Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 10997–11014.
DOI:10.3934/math.2023557
Received: 13 January 2023
Revised: 14 February 2023
Accepted: 20 February 2023
Published: 08 March 2023

Research article

Projective class rings of a kind of category of Yetter-Drinfeld modules

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Abstract: In this paper, all simple Yetter-Drinfeld modules and indecomposable projective Yetter-Drinfeld modules over a family of non-pointed $8m$-dimension Hopf algebras of tame type with rank two, are construted and classified. The technique is Radford's method of constructing Yetter-Drinfeld modules over a Hopf algebra. Furthermore, the projective class rings of the category of Yetter-Drinfeld modules over this class of Hopf algebras are described explicitly by generators and relations.

Keywords: simple subcomodule; Yetter-Drinfeld module; tensor product; the projective class ring
Mathematics Subject Classification: 16D70, 16T05, 16T99

1. Introduction

The category of Yetter-Drinfeld modules over a Hopf algebra H was introduced firstly in [1], which provides a solution to the Yang-Baxter equation [2] when the antipode of H is bijective. In 1998, Andruskiewitsch and Schneider [3] introduced the *lifting method* which was extensively used in the classification of finite dimensional pointed and copointed Hopf algebras. It is remarked that the Yetter-Drinfeld modules play an important role in this process. More precisely, by determining the braiding in ${}^H_H\mathcal{YD}$ and indecomposable objects in the category of Yetter-Drinfeld modules over a Hopf algebra H, one can construct all finite dimensional Nichols algebras in ${}^H_H\mathcal{YD}$, and then all finite dimensional Hopf algebras over H by the lifting method. There are a lot of works to classify finite dimensional Hopf algebras by lifting method, see for example [4–13]. Therefore, it is important to understand the structures of Yetter-Drinfeld modules for a finite dimensional Hopf algebra.

In 2003, Radford [14] gave an idea of constructing the simple Yetter-Drinfeld modules: any Yetter-Drinfeld H-module M is the form $M = H \cdot N$ for some simple subcomodule N of $H \otimes L$, where L is a left H-module. In 2012, Zhu and Chen [15] gave the classification of all simple Yetter-Drinfeld modules over the Hopf-Ore extension $A(n, 0)$ of the dihedral group $\mathcal{D}_n$ for even or odd nummber n. In 2020, Yang and Zhang [16] classified all Hopf algebra structures on the quotient of Ore extensions $H_4[z;\sigma]$ of automorphism type for the Sweedler's 4-dimension Hopf algebra H_4, thereby obtaining a family of non-pointed and non-seimsimple Hopf algebras H_{4n} of rank one. The classifications of finite dimensional Hopf algebras over H_8 and H_{12} were given in [8] and [9] respectively. The Green ring of H_{4n} was determined by Chen and Yang et al. [17]. Xiong [18] (see also Zhang [19] in somewhat different idea) classified all simple Yetter-Drinfeld modules over H_{4n} and gave the structures of projective class rings of the category of the Yetter-Drinfeld modules of H_{4n}. In [20], we classified the Yetter-Drinfeld modules over H_{2n^2} and gave the Grothendieck rings of the category of the Yetter-Drinfeld modules of H_{2n^2}, where H_{2n^2} is a family of Kac-Paljutkin semisimple Hopf algebra of dimension $2n^2$.

Motivated by the above works, a family of non-pointed $8m$-dimension Hopf algebras H_{8m} of tame type with rank two are studied in [21], where m is even. It is pointed that H_{8m} is a special biserial algebra, which is just one subclass of basic tame Hopf algebras with only one block in [22]. By the technique of special biserial algebras (see for example [23]), we construct and classify the isomorphism classes of all indecomposable modules of H_{8m}, and determine the components of Auslander-Reiten quivers. Furthermore, we establish the tensor product of arbitrary simple (or projective) modules and indecomposable modules, and characterize the projective class rings and Grothendieck rings of H_{8m}.

In this paper, we focus on the classification of simple (or indecomposable projective) Yetter-Drinfeld modules over H_{8m} by Radford's method and the description of the projective class rings of the category of the Yetter-Drinfeld modules over H_{8m}. In our further works, we hope that the classification of finite dimensional Hopf algebras over H_{8m} is established.

Throughout this paper, $\mathbb{K}$ is assumed to be an algebraically closed field of characteristic zero, $m \geq 2$ is even and ω the $2m$-th primitive root of unity. The Sweedler's notation

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

for a Hopf algebra is used, and some other notations see [24].

2. Preliminaries

In this section, let us recall some basic definition and results for the Hopf algebra H_{8m}.

By definition, the Hopf algebra H_{8m} as an algebra is generated by $x_i (i = 1, 2), z$ with the following relations

$$z^{2m} = 1, \quad x_i^2 = 0, \quad x_1x_2 + x_2x_1 = 0, \quad x_iz = (-1)^i \omega z x_i.$$

The co-multiplication, counit and antipode are given as follows:

$$\begin{aligned} &\Delta(x_i) = 1 \otimes x_i + x_i \otimes z^m, \\ &\Delta(z) = (1 \otimes 1 + x_2 \otimes x_2 z^m)(z \otimes z), \\ &\varepsilon(x_i) = 0, \quad \varepsilon(z) = 1, \\ &S(x_i) = -x_i z^m, \quad S(z) = z^{-1}. \end{aligned}$$

for $i = 1, 2$. It is easy to see that

$$\{x_1^s x_2^t z^i | i \in \mathbb{Z}_{2m}, s, t \in [0, 1]\}$$

is a basis of H_{8m}, and

$$\Delta(z^l) = z^l \otimes z^l + a_l x_2 z^l \otimes x_2 z^{m+l} \tag{2.1}$$

for $l \in \mathbb{Z}_{2m}$, where $a_l = \frac{1-\omega^{-2l}}{1-\omega^{-2}}$. In particular,

$$\Delta(z^m) = z^m \otimes z^m.$$

In [21], all finite dimensional simple modules of H_{8m} were constructed and classified. There exist exactly $2m$ pairwise non-isomorphic 1-dimension simple H_{8m}-modules S_i with the basis $\{\upsilon^i\}$ for $i \in \mathbb{Z}_{2m}$. The actions are given by

$$x_1 \cdot \upsilon^i = 0,\ x_2 \cdot \upsilon^i = 0,\ z \cdot \upsilon^i = \omega^i \upsilon^i.$$

Let ${}_H\mathcal{M}$ be the category of left H-modules. Recall that a left Yetter-Drinfeld H-module M for a finite dimensional Hopf algebra H is a left H-module $(M, \cdot)$ and a left H-comodule (M, ρ) satisfying

$$\rho(h \cdot m) = \sum h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \quad \forall m \in M, h \in H,$$

where S is the antipode of H and $\rho(m) = \sum\limits_{(m)} m_{(-1)} \otimes m_{(0)}$. The category of left Yetter-Drinfeld H-modules is denoted by ${}_H^H\mathcal{YD}$, whose morphisms are both H-linear and H-colinear maps (see [1]). Let $V \in {}_H^H\mathcal{YD}$, the left dual V^* is defined by

$$\langle h \cdot f, v \rangle = \langle f, S(h)v \rangle, \quad f_{(-1)} \langle f_{(0)}, v \rangle = S^{-1}(v_{(-1)}) \langle f, v_{(0)} \rangle. \tag{2.2}$$

According to Radford's results in [14], we have the following results.

Proposition 2.1. *If $V, W \in {}_H^H\mathcal{YD}$, then $V \otimes W \in {}_H^H\mathcal{YD}$. The actions and coactions are as follows:*

$$h \cdot (\upsilon \otimes \omega) = \sum_{(h)} h_{(1)} \cdot \upsilon \otimes h_{(2)} \cdot \omega, \quad \rho(\upsilon \otimes \omega) = \sum_{(\upsilon),(\omega)} \upsilon_{(-1)} \omega_{(-1)} \otimes \upsilon_{(0)} \otimes \omega_{(0)},$$

where $\upsilon \in V, \omega \in W, h \in H$.

Lemma 2.2. *Let $L \in {}_H\mathcal{M}$. Then, we have*

(1) $H \otimes L \in {}_H^H\mathcal{YD}$; the module and comodule actions are given by

$$g \cdot (h \otimes l) = \sum_{(g)} g_{(1)} h S(g_{(3)}) \otimes g_{(2)} \cdot l, \tag{2.3}$$

$$\rho(h \otimes l) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes l, \quad \forall h, g \in H, l \in L. \tag{2.4}$$

(2) If M is a simple Yetter-Drinfeld H-module, then $M = H \cdot N$ for some simple subcomodule N of $H \otimes L$ where L is a simple left H-module.

By Proposition 2.1 and Lemma 2.2, we can construct all simple Yetter-Drinfeld modules over any Hopf algebra H.

3. Simple and indecomposable projective Yetter-Drinfeld modules over H_{8m}

To classify the simple Yetter-Drinfeld modules over the Hopf algebra H_{8m}, we firstly give all simple subcomodules of $H_{8m} \otimes S_i$ for all $i \in \mathbb{Z}_{2m}$.

Lemma 3.1. *Let $0 \neq L$ be a subcomodule of $H_{8m} \otimes S_i$, then there exists $j \in \mathbb{Z}_{2m}$ such that $z^j \otimes v^i \in L$ or $(z^j + lx_2z^j) \otimes v^i \in L$, where $0 \neq l \in \mathbb{K}$.*

Proof. Let $0 \neq L$ be a subcomodule of $H_{8m} \otimes S_i$ and $0 \neq u = (\sum_{j=0}^{2m-1} \sum_{s,t=0}^{1} k_{s,t,i} x_1^s x_2^t z^i) \otimes v^i \in L$, where $k_{s,t,j} \in \mathbb{K}$. By (2.4), we have

$$\rho(u) = \sum_{j=0}^{2m-1} (\alpha_{0,j} \otimes z^j + \alpha_{1,j} \otimes x_1 z^j + \alpha_{2,j} \otimes x_2 z^j + \alpha_{3,j} \otimes x_1 x_2 z^j) \otimes v^i$$

where

$$\begin{aligned}
\alpha_{0,j} &= k_{0,0,j} z^j + k_{1,1,j} x_1 x_2 z^j + k_{1,0,m+j} x_1 z^{m+j} + k_{0,1,m+j} x_2 z^{m+j}, \\
\alpha_{1,j} &= k_{1,0,j} z^j + k_{1,1,m+j} x_2 z^{m+j}, \\
\alpha_{2,j} &= k_{0,1,j} z^j - k_{1,0,j} a_j x_1 x_2 z^j + k_{0,0,m+j} a_{m+j} x_2 z^{m+j} - k_{1,1,m+j} x_1 z^{m+j}, \\
\alpha_{3,j} &= k_{1,1,j} z^j + k_{1,0,m+j} a_{m+j} x_2 z^{m+j}.
\end{aligned}$$

Assume that $j \neq 0, m$ (the discussions when $j = 0, m$ are similar). Note that there exist some j such that the vector $\theta_j = (k_{0,0,j}, k_{1,0,j}, k_{0,1,j}, k_{1,1,j}) \neq 0$ and $a_j \neq 0$. Now, we complete the proof by discussing the cases (i)-(vi).

(i) $k_{1,1,j} \neq 0, k_{1,0,j} \neq 0$.

(a) If $\alpha_{0,j}$ and $\alpha_{2,j}$ are linearly independent, we have $z^j \otimes v^i \in L$. Moreover, $x_2 z^{m+j} \otimes v^i \in L$ since

$$\rho(z^j \otimes v^i) = (z^j \otimes z^j + a_j x_2 z^j \otimes x_2 z^{m+j}) \otimes v^i;$$

(b) If $\alpha_{0,j}$ and $\alpha_{2,j}$ are linearly dependent, there exist a $0 \neq l \in \mathbb{K}$ such that $(z^j + lx_2z^j) \otimes v^i \in L$. Moreover, $(a_j x_2 z^{m+j} + l z^{m+j}) \otimes v^i \in L$ since

$$\rho\Big((z^j + lx_2z^j) \otimes v^i\Big) = \Big(z^j \otimes (z^j + lx_2z^j) + x_2z^j \otimes (a_j x_2 z^{m+j} + l z^{m+j})\Big) \otimes v^i;$$

(ii) $k_{1,1,j} \neq 0, k_{1,0,j} = 0$, we have $z^j \otimes v^i \in L$;

(iii) $k_{1,1,j} = 0, k_{1,0,j} \neq 0$, we have $x_2 z^j \in L$. Moreover, $z^{m+j} \otimes v^i \in L$ since

$$\rho(x_2 z^j \otimes v^i) = (z^j \otimes x_2 z^j + x_2 z^j \otimes z^{m+j}) \otimes v^i;$$

(iv) $k_{1,1,j} = 0, k_{1,0,j} = 0, k_{0,0,j} \neq 0, k_{0,1,j} \neq 0$, similar to (i);

(v) $k_{1,1,j} = 0, k_{1,0,j} = 0, k_{0,0,j} \neq 0, k_{0,1,j} = 0$, similar to (ii);

(vi) $k_{1,1,j} = 0, k_{1,0,j} = 0, k_{0,0,j} = 0, k_{0,1,j} \neq 0$, similar to (iii).

□

Lemma 3.2. *For $i \in \mathbb{Z}_{2m}$, all simple subcomodules of $H_{8m} \otimes S_i$ are as follows:*

(1) $U(i,j) := \mathbb{K}\{z^j \otimes \upsilon^i\}$, *where* $j = 0, m$;
(2) $V(i,j) := \mathbb{K}\{z^j \otimes \upsilon^i, x_2 z^{m+j} \otimes \upsilon^i\}$, *where* $j \in \mathbb{Z}_{2m}$ *and* $j \neq 0, m$;
(3) $W(i,j,l) := \mathbb{K}\{(z^j + l x_2 z^j) \otimes \upsilon^i, (a_j x_2 z^{m+j} + l z^{m+j}) \otimes \upsilon^i\}$, *where* $0 \neq l \in \mathbb{K}$, $j \in \mathbb{Z}_{2m}$ *and* $j \neq 0, m$.

Proof. It is easy to see that $U(i,j)$ is a simple subcomodule of $H_{8m} \otimes S_i$ for $j = 0, m$. One also see that $V(i,j)$ is a subcomodule of $H_{8m} \otimes S_i$ by (2.1). We show that $V(i,j)$ is a simple. Indeed, let $0 \neq V$ be a subcomodule of $V(i,j)$ and $0 \neq \upsilon = (k_{i,j} z^j + l_{i,j} x_2 z^{m+j}) \otimes \upsilon^i \in V$, where $k_{i,j}, l_{i,j} \in \mathbb{K}$ and $(k_{i,j}, l_{i,j}) \neq 0$. By (2.4), we have

$$\rho(\upsilon) = (k_{i,j} z^j + l_{i,j} x_2 z^{m+j}) \otimes z^j \otimes \upsilon^i + (k_{i,j} a_j x_2 z^j + l_{i,j} z^{m+j}) \otimes x_2 z^{m+j} \otimes \upsilon^i.$$

One can check that $k_{i,j} z^j + l_{i,j} x_2 z^{m+j}, k_{i,j} a_j x_2 z^j + l_{i,j} z^{m+j}$ are linearly independent. It follows that $z^j \otimes \upsilon^i, x_2 z^{m+j} \otimes \upsilon^i \in V$ and $V = V(i,j)$. Hence, $V(i,j)$ is a simple subcomodule of $H_{8m} \otimes S_i$. Similarly, we have $W(i,j,l)$ is a simple subcomodule of $H_{8m} \otimes S_i$ for $0 \neq l \in \mathbb{K}$, $j \in \mathbb{Z}_{2m}$ and $j \neq 0, m$. By Lemma3.1, the vector spaces defined in (1)–(3) are all simple subcomodules of $H_{8m} \otimes S_i$. □

Lemma 3.3. *As Yetter-Drinfeld modules,* $H_{8m} \cdot V(i,j) = H_{8m} \cdot W(i,j,l)$*, where* $0 \neq l \in \mathbb{K}$, $j \in \mathbb{Z}_{2m}$ *and* $j \neq 0, m$.

Proof. By (2.3), we have

$$x_2 \cdot ((z^j + l x_2 z^j) \otimes \upsilon^i) = ((-1)^i - \omega^{-j}) x_2 z^{m+j} \otimes \upsilon^i \in H_{8m} \cdot W(i,j,l).$$

By (2.4), we have

$$\rho(x_2 z^{m+j} \otimes \upsilon^i) = x_2 z^{m+j} \otimes z^j \otimes \upsilon^i + z^{m+j} \otimes x_2 z^{m+j} \otimes \upsilon^i.$$

Then $z^j \otimes \upsilon^i \in H_{8m} \cdot W(i,j,l)$. Hence

$$x_1 \cdot (z^j \otimes \upsilon^i) = ((-1)^i + (-1)^{j+1} \omega^{-j}) x_1 z^{m+j} \otimes \upsilon^i \in H_{8m} \cdot W(i,j,l),$$
$$x_1 \cdot (x_2 z^{m+j} \otimes \upsilon^i) = ((-1)^i + (-1)^{j+1} \omega^{-j}) x_1 x_2 z^j \otimes \upsilon^i \in H_{8m} \cdot W(i,j,l).$$

Therefore, as Yetter-Drinfeld modules, we have $H_{8m} \cdot V(i,j) = H_{8m} \cdot W(i,j,l)$, where $0 \neq l \in \mathbb{K}$, $j \in \mathbb{Z}_{2m}$ and $j \neq 0, m$. □

By Lemma 3.2 and Lemma 3.3, we just need to consider the following Yetter-Drinfeld modules:

- $H_{8m} \cdot U(i,j)$, where $i \in \mathbb{Z}_{2m}$, $j = 0, m$;
- $H_{8m} \cdot V(i,j)$, where $i, j \in \mathbb{Z}_{2m}$, $j \neq 0, m$.

Now, we consider $M(i,j) := H_{8m} \cdot U(i,j)$, where $i \in \mathbb{Z}_{2m}$, $j = 0, m$. There are two cases:

Case 1: $M_i := M(i,j)$ for $i \in \mathbb{Z}_{2m}$, where $j = 0$ if i is even and $j = m$ if i is odd. M_i are 1-dimension simple Yetter-Drinfeld modules. The actions on the basis $\{v^i = z^{im} \otimes \upsilon^i\}$ and the coactions are given by

$$x_1 \cdot v^i = 0, \quad x_2 \cdot v^i = 0, \quad z \cdot v^i = \omega^i v^i, \quad \rho(v^i) = z^{im} \otimes v^i.$$

Case 2: $M(i,j)$ for $i \in \mathbb{Z}_{2m}$, where $j = 0$ if i is odd and $j = m$ if i is even. $M(i,j)$ is a 4-dimension simple Yetter-Drinfeld module. The action on the basis $\{v_0^{i,j} = z^j \otimes \upsilon^i, v_1^{i,j} = x_1 z^{m+j} \otimes \upsilon^i, v_2^{i,j} = x_2 z^{m+j} \otimes \upsilon^i, v_3^{i,j} = x_1 x_2 z^j \otimes \upsilon^i\}$ and the coaction are given by

$$x_1 \cdot v_0^{i,j} = (-1)^i 2 v_1^{i,j}, \quad x_2 \cdot v_0^{i,j} = (-1)^i 2 v_2^{i,j}, \qquad z \cdot v_0^{i,j} = \omega^i v_0^{i,j},$$

$$\begin{aligned}
&x_1\cdot v_1^{i,j}=0, && x_2\cdot v_1^{i,j}=(-1)^{i+1}2v_3^{i,j}, && z\cdot v_1^{i,j}=-\omega^{i-1}v_1^{i,j},\\
&x_1\cdot v_2^{i,j}=(-1)^{i}2v_3^{i,j}, && x_2\cdot v_2^{i,j}=0, && z\cdot v_2^{i,j}=\omega^{i-1}v_2^{i,j},\\
&x_1\cdot v_3^{i,j}=0, && x_2\cdot v_3^{i,j}=0, && z\cdot v_3^{i,j}=-\omega^{i-2}v_3^{i,j}.
\end{aligned}$$

$$\begin{aligned}
\rho(v_0^{i,j})&=z^j\otimes v_0^{i,j},\\
\rho(v_1^{i,j})&=x_1z^{m+j}\otimes v_0^{i,j}+z^{m+j}\otimes v_1^{i,j},\\
\rho(v_2^{i,j})&=x_2z^{m+j}\otimes v_0^{i,j}+z^{m+j}\otimes v_2^{i,j},\\
\rho(v_3^{i,j})&=x_1x_2z^j\otimes v_0^{i,j}+x_2z^j\otimes v_1^{i,j}-x_1z^j\otimes v_2^{i,j}+z^j\otimes v_3^{i,j}.
\end{aligned}$$

For $N(i,j):=H_{8m}\cdot V(i,j)$, $i,j\in\mathbb{Z}_{2m}, j\neq 0,m$. $N(i,j)$ are 4-dimension simple Yetter-Drinfeld modules. The action on the basis $\{v_0^{i,j}=z^j\otimes v^i, v_1^{i,j}=x_1z^{m+j}\otimes v^i, v_2^{i,j}=x_2z^{m+j}\otimes v^i, v_3^{i,j}=x_1x_2z^j\otimes v^i\}$ and the coaction are given by

$$\begin{aligned}
&x_1\cdot v_0^{i,j}=a_{i,j}v_1^{i,j}, && x_2\cdot v_0^{i,j}=b_{i,j}v_2^{i,j}, && z\cdot v_0^{i,j}=\omega^{i}v_0^{i,j},\\
&x_1\cdot v_1^{i,j}=0, && x_2\cdot v_1^{i,j}=-b_{i,j}v_3^{i,j}, && z\cdot v_1^{i,j}=-\omega^{i-1}v_1^{i,j},\\
&x_1\cdot v_2^{i,j}=a_{i,j}v_3^{i,j}, && x_2\cdot v_2^{i,j}=0, && z\cdot v_2^{i,j}=\omega^{i-1}v_2^{i,j},\\
&x_1\cdot v_3^{i,j}=0, && x_2\cdot v_3^{i,j}=0, && z\cdot v_3^{i,j}=-\omega^{i-2}v_3^{i,j}.
\end{aligned}$$

$$\begin{aligned}
\rho(v_0^{i,j})&=z^j\otimes v_0^{i,j}+a_jx_2z^j\otimes v_2^{i,j},\\
\rho(v_1^{i,j})&=x_1z^{m+j}\otimes v_0^{i,j}+z^{m+j}\otimes v_1^{i,j}-a_jx_1x_2z^{m+j}\otimes v_2^{i,j}+a_jx_2z^{m+j}\otimes v_3^{i,j},\\
\rho(v_2^{i,j})&=x_2z^{m+j}\otimes v_0^{i,j}+z^{m+j}\otimes v_2^{i,j},\\
\rho(v_3^{i,j})&=x_1x_2z^j\otimes v_0^{i,j}+x_2z^j\otimes v_1^{i,j}-x_1z^j\otimes v_2^{i,j}+z^j\otimes v_3^{i,j}.
\end{aligned}$$

Here $a_{i,j}=(-1)^i+(-1)^{j+1}\omega^{-j}, b_{i,j}=(-1)^i-\omega^{-j}, a_j=\frac{1-\omega^{-2j}}{1-\omega^{-2}}$.

Let $\Lambda=\{(i,j)|i,j\in\mathbb{Z}_{2m}$ and $j\not\equiv im \pmod{2m}\}$ and

$$N_{i,j}:=\mathbb{K}\left\{v_0^{i,j}=z^j\otimes v^i, v_1^{i,j}=x_1z^{m+j}\otimes v^i, v_2^{i,j}=x_2z^{m+j}\otimes v^i, v_3^{i,j}=x_1x_2z^j\otimes v^i\right\},$$

where $(i,j)\in\Lambda$. Note that

$$\left\{N_{i,j}|(i,j)\in\Lambda\right\}=\{M(i,j)|i\in\mathbb{Z}_{2m}, j=0,m, j\not\equiv im \pmod{2m}\}\cup\{N(i,j)|i,j\in\mathbb{Z}_{2m}, j\neq 0,m\}.$$

In the sequel the operations of subscripts are in $\mathbb{Z}_{2m}$.

We get the first result of this paper.

Theorem 3.4. *The set*

$$\{M_i|i\in\mathbb{Z}_{2m}\}\bigcup\left\{N_{i,j}|(i,j)\in\Lambda\right\}$$

forms a complete list of non-isomorphic simple Yetter-Drinfeld modules over H_{8m}.

Proof. Firstly, we show that

$$M_i \cong M_{i'} \text{ if and only if } i = i' \text{ for } i, i' \in \mathbb{Z}_{2m}.$$

Let $f : M_i \to M_{i'}, v^i \mapsto av^{i'}$ be a Yetter-Drinfeld module isomorphism, where $0 \neq a \in \mathbb{K}$. Then, we have

$$z \cdot f(v^i) = az \cdot v^{i'} = a\omega^{i'} v^{i'} = f(z \cdot v^i) = \omega^i f(v^i) = a\omega^i v^{i'},$$
$$(\mathrm{id} \otimes f)\rho(v^i) = (id \otimes f)(z^{im} \otimes v^i) = az^{im} \otimes v^{i'} = \rho(f(v^i)) = \rho(v^{i'}) = az^{i'm} \otimes v^{i'}.$$

Hence $i = i'$.

Now we show that each $N_{i,j}$ for $(i, j) \in \Lambda$ is simple Yetter-Drinfeld module.

Let $0 \neq V$ be a Yetter-Drinfeld submodule of $N(i, j)$ and $0 \neq \upsilon = \sum_{k=0}^{3} \alpha_k^{i,j} v_k^{i,j} \in V$, where $\alpha_k^{i,j} \in \mathbb{K}$ and $(\alpha_0^{i,j}, \alpha_1^{i,j}, \alpha_2^{i,j}, \alpha_3^{i,j}) \neq 0$. By (2.4), we have

$$\begin{aligned}\rho(\upsilon) =&(\alpha_0^{i,j} z^j + \alpha_1^{i,j} x_1 z^{m+j} + \alpha_2^{i,j} x_2 z^{m+j} + \alpha_3^{i,j} x_1 x_2 z^j) \otimes v_0^{i,j}\\ &+ (\alpha_1^{i,j} z^{m+j} + \alpha_3^{i,j} x_2 z^j) \otimes v_1^{i,j}\\ &+ (\alpha_0^{i,j} a_j x_2 z^j - \alpha_1^{i,j} a_j x_1 x_2 z^{m+j} + \alpha_2^{i,j} z^{m+j} - \alpha_3^{i,j} x_1 z^j) \otimes v_2^{i,j}\\ &+ (\alpha_1^{i,j} a_j x_2 z^{m+j} + \alpha_3^{i,j} z^j) \otimes v_3^{i,j}.\end{aligned}$$

If $(\alpha_1^{i,j}, \alpha_3^{i,j}) = 0$, it is easy to see that $v_0^{i,j} \in V$. If $(\alpha_1^{i,j}, \alpha_3^{i,j}) \neq 0$, $\alpha_0^{i,j} z^j + \alpha_1^{i,j} x_1 z^{m+j} + \alpha_2^{i,j} x_2 z^{m+j} + \alpha_3^{i,j} x_1 x_2 z^j$ and $\alpha_1^{i,j} a_j x_2 z^{m+j} + \alpha_3^{i,j} z^j$ are linearly independent. We also get that $v_0^{i,j} \in V$. It follows that

$$\begin{aligned}v_1^{i,j} &= \frac{1}{a_{i,j}} x_1 \cdot v_0^{i,j} \in V,\\ v_2^{i,j} &= \frac{1}{b_{i,j}} x_2 \cdot v_0^{i,j} \in V,\\ v_3^{i,j} &= \frac{1}{a_{i,j} b_{i,j}} x_1 x_2 \cdot v_0^{i,j} \in V.\end{aligned}$$

Hence, $V = N(i, j)$ and $N(i, j)$ is a simple Yetter-Drinfeld module.

Finally we show that

$$N_{i,j} \cong N_{i',j'} \text{ if and only if } (i, j) = (i', j') \text{ for } (i, j), (i', j') \in \Lambda.$$

Let $g : N_{i,j} \to N_{i',j'}, v_k^{i,j} \mapsto \sum_{l=0}^{3} \alpha_{k,l} v_l^{i',j'}$, where $k \in \{0, 1, 2, 3\}$, $\alpha_{k,l} \in \mathbb{K}$ and there exists $l \in \{0, 1, 2, 3\}$ such that $\alpha_{k,l} \neq 0$ for each $k \in \{0, 1, 2, 3\}$. Then we have

$$x_1 \cdot g(v_0^{i,j}) = \alpha_{0,0} a_{i',j'} v_1^{i',j'} + \alpha_{0,2} a_{i',j'} v_3^{i,j} = g(x_1 \cdot v_0^{i,j}) = a_{i,j} \sum_{l=0}^{3} \alpha_{1,l} v_l^{i',j'},$$

$$x_2 \cdot g(v_0^{i,j}) = \alpha_{0,0} b_{i',j'} v_2^{i',j'} - \alpha_{0,1} b_{i',j'} v_3^{i,j} = g(x_2 \cdot v_0^{i,j}) = b_{i,j} \sum_{l=0}^{3} \alpha_{2,l} v_l^{i',j'},$$

$$
\begin{aligned}
z\cdot g(v_0^{i,j}) &= \alpha_{0,0}\omega^{i'}v_0^{i',j'} - \alpha_{0,1}\omega^{i'-1}v_1^{i,j} + \alpha_{0,2}\omega^{i'-1}v_2^{i',j'} - \alpha_{0,3}\omega^{i'-2}v_3^{i,j}\\
&= g(z\cdot v_0^{i,j}) = \omega^{i}\sum_{l=0}^{3}\alpha_{0,l}v_l^{i',j'},\\
x_1\cdot g(v_1^{i,j}) &= \alpha_{1,0}a_{i',j'}v_1^{i',j'} + \alpha_{1,2}a_{i',j'}v_3^{i,j} = g(x_1\cdot v_1^{i,j}) = 0,\\
x_2\cdot g(v_1^{i,j}) &= \alpha_{1,0}b_{i',j'}v_2^{i',j'} - \alpha_{1,1}b_{i',j'}v_3^{i,j} = g(x_2\cdot v_1^{i,j}) = -b_{i,j}\sum_{l=0}^{3}\alpha_{3,l}v_l^{i',j'},\\
z\cdot g(v_1^{i,j}) &= \alpha_{1,0}\omega^{i'}v_0^{i',j'} - \alpha_{1,1}\omega^{i'-1}v_1^{i,j} + \alpha_{1,2}\omega^{i'-1}v_2^{i',j'} - \alpha_{1,3}\omega^{i'-2}v_3^{i,j}\\
&= g(z\cdot v_1^{i,j}) = -\omega^{i-1}\sum_{l=0}^{3}\alpha_{1,l}v_l^{i',j'},\\
x_1\cdot g(v_2^{i,j}) &= \alpha_{2,0}a_{i',j'}v_1^{i',j'} + \alpha_{2,2}a_{i',j'}v_3^{i,j} = g(x_1\cdot v_2^{i,j}) = a_{i,j}\sum_{l=0}^{3}\alpha_{3,l}v_l^{i',j'},\\
x_2\cdot g(v_2^{i,j}) &= \alpha_{2,0}b_{i',j'}v_2^{i',j'} - \alpha_{2,1}b_{i',j'}v_3^{i,j} = g(x_2\cdot v_2^{i,j}) = 0,\\
z\cdot g(v_2^{i,j}) &= \alpha_{2,0}\omega^{i'}v_0^{i',j'} - \alpha_{2,1}\omega^{i'-1}v_1^{i,j} + \alpha_{2,2}\omega^{i'-1}v_2^{i',j'} - \alpha_{2,3}\omega^{i'-2}v_3^{i,j}\\
&= g(z\cdot v_2^{i,j}) = \omega^{i-1}\sum_{l=0}^{3}\alpha_{2,l}v_l^{i',j'},\\
x_1\cdot g(v_3^{i,j}) &= \alpha_{3,0}a_{i',j'}v_1^{i',j'} + \alpha_{3,2}a_{i',j'}v_3^{i,j} = g(x_1\cdot v_3^{i,j}) = 0,\\
x_2\cdot g(v_3^{i,j}) &= \alpha_{3,0}b_{i',j'}v_2^{i',j'} - \alpha_{3,1}b_{i',j'}v_3^{i,j} = g(x_2\cdot v_3^{i,j}) = 0,\\
z\cdot g(v_3^{i,j}) &= \alpha_{3,0}\omega^{i'}v_0^{i',j'} - \alpha_{3,1}\omega^{i'-1}v_1^{i,j} + \alpha_{3,2}\omega^{i'-1}v_2^{i',j'} - \alpha_{3,3}\omega^{i'-2}v_3^{i,j}\\
&= g(z\cdot v_3^{i,j}) = -\omega^{i-2}\sum_{l=0}^{3}\alpha_{3,l}v_l^{i',j'}.
\end{aligned}
$$

Hence $i = i'$ and $\alpha_{k,l} = 0$ for $k \neq l$. Also, we have

$$
\begin{aligned}
(\mathrm{id}\otimes g)\rho(v_3^{i,j}) &= (id\otimes g)(x_1x_2z^j\otimes v_0^{i,j} + x_2z^j\otimes v_1^{i,j} - x_1z^j\otimes v_2^{i,j} + z^j\otimes v_3^{i,j})\\
&= \alpha_{0,0}x_1x_2z^j\otimes v_0^{i',j'} + \alpha_{1,1}x_2z^j\otimes v_1^{i',j'} - \alpha_{2,2}x_1z^j\otimes v_2^{i',j'} + \alpha_{3,3}z^j\otimes v_3^{i',j'}\\
&= \rho(g(v_3^{i,j})) = \alpha_{3,3}\rho(v_3^{i',j'})\\
&= \alpha_{3,3}x_1x_2z^{j'}\otimes v_0^{i',j'} + \alpha_{3,3}x_2z^{j'}\otimes v_1^{i',j'} - \alpha_{3,3}x_1z^{j'}\otimes v_2^{i',j'} + \alpha_{3,3}z^{j'}\otimes v_3^{i',j'}.
\end{aligned}
$$

Hence $j = j'$ and $\alpha_{0,0} = \alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$. Therefore,

$$N_{i,j} \cong N_{i',j'} \text{ if and only if } (i,j) = (i',j') \text{ for } (i,j),(i',j') \in \Lambda.$$

The proof is finished. □

Corollary 3.5. *$N_{i,j}^* \cong N_{m+2-i,-j}$ for $(i,j) \in \Lambda$.*

Proof. Let $\{\mu_k^{i,j} | k \in \{0,1,2,3\}\}$ be the dual basis of $\{v_k^{i,j} | k \in \{0,1,2,3\}\}$ such that

$$\mu_0^{i,j}(v_0^{i,j}) = 0,\quad \mu_0^{i,j}(v_1^{i,j}) = 0,\qquad \mu_0^{i,j}(v_2^{i,j}) = 0,\qquad \mu_0^{i,j}(v_3^{i,j}) = (-1)^j\omega^{2j},$$

$$\begin{aligned}
&\mu_1^{i,j}(v_0^{i,j}) = 0, \quad \mu_1^{i,j}(v_1^{i,j}) = 0, \quad \mu_1^{i,j}(v_2^{i,j}) = -\omega^j, \quad \mu_0^{i,j}(v_3^{i,j}) = 0,\\
&\mu_2^{i,j}(v_0^{i,j}) = 0, \quad \mu_2^{i,j}(v_1^{i,j}) = (-1)^j\omega^j, \quad \mu_2^{i,j}(v_2^{i,j}) = 0, \quad \mu_2^{i,j}(v_3^{i,j}) = 0,\\
&\mu_3^{i,j}(v_0^{i,j}) = 1, \quad \mu_2^{i,j}(v_1^{i,j}) = 0, \quad \mu_3^{i,j}(v_2^{i,j}) = 0, \quad \mu_3^{i,j}(v_3^{i,j}) = 0.
\end{aligned}$$

By (2.2), we have

$$\begin{aligned}
&x_1 \cdot \mu_0^{i,j} = a_{m+2-i,-j}\mu_1^{i,j}, \quad x_2 \cdot \mu_0^{i,j} = b_{m+2-i,-j}\mu_2^{i,j}, \quad z \cdot \mu_0^{i,j} = -\omega^{2-i}\mu_0^{i,j},\\
&x_1 \cdot \mu_1^{i,j} = 0, \quad x_2 \cdot \mu_1^{i,j} = -b_{m+2-i,-j}\mu_3^{i,j}, \quad z \cdot \mu_1^{i,j} = \omega^{1-i}\mu_1^{i,j},\\
&x_1 \cdot \mu_2^{i,j} = a_{m+2-i,-j}\mu_3^{i,j}, \quad x_2 \cdot \mu_2^{i,j} = 0, \quad z \cdot \mu_2^{i,j} = -\omega^{1-i}\mu_2^{i,j},\\
&x_1 \cdot \mu_3^{i,j} = 0, \quad x_2 \cdot \mu_3^{i,j} = 0, \quad z \cdot \mu_3^{i,j} = \omega^{-i}\mu_3^{i,j}.
\end{aligned}$$

and

$$\begin{aligned}
\rho(\mu_0^{i,j}) &= z^{-j} \otimes \mu_0^{i,j} + a_{-j}x_2z^{-j} \otimes \mu_2^{i,j},\\
\rho(\mu_1^{i,j}) &= x_1z^{m-j} \otimes \mu_0^{i,j} + z^{m-j} \otimes \mu_1^{i,j} - a_{-j}x_1x_2z^{m-j} \otimes \mu_2^{i,j} + a_{-j}x_2z^{m-j} \otimes \mu_3^{i,j},\\
\rho(\mu_2^{i,j}) &= x_2z^{m-j} \otimes \mu_0^{i,j} + z^{m-j} \otimes \mu_2^{i,j},\\
\rho(\mu_3^{i,j}) &= x_1x_2z^{-j} \otimes \mu_0^{i,j} + x_2z^{-j} \otimes \mu_1^{i,j} - x_1z^{-j} \otimes \mu_2^{i,j} + z^{-j} \otimes \mu_3^{i,j}.
\end{aligned}$$

Hence $N_{i,j}^* \cong N_{m+2-i,-j}$. □

4. Projective class rings of ${}^{H_{8m}}_{H_{8m}}\mathcal{YD}$

Assume that H is a finite dimensional Hopf algebra. Let $F(H)$ be the free abelian group generated by the isomorphic classes $[M]$ of H-modules M. Then the abelian group $F(H)$ becomes a ring equipped with a multiplication given by the tensor product $[M][N] = [M \otimes N]$. The Green ring $r(H)$ is defined to be the quotient ring of $F(H)$ module the relations $[M \oplus N] = [M] + [N]$. The projective class ring of H is the subring of $r(H)$ generated by their projective and simple representations of H. As is known, ${}^H_H\mathcal{YD} \cong {}_{\mathcal{D}(H^{cop})}\mathcal{M}$. where ${}_{\mathcal{D}(H^{cop})}\mathcal{M}$ is the category of the left modules of Drinfeld double $\mathcal{D}(H^{\rm cop})$. The projective class ring of $\mathcal{D}(H^{\rm cop})$, or equivalently, the projective class ring of ${}^H_H\mathcal{YD}$ is denoted by $r_P(\mathcal{D}(H^{\rm cop}))$.

In this section, we denote $\mathcal{D} := \mathcal{D}(H_{8m}^{\rm cop})$. Let $\mathcal{P}(V)$ be the projective cover of a simple $\mathcal{D}$-module V, or equivalently, a simple Yetter-Drinfeld module $V \in {}^{H_{8m}}_{H_{8m}}\mathcal{YD}$. It is well-known that $\mathcal{P}(V)$ is unique (up to isomorphism) indecomposable projective $\mathcal{D}$-module which maps onto V. Let $\mathrm{Irr}(\mathcal{D})$ be the set of isomorphism classes of simple $\mathcal{D}$-modules. One sees that

$$\mathcal{D} \cong \bigoplus_{V \in \mathrm{Irr}(\mathcal{D})} \mathcal{P}(V)^{\oplus \dim V},$$

and $\mathcal{D}$ is unimodular and quasi-triangular (see [24, 25]).

Let us determine the projective class ring $r_P(\mathcal{D})$. For this purpose, we denote $\mathcal{P}_i := \mathcal{P}(M_i)$, $i \in \mathbb{Z}_{2m}$ and firstly introduce some lemmas.

Lemma 4.1. *(1) $M_i \otimes M_j \cong M_{i+j} \cong M_j \otimes M_i$ and $M_i \otimes N_{k,l} \cong N_{i+k,im+l} \cong N_{k,l} \otimes M_i$ for $i, j \in \mathbb{Z}_{2m}$, $(k,l) \in \Lambda$;*

(2) $\mathcal{P}(N_{i,j}) \cong N_{i,j}$, $(i,j) \in \Lambda$;
(3) $M_i \otimes \mathcal{P}_j \cong \mathcal{P}_{i+j} \cong \mathcal{P}_j \otimes M_i$ *and* $\dim \mathcal{P}_i = 16$ *for* $i, j \in \mathbb{Z}_{2m}$;
(4) *For* $(i,j),(k,l) \in \Lambda, h \in \mathbb{Z}_{2m}, \mathrm{Hom}(N_{i,j} \otimes N_{k,l}, M_h) \neq 0$ *if and only if* $h \equiv i+m+k-2 \pmod{2m}$ *and* $j \equiv (i+k)m - l \pmod{2m}$.

Proof. (1) It follows from a direct computation.
(2) Suppose that $\mathcal{P}(N_{i,j}) \ncong N_{i,j}$ for some $(i,j) \in \Lambda$. Since $\mathcal{D}$ is unimodular, $\mathrm{Soc}\mathcal{P}(N_{i,j}) \cong N_{i,j}$, and $\dim \mathcal{P}(N_{i,j}) \geq 2 \dim N_{i,j} = 8$. Since $\mathcal{P}(N_{i-k,j-mk}) \otimes M_k$ is projective and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}(N_{i-k,j-mk}) \otimes M_k, N_{i-k,j-mk} \otimes M_k) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}(N_{i-k,j-mk}), N_{i-k,j-mk} \otimes M_k \otimes M_k^*) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}(N_{i-k,j-mk}), N_{i-k,j-mk}) \neq 0,\end{aligned}$$

we have $\mathcal{P}(N_{i,j}) \cong \mathcal{P}(N_{i-k,j-mk} \otimes M_k) \subset \mathcal{P}(N_{i-k,j-mk}) \otimes M_k$, which implies that

$$\dim\mathcal{P}(N_{i-k,j-mk}) \geq \dim\mathcal{P}(N_{i,j}) \geq 8.$$

Let $J = \{(s,t) \in \Lambda | (s,t) = (i-k, j-mk) \text{ for } k \in \mathbb{Z}_{2m}\}$. It is obvious that

$$|J| = 2m \text{ and } |\Lambda - J| = 4m^2 - 4m.$$

Then

$$\begin{aligned}\dim\mathcal{D} &= \sum_{i=0}^{2m-1} \dim\mathcal{P}(M_i) + \sum_{(s,t)\in J} 4\dim\mathcal{P}(N_{s,t}) + \sum_{(s,t)\in\Lambda-J} 4\dim\mathcal{P}(N_{s,t}) \\ &\geq \sum_{i=0}^{2m-1} \dim\mathcal{P}(M_i) + 32|J| + 16|\Lambda - J| > 32|J| + 16|\Lambda - J| = 64m^2.\end{aligned}$$

It is a contradiction. Hence $\mathcal{P}(N_{i,j}) \cong N_{i,j}$ for $(i,j) \in \Lambda$.
(3) Since $\mathcal{P}(N_{i,j}) \cong N_{i,j}$ for any fixed $(i,j) \in \Lambda$, it follows that $2m \dim \mathcal{P}_i = \dim \mathcal{D} - 4|\Lambda| \dim N_{i,j} = 32m$ and hence $\dim \mathcal{P}_i = 16$. Similar to (2), we have $\mathcal{P}_i \subseteq \mathcal{P}_0 \otimes M_i$. Then, $\mathcal{P}_i \cong \mathcal{P}_0 \otimes M_i$. Hence, $\mathcal{P}_j \otimes M_i \cong \mathcal{P}_0 \otimes M_j \otimes M_i \cong \mathcal{P}_0 \otimes M_{i+j} \cong \mathcal{P}_{i+j}$.
(4) Since $\mathcal{D}$ is quasi-triangular, $M \otimes_{\mathbb{K}} N \cong N \otimes_{\mathbb{K}} M$ for $M, N \in {}_{\mathcal{D}}\mathcal{M}$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(N_{i,j} \otimes N_{k,l}, M_h) &\cong \mathrm{Hom}_{\mathcal{D}}\left(N_{i,j}, \mathrm{Hom}_{\mathbb{K}}(N_{k,l}, M_h)\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(N_{i,j}, N_{k,l}^* \otimes M_h\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(N_{i,j}, M_h \otimes N_{k,l}^*) \cong \mathrm{Hom}_{\mathcal{D}}(N_{i,j}, M_h \otimes N_{m+2-k,-l}) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(N_{i,j}, N_{h+m+2-k,hm-l}).\end{aligned}$$

Then by Schur's lemma, $\mathrm{Hom}_{\mathcal{D}}(N_{i,j} \otimes N_{k,l}, M_h) \neq 0$ if and only if $i \equiv h+m+2-k \pmod{2m}$ and $j \equiv hm - l \pmod{2m}$, if and only if $h \equiv i+m+k-2 \pmod{2m}$ and $j \equiv (i+k)m - l \pmod{2m}$. □

Corollary 4.2. *We have*

$${}_{\mathcal{D}}\mathcal{D} \cong \left(\bigoplus_{i=0}^{2m-1} \mathcal{P}_i\right) \bigoplus \left(\bigoplus_{(j,k)\in\Lambda} N_{j,k}^{\oplus 4}\right).$$

Now we describe the projective cover $\mathcal{P}_i$ of the simple module M_i for $i \in \mathbb{Z}_{2m}$. For convenience, we let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\omega^{-1} & 0 & 0 \\ 0 & 0 & \omega^{-1} & 0 \\ 0 & 0 & 0 & -\omega^{-2} \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^m & 0 & 0 \\ 0 & 0 & z^m & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2x_1z^m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2x_1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2x_2z^m & 0 & 0 & 0 \\ 0 & 2x_2 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4x_1x_2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{P}$ be a vector space with a basis $\{p_0, p_1, \cdots, p_{15}\}$ and the action and coaction of H_{8m} on $\mathcal{P}$ be the form:

$$[x_1] = \begin{pmatrix} A & 0 & 0 & 0 \\ B & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & B & A \end{pmatrix}, \quad [x_2] = \begin{pmatrix} C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ B & 0 & C & 0 \\ 0 & -B & 0 & C \end{pmatrix}, \quad [z] = \begin{pmatrix} -\omega^2 D & 0 & 0 & 0 \\ 0 & \omega D & 0 & 0 \\ \omega E & 0 & -\omega D & 0 \\ 0 & E & 0 & D \end{pmatrix},$$

$$\rho(P) = \begin{pmatrix} F-G-H+K & 0 & 0 & 0 \\ 2x_1z^m(H+F) & z^m(F+G+H+K) & 0 & 0 \\ 2x_2z^m(F-G) & 0 & z^m(F+G+H+K) & 0 \\ 4x_1x_2F & 2x_2(F+G) & 2x_1(H-F) & F-G-H+K \end{pmatrix} \otimes P,$$

where $P = (p_0, p_1, \cdots, p_{15})^T$. By a complex computation, we know that $\mathcal{P}$ is a left Yetter-Drinfeld module.

Lemma 4.3. *$\mathcal{P}$ is an indecomposable Yetter-Drinfeld module over H_{8m}.*

Proof. Suppose that $\mathcal{P}$ is not indecomposable. Then there exist two non-trivial submodules M and N such that $\mathcal{P} = M \oplus N$. Let

$$\begin{aligned}
&\kappa_0 = p_3 + p_{12} + p_9 - p_6, \quad &&\kappa_1 = p_7 - p_{13}, \quad &&\kappa_2 = p_{11} - p_{14}, \quad &&\kappa_3 = p_{15}, \\
&\kappa_4 = p_1 + p_4, &&\kappa_5 = p_5, &&\kappa_6 = p_2 + p_8, &&\kappa_7 = p_{10}, \\
&\kappa_8 = p_0, &&\kappa_9 = p_3 - p_{12}, &&\kappa_{10} = p_6 + p_9, &&\kappa_{11} = p_{11} + p_{14}, \\
&\kappa_{12} = p_1, &&\kappa_{13} = p_2, &&\kappa_{14} = p_7, &&\kappa_{15} = p_6,
\end{aligned}$$

and $K_{-1} = 0, K_l = \sum_{i=0}^{l} \mathbb{K}\kappa_i$ for $l \in \{0, \cdots, 15\}$. One can check that

$$0 = K_{-1} \subset K_0 \subset K_1 \subset \cdots \subset K_{15} = \mathcal{P}$$

is a Yetter-Drinfeld submodules chain of $\mathcal{P}$ such that K_l/K_{l-1} is a one dimensional Yetter-Drinfeld module. And

$$
\begin{aligned}
&K_l/K_{l-1} \cong M_{-1}, \ \text{if } l = 2, 11; && K_l/K_{l-1} \cong M_0, \ \text{if } l = 0, 9, 10, 15;\\
&K_l/K_{l-1} \cong M_1, \ \text{if } l = 4, 12; && K_l/K_{l-1} \cong M_{m-2}, \text{if } l = 3;\\
&K_l/K_{l-1} \cong M_{m-1}, \text{if } l = 1, 14; && K_l/K_{l-1} \cong M_m, \ \text{if } l = 5, 7;\\
&K_l/K_{l-1} \cong M_{m+1}, \text{if } l = 6, 13; && K_l/K_{l-1} \cong M_{m+2}, \text{if } l = 8.
\end{aligned}
$$

We claim that $\kappa_{15} \notin M$ and $\kappa_{15} \notin N$. If $\kappa_{15} \in M$, then $\kappa_{11} - \kappa_2, \kappa_4 - \kappa_{12}, \kappa_8, \kappa_{13}, \kappa_{14} \in M$ since

$$
\begin{aligned}
&x_1 \cdot \kappa_{15} = \kappa_{14},\\
&x_2 \cdot \kappa_{15} = \frac{\kappa_{11} - \kappa_2}{2},\\
&\rho(\kappa_{15}) = 4x_1x_2 \otimes \kappa_8 + 2x_2 \otimes (\kappa_4 - \kappa_{12}) + 2x_1 \otimes \kappa_{13} + 1 \otimes \kappa_{15},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\kappa_0 = x_1x_2 \cdot \kappa_8 \in M, && \kappa_1 = x_1x_2 \cdot (\kappa_4 - \kappa_{12}) \in M,\\
&\kappa_2 = x_2x_1 \cdot \kappa_{13} \in M, && \kappa_3 = -x_2 \cdot \kappa_{14} \in M,\\
&\kappa_4 = x_1 \cdot \kappa_8 \in M, && \kappa_5 = x_1 \cdot (\kappa_4 - \kappa_{12}) \in M,\\
&\kappa_6 = x_2 \cdot \kappa_8 \in M, && \kappa_7 = -x_2 \cdot \kappa_{13} \in M,\\
&\kappa_9 = x_1 \cdot \kappa_{13} + x_2 \cdot (\kappa_4 - \kappa_{12}) \in M, && \kappa_{10} = x_1x_2 \cdot \kappa_8 + x_2 \cdot (\kappa_4 - \kappa_{12}) - x_1 \cdot \kappa_{13} \in M,\\
&\kappa_{11} = x_2x_1 \cdot \kappa_{13} + 2x_2 \cdot \kappa_{15} \in M, && \kappa_{12} = x_1 \cdot \kappa_8 + \kappa_{12} - \kappa_4 \in M.
\end{aligned}
$$

Then $M = \mathcal{P}$. It's a contradiction. Similarly, if $\kappa_{15} \in N$, then $N = \mathcal{P}$ and also a contradiction. Hence the claim follows. Therefore, there exist some $\alpha_i \in \mathbb{K}$, for $i \in \{0, 1, \cdots, 14\}$, such that

$$\kappa = \sum_{i=0}^{14} \alpha_i\kappa_i + \kappa_{15} \in M.$$

Then

$$
\begin{aligned}
\rho(\kappa) = &(\alpha_0 + (2\alpha_1 + \alpha_{14})x_1z^m + 2\alpha_2x_2z^m + 4\alpha_3x_1x_2) \otimes \kappa_0\\
&+ (\alpha_1z^m + 2\alpha_3x_2) \otimes \kappa_1 + (\alpha_2z^m - 2\alpha_3x_1) \otimes \kappa_2 + \alpha_3 1 \otimes \kappa_3\\
&+ (\alpha_4z^m + 2\alpha_5x_1 + 2(\alpha_{10} - \alpha_9 + \alpha_{15})x_2 + 4\alpha_{14}x_1x_2z^m) \otimes \kappa_4\\
&+ (\alpha_5 + 2\alpha_{14}x_2z^m) \otimes \kappa_5\\
&+ (\alpha_6z^m + 2\alpha_7x_2 + 2(\alpha_9 + \alpha_{10})x_1 + 8\alpha_{11}x_1x_2z^m) \otimes \kappa_6\\
&+ (\alpha_7 - 4\alpha_{11}x_1z^m) \otimes \kappa_7\\
&+ (\alpha_8 - 2\alpha_{12}x_1z^m - 2\alpha_{13}x_2z^m + 4\alpha_{15}x_1x_2) \otimes \kappa_8\\
&+ (\alpha_9 + 2\alpha_{11}x_2z^m + \alpha_{14}x_1z^m) \otimes \kappa_9 + (\alpha_{10} + 2\alpha_{11}x_2z^m - \alpha_{14}x_1z^m) \otimes \kappa_{10}\\
&+ \alpha_{11}z^m \otimes \kappa_{11} + (\alpha_{12}z^m - 2\alpha_{15}x_2) \otimes \kappa_{12} + (\alpha_{13}z^m + 2\alpha_{15}x_1) \otimes \kappa_{13}\\
&+ \alpha_{14}z^m \otimes \kappa_{14} + 1 \otimes \kappa_{15}.
\end{aligned}
$$

It is observe that $\hat{\kappa}_1 = \alpha_0\kappa_0 + \alpha_3\kappa_3 + \alpha_5\kappa_5 + \alpha_7\kappa_7 + \alpha_8\kappa_8 + \alpha_9\kappa_9 + \alpha_{10}\kappa_{10} + \kappa_{15} \in M$. Hence

$$\begin{aligned} x_1 \cdot \hat{\kappa}_1 &= (\alpha_9 + \alpha_{10})\kappa_1 + \alpha_7\kappa_2 + \alpha_8\kappa_4 + \kappa_{14} \in M, \\ x_2 \cdot \hat{\kappa}_1 &= -\alpha_5\kappa_1 + (\alpha_9 - \alpha_{10} - \frac{1}{2}) \otimes \kappa_2 + \alpha_8\kappa_6 + \frac{\kappa_{11}}{2} \in M, \\ x_1x_2 \cdot \hat{\kappa}_1 &= \alpha_8\kappa_0 + \frac{\kappa_3}{2} \in M, \end{aligned}$$

and

$$\rho(\alpha_8\kappa_0 + \frac{\kappa_3}{2}) = (\alpha_8 + 2x_1x_2) \otimes \kappa_0 + x_2 \otimes \kappa_1 - x_1 \otimes \kappa_2 + \frac{1}{2}1 \otimes \kappa_3.$$

Then $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ and $\alpha_8\kappa_4 + \kappa_{14}, \alpha_8\kappa_6 + \frac{\kappa_{11}}{2} \in M$. Note that

$$\begin{aligned} \rho(\alpha_8\kappa_4 + \kappa_{14}) &= (\alpha_8 z^m + 4x_1x_2z^m) \otimes \kappa_4 + 2x_2z^m \otimes \kappa_5 + x_1z^m \otimes (\kappa_0 + \kappa_9 - \kappa_{10}) + z^m \otimes \kappa_{14}, \\ \rho(\alpha_8\kappa_6 + \frac{\kappa_{11}}{2}) &= (\alpha_8 + 4x_1x_2z^m) \otimes \kappa_6 - 2x_1z^m \otimes \kappa_7 + x_2z^m \otimes (\kappa_9 + \kappa_{10}) + \frac{z^m}{2} \otimes \kappa_{11}. \end{aligned}$$

Then $\kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_9, \kappa_{10}, \kappa_{11}, \kappa_{14} \in M$. Hence $\hat{\kappa}_2 = \alpha_8\kappa_8 + \alpha_{12}\kappa_{12} + \alpha_{13}\kappa_{13} + \kappa_{15} \in M$, and

$$\begin{aligned} \rho(\hat{\kappa}_2) = {} & 2x_2 \otimes \kappa_4 + (\alpha_8 - 2\alpha_{12}x_1z^m - 2\alpha_{13}x_2z^m + 4x_1x_2) \otimes \kappa_8 + (\alpha_{12}z^m - 2x_2) \otimes \kappa_{12} \\ & + (\alpha_{13}z^m + 2x_1) \otimes \kappa_{13} + 1 \otimes \kappa_{15}. \end{aligned}$$

Thus $\kappa_{15} \in M$. It's a contradiction. Consequently, $\mathcal{P}$ is indecomposable. □

Lemma 4.4. *$\mathcal{P} \cong \mathcal{P}_0$ as $\mathcal{D}$-modules.*

Proof. It is well known that $\mathcal{D}$ is a symmetric algebra [26] and every projective module is injective. In particular, $\mathcal{P}_0 = E(M_i)$ for some $i \in \mathbb{Z}_{2m}$ and the socle and top of $\mathcal{P}_0$ coincide. Therefore $\mathcal{P}_0 \cong E(M_0)$. On the other hand, by Lemma 4.3, we know that $\mathcal{P}$ is an indecomposable module with Soc$\mathcal{P} \cong M_0$. Thus, $\mathcal{P}$ embeds in $E(M_0)$, which implies that $\mathcal{P} \cong E(M_0)$, since they have the same dimension. Hence, $\mathcal{P} \cong \mathcal{P}_0$. □

Now we calculate the tensor decompositions of the simple and indecomposable projective Yetter-Drinfeld modules. Denote $\bigoplus_{\pm} V_{a\pm b} := V_{a+b} \oplus V_{a-b}$, for $a, b \in \mathbb{Z}_{2m}, V \in {}^{H_{8m}}_{H_{8m}}\mathcal{YD}$.

Lemma 4.5. *(1) For $i, j \in \mathbb{Z}_{2m}$,*

$$\mathcal{P}_i \otimes \mathcal{P}_j \cong \bigoplus_{\pm} \left(\mathcal{P}^{\oplus 2}_{i+j\pm 1} \oplus \mathcal{P}^{\oplus 2}_{i+j+m\pm 1} \oplus \mathcal{P}_{i+j+m\pm 2}\right) \oplus \mathcal{P}^{\oplus 4}_{i+j} \oplus \mathcal{P}^{\oplus 2}_{i+j+m}.$$

(2) For $i, j \in \Lambda, k \in \mathbb{Z}_{2m}$,

$$\begin{aligned} N_{i,j} \otimes \mathcal{P}_k \cong & \mathcal{P}_k \otimes N_{i,j} \\ \cong & \bigoplus_{\pm} \left(N^{\oplus 2}_{i+k\pm 1, j+km+m} \oplus N^{\oplus 2}_{i+k+m\pm 1, j+km+m} \oplus N_{i+k+m\pm 2, j+km}\right) \\ & \oplus N^{\oplus 4}_{i+k, j+km} \oplus N^{\oplus 2}_{i+k+m, j+km}. \end{aligned}$$

Proof. It suffices to prove the lemma for $\mathcal{P}_0 \otimes \mathcal{P}_0$ and $\mathcal{P}_0 \otimes N_{i,j}$ by Lemma 4.1 (1)(3).

By the proof of Lemma 4.3,

$$[\mathcal{P}_0] = 2[M_1] + 2[M_{-1}] + 2[M_{m+1}] + 2[M_{m-1}] + [M_{m+2}] + [M_{m-2}] + 4[M_0] + 2[M_m]$$

in the Grothendieck ring of the category of the Yetter-Drifeld modules over H_{8m}. It follows that

$$\begin{aligned}\mathcal{P}_0 \otimes \mathcal{P}_0 &\cong \bigoplus_{\pm} \Big((M_{\pm 1} \otimes \mathcal{P}_0)^{\oplus 2} \oplus (M_{m\pm 1} \otimes \mathcal{P}_0)^{\oplus 2} \oplus (M_{m\pm 2} \otimes \mathcal{P}_0) \Big) \\ &\quad \oplus (M_0 \otimes \mathcal{P}_0)^{\oplus 4} \oplus (M_m \otimes \mathcal{P}_0)^{\oplus 2} \\ &\cong \bigoplus_{\pm} \Big(\mathcal{P}_{\pm 1}^{\oplus 2} \oplus \mathcal{P}_{m\pm 1}^{\oplus 2} \oplus \mathcal{P}_{m\pm 2} \Big) \oplus \mathcal{P}_0^{\oplus 4} \oplus \mathcal{P}_m^{\oplus 2},\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_0 \otimes N_{i,j} &\cong \bigoplus_{\pm} \Big((M_{\pm 1} \otimes N_{i,j})^{\oplus 2} \oplus (M_{m\pm 1} \otimes N_{i,j})^{\oplus 2} \oplus (M_{m\pm 2} \otimes N_{i,j}) \Big) \\ &\quad \oplus (M_0 \otimes N_{i,j})^{\oplus 4} \oplus (M_m \otimes N_{i,j})^{\oplus 2} \\ &\cong \bigoplus_{\pm} \Big(N_{i\pm 1, j+m}^{\oplus 2} \oplus N_{i+m\pm 1, j+m}^{\oplus 2} \oplus N_{i+m\pm 2, j} \Big) \oplus N_{i,j}^{\oplus 4} \oplus N_{i+m,j}^{\oplus 2},\end{aligned}$$

The proof is finished. □

Lemma 4.6. *For* $(i, j), (k, l) \in \Lambda$, *we have*

$$N_{i,j} \otimes N_{k,l} \cong \begin{cases} \mathcal{P}_{i+m+k-2}, & \text{if } j \equiv (i+k)m - l \pmod{2m}; \\ \bigoplus_{t=0}^{1} \Big(N_{i+k+(m-1)t,\, j+l+mt} \oplus N_{i+k+(m-1)t-1,\, j+l+(t+1)m} \Big), & \text{otherwise}. \end{cases}$$

Proof. If $j \equiv (i+k)m - l \pmod{2m}$, then by Lemma 4.1(4), $\mathrm{Hom}(N_{i,j} \otimes N_{k,l}, M_h) \neq 0$ if and only if $h \equiv i+m+k-2 \pmod{2m}$. Since $N_{i,j} \otimes N_{k,l}$ is projective and $\dim \mathcal{P}_{i+m+k-2} = \dim(N_{i,j} \otimes N_{k,l}) = 16$, we get that $\mathcal{P}_{i+m+k-2} \cong N_{i,j} \otimes N_{k,l}$.

If $j \not\equiv (i+k)m - l \pmod{2m}$, then by Lemma 4.1(4), $\mathrm{Hom}(N_{i,j} \otimes N_{k,l}, M_h) = 0$ for any $h \in \mathbb{Z}_{2m}$, which implies that $\mathcal{P}_h$ can not the direct summand of $N_{i,j} \otimes N_{k,l}$. Hence $N_{i,j} \otimes N_{k,l}$ has to be the direct sum of four 4-dimensional simple projective modules.

Let

$$t = \frac{(-1)^k b_{i,j} b_{k,l}}{1 - \omega^{-2}}, \quad \theta_{r,s} = \mu_r^{i,j} \otimes \nu_s^{k,l}$$

for $r, s \in \{0, 1, 2, 3\}$ and

$$\begin{aligned}\beta_{0,0} &= a_{i+k,j+l} b_{i+k,j+l} (\theta_{0,0} + t\theta_{2,2}), \\ \beta_{0,1} &= b_{i+k,j+l} (a_{k,l} (\theta_{0,1} + t\theta_{2,3}) + (-1)^k a_{i,j} (\theta_{1,0} - t\theta_{3,2})), \\ \beta_{0,2} &= a_{i+k,j+l} (b_{k,l} \theta_{0,2} + (-1)^k b_{i,j} \theta_{2,0}), \\ \beta_{0,3} &= a_{k,l} b_{k,l} \theta_{0,3} + a_{i,j} b_{i,j} \theta_{3,0} + (-1)^{k-1} a_{i,j} b_{k,l} \theta_{1,2} + (-1)^k a_{k,l} b_{i,j} \theta_{2,1}, \\ \beta_{1,0} &= b_{i+k+m-1,j+l+m} (\theta_{0,1} + t\theta_{2,3} - (-1)^j \omega^{-j} (\theta_{1,0} - t\theta_{3,2})),\end{aligned}$$

$$
\begin{aligned}
\beta_{1,1} &= b_{i+k+m-1,j+l+m}(\theta_{1,1} - t\theta_{3,3}),\\
\beta_{1,2} &= (-1)^k b_{i,j}((-1)^j\omega^{-j}\theta_{3,0} - \theta_{2,1}) - b_{k,l}(\theta_{0,3} + (-1)^j\omega^{-j}\theta_{1,2}),\\
\beta_{1,3} &= b_{k,l}\theta_{1,3} - (-1)^k b_{i,j}\theta_{3,1},\\
\beta_{2,0} &= a_{i+k-1,j+l+m}(\theta_{0,2} - \omega^{-j}\theta_{2,0}),\\
\beta_{2,1} &= a_{k,l}(\theta_{0,3} - \omega^{-j}\theta_{2,1}) - (-1)^k a_{i,j}(\theta_{1,3} + \omega^{-j}\theta_{3,0}),\\
\beta_{2,2} &= a_{i+k-1,j+l+m}\theta_{2,2},\\
\beta_{2,3} &= a_{k,l}\theta_{2,3} - (-1)^k a_{i,j}\theta_{3,2},\\
\beta_{3,0} &= \theta_{0,3} - \omega^{-j}\theta_{2,1} + (-1)^j\omega^{-j}(\theta_{1,2} + \omega^{-j}\theta_{3,0}),\\
\beta_{3,1} &= \theta_{1,3} + \omega^{-j}\theta_{3,1},\\
\beta_{3,2} &= \theta_{2,3} + (-1)^j\omega^{-j}\theta_{3,2},\\
\beta_{3,3} &= \theta_{3,3}.
\end{aligned}
$$

A direct computation shows that

$$
\begin{aligned}
\mathbb{K}\{\beta_{0,0}, \beta_{0,1}, \beta_{0,2}, \beta_{0,3}\} &\cong N_{i+k,j+l},\\
\mathbb{K}\{\beta_{1,0}, \beta_{1,1}, \beta_{1,2}, \beta_{1,3}\} &\cong N_{i+k+m-1,j+l+m},\\
\mathbb{K}\{\beta_{2,0}, \beta_{2,1}, \beta_{2,2}, \beta_{2,3}\} &\cong N_{i+k-1,j+l+m},\\
\mathbb{K}\{\beta_{3,0}, \beta_{3,1}, \beta_{3,2}, \beta_{3,3}\} &\cong N_{i+k+m-2,j+l}.
\end{aligned}
$$

Hence,

$$N_{i,j} \otimes N_{k,l} \cong N_{i+k,j+l} \oplus N_{i+k+m-1,j+l+m} \oplus N_{i+k-1,j+l+m} \oplus N_{i+k+m-2,j+l}.$$

□

By Lemma 4.1, Lemma 4.5, Lemma 4.6, the projective class ring $r_P(\mathcal{D})$ is a commutative ring. Let $\dot{y}_0 = [M_1], \dot{y}_i = [N_{0,i}], i \in \{1, \cdots, 2m-1\}$

Lemma 4.7. *The following statements hold in* $r_P(\mathcal{D})$*.*

(1) $[M_i] = \dot{y}_0^i$ *for* $i \in \mathbb{Z}_{2m}$;
(2) $[N_{i,j}] = \dot{y}_0^i \dot{y}_{j+im}$ *for* $(i, j) \in \Lambda$;
(3) $[P_i] = \dot{y}_0^{i+m+2} \dot{y}_m^2$ *for* $i \in \mathbb{Z}_{2m}$;
(4) For $i, j, k, j+k \in \{1, \cdots, 2m-1\}$,

$$
\begin{aligned}
&\dot{y}_0^{2m} = 1, \quad \dot{y}_i \dot{y}_{2m-i} = \dot{y}_m^2,\\
&\dot{y}_j \dot{y}_k = (1 + \dot{y}_0^{2m-1} + \dot{y}_0^{m-1} + \dot{y}_0^{m-2})\dot{y}_{j+k},\\
&\dot{y}_i \dot{y}_m^2 = (1 + 2\dot{y}_0^{2m-1} + 2\dot{y}_0^{2m-2} + 2\dot{y}_0^{2m-3} + 2\dot{y}_0^{m-1} + 4\dot{y}_0^{m-2} + 2\dot{y}_0^{m-3} + \dot{y}_0^{m-4})\dot{y}_i.
\end{aligned}
$$

Proof. The results are easy to get from Lemma 4.1, Lemma 4.5, Lemma 4.6. □

Corollary 4.8. *The following set is a* $\mathbb{Z}$*-basis of* $r_P(\mathcal{D})$*:*

$$\{\dot{y}_0^i, \dot{y}_0^i \dot{y}_j, \dot{y}_0^i \dot{y}_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m-1\}\}.$$

Proof. By Lemma 4.7(4), $\dot{y}_0^{2m} = 1$, and for $k, l, k + l \in \{1, \cdots, 2m - 1\}$, $\dot{y}_k\dot{y}_m^2, \dot{y}_k\dot{y}_l$ can be expressed as a linear combination of $\{\dot{y}_0^i, \dot{y}_0^i\dot{y}_j, \dot{y}_0^i\dot{y}_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m - 1\}\}$. It is easy to check that the set

$$\{\dot{y}_0^i, \dot{y}_0^i\dot{y}_j, \dot{y}_0^i\dot{y}_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m - 1\}\}$$

is a independent set since $\#\{\dot{y}_0^i, \dot{y}_0^i\dot{y}_j, \dot{y}_0^i\dot{y}_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m - 1\}\} = 4m^2 + 2m$, the number of $\mathbb{Z}$-basis of $r_P(\mathcal{D})$.

Hence, $\{\dot{y}_0^i, \dot{y}_0^i\dot{y}_j, \dot{y}_0^i\dot{y}_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m - 1\}\}$ is a $\mathbb{Z}$-basis of $r_P(\mathcal{D})$. □

The results of this section is as follows.

Theorem 4.9. *The projective class ring $r_P(\mathcal{D})$ is isomorphic to the quotient ring of the ring $\mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}]$ modulo the ideal I generated by the following elements*

$$\begin{aligned} &y_0^{2m} - 1, \quad y_i y_{2m-i} - y_m^2, \quad y_j y_k - (1 + y_0^{2m-1} + y_0^{m-1} + y_0^{m-2})y_{j+k}, \\ &y_i y_m^2 - (1 + 2y_0^{2m-1} + 2y_0^{2m-2} + 2y_0^{2m-3} + 2y_0^{m-1} + 4y_0^{m-2} + 2y_0^{m-3} + y_0^{m-4})y_i, \end{aligned} \tag{4.1}$$

where $i, j, k, j + k \in \{1, \cdots, 2m - 1\}$.

Proof. By Corollary 4.8, there is a unique ring epimorphism

$$\Phi : \mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}] \to r_P(H_{8m})$$

such that $\Phi(y_i) = \dot{y}_i$ for $i \in \mathbb{Z}_{2m}$. By Lemma 4.7(4), we have

$$\begin{aligned} &\Phi(y_0^{2m} - 1) = \Phi(y_0)^{2m} - 1 = \dot{y}_0^{2m} - 1 = 0, \\ &\Phi(y_i y_{2m-i} - y_m^2) = \Phi(y_i)\Phi(y_{2m-i}) - \Phi(y_m)^2 = \dot{y}_i\dot{y}_{2m-i} - \dot{y}_m^2 = 0, \\ &\Phi(y_j y_k - (1 + y_0^{2m-1} + y_0^{m-1} + y_0^{m-2})y_{j+k}) \\ &\qquad = \Phi(y_j)\Phi(y_k) - (1 + \Phi(y_0)^{2m-1} + \Phi(y_0)^{m-1} + \Phi(y_0)^{m-2})\Phi(y_{j+k}) \\ &\qquad = \dot{y}_j\dot{y}_k - (1 + \dot{y}_0^{2m-1} + \dot{y}_0^{m-1} + \dot{y}_0^{m-2})\dot{y}_{j+k} = 0, \\ &\Phi(y_i y_m^2 - (1 + 2y_0^{2m-1} + 2y_0^{2m-2} + 2y_0^{2m-3} + 2y_0^{m-1} + 4y_0^{m-2} + 2y_0^{m-3} + y_0^{m-4})y_i) \\ &\qquad = \dot{y}_i\dot{y}_m^2 - (1 + 2\dot{y}_0^{2m-1} + 2\dot{y}_0^{2m-2} + 2\dot{y}_0^{2m-3} + 2\dot{y}_0^{m-1} + 4\dot{y}_0^{m-2} + 2\dot{y}_0^{m-3} + \dot{y}_0^{m-4})\dot{y}_i = 0. \end{aligned}$$

Let I is the ideal generated by the elements (4.1). It follows that, $\Phi(I) = 0$ and Φ induces a natural ring epimorphism

$$\overline{\Phi} : \mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}]/I \to r_P(H_{8m})$$

such that $\overline{\Phi}(\overline{v}) = \Phi(v)$ for all $v \in \mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}]$, where $\overline{v} = v + I$.

It is straightforward to check that the ring $\mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}]/I$ is $\mathbb{Z}$-spanned by

$$\{y_0^i, y_0^i y_j, y_0^i y_m^2 | i \in \mathbb{Z}_{2m}, j \in \{1, 2, \cdots 2m - 1\}\}.$$

This means the $\mathbb{Z}$-rank of $\mathbb{Z}[y_0, y_1, \cdots, y_{2m-1}]/I$ is $4m^2+2m$. Hence we get the ring isomorphism $\overline{\Phi}$. □

5. Conclusions

It's a challenging work to classify all indecomposable modules or indecomposable Yetter-Drinfeld modules over the finite dimensional Hopf algebras over an algebraically closed field of characteristic $p > 0$, and to classify all finite dimensional (Nichols) Hopf algebras over H_{8m} over any field. This is our future attempt.

Acknowledgments

The work is supported by National Natural Science Foundation of China (Grant No. 11671024). The authors are grateful to thank the referee for careful reading and helpful suggestions.

Conflict of interest

The authors declare that they have no competing interests.

References

1. D. N. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge*, **108** (1990), 261–290. http://doi.org/10.1017/S0305004100069139
2. C. Kassel, *Quantum groups*, New York: Springer, 1995. https://doi.org/10.1007/978-1-4612-0783-2
3. N. Andruskiewitsch, H. J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of orderp^3, *J. Algebra*, **209** (1998), 658–691. http://doi.org/10.1006/jabr.1998.7643
4. N. Andruskiewitsch, H. J. Schneider, Finite quantum groups and Cartan matrices, *Adv. Math.*, **154** (2000), 1–45. http://doi.org/10.1006/aima.1999.1880
5. N. Andruskiewitsch, H. J. Schneider, Pointed Hopf algebras, In: *New Directions in Hopf algebras*, Cambridge: Cambridge University Press, 2002, 1–68.
6. N. Andruskiewitsch, H. J. Schneider, On the classification of finite dimensional pointed Hopf algebras, *Ann. Math.*, **171** (2010), 375–417. http://doi.org/10.4007/annals.2010.171.375
7. N. Andruskiewitsch, G. Carnovale, G. A. García, Finite dimensional pointed Hopf algebras over finite simple groups of Lie type I. Non-semisimple classes in $PSL_n(q)$, *J. Algebra*, **442** (2015), 36–65. http://doi.org/10.1016/j.jalgebra.2014.06.019
8. G. A. García, J. M. J. Giraldi, On Hopf algebras over quantum subgroups, *J. Pure Appl. Algebra*, **223** (2019), 738–768. https://doi.org/10.1016/j.jpaa.2018.04.018
9. N. Hu, R. Xiong, Some Hopf algebras of dimension 72 without the Chevalley property, 2016, arXiv: 1612.04987. http://doi.org/10.48550/arXiv.1612.04987
10. Y. X. Shi, Finite dimensional Nichols algebras over Kac-Paljutkin algebra H_8, *Rev. Unión Mat. Argent.*, **60** (2019), 265–298. http://doi.org/10.33044/revuma.v60n1a17
11. R. C. Xiong, On Hopf algebras over the unique 12-dimensional Hopf algebra without the dual Chevalley property, *Commun. Algebra*, **47** (2019) 1516–1540. http://doi.org/10.1080/00927872.2018.1508582
12. Y. Zheng, Y. Gao, N. H. Hu, Finite dimensional Hopf algebras over the Hopf algebra $H_{b:1}$ of Kashina, *J. Algebra*, **567** (2021), 613–659. http://doi.org/10.1016/j.jalgebra.2020.09.035
13. Y. Zheng, Y. Gao, N. H. Hu, Finite dimensional Hopf algebras over the Hopf algebra $H_{d:-1,1}$ of Kashina, *J. Pure Appl. Algebra*, **225** (2021), 106527. https://doi.org/10.1016/j.jpaa.2020.106527

14. D. E. Radford, On oriented quantum algebras derived from representations of the quantum double of a finite dimensional Hopf algebra, *J. Algebra*, **270** (2003), 670–695. http://doi.org/10.1016/j.jalgebra.2003.07.006

15. H. Zhu, H. X. Chen, Yetter-Drinfeld modules over the Hopf-Ore extension of the group algebra of dihedral group, *Acta Math. Sin.*, **28** (2012), 487–502. http://doi.org/10.1007/s10114-011-9777-4

16. S. L. Yang, Y. F. Zhang, Ore extensions for the Sweedler's Hopf algebra H_4, *Mathematics*, **8** (2020), 1293. http://doi.org/10.3390/math8081293

17. J. Chen, S. Yang, D. Wang, Y. Xu, On $4n$-dimension neither pointed nor semisimple Hopf algebras and the associated weak Hopf algebras, 2018, arXiv: 1809.00514. http://doi.org/10.48550/arXiv.1809.00514

18. R. C. Xiong, Some classification results on finite dimensional Hopf algebras, PhD thesis, East China Normal University, 2019.

19. Y. Zhang, The Ore extensions of Hopf algebras and their related topics, PhD thesis, Beijing University of Technology, 2020.

20. Y. Guo, S. Yang, The Grothendieck ring of Yetter-Drinfeld modules over a class of $2n^2$-dimension Kac-Paljutkin Hopf algebras, submmited for publication.

21. Y. Guo, S. Yang, Representations of a family of $8m$-dimension Hopf algebras of rank two, submmited for publication.

22. G. Liu, Basic Hopf algebras of tame type, *Algebr. Represent. Theory*, **16** (2013), 771–791. https://doi.org/10.1007/s10468-011-9331-1

23. B. Wald, J. Waschbüsch, Tame biserial algebras, *J. Algebra*, **95** (1985), 480–500. http://doi.org/10.1016/0021-8693(85)90119-X

24. S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, 1993. http://doi.org/10.1090/cbms/082

25. M. Auslander, I. Reiten, S. O. Smalø, *Representation theory of Artin algebras*, Cambridge: Cambridge University Press, 1995. http://doi.org/10.1017/CBO9780511623608

26. M. Lorenz, Representations of finite dimensional Hopf algebra, *J. Algebra*, **188** (1997), 476–505. https://doi.org/10.1006/jabr.1996.6827