d_{j}^{n+1}\left[\frac{66a_2}{120}\right] + d_{j+1}^{n+1}\left[\frac{26a_2}{120}\right] + d_{j+2}^{n+1}\left[\frac{a_2}{120}\right] = m\left(x_j, t_n\right).
\end{aligned}
\tag{3.3}
$$

For $j = N-1$,

$$
\begin{aligned}
& c_{N-7}^{n+1}\left[-\frac{\theta_3}{4320h^2}\right] + c_{N-6}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right] + c_{N-5}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right] + c_{N-4}^{n+1}\left[-\frac{50\theta_3}{4320h^2}\right] \\
& + c_{N-3}^{n+1}\left[\frac{a_1}{120} - \frac{602\theta_3}{4320h^2}\right] + c_{N-2}^{n+1}\left[\frac{26a_1}{120} - \frac{1574\theta_3}{4320h^2}\right] + c_{N-1}^{n+1}\left[\frac{66a_1}{120} + \frac{4396\theta_3}{4320h^2}\right] \\
& + c_{N}^{n+1}\left[\frac{26a_1}{120} - \frac{1454\theta_3}{4320h^2}\right] + c_{N+1}^{n+1}\left[\frac{a_1}{120} - \frac{725\theta_3}{4320h^2}\right] + c_{N+2}^{n+1}\left[\frac{2\theta_3}{4320h^2}\right] + d_{N-3}^{n+1}\left[\frac{a_2}{120}\right] \\
& + d_{N-2}^{n+1}\left[\frac{26a_2}{120}\right] + d_{N-1}^{n+1}\left[\frac{66a_2}{120}\right] + d_{N}^{n+1}\left[\frac{26a_2}{120}\right] + d_{N+1}^{n+1}\left[\frac{a_2}{120}\right] = m\left(x_{N-1}, t_n\right).
\end{aligned}
\tag{3.4}
$$

For $j = N$,

$$
\begin{aligned}
& c_{N-7}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right] + c_{N-6}^{n+1}\left[\frac{7\theta_3}{4320h^2}\right] + c_{N-5}^{n+1}\left[\frac{10\theta_3}{4320h^2}\right] + c_{N-4}^{n+1}\left[-\frac{92\theta_3}{4320h^2}\right] \\
& + c_{N-3}^{n+1}\left[\frac{202\theta_3}{4320h^2}\right] + c_{N-2}^{n+1}\left[\frac{a_1}{120} - \frac{938\theta_3}{4320h^2}\right] + c_{N-1}^{n+1}\left[\frac{26a_1}{120} - \frac{1322\theta_3}{4320h^2}\right] \\
& + c_{N}^{n+1}\left[\frac{66a_1}{120} + \frac{4300\theta_3}{4320h^2}\right] + c_{N+1}^{n+1}\left[\frac{26a_1}{120} - \frac{1448\theta_3}{4320h^2}\right] + c_{N+2}^{n+1}\left[\frac{a_1}{120} - \frac{717\theta_3}{4320h^2}\right] \\
& + d_{N-2}^{n+1}\left[\frac{a_2}{120}\right] + d_{N-1}^{n+1}\left[\frac{26a_2}{120}\right] + d_{N}^{n+1}\left[\frac{66a_2}{120}\right] + d_{N+1}^{n+1}\left[\frac{26a_2}{120}\right] + d_{N+2}^{n+1}\left[\frac{a_2}{120}\right] \\
& = m\left(x_N, t_n\right).
\end{aligned}
\tag{3.5}
$$

And for $j = 0$,

$$
\begin{aligned}
& c_{-2}^{n+1}\left[\frac{a_5}{120} + \frac{717a_7}{4320h^2}\right] + c_{-1}^{n+1}\left[\frac{26a_5}{120} + \frac{1448a_7}{4320h^2}\right] + c_{0}^{n+1}\left[\frac{66a_5}{120} - \frac{4300a_7}{4320h^2}\right] + \\
& + c_{1}^{n+1}\left[\frac{26a_5}{120} + \frac{1322a_7}{4320h^2}\right] + c_{2}^{n+1}\left[\frac{a_5}{120} + \frac{938a_7}{4320h^2}\right] + c_{3}^{n+1}\left[-\frac{202a_7}{4320h^2}\right] + c_{4}^{n+1}\left[\frac{92a_7}{4320h^2}\right] \\
& + c_{5}^{n+1}\left[-\frac{10a_7}{4320h^2}\right] + c_{6}^{n+1}\left[-\frac{7a_7}{4320h^2}\right] + c_{7}^{n+1}\left[\frac{2a_7}{4320h^2}\right] + d_{-2}^{n+1}\left[\frac{a_6}{120} - \frac{717\theta_3}{4320h^2}\right] \\
& + d_{-1}^{n+1}\left[\frac{26a_6}{120} - \frac{1448\theta_3}{4320h^2}\right] + d_{0}^{n+1}\left[\frac{66a_6}{120} + \frac{4300\theta_3}{4320h^2}\right] + d_{1}^{n+1}\left[\frac{26a_6}{120} - \frac{1322\theta_3}{4320h^2}\right] \\
& + d_{2}^{n+1}\left[\frac{a_6}{120} - \frac{938\theta_3}{4320h^2}\right] + d_{3}^{n+1}\left[\frac{202\theta_3}{4320h^2}\right] + d_{4}^{n+1}\left[-\frac{92\theta_3}{4320h^2}\right] \\
& + d_{5}^{n+1}\left[\frac{10\theta_3}{4320h^2}\right] + d_{6}^{n+1}\left[\frac{7\theta_3}{4320h^2}\right] + d_{7}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right] = k\left(x_0, t_n\right).
\end{aligned}
\tag{3.6}
$$

For $j = 1$,

$$\begin{aligned}
&c_{-2}^{n+1}\left[-\frac{2a_7}{4320h^2}\right]+c_{-1}^{n+1}\left[\frac{a_5}{120}+\frac{725a_7}{4320h^2}\right]+c_{0}^{n+1}\left[\frac{26a_5}{120}+\frac{1454a_7}{4320h^2}\right]\\
&+c_{1}^{n+1}\left[\frac{66a_5}{120}-\frac{4396a_7}{4320h^2}\right]+c_{2}^{n+1}\left[\frac{26a_5}{120}+\frac{1574a_7}{4320h^2}\right]+c_{3}^{n+1}\left[\frac{a_5}{120}+\frac{602a_7}{4320h^2}\right]\\
&+c_{4}^{n+1}\left[\frac{50a_7}{4320h^2}\right]+c_{5}^{n+1}\left[-\frac{4a_7}{4320h^2}\right]+c_{6}^{n+1}\left[-\frac{4a_7}{4320h^2}\right]+c_{7}^{n+1}\left[\frac{a_7}{4320h^2}\right]\\
&+d_{-2}^{n+1}\left[\frac{2\theta_3}{4320h^2}\right]+d_{-1}^{n+1}\left[\frac{a_6}{120}-\frac{725\theta_3}{4320h^2}\right]+d_{0}^{n+1}\left[\frac{26a_6}{120}-\frac{1454\theta_3}{4320h^2}\right]\\
&+d_{1}^{n+1}\left[\frac{66a_6}{120}+\frac{4396\theta_3}{4320h^2}\right]+d_{2}^{n+1}\left[\frac{26a_6}{120}-\frac{1574\theta_3}{4320h^2}\right]+d_{3}^{n+1}\left[\frac{a_6}{120}-\frac{602\theta_3}{4320h^2}\right]\\
&+d_{4}^{n+1}\left[-\frac{50\theta_3}{4320h^2}\right]+d_{5}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right]+d_{6}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right]+d_{7}^{n+1}\left[-\frac{\theta_3}{4320h^2}\right]=k\left(x_1,t_n\right).
\end{aligned}\tag{3.7}$$

For $2 \leq j \leq N-2$

$$\begin{aligned}
&c_{j-4}^{n+1}\left[-\frac{a_7}{4320h^2}\right]+c_{j-3}^{n+1}\left[\frac{2a_7}{4320h^2}\right]+c_{j-2}^{n+1}\left[\frac{a_5}{120}+\frac{728a_7}{4320h^2}\right]+c_{j-1}^{n+1}\left[\frac{26a_5}{120}+\frac{1406a_7}{4320h^2}\right]\\
&+c_{j}^{n+1}\left[\frac{66a_5}{120}-\frac{4270a_7}{4320h^2}\right]+c_{j+1}^{n+1}\left[\frac{26a_5}{120}+\frac{1406a_7}{4320h^2}\right]+c_{j+2}^{n+1}\left[\frac{a_5}{120}+\frac{728a_7}{4320h^2}\right]+c_{j+3}^{n+1}\left[\frac{2a_7}{4320h^2}\right]\\
&+c_{j+4}^{n+1}\left[-\frac{a_7}{4320h^2}\right]+d_{j-4}^{n+1}\left[\frac{\theta_3}{4320h^2}\right]+d_{j-3}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right]+d_{j-2}^{n+1}\left[\frac{a_6}{120}-\frac{728\theta_3}{4320h^2}\right]\\
&+d_{j-1}^{n+1}\left[\frac{26a_6}{120}-\frac{1406\theta_3}{4320h^2}\right]+d_{j}^{n+1}\left[\frac{66a_6}{120}+\frac{4270\theta_3}{4320h^2}\right]+d_{j+1}^{n+1}\left[\frac{26a_6}{120}-\frac{1406\theta_3}{4320h^2}\right]\\
&+d_{j+2}^{n+1}\left[\frac{a_6}{120}-\frac{728\theta_3}{4320h^2}\right]+d_{j+3}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right]+d_{j+4}^{n+1}\left[\frac{\theta_3}{4320h^2}\right]=k\left(x_j,t_n\right).
\end{aligned}\tag{3.8}$$

For $j = N-1$,

$$\begin{aligned}
&c_{N-7}^{n+1}\left[\frac{a_7}{4320h^2}\right]+c_{N-6}^{n+1}\left[-\frac{4a_7}{4320h^2}\right]+c_{N-5}^{n+1}\left[-\frac{4a_7}{4320h^2}\right]+c_{N-4}^{n+1}\left[\frac{50a_7}{4320h^2}\right]\\
&+c_{N-3}^{n+1}\left[\frac{a_5}{120}+\frac{602a_7}{4320h^2}\right]+c_{N-2}^{n+1}\left[\frac{26a_5}{120}+\frac{1574a_7}{4320h^2}\right]+c_{N-1}^{n+1}\left[\frac{66a_5}{120}-\frac{4396a_7}{4320h^2}\right]\\
&+c_{N}^{n+1}\left[\frac{26a_5}{120}+\frac{1454a_7}{4320h^2}\right]+c_{N+1}^{n+1}\left[\frac{a_5}{120}+\frac{725a_7}{4320h^2}\right]+c_{N+2}^{n+1}\left[-\frac{2a_7}{4320h^2}\right]\\
&+d_{N-7}^{n+1}\left[-\frac{\theta_3}{4320h^2}\right]+d_{N-6}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right]+d_{N-5}^{n+1}\left[\frac{4\theta_3}{4320h^2}\right]+d_{N-4}^{n+1}\left[-\frac{50\theta_3}{4320h^2}\right]\\
&+d_{N-3}^{n+1}\left[\frac{a_6}{120}-\frac{602\theta_3}{4320h^2}\right]+d_{N-2}^{n+1}\left[\frac{26a_6}{120}-\frac{1574\theta_3}{4320h^2}\right]+d_{N-1}^{n+1}\left[\frac{66a_6}{120}+\frac{4396\theta_3}{4320h^2}\right]\\
&+d_{N}^{n+1}\left[\frac{26a_6}{120}-\frac{1454\theta_3}{4320h^2}\right]+d_{N+1}^{n+1}\left[\frac{a_6}{120}-\frac{725\theta_3}{4320h^2}\right]+d_{N+2}^{n+1}\left[\frac{2\theta_3}{4320h^2}\right]=k\left(x_{N-1},t_n\right).
\end{aligned}\tag{3.9}$$

For $j = N$,

$$\begin{aligned}
&c_{N-7}^{n+1}\left[\frac{2a_7}{4320h^2}\right]+c_{N-6}^{n+1}\left[-\frac{7a_7}{4320h^2}\right]+c_{N-5}^{n+1}\left[-\frac{10a_7}{4320h^2}\right]+c_{N-4}^{n+1}\left[\frac{92a_7}{4320h^2}\right]\\
&+c_{N-3}^{n+1}\left[-\frac{202a_7}{4320h^2}\right]+c_{N-2}^{n+1}\left[\frac{a_5}{120}+\frac{938a_7}{4320h^2}\right]+c_{N-1}^{n+1}\left[\frac{26a_5}{120}+\frac{1322a_7}{4320h^2}\right]\\
&+c_{N}^{n+1}\left[\frac{66a_5}{120}-\frac{4300a_7}{4320h^2}\right]+c_{N+1}^{n+1}\left[\frac{26a_5}{120}+\frac{1448a_7}{4320h^2}\right]+c_{N+2}^{n+1}\left[\frac{a_5}{120}+\frac{717a_7}{4320h^2}\right]\\
&+d_{N-7}^{n+1}\left[-\frac{2\theta_3}{4320h^2}\right]+d_{N-6}^{n+1}\left[\frac{7\theta_3}{4320h^2}\right]+d_{N-5}^{n+1}\left[\frac{10\theta_3}{4320h^2}\right]+d_{N-4}^{n+1}\left[-\frac{92\theta_3}{4320h^2}\right]\\
&+d_{N-3}^{n+1}\left[\frac{202\theta_3}{4320h^2}\right]+d_{N-2}^{n+1}\left[\frac{a_6}{120}-\frac{938\theta_3}{4320h^2}\right]+d_{N-1}^{n+1}\left[\frac{26a_6}{120}-\frac{1322\theta_3}{4320h^2}\right]\\
&+d_{N}^{n+1}\left[\frac{66a_6}{120}+\frac{4300\theta_3}{4320h^2}\right]+d_{N+1}^{n+1}\left[\frac{26a_6}{120}-\frac{1448\theta_3}{4320h^2}\right]+d_{N+2}^{n+1}\left[\frac{a_6}{120}-\frac{717\theta_3}{4320h^2}\right]=k\left(x_N,t_n\right),
\end{aligned}\tag{3.10}$$

where

$$\begin{array}{lll}
a_1 = 1+\theta_3\lambda^2 & a_2 = 2\alpha\theta_3-\theta_1 & a_3 = 1-\theta_4\lambda^2\\
a_4 = -2\alpha\theta_4+\theta_2 & a_5 = \lambda^2\theta_1-2\alpha\lambda^2\theta_3 & a_6 = 1+2\alpha\theta_1+\lambda^2\theta_3-4\alpha^2\theta_4\\
a_7 = -\theta_1+2\alpha\theta_3 & a_8 = -\lambda^2\theta_2+2\alpha\lambda^2\theta_4 & a_9 = 1-2\alpha\theta_2-\lambda^2\theta_4+4\alpha^2\theta_4\\
a_{10} = \theta_2-2\alpha\theta_4 & a_{11} = -2\alpha\theta_4+\theta_2 & a_{12} = \theta_1-2\alpha\theta_3
\end{array}$$

$$m(x_j,t_n) = a_3S_j^n + R_j^n + \theta_4\left(S''\right)_j^n + \theta_4 g\left(x_j,t_n\right) + \theta_3 g\left(x_j,t_{n+1}\right),\ 0\le j\le N$$

and for $0 \le j \le N$

$$\begin{aligned}
k(x_j,t_n) \;=\;& a_8S_j^n + a_9R_j^n + a_{10}\left(S''\right)_j^n + \theta_4\left(R''\right)_j^n + a_{11}g\left(x_j,t_n\right)\\
&+a_{12}g\left(x_j,t_{n+1}\right) + \theta_4 g_t(x_j,t_n) + \theta_3 g_t(x_j,t_{n+1}).
\end{aligned}$$

The final system obtained by combining Eqs (3.1)–(3.5) with Eqs (3.6)–(3.10) consists of $2N+2$ linear equations involving $2N+10$ unknown parameters, $\mathbf{c} = \left(c_{-2}^{n+1}, c_{-1}^{n+1}, ..., c_{N+2}^{n+1}\right)$ and $\mathbf{d} = \left(d_{-2}^{n+1}, d_{-1}^{n+1}, ..., d_{N+2}^{n+1}\right)$. To set the number of unknown parameters equal to the number of equations, BCs are implemented to eliminate unknown parameters c_{-2}^{n+1}, d_{-2}^{n+1}, c_{-1}^{n+1}, d_{-1}^{n+1}, c_{N+1}^{n+1}, d_{N+1}^{n+1}, c_{N+2}^{n+1}, d_{N+2}^{n+1} from the system. Therefore, the resultant system can be written in the following form

$$AX = B \tag{3.11}$$

where A is $(2N+2)\times(2N+2)$ matrix, B and X are $(2N+2)\times 1$ column matrices. In order to initiate the time evolution process, the initial vector

$$\left(c_{-2}^0, d_{-2}^0, c_{-1}^0, d_{-1}^0, ..., c_{N+2}^0, d_{N+2}^0\right)^T$$

is first determined by utilizing ICs and BCs of the problem. After getting the values of the initial parameters $\mathbf{c}^0$ and $\mathbf{d}^0$, the unknown vector

$$\left(c_{-2}^{n+1}, d_{-2}^{n+1}, ..., c_{N+2}^{n+1}, d_{N+2}^{n+1}\right)^T$$

for $n = 0, 1, 2, ...$ is iteratively calculated at the required any time level. Pseudo code for the numerical algorithm is given below.

Numerical Algorithm
Input $\alpha, \lambda, \theta_1, \theta_2, \theta_3, \theta_4, \Delta t, t, N, h$
Output $S(x,t), R(x,t)$
Determine $u_0(x,0)$ and $v_0(x,0)$
Obtain the initial parameters $\left(c_m^0\right)_{m=-2}^{N+2}$ and $\left(d_m^0\right)_{m=-2}^{N+2}$
for $i = \Delta t : \Delta t : t$ do
Define $\{a_k\}_{k=1}^{12}$
Form the matrix A and column matrix B stated in Eq 3.11
Find $\left\{x_k^i\right\}_{k=-2}^{2N+2}$ by solving the system $AX = B$
end
Obtain the unknowns $S(x,t)$ and $R(x,t)$

4. Numerical results

In this section, three numerical examples are presented to validate the applicability and efficiency of the suggested technique. We compute the L_∞ error norm defined by the following formula:

$$L_\infty = \|u - U_N\|_\infty = \max_j \left|u_j - U_j\right|, \tag{4.1}$$

where u and U_N denote analytical and approximate solutions, respectively. The convergence orders in time and space are calculated by the following formulas:

$$\text{order} = \frac{\log\left|\dfrac{(L_\infty)_{h_i}}{(L_\infty)_{h_{i+1}}}\right|}{\log\left|\dfrac{h_i}{h_{i+1}}\right|}, \tag{4.2}$$

$$\text{order} = \frac{\log\left|\dfrac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}}\right|}{\log\left|\dfrac{\Delta t_i}{\Delta t_{i+1}}\right|} \tag{4.3}$$

where $(L_\infty)_{h_i}$and $(L_\infty)_{\Delta t_i}$ are the error norm L_∞ for space step h_i and time step Δt_i, respectively.

4.1. Test problem 1

Consider the following TE

$$u_{tt} + 20u_t + 25u = u_{xx} + g(x,t) \quad x \in [0,2]$$

with BCs and ICs

$$u(0,t) = \tan\left(\tfrac{t}{2}\right), \qquad u(2,t) = \tan\left(\tfrac{2+t}{2}\right) \quad t \geq 0$$
$$u(x,0) = \tan\left(\tfrac{x}{2}\right), \qquad u_t(x,0) = \tfrac{1}{2}\left(1 + \tan^2\left(\tfrac{x}{2}\right)\right)$$

where

$$g(x,t) = 10\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + 25\tan\left(\frac{x+t}{2}\right).$$

The analytical solution of this problem is also taken as

$$u(x,t) = \tan\left(\frac{x+t}{2}\right).$$

The computational process is carried out at different levels of time for the step sizes $h = 0.001$, $h = 0.01$, $h = 0.02$, $\Delta t = 0.001$ and $\Delta t = 0.01$. The error L_∞ is reported in Table 1 along with the results obtained by the methods in [8, 14, 20, 23, 25, 26]. From Table 1, we can observe that our method provides far better results than the methods in [8, 14, 20, 23, 25, 26]. Moreover, Table 1 demonstrates the improvement in results even with fewer collocation points and number of divisions of the temporal domain, which makes the proposed method computationally efficient. The spatial and temporal orders of the convergence along with the error norm L_∞ are listed in Tables 2 and 3, respectively. As expected from the theoretical analysis, the used technique approaches fourth-order accuracy in time and nearly sixth-order accuracy in space. The propagation of the approximate solution obtained for $\Delta t = 0.01$ and $h = 0.01$ at different times up to $t = 1$ is plotted in Figure 1. Furthermore, the absolute error at $t = 1$ is given in Figure 2.

Table 1. Comparison of error norms L_∞.

Method	h	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1$
Present ($\Delta t = 0.01$)	0.01	2.82×10^{-10}	1.04×10^{-9}	5.02×10^{-9}	4.90×10^{-8}	3.72×10^{-6}
Present ($\Delta t = 0.01$)	0.02	2.83×10^{-10}	1.05×10^{-9}	5.16×10^{-9}	5.23×10^{-8}	–
Present ($\Delta t = 0.001$)	0.001	3.79×10^{-13}	7.24×10^{-13}	1.21×10^{-12}	5.78×10^{-12}	4.00×10^{-10}
Present ($\Delta t = 0.001$)	0.02	4.25×10^{-12}	2.68×10^{-11}	2.53×10^{-10}	6.03×10^{-9}	–
[26] ($\Delta t = 0.001$)	0.001	1.20×10^{-8}	3.23×10^{-8}	1.23×10^{-7}	6.35×10^{-7}	1.26×10^{-5}
[20] ($\Delta t = 0.001$)	0.001	3.61×10^{-4}	1.04×10^{-4}	2.60×10^{-3}	7.43×10^{-3}	4.66×10^{-2}
[23] ($\Delta t = 0.001$)	0.001	6.83×10^{-5}	4.28×10^{-5}	–	–	–
[8] ($\Delta t = 0.001$)	0.001	2.56×10^{-7}	4.93×10^{-7}	–	–	–
[14] ($\Delta t = 0.001$)	0.01	5.11×10^{-7}	1.60×10^{-6}	7.03×10^{-6}	5.79×10^{-5}	2.46×10^{-3}
[25] ($\Delta t = 0.001$)	0.02	1.69×10^{-7}	4.79×10^{-7}	1.37×10^{-6}	6.39×10^{-6}	–

Table 2. Spatial order of convergence and the error norm with $\Delta t = 0.0001$ at $t = 1$.

h	L_∞	Order
0.0125	1.66×10^{-7}	-
0.00625	3.97×10^{-9}	5.39
0.003125	7.67×10^{-11}	5.69

Table 3. Temporal order of convergence and error norm with $h = 0.001$ at $t = 1$.

Δt	L_∞	Order
0.01	2.42×10^{-2}	-
0.05	2.21×10^{-3}	3.45
0.025	1.53×10^{-4}	3.85
0.0125	9.72×10^{-6}	3.98

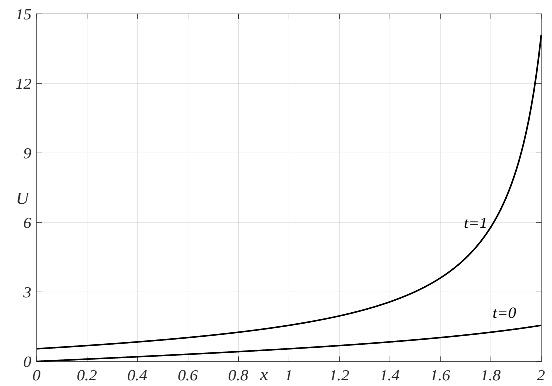

Figure 1. The simulation of approximate solution at $t = 0$ and $t = 1$.

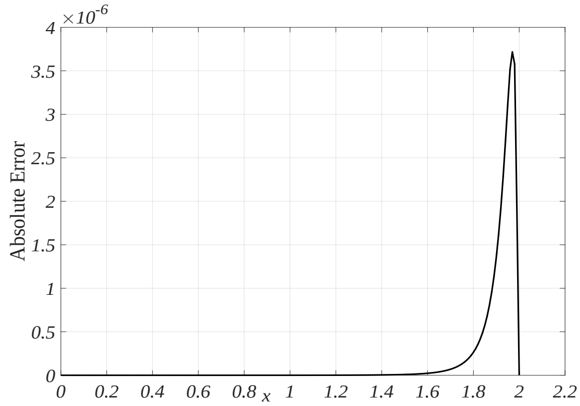

Figure 2. Absolute error of the proposed method at $t = 1$.

4.2. Test problem 2

Consider the TE given by

$$u_{tt} + 40u_t + 100u = u_{xx} + g(x, t)$$

with $g(x,t) = 23\exp(-2t)\sinh(x)$ over interval $[0,1]$ for space variable. The ICs and BCs are considered as:

$$\begin{aligned} &u(x,0) = \sinh(x), && u_t(x,0) = -2\sinh(x) \\ &u(0,t) = \exp(-2t)\sinh(0) && u(1,t) = \exp(-2t)\sinh(1) \quad t \geq 0. \end{aligned}$$

The analytical solution of the above test problem is

$$u(x,t) = \exp(-2t)\sinh(x).$$

The computations are performed at various times by taking $h = 0.05$, $h = 0.01$, $\Delta t = 0.01$ and $\Delta t = 0.001$. The error L_∞ is presented in Table 4. It is seen from Table 4 that the obtained error L_∞ is satisfactorily better and smaller than those of the methods given in [10, 12] even with less number of divisions of the temporal domain. We have listed the spatial and temporal order of convergence along with errors in Tables 5 and 6, respectively. From the Tables 5 and 6, it can be said that the proposed scheme has fourth-order temporal accuracy and sixth-order rate of convergence in space. The simulation of the approximate solution for $h = 0.05, \Delta t = 0.01$ at various values of t is shown graphically in Figure 3. The absolute error of the presented method for $h = 0.05$ and $\Delta t = 0.01$ is drawn in Figure 4.

Table 4. Comparison of error norm L_∞ with $h = 0.05$.

Method	Δt	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$	$t = 3$
Present	0.01	5.53×10^{-11}	3.05×10^{-11}	1.34×10^{-11}	5.38×10^{-12}	7.88×10^{-13}
Present	0.001	2.63×10^{-14}	1.14×10^{-14}	4.38×10^{-15}	1.75×10^{-15}	2.45×10^{-16}
[10]	0.01	2.18×10^{-9}	8.73×10^{-10}	3.32×10^{-10}	–	–
[10]	0.001	2.15×10^{-9}	8.59×10^{-10}	3.26×10^{-10}	–	–
[12]	0.001	2.24×10^{-5}	3.57×10^{-4}	1.95×10^{-2}	–	–

Table 5. Spatial order of convergence and the error norm with $\Delta t = 0.001$ at $t = 1.5$.

h	L_∞	Order
0.2	3.43×10^{-11}	-
0.1	3.19×10^{-13}	6.75
0.05	4.38×10^{-15}	6.19

Table 6. Temporal order of convergence and error norm with $h = 0.01$ at $t = 1.5$.

Δt	L_∞	Order
0.1	1.37×10^{-7}	-
0.05	8.51×10^{-9}	4.00
0.025	5.32×10^{-10}	4.00
0.0125	3.32×10^{-11}	4.00

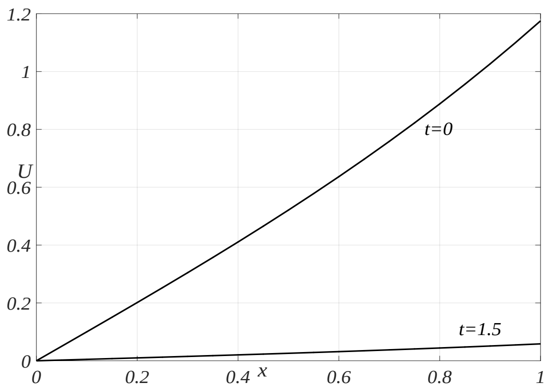

Figure 3. The simulation of approximate solution at $t = 0$ and $t = 1.5$.

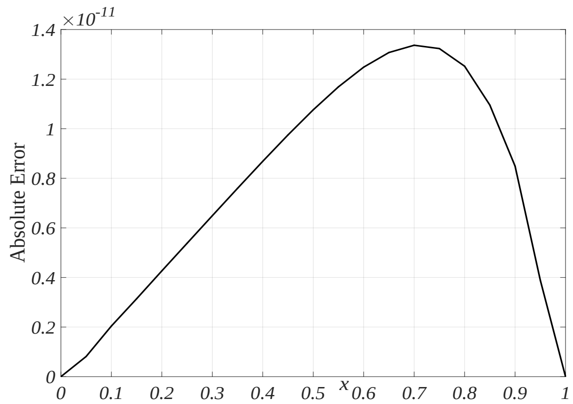

Figure 4. Absolute error of the proposed method at $t = 1.5$.

4.3. Test problem 3

Consider the following TE

$$u_{tt} + 12u_t + 4u = u_{xx} + g(x,t), \quad x \in [0,1]$$

with ICs and BCs

$$u(x,0) = \sin(x), \qquad u_t(x,0) = 0$$
$$u(0,t) = 0, \qquad u(1,t) = \cos(t)\sin(1), \quad t \geq 0$$

where

$$g(x,t) = -12\sin(t)\sin(x) + 4\cos(t)\sin(x).$$

The analytical solution of this problem is considered as

$$u(x,t) = \cos(t)\sin(x).$$

This problem has been solved by choosing $\Delta t = 0.01$, $\Delta t = 0.001$, $h = 0.05$ and $h = 0.005$. The error L_∞ at various values of t is reported in Table 7 which confirms that the present method produces far better results than the methods in [14, 17, 20, 24, 25]. In addition, Table 7 shows the improvement in results even with small partitions in space and time domains. Table 8 displays the error L_∞ and order of spatial convergence with different spatial steps h at $t = 1$. In Table 9, the error L_∞ and the order of temporal convergence are calculated with different time steps Δt at $t = 1$. When examined the results in Tables 8 and 9, it can be seen that our method has almost accuracy of $O\left(h^6\right)$ in the spatial direction and an accuracy of $O\left(\Delta t^4\right)$ in temporal direction. This shows that the orders of the temporal and spatial convergence obtained by our method are compatible with the theoretical analysis. The simulation of the approximate solution obtained for $h = 0.05$ and $\Delta t = 0.01$ at various values of t is given in Figure 5. Figure 6 displays the absolute error at $t = 1$.

Table 7. Comparison of error norms L_∞ with $\Delta t = 0.001$.

Method	h	$t = 0.2$	$t = 0.4$	$t = 1$	$t = 2$	$t = 3$
Present ($\Delta t = 0.01$)	0.05	2.35×10^{-13}	7.67×10^{-13}	3.05×10^{-12}	5.73×10^{-12}	3.86×10^{-12}
Present	0.005	3.54×10^{-14}	4.51×10^{-14}	5.16×10^{-14}	1.59×10^{-14}	5.42×10^{-14}
[25]	0.005	1.78×10^{-9}	7.00×10^{-9}	3.34×10^{-8}	–	–
[17]	0.005	2.42×10^{-5}	7.93×10^{-5}	1.64×10^{-4}	–	–
[24]	0.005	6.83×10^{-5}	1.48×10^{-4}	3.43×10^{-4}	–	–
[14]	0.005	3.73×10^{-11}	3.99×10^{-11}	5.42×10^{-11}	–	–
[20]	0.005	2.00×10^{-7}	4.30×10^{-7}	–	–	–

Table 8. Spatial order of convergence and the error norm with $\Delta t = 0.001$ at $t = 1$.

h	L_∞	Order
0.2	4.39×10^{-10}	-
0.1	1.06×10^{-11}	5.37
0.05	1.14×10^{-13}	6.53

Table 9. Temporal order of convergence and error norm with $h = 0.01$ at $t = 1$.

Δt	L_∞	Order
0.1	3.28×10^{-8}	-
0.05	2.05×10^{-9}	4.00
0.025	1.28×10^{-10}	4.00
0.0125	8.04×10^{-12}	4.00

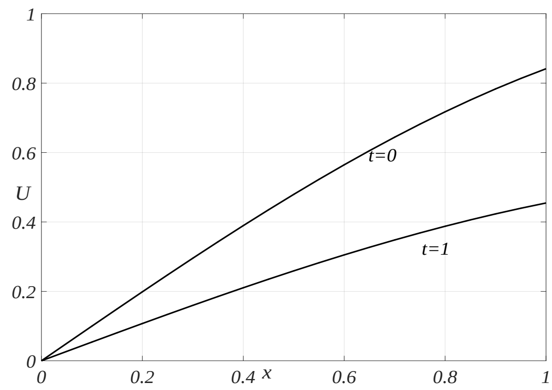

Figure 5. The simulation of approximate solution at $t = 0$ and $t = 1$.

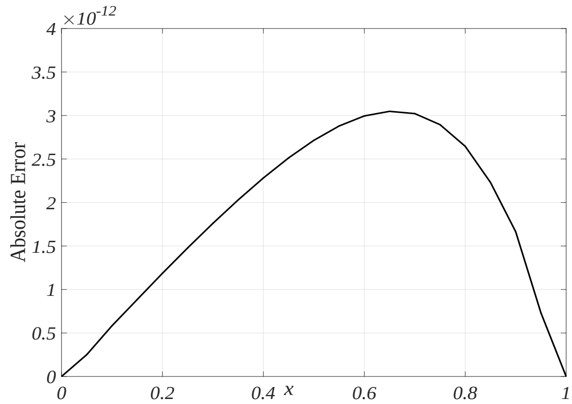

Figure 6. Absolute error of the proposed method at $t = 1$.

5. Conclusions

In this study, we have constructed a new high-order numerical scheme with fourth-order accuracy in time and sixth-order accuracy in space. A new approximation for the second-order spatial derivative is developed. The local truncation error analysis is discussed for time discretization. The error estimation of optimal QBS technique is carried out. Three test problems are provided to justify the effectiveness and improvement in the obtained numerical results and to check the theoretical rate of convergence. It is obviously observed that theoretical prediction is consistent with the numerical calculation. Moreover, the obtained approximate results are compared with other some existing methods applied to find the numerical solution of the TE. Comparison confirms that the present scheme provides far better results than the methods given in [8,10,12,14,17,20,23–26]. A major improvement of the proposed technique is that satisfactory results are provided even with fewer points in space and time domains. To conclude, the proposed method approximates very well solution of the TE and is computationally efficient for solving the TE. Also, as a future direction, the technique suggested for the spatial discretization can also be applied to two-dimensional problems in space [30] and time fractional-type problems [31].

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Conflict of interest

The author declares that he has no competing interest.

References

1. M. El-Azab, M. El-Gamel, A numerical algorithm for the solution of telegraph equations, *Appl. Math. Comput.*, **190** (2007), 757–764. //doi.org/10.1016/j.amc.2007.01.091
2. S. A. Yousefi, Legendre multiwavelet Galerkin method for solving the hyperbolic telegraph equation, *Numer. Methods Partial Differ. Equ.*, **26** (2010), 535–543. https://doi.org/10.1002/num.20445
3. M. M. Hosseini, S. T. Mohyud-Din, A. Nakhaeei, New Rothe-wavelet method for solving telegraph equations, *Int. J. Syst. Sci.*, **43** (2012), 1171–1176. https://doi.org/10.1080/00207721.2010.547626
4. M. Inc, A. Akgul, A. Kilicman, Numerical solutions of the second-order one-dimensional telegraph equation based on reproducing kernel Hilbert space, *Abstr. Appl. Anal.*, **2013** (2013), 768963. https://doi.org/10.1155/2013/768963
5. M. H. Heydari, M. R. Hooshmandasl F. M. Ghaini, A new approach of the Chebyshev wavelets method of partial differential equations with boundary conditions of the telegraph type, *Appl. Math. Model.*, **38** (2014), 1597–1606. https://doi.org/10.1016/j.apm.2013.09.013
6. S. Abbasbandy, H. R. Ghehsareh, I. Haskim, A. Alsaedi, A comparison study of meshfree techniques for solving the two-dimensional linear hyperbolic telegraph equation, *Eng. Anal. Bound. Elem.*, **47** (2014), 10–20. https://doi.org/10.1016/j.enganabound.2014.04.006
7. J. Rashidinia, M. Jokar, Application of polynomial scaling functions for numerical solution of telegraph equation, *Appl. Anal.*, **95** (2016), 105–123. https://doi.org/10.1080/00036811.2014.998654
8. D. Zhang, F. Peng, X. Miao, A new unconditionally stable method of telegraph equation based on associated hermite orthogonal functions, *Adv. Math. Phys.*, **2016** (2016), 7045657. https://doi.org/10.1155/2016/7045657
9. S. Yuzbasi, Numerical solutions of hyperbolic telegraph equation by using the Bessel functions of first kind and residual correction, *Appl. Math. Comput.*, **287** (2016), 83–93. https://doi.org/10.1016/j.amc.2016.04.036
10. E. Kirli, D. Irk, M. Z. Gorgulu, High order accurate method for the numerical solution of the second order linear hyperbolic telegraph equation, *Numer. Methods Partial Differ. Equ.*, 2022. https://doi.org/10.1002/num.22957

11. R. K. Mohanty, An unconditionally stable difference scheme for the one-space dimensional linear hyperbolic equation, *Appl. Math. Lett.*, **13** (2013), 101–105. https://doi.org/10.1016/S0893-9659(04)90019-5
12. R. Jiwari, S. Pandit, R. C. Mittal, A differential quadrature algorithm for the numerical solution of the second-order one dimensional hyperbolic telegraph equation, *Int. J. Nonlinear Sci.*, **13** (2012), 259–266.
13. B. Pekmen, M. T. Sezgin, Differential quadrature solution of hyperbolic telegraph equation, *J. Appl. Math.*, **2012** (2012), 924765. https://doi.org/10.1155/2012/924765
14. A. Babu, B. Han, N. Asharaf, Numerical solution of the hyperbolic telegraph equation using cubic B-spline based differential quadrature of high accuracy, *Comput. Methods Differ. Equ.*, **10** (2022), 837–859. https://doi.org/10.22034/cmde.2022.47744.1997
15. A. S. Alshomrani, S. Pandit, A. K. Alzahrani, M. S. Alghamdi, R. Jiwari, A numerical algorithm based on modified cubic trigonometric B-spline functions for computational modelling of hyperbolic-type wave equations, *Eng. Comput.*, **34** (2017), 1257–1276. https://doi.org/10.1108/EC-05-2016-0179
16. M. Dehghan, A. Shokri, A numerical method for solving the hyperbolic telegraph equation, *Numer. Methods Partial Differ. Equ.*, **24** (2008), 1080–1093. https://doi.org/10.1002/num.20306
17. M. Dosti, A. Nazemi, Quartic B-spline collocation method for solving one dimensional hyperbolic telegraph equation, *J. Inf. Sci. Eng.*, **7** (2012), 83–90.
18. M. Dosti, A. Nazemi, Septic B-spline collocation method for solving one dimensional hyperbolic telegraph equation, *World Acad. Sci. Eng. Technol.*, **5** (2011), 1192–1196. https://doi.org/10.5281/zenodo.1331893
19. M. Dosti, A. Nazemi, Solving one-dimensional hyperbolic telegraph equation using cubic B-spline quasi-interpolation, *World Acad. Sci. Eng. Technol.*, **5** (2011), 674–679. https://doi.org/10.5281/zenodo.1331887
20. R. C. Mittal, R. Bhatia, Numerical solution of second order one dimensional hyperbolic telegraph equation by cubic B-spline collocation method, *Appl. Math. Comput.*, **220** (2013), 496–506. https://doi.org/10.1016/j.amc.2013.05.081
21. J. Rashidinia, S. Jamalzadeh, F. Esfahani, Numerical solution of one-dimensional telegraph equation using cubic B-spline collocation method, *J. Interpolat. Approx. Sci. Comput.*, **2014** (2014), 1–8. https://doi.org/10.5899/2014/jiasc-00042
22. T. Nazir, M. Abbas, M. Yaseen, Numerical solution of second-order hyperbolic telegraph equation via new cubic trigonometric B-spline approach, *Cogent Math. Stat.*, **4** (2017), 138206. https://doi.org/10.1080/23311835.2017.1382061
23. S. Sharifi, J. Rashidinia, Numerical solution of hyperbolic telegraph equation by cubic B-spline collocation method, *Appl. Math. Comput.*, **281** (2016), 28–38. https://doi.org/10.1016/j.amc.2016.01.049
24. S. Singh, S. Singh, R. Arora, Numerical solution of second order one-dimensional hyperbolic equation by exponential B-spline collocation method, *Numer. Anal. Appl.*, **7** (2017), 164–176. https://doi.org/10.1134/S1995423917020070

25. S. Singh, A. Aggarwal, Fourth-order cubic B-spline collocation method for hyperbolic telegraph equation, *Math. Sci.*, **16** (2022), 389–400. https://doi.org/10.1007/s40096-021-00428-y
26. E. Kırlı, D. Irk, M. Z. Gorgulu, Numerical solution of second order linear hyperbolic telegraph equation, *TWMS. J. Appl. Eng.*, **12** (2022), 919–930.
27. C. De Boor, *A practical guide to splines*, New York: Springer, 1978.
28. D. J. Fyfe, Linear dependence relations connecting equal interval Nth degree splines and their derivatives, *J. Inst.Math. Appl.*, **7** (1971), 398–407. https://doi.org/10.1093/imamat/7.3.398
29. R. K. Lodhi, S. F. Aldosary, K. S. Nisar, A. Alsaadi, Numerical solution of non-linear Bratu-type boundary value problems via quintic B-spline collocation method, *Math. Sci.*, **7** (2022), 7257–7273. https://doi.org/10.3934/math.2022405
30. Y. Zhou, W. Qu, Y. Gu, H. Gao, A hybrid meshless method for the solution of the second order hyperbolic telegraph equation in two space dimensions, *Eng. Anal. Bound. Elem.*, **115** (2020), 21–27. https://doi.org/10.1016/j.enganabound.2020.02.015
31. F. Z. Wang, E. R. Hou, S. A. Salama, M. M. A. Khater, Numerical investigation of the nonlinear fractional Ostrovsky equation, *Fractals*, **30** (2022), 22401429. https://doi.org/10.1142/S0218348X22401429

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