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Research article

On a coupled system of generalized hybrid pantograph equations involving fractional deformable derivatives

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Abstract: The goal of this work is to study the existence of a unique solution and the Ulam-Hyers stability of a coupled system of generalized hybrid pantograph equations with fractional deformable derivatives. Our main tool is Banach's contraction principle. The paper ends with an example to support our results.

Keywords: deformable derivative; pantograph equations; Banach contraction principle; hybrid coupled system; Ulam-Hyers stability
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1. Introduction

Grünwad, Letnikov, Riemann, and Liouville are the famous mathematicians who dealt with fractional derivatives whose concept dates from 1695. Since then, fractional calculus has proven its effectiveness as a relevant tool in the study of differential equations [1] and systems of equations involving fractional derivatives, whether from a theoretical point of view or in the modeling of several phenomena in different disciplines such as engineering, control theory, image processing, quantum mechanics, solid-state physics, optical physics, chemical engineering, population dynamics, control systems, fractional multi-pantograph systems, diffusion models and astronomy [2–4]. New progress in the field of fractional calculus and the remarkable evolution of different types of fractional derivatives (conformable derivatives [5], M-conformable fractional derivatives [6] and deformable fractional derivatives [7]), widened the field of research to address several problems in different disciplines especially since the maximum information on the phenomena under study will be incorporated in the mathematical models more realistically.

The deformable derivative was developed in [7] to remedy the lack of the conformable derivative designed by R. Khalil in [5] and which does not include zero and negative numbers. Although the deformable derivative is defined by a limit-based approach to the ordinary derivative, but the difference lies in the fact that the range of the parameters varies over a unit interval which makes it lose the notion of locality. Fundamental notions and results of existence and uniqueness concerning deformable derivatives can be consulted in [5, 7–10] and the references therein. Delay differential equations are powerful mathematical tools for modeling many delay phenomena in physics and engineering and other fields of science since the present depended on past history. Pantograph is an articulated device that allows an electric locomotive or tram or other electric self-propelled system to pick up current by friction on a catenary. The pantograph differential equation is a mathematical model used to describe the behavior of the mechanical system with a pantograph linkage, such as in trains or trams. It can also be used to model the dynamic behavior of mechanical systems and provide valuable information about the performance of such systems. Its origin comes from the work of Ockendon and Tayler [11] on the dynamics of a current collection system for an electric locomotive. This kind of equation appears in many domains of science where authors used them to model several problems [12–19].

Many scientists have investigated the pantograph equations with fractional order considering various aspects and different derivative operators [20–25]. The existence results for the solution of the hybrid pantograph equation with fractional order were studied in [24]. Later, Karimov et al. have established existence results for a generalized hybrid type pantograph equation with Riemann-Liouville fractional derivative in [26]. Afterwards, existence results were explored for a coupled system of fractional order differential equation with ψ-Hilfer derivative in [27]. Recently in 2020, using degree theory and some tools from nonlinear analysis, Ahmad et al. [28] have established existence and stabilities results for a coupled system of pantograph fractional differential equation involving Caputo fractional derivatives. In 2022, a more general coupled system of pantograph problem with three sequential fractional derivatives was considered in the work of George et al. [29]. Using the Leray-Schauder and Banach fixed point theorems and positive contraction-type inequalities, two results on the uniqueness and existence were proved. In recent papers [30, 31], and also in most of the works mentioned above, fixed point theorems are the basic tool for retrieving existence results. Unfortunately, even if the solution exists, its analytical calculation is not obvious in most cases, it is the researchers are generally content with an approximate solution to the problem under study. But, when we have to deal with approximate solutions, the problems of convergence towards the exact solution and the reduction of the calculation error appear. A technique to avoid the problem of convergence is to study stability. In the last two decades, the stability of fractional differential equation and systems have been considered in many work, see for example [28, 29, 32–36]. In 2020, Derakhshan [37] has investigated Ulam-Hyers stability results of a time-fractional linear differential equation arising in fluid mechanics and involving Caputo fractional derivative. Very recent work in 2023 goes to Kahouli et al. [38] who have proven the Ulam-Hyers stability for a class of Itô-Doob Stochastic integral equations with Hadamard fractional derivative.

We point out that the stability of a coupled system was respectively in 2019, 2020, 2022 in the respective works: Zada et al. [39], Ahmad et al. [28], and George et al. [29]. The concept of stability forms part of the quality aspect of dynamic systems. The Ulam and Ulam-Hyers are two types of stabilities which have contributed a great when we deal with the approximate solution of differential equations. Indeed if an equation is Ulam or Ulam-Hyers stable then for each approximate solution there is an exact solution that satisfies certain criteria. In fact, this replaces in a way the study of the convergence of the approximate solution toward the exact solution.

Motivated by the large source of works on deformable fractional derivatives and their applications associated with the works mentioned above on coupled systems of pantograph fractional differential equation and combined with the notion of generalized hybrid-type pantograph equation, the study of a coupled system of two generalized hybrid type pantograph equations involving deformable is investigated. Our contribution is to prove the existence of a unique solution and the Ulam-Hyers stability of the following system

$$\begin{cases} D^{\tau}\Big(\dfrac{\upsilon_1}{h_1\left(.,\upsilon_1(.),\upsilon_2(g_1(.))\right)}\Big)(x) = f_1\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big), \\ \\ D^{\tau}\Big(\dfrac{\upsilon_2}{h_2\left(.,\upsilon_1(.),\upsilon_2(g_1(.))\right)}\Big)(x) = f_2\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big), \\ \\ x \in I = [a,b], \\ \\ \upsilon_1(a) = \upsilon_2(a) = \lambda_1,\ \upsilon_2\left(g_1(a)\right) = \lambda_2,\ \upsilon = (\upsilon_1\upsilon_2),\quad 0 < a < b, \lambda_i > 0,\ i = 1,2, \end{cases} \tag{1.1}$$

where D^{τ} is the deformable fractional derivative of order τ with $\tau + \alpha = 1, 0 \leq \tau \leq 1,$ and $\alpha > 0$. The functions $h_i, f_i, g_i,\ i = 1,2$ will be defined later.

This article is composed of three sections, the second section is devoted to mathematical tools that we will need in the sequel and the last section is intended for our existence and stability results and of course an example to close the work.

2. Preliminaries

In this section, we present the most relevant notions concerning deformable fractional derivatives by referring to [5, 7–10, 40].

Let $C = C([a,\ b], \mathbb{R})$ denote the Banach space of continuous functions from $[a, b]$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_C = \sup_{x\in[a,\ b]} |u(x)|.$$

Definition 2.1. [7, 8] Let $u : [a,\ b] \to \mathbb{R}$ be a continuous function and τ, α positive numbers with $0 \leq \tau \leq 1$ and $\tau + \alpha = 1$. The deformable derivative of u of order τ at $x \in I = [a,\ b]$ is defined by

$$\left(D^{\tau}u\right)(x) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\alpha)\,u\,(x+\varepsilon\tau) - u(x)}{\varepsilon}. \tag{2.1}$$

If the limit exists, u is $\tau-$differentiable at x. If $\tau = 1$, then $\alpha = 0$, we recover the usual derivative. Therefore, the deformable derivative is more general than the usual derivative.

Definition 2.2. [7,8] For $\tau \in (0,\ 1]$, the $\tau-$integral of the function $u \in L^1([a,\ b], \mathbb{R}_+)$ is defined by

$$(I_a^\tau u)(x) = \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} u(t)\, dt, \quad x \in [a,\ b], \tag{2.2}$$

where $\tau + \alpha = 1$. When $a = 0$, we write $(I^\tau u)$ instead of writing $\left(I_0^\tau u\right)$.

The following theorem gathers the most important properties of the operators D^τ, I_a^τ which can be useful in the paper.

Theorem 2.3. *[7] Let $\tau, \tau_1, \tau_2 \in (0,\ 1]$ be such that $\tau + \alpha = 1$ and $\tau_i + \alpha_i = 1$ for $i = 1, 2$. Then*

(1) The operators D^τ and I_a^τ are linear.
(2) The operators D^τ and I_a^τ are commutative.
(3) $D^\tau(\sigma) = \alpha\sigma$ for all constant $\sigma \in \mathbb{R}$.
(4) $D^\tau(uv) = (D^\tau u)\, v + \tau h D v$.
(5) Let u be continuous function on $[a,\ b]$. Then $I_a^\tau u$ is $\tau-$differentiable in (a, b) and we have

$$D^\tau \left(I_a^\tau u\right)(x) = u(x), \tag{2.3}$$

$$I_a^\tau \left(D^\tau u\right)(x) = u(t) - e^{\frac{\alpha}{\tau}(a-x)} u(a). \tag{2.4}$$

Lemma 2.4. *[12] Let $\tau \in (0,\ 1]$. The differential equation $(D^\tau u)(x) = 0$ has solutions*

$$u(x) = \sigma e^{-\frac{\alpha}{\tau}t},$$

where $\sigma \in \mathbb{R}$ is a constant.

3. Main results

3.1. Existence results

Lemma 3.1. *Let $f \in C([a,\ b], \mathbb{R})$ and $h \in C([a,\ b], \mathbb{R}^*)$. The function $u \in C([a, b], \mathbb{R})$, such that*

$$u(x) = h(x)\Big[\frac{\lambda}{h(a)} e^{\frac{\alpha}{\tau}(a-x)} + \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} f(t)\, dt\Big],$$

is a solution of the fractional initial value problem

$$\begin{cases} D^\tau \left(\dfrac{u}{h}\right)(x) = f(x), & x \in [a,\ b] \\ u(a) = \lambda > 0, \end{cases}$$

where D^τ is the deformable fractional derivative of order τ with $\tau + \alpha = 1,\ 0 \le \tau \le 1,$ and $\alpha \neq 0$.

Proof. Since $\dfrac{u}{h}$ is continuous on $[a, b]$ and f is a continuous anti-τ-derivative of $\dfrac{u}{h}$ over $[a, b]$, we have

$$\left[I_{a^+}^\tau \left(D^\tau \left(\frac{u}{h}\right)\right)\right](x) = (I_{a^+}^\tau f)(x).$$

Using (2.4), we obtain

$$\frac{u}{g}(x) - \frac{u(a)}{h(a)} e^{\frac{\alpha}{\tau}(a-x)} = \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}s} f(t) dt,$$

$$\begin{aligned} u(x) &= \frac{u(a)}{h(a)} e^{\frac{\alpha}{\tau}(a-x)} h(x) + \frac{1}{\tau} h(x) e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}s} f(t)\, dt \\ &= \frac{\lambda}{h(a)} e^{\frac{\alpha}{\tau}(a-x)} h(x) + \frac{1}{\tau} h(x) e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}s} f(t)\, dt. \end{aligned}$$

The proof is completed. □

Now, we will reconsider our initial coupled system (1.1), where

$$h_i \in C\Big([0,\ 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} - \{0\}\Big), f_i \in C\Big([a,\ b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big) \text{ and } g_i \in C\Big([a,\ b],\ [a,\ b]\Big). \tag{3.1}$$

Let $\varSigma$ be the Banach space defined by

$$\varSigma = \{\upsilon = (\upsilon_1, \upsilon_2) \in C \times C \ / \ \upsilon_1, \upsilon_2 \in C\}$$

endowed with the norm

$$\|\upsilon\|_{\varSigma} = \|\upsilon_1\|_C + \|\upsilon_2\|_C.$$

$\varLambda$ denotes the following space

$$\varLambda = \{\upsilon = (\upsilon_1, \upsilon_2) \in \varSigma \ / \ \upsilon_1, \upsilon_2 \in C, \text{ with } D^\tau \left(\frac{\upsilon_i}{h_i}\right) \in C,\ i = 1, 2\}$$

with D^τ is the deformable fractional derivative, $\tau \in (0,\ 1)$ satisfies $\alpha + \tau = 1$ for some $\alpha > 0$.

Definition 3.2. $\upsilon = (\upsilon_1, \upsilon_2) \in \varLambda$ is called a solution of the coupled system (1.1) if $\upsilon_1, \upsilon_2 \in C$, respectively are solutions of the hybrid nonlinear fractional pantograph equations of the coupled system (1.1).

T_1, T_2, T are three operators defined as follows:

$$\begin{array}{rcccc} T_i & : & C & \longrightarrow & C \\ & & \upsilon_i & \longmapsto & T_i\upsilon_i \end{array} \qquad \begin{array}{rcccc} T_i\upsilon_i & : & [a,\ b] & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & T_i\upsilon_i(x) \end{array}$$

$$\begin{aligned} T_i\upsilon_i(x) =& h_i\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right) \Big[\frac{\lambda_i}{h_i\left(a, \upsilon_1(a), \upsilon_2(g_1(a))\right)} e^{\frac{\alpha}{\tau}(a-x)} \\ & + \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} f_i\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\, dt \Big],\quad i = 1, 2, \end{aligned} \tag{3.2}$$

and

$$\begin{array}{rcccc} T & : & \varLambda & \longrightarrow & \varLambda \\ & & (\upsilon_1, \upsilon_2) & \longmapsto & T(\upsilon_1, \upsilon_2) = (T_1\upsilon_1, T_2\upsilon_2) \end{array} \qquad \begin{array}{rcccc} T(\upsilon_1, \upsilon_2) & : & [a,\ b] & \longrightarrow & \mathbb{R} \times \mathbb{R} \\ & & x & \longmapsto & T(\upsilon_1, \upsilon_2)(x) \end{array}$$

$$T(\upsilon_1, \upsilon_2)(x) = \left(T_1\left(\upsilon_1(x), \upsilon_2(x)\right), T_2\left(\upsilon_1(x), \upsilon_2(x)\right)\right) \tag{3.3}$$

with

$$\begin{cases} T_1\left(\upsilon_1(x), \upsilon_2(x)\right) = T_1\upsilon_1(x), \\ T_2\left(\upsilon_1(x), \upsilon_2(x)\right) = T_2\upsilon_2(x), \\ \upsilon = \left(\upsilon_1, \upsilon_2\right). \end{cases}$$

In order to carry out existence results for the coupled system (1.1), additional assumptions are made on h_i, f_i, g_i for $i = 1, 2$. Let $\delta_i > 0$ be positive numbers satisfying

$$\delta_i = \sup_{x\in[a,\, b]} |h_i\,(x, 0, 0)\,|$$

and assume
(P_1) $\exists k_i > 0$, such that

$$|h_1\,(x, x_1, x_2) - h_1\,(x, y_1, y_2)\,| \leq k_1\Big(|x_1 - y_1| + |x_2 - y_2|\Big),$$
$$|h_2\,(x, x_1, x_2) - h_2\,(x, y_1, y_2)\,| \leq k_2\Big(|x_1 - y_1| + |x_2 - y_2|\Big),$$

for all $x \in [0,\ 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(P_2) $\exists\theta > 2\lambda_1 > 0,$ such that

$$|h_i\,(a, \lambda_1, \lambda_2)\,| \geq \theta,\ \ i = 1, 2.$$

(P_3) $\exists q_i > 0,$ such that

$$|f_1\,(x, x_1, x_2) - f_1\,(x, y_1, y_2)\,| \leq q_1\Big(|x_1 - y_1| + |x_2 - y_2|\Big),$$
$$|f_2\,(x, x_1, x_2) - f_2\,(x, y_1, y_2)\,| \leq q_2\Big(|x_1 - y_1| + |x_2 - y_2|\Big),$$

for all $x \in [0,\ 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(P_4) $\exists\kappa_i > 0,$ such that

$$\sup_{x\in[a,\, b]} |f_1\,(x, 0, 0)\,| \leq \kappa_1,\ \text{ and } \sup_{x\in[a,\, b]} |f_2\,(x, 0, 0)\,| \leq \kappa_2, \qquad \forall x \in [a,\ b]\,.$$

(P_5) $\exists M_i > 0,\ M_i^* > 0,$ such that

$$\|f_i\| \leq M_i\ \text{ and }\ \|h_i\| \leq M_i^*,\ i = 1, 2.$$

To make the computation simple, we use the following notations

$$\begin{cases} \Theta = \frac{1}{\alpha}\Big(e^{\frac{\alpha}{\tau}(b-a)} - 1\Big), \\ A_i = q_i\Theta, \\ B_i = \kappa_i\Theta + \frac{\lambda_1}{\theta}, \\ \xi = \sum_{i=1}^{2} \left(k_i B_i + \delta_i A_i\right), \\ \gamma = \sum_{i=1}^{2} k_i A_i, \\ \nu = \sum_{i=1}^{2} \delta_i B_i, \\ \sigma = \nu\gamma, \\ X_i = k_i M_i + q_i M_i^*, \quad i = 1, 2. \end{cases} \tag{3.4}$$

Lemma 3.3. *The operator T defined on Λ by* (3.3) *is well defined.*

Proof. We will prove that T_1 and T_1 are well defined on C, moreover $D^\tau\left(\frac{T_1 \upsilon_1}{h_1}\right)$ and $D^\tau\left(\frac{T_2 \upsilon_2}{h_2}\right)$ also must be in C. For any $\upsilon_1, \upsilon_2 \in C$ and for $x \in [a, b]$, we have

$$T_1\upsilon_1 = (\varphi_1\upsilon_1)(\psi_1\upsilon_1), \quad T_2\upsilon_2 = (\varphi_2\upsilon_2)(\psi_2\upsilon_2) \tag{3.5}$$

where

$$\begin{cases} \varphi_i\upsilon_i(x) = h_i\Big(x, \upsilon_1(x), \upsilon_2(g_1(x))\Big) \\ \psi_i\upsilon_i(x) = \dfrac{\lambda_i}{h_i\Big(a, \upsilon_1(a), \upsilon_2(g_1(a))\Big)} e^{\frac{\alpha}{\tau}(a-x)} + \dfrac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \displaystyle\int_a^x e^{\frac{\alpha}{\tau}t} f_i\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\, dt, \\ i = 1, 2. \end{cases}$$

Let (x_n) a sequence in $[a, b]$ which converges to $x_0 \in [a, b]$ when $n \to +\infty$. For any $\upsilon_1 \in C$, we have

$$\Big|\varphi_1\upsilon_1(x_n) - \varphi_1\upsilon_1(x_0)\Big| = \Big|h_1\Big(x_n, \upsilon_1(x_n), \upsilon_2(g_1(x_n))\Big) - h_1\Big(x_0, \upsilon_1(x_0), \upsilon_2(g_1(x_0))\Big)\Big| \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.6}$$

It yields that $\varphi_1\upsilon_1$ is continuous on $[a, b]$. On the other hand, taking into consideration that $x_n \geq x_0$, we have

$$\left|\psi_1\upsilon_1(x_n)-\psi_1\upsilon_1(x_0)\right| \leq \left|\frac{\lambda_1}{h_1\Big(a,\upsilon_1(a),\upsilon_2(g_1(a))\Big)}e^{\frac{\alpha}{\tau}(a-x_n)}-\frac{\lambda_1}{h_1\Big(a,\upsilon_1(a),\upsilon_2(g_1(a))\Big)}e^{\frac{\alpha}{\tau}(a-x_0)}\right|$$
$$\left|\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_n}\int_a^{x_n}e^{\frac{\alpha}{\tau}t}f_1\Big(t,\upsilon_1(t),\upsilon_2(g_2(t))\Big)\,dt-\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_0}\int_a^{x_0}e^{\frac{\alpha}{\tau}t}f_1\Big(t,\upsilon_1(t),\upsilon_2(g_2(t))\Big)\,dt\right|,$$

$$\begin{aligned}\left|\psi_1\upsilon_1(x_n)-\psi_1\upsilon_1(x_0)\right| &\leq \frac{\lambda_1}{\theta}\left|e^{\frac{\alpha}{\tau}(a-x_n)}-e^{\frac{\alpha}{\tau}(a-x_0)}\right|\\ &+\left|\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_n}\int_a^{x_n}e^{\frac{\alpha}{\tau}s}f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\,dt-\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_0}\int_a^{x_n}e^{\frac{\alpha}{\tau}t}f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\,dt\right|\\ &+\left|\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_0}\int_a^{x_n}e^{\frac{\alpha}{\tau}t}f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\,dt-\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_0}\int_a^{x_0}e^{\frac{\alpha}{\tau}s}f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\,dt\right|,\end{aligned}$$

$$\begin{aligned}\left|\psi_1\upsilon_1(x_n)-\psi_1\upsilon_1(x_0)\right| &\leq \frac{\lambda_1}{\theta}\left|e^{\frac{\alpha}{\tau}(a-x_n)}-e^{\frac{\alpha}{\tau}(a-x_0)}\right|\\ &+\frac{1}{\tau}\left(e^{-\frac{\alpha}{\tau}x_n}-e^{-\frac{\alpha}{\tau}x_0}\right)\int_a^{x_n}e^{\frac{\alpha}{\tau}t}\left|f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\right|dt\\ &+\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x_0}\int_{x_0}^{x_n}e^{\frac{\alpha}{\tau}t}\left|f_1\left(t,\upsilon_1(t),\upsilon_2(g_2(t))\right)\right|dt,\end{aligned}$$

by (P_5), we obtain

$$\left|\psi_1\upsilon_1(x_n)-\psi_1\upsilon_1(x_0)\right| \leq \left|e^{\frac{\alpha}{\tau}(x_0-x_n)}-1\right|\left(\frac{\lambda_1}{\theta}e^{\frac{\alpha}{\tau}(a-x_0)}+\frac{M_1}{\alpha}e^{\frac{\alpha}{\tau}(a-x_0)}\left|e^{\frac{\alpha}{\tau}(x_n-a)}-1\right|-\frac{M_1}{\alpha}e^{\frac{\alpha}{\tau}(x_n-x_0)}\right)\underset{n\to+\infty}{\longrightarrow}0.$$

Then, $\psi_1\upsilon_1$ is continuous on $[a,b]$. □

Besides, since $T_1\upsilon_1(a)=\lambda$, it can be easily checked that

$$D^{\tau}\left(\frac{T_1\upsilon_1}{h_1\Big(.,\upsilon_1(.),\upsilon_2(g_1(.))\Big)}\right)(x)=f_1\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big),\quad x\in[a,b],$$

which means that $D^{\tau}\left(\frac{T_1\upsilon_1}{h_1\Big(.,\upsilon_1(.),\upsilon_2(g_1(.))\Big)}\right)$ is continuous on $[a,b]$.

In a same way, we prove that $T_2\upsilon_2$, and $D^{\tau}\left(\frac{T_2\upsilon_{12}}{h_2\Big(.,\upsilon_1(.),\upsilon_2(g_1(.))\Big)}\right)$ are in C. Therefore T is well defined on Λ. The following theorem is devoted to our existence result.

Theorem 3.4. *If* $(P_1)-(P_5)$ *are hold and if*

$$0<\xi\leq 1-2\sqrt{\sigma},\quad \textit{with } 0<\sigma<\frac{1}{4} \tag{3.7}$$

and

$$0<X_1+X_2<\frac{1}{\Theta\theta}\left(\theta-2\lambda_1\right), \tag{3.8}$$

then, the coupled system (1.1) *has a unique solution.*

Our tool for the proof is Banach's contraction principle.

Proof. The proof is done in two steps.

(1) T maps bounded sets into bounded sets in Λ.

Proof. r_2, r_1 are two real numbers satisfying $r_1 + r_2 = \dfrac{1-\xi}{\gamma}$, and $r_1 r_2 = \dfrac{\nu}{\gamma}$. Regarding (3.7), it's obvious that $r_1 > 0$ and $r_2 > 0$. We assume that $r_2 > r_1$and we consider the set

$$B_\rho = \{\upsilon \in \Lambda / \|\upsilon\|_\Lambda \leq \rho\}$$

where ρ is a positive real number such that $\rho \in [r_1, r_2]$. We claim that $T\left(B_\rho\right) \subset \rho$. Indeed, for any $\upsilon \in B_\rho$, we have

$$\|T\upsilon\|_\Lambda = \|T_1\upsilon_1\|_c + \|T_2\upsilon_2\|_c, \text{ with } \upsilon = (\upsilon_1, \upsilon_2).$$

For any $x \in [a, b]$ and $\upsilon_1,\ \upsilon_2 \in C$, we have

$$\left|T_1\upsilon_1(x)\right| = \left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\right| \left| \frac{\lambda_1}{h_1\left(a, \upsilon_1(a), \upsilon_2(g_1(a))\right)} e^{\frac{\alpha}{\tau}(a-x)} + \frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\, dt\right|$$

$$\begin{aligned}\left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\right| &\leq h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right) - h_1(x, 0, 0))\Big| + \Big|h_1(x, 0, 0))\Big| \\ &\leq k_1\Big(|\upsilon_1| + |\upsilon_2|\Big) + \Big|h_1(x, 0, 0))\Big|\end{aligned}$$

then,

$$\sup_{x\in[a,\,b]} \left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\right| = k_1 \sup_{x\in[a,\,b]} \Big(\left|\upsilon_1(x)\right| + \left|\upsilon_2(g_1(x))\right|\Big) + \sup_{x\in[a,\,b]} |h_1(x, 0, 0)|$$

since for any $x \in [a,\ b]$ it yields that $g_1(x) \in [a,\ b]$, we have

$$\sup_{x\in[a,\,b]} \Big(\left|\upsilon_1(x)\right| + \left|\upsilon_2(g_1(x))\right|\Big) = \|\upsilon_1\|_C + \|\upsilon_2\|_C = \|\upsilon\|_\Lambda,$$

hence,

$$\sup_{x\in[a,\,b]} \left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\right| \leq k_1\|\upsilon\|_\Lambda + \delta_1$$

$$\sup_{x\in[a,\,b]} \left|\frac{\lambda_1}{h_1\left(a, \upsilon_1(a), \upsilon_2(g_1(a))\right)} e^{\frac{\alpha}{\tau}(a-x)}\right| \leq \frac{\lambda_1}{\theta}$$

$$\left|\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\, dt\right| \leq \int_a^x e^{\frac{\alpha}{\tau}t} \Big|f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big) - f_1\Big(t, 0, 0\Big)\Big|\, dt + \int_a^x e^{\frac{\alpha}{\tau}t} \Big|f_1\Big(t, 0, 0\Big)\Big|\, dt$$

and using (P_4), we have

$$\sup_{x\in[a,\,b]} \left|\frac{1}{\tau} e^{-\frac{\alpha}{\tau}x} \int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\, dt\right| \quad \leq \quad \frac{q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\|\upsilon\|_\Lambda \ + \ \frac{\kappa_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)$$

$$\|T_1\upsilon_1\| \leq \left(k_1\|\upsilon\|_\Lambda + \delta_1\right)\left(\frac{\lambda_1}{\theta} + \frac{q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\|\upsilon\|_\Lambda + \frac{\kappa_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\right).$$

In a similar way, we obtain

$$\|T_2\upsilon_2\| \leq \left(k_2\|\upsilon\|_\Lambda + \delta_2\right)\left(\frac{\lambda_1}{\theta} + \frac{q_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\|\upsilon\|_\Lambda + \frac{\kappa_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\right).$$

$$\begin{aligned}\|T\upsilon\|_\Lambda &\leq \|T_1\upsilon_1\|_C + \|T_2\upsilon_2\|_C,\\ &\leq \gamma\|\upsilon\|_\Lambda^2 + \xi\|\upsilon\|_\Lambda + \nu,\\ &\leq \gamma\rho^2 + \xi\rho + \nu,\\ &\leq \rho,\end{aligned}$$

where we have used (3.7) with the fact that $\rho \in [r_1, r_2]$ to deduce that

$$\gamma\rho^2 + \xi\rho + \nu \leq \rho.$$

Then the proof $T\left(B_\rho\right) \subset B_\rho$ is achieved. □

(2) Now, we show that T is a contraction.

Proof. Let $\upsilon = (\upsilon_1, \upsilon_2)$, $\upsilon^* = \left(\upsilon_1^*, \upsilon_2^*\right) \in \Lambda$, with $\upsilon_1, \upsilon_2, \upsilon_1^*, \upsilon_2^* \in C$. For any $x \in [a, b]$, we have

$$\begin{aligned}T_1\upsilon_1(x) - T_1\upsilon_1^*(x) &= h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\left(\frac{\lambda_1}{h_1\left(a, \upsilon_1(a), \upsilon_2(g_1(a))\right)}e^{\frac{\alpha}{\tau}(a-x)}\right)\\ &\quad + h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\left(t, \upsilon_1(t), \upsilon_2(g_2(t))\right)dt\right)\\ &\quad - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{\lambda_1}{h_1\left(a, \upsilon_1^*(a), \upsilon_2^*(g_1(a))\right)}e^{\frac{\alpha}{\tau}(a-x)}\right)\\ &\quad - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\left(t, \upsilon_1^*(t), \upsilon_2^*(g_2(t))\right)dt\right)\end{aligned}$$

$$\sup_{x\in[a,\,b]}\left(\left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right) - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\right|\left(\frac{\lambda_1}{h_1\left(a, \lambda_1, \lambda_2\right)}e^{\frac{\alpha}{\tau}(a-x)}\right)\right) \leq \frac{\lambda_1}{\theta}\left\|\upsilon - \upsilon^*\right\|_\Lambda \quad (3.9)$$

$$\begin{aligned}&h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\left(t, \upsilon_1(t), \upsilon_2(g_2(t))\right)dt\right)\\ &\qquad - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\left(t, \upsilon_1^*(t), \upsilon_2^*(g_2(t))\right)dt\right)\end{aligned}$$

$$+ h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\,dt\right)$$
$$- h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\,dt\right) =$$
$$\Big(h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right) - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\Big)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\,dt\right)$$
$$+ h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t}\Big[f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big) - f_1\Big(t, \upsilon_1^*(t), \upsilon_2^*(g_2(t))\Big)\Big]\,dt\right). \quad (3.10)$$

By $(P_1), (P_3)$, and (P_5), we find

$$\sup_{x\in[a,b]}\left|h_1\left(x, \upsilon_1(x), \upsilon_2(g_1(x))\right) - h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\Big)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\,dt\right)\right|$$
$$\leq \frac{k_1 M_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\|\upsilon - \upsilon^*\|_\Lambda, \quad (3.11)$$

$$\sup_{x\in[a,b]}\left|h_1\left(x, \upsilon_1^*(x), \upsilon_2^*(g_1(x))\right)\left(\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t}\Big[f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big) - f_1\Big(t, \upsilon_1^*(t), \upsilon_2^*(g_2(t))\Big)\Big]\,dt\right)\right|$$
$$\leq \frac{M_1^* q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\|\upsilon - \upsilon^*\|_\Lambda. \quad (3.12)$$

From (3.9),(3.11),(3.12), we obtain

$$\|T_1\upsilon_1 - T_1\upsilon_1\|_C \leq \frac{\lambda_1}{\theta}\left\|\upsilon - \upsilon^*\right\|_\Lambda + \frac{k_1 M_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\|\upsilon - \upsilon^*\|_\Lambda + \frac{M_1^* q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\|\upsilon - \upsilon^*\|_\Lambda,$$
$$\leq \left(\frac{\lambda_1}{\theta} + \frac{k_1 M_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right) + \frac{M_1^* q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\right)\|\upsilon - \upsilon^*\|_\Lambda. \quad (3.13)$$

Therefore,

$$\|T_2\upsilon_2 - T_2\upsilon_2^*\|_C \leq \left(\frac{\lambda_1}{\theta} + \frac{k_2 M_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right) + \frac{M_2^* q_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\right)\|\upsilon - \upsilon^*\|_\Lambda. \quad (3.14)$$

On the other hand, using the above inequalities (3.13) and (3.14), we get

$$\|T\upsilon - T\upsilon^*\|_\Lambda = \|T_1\upsilon_1 - T_1\upsilon_1^*\|_C + \|T_2\upsilon_2 - T_2\upsilon_2^*\|_C,$$
$$\leq \left(\frac{\lambda_1}{\theta} + \frac{k_1 M_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right) + \frac{M_1^* q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\right)\|\upsilon - \upsilon^*\|_\Lambda$$
$$+ \left(\frac{\lambda_1}{\theta} + \frac{k_2 M_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right) + \frac{M_2^* q_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right)\right)\|\upsilon - \upsilon^*\|_\Lambda$$
$$\leq \left[\left(\frac{\lambda_1}{\theta} + \frac{k_1 M_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)} - 1\right) + \right.\right.$$

$$\frac{M_1^* q_1}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\Bigg)+\left(\frac{\lambda_1}{\theta}+\frac{k_2 M_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)+\frac{M_2^* q_2}{\alpha}\left(e^{\frac{\alpha}{\tau}(b-a)}-1\right)\right)\Bigg]\|\upsilon-\upsilon^*\|_\Lambda, \quad (3.15)$$

$$\begin{aligned}\|T\upsilon - T\upsilon^*\|_\Lambda &\leq \left(\frac{2\lambda_1}{\theta}+(X_1+X_2)\,\Theta\right)\|\upsilon-\upsilon^*\|_\Lambda,\\ &\leq \beta\|\upsilon-\upsilon^*\|_\Lambda,\end{aligned}$$

with $0 < \beta = \dfrac{2\lambda_1}{\theta} + (X_1 + X_2)\,\Theta < 1$, where we have used (3.8) for this deduction. Hence T is a contraction and Banach fixed point theorem ensures the existence of a unique solution of the coupled system (1.1) in B_ρ. □

□

Remark 3.5. We can prove that T maps bounded sets into bounded sets even if (P_4) is not carried out and we have the theorem below.

By Y_1 and Y_2, we denote the following real numbers

$$Y_1 = \frac{\lambda_1}{\theta} + M_1\Theta, \quad Y_2 = \frac{\lambda_1}{\theta} + M_2\Theta.$$

Theorem 3.6. *If* $(P_1), (P_2), (P_3), (P_5)$ *and* (3.8) *are satisfied and if*

$$0 < k_1 Y_1 + k_2 Y_2 < 1 \tag{3.16}$$

then, the Problem (1.1) *has a unique solution.*

In fact, the condition (P_4) has not intervened in the demonstration that T is a contraction.

Let us consider the bounded set $B_R = \{\upsilon \in \Lambda / \|\upsilon\| \leq R\}$, where R is a real positive number selected as follows:

$$R \geq \frac{\delta_1\left(\frac{\lambda_1}{\theta}+M_1\Theta\right)+\delta_2\left(\frac{\lambda_1}{\theta}+M_2\Theta\right)}{1-\left(k_1\left(\frac{\lambda_1}{\theta}+M_1\Theta\right)+k_2\left(\frac{\lambda_1}{\theta}+M_2\Theta\right)\right)}.$$

Therefore, we have to prove that $T(B_R) \subset R$ without using assumption (P_4). Indeed, using (P_5) and taking $\upsilon = (\upsilon_1, \upsilon_2) \in B_R$, it yields:

$$\sup_{x\in[a,b]}\left|\frac{1}{\tau}e^{-\frac{\alpha}{\tau}x}\int_a^x e^{\frac{\alpha}{\tau}t} f_1\Big(t, \upsilon_1(t), \upsilon_2(g_2(t))\Big)\,dt\right| \leq M_1\Theta, \tag{3.17}$$

$$\|T_1\upsilon_1\| \leq \Big(k_1\|\upsilon\|_\Lambda + \delta_1\Big)\left(\frac{\lambda_1}{\theta}+M_1\Theta\right). \tag{3.18}$$

In the same manner, we get

$$\|T_2\upsilon_2\| \leq \Big(k_2\|\upsilon\|_\Lambda + \delta_2\Big)\left(\frac{\lambda_1}{\theta}+M_2\Theta\right). \tag{3.19}$$

By summing (3.18) and (3.19), we obtain

$$\begin{aligned}
\|T\upsilon\|_{\Lambda} &\leq \|T_1\upsilon_1\|_C + \|T_2\upsilon_2\|_C, \\
&\leq R\Big(k_1Y_1 + k_2Y_2\Big) + \Big(\delta_1Y_1 + \delta_2Y_2\Big), \\
&\leq R.
\end{aligned}$$

This last result is valid thanks to (3.16) and the selection of R. Since, the operator T remains a contraction even if we delete (P_4), then Theorem 3.6 ensures the existence of a unique solution of the coupled system (1.1) in the bounded set B_R.

3.2. Ulam-Hyers stability

Definition 3.7. The fractional boundary value problem (1.1) is generalized Ulam-Hyers stable if there exists $\Upsilon_{(f_1,f_2)} \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\Upsilon_{(f_1,f_2)}(0) = 0$, such that for each $\varrho > 0$ and for each solution $\omega = (\omega_1,\omega_2) \in \Lambda$ of the inequality

$$\left|D^{\tau}\left(\frac{\upsilon_i}{h_i\left(.,\upsilon_1(.),\upsilon_2(g_1(.))\right)}\right)(x) - f_i\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big)\right| \leq \varrho,\ x\in I, i = 1,2,$$

there exists a solution $\upsilon = (\upsilon_1,\upsilon_2) \in \Lambda$ of the fractional boundary value problem (1.1) with

$$\|\omega - \upsilon\| \leq \Upsilon_{(f_1,f_2)}(\varrho),\ x \in I.$$

If $\Upsilon_{(f_1,f_2)}(\varrho) = \nu\varrho$ with $\nu > 0$, then the fractional boundary value problem (1.1) is Ulam-Hyers stable.

Theorem 3.8. *If all assumptions of Theorems 3.4 or 3.6 are hold, then the Problem* (1.1) *is Ulam-Hyers stable.*

Proof. Let ϱ be a real positive number and $\upsilon = (\upsilon_1,\upsilon_2)$ the unique solution of the Problem (1.1) in Λ. Let $\omega = (\omega_1,\omega_2) \in \Lambda$ be a solution of the coupled system of inequalities

$$\begin{cases}
\left|D^{\tau}\left(\dfrac{\omega_1}{h_1\left(.,\omega_1(.),\omega_2(g_1(.))\right)}\right)(x) - f_1\Big(x,\upsilon\omega_1(x),\omega_2(g_2(x))\Big)\right| \leq \varrho, \\[2ex]
\left|D^{\tau}\left(\dfrac{\omega_2}{h_2\left(.,\omega_1(.),\omega_2(g_1(.))\right)}\right)(x) - f_2\Big(x,\omega_1(x),\omega_2(g_2(x))\Big)\right| \leq \varrho, \\[2ex]
x \in I = [a,b], \\[2ex]
\omega_1(a) = \omega_2(a) = \upsilon_1(a) = \lambda_1,\ \omega_2\left(g_1(a)\right) = \upsilon_2\left(g_1(a)\right) = \lambda_2, \\
\omega = (\omega_1,\omega_2),\quad 0 < a < b, \lambda_i > 0,\ i = 1,2.
\end{cases} \tag{3.20}$$

By integrating the inequalities in the coupled system (3.20), we obtain

$$\begin{cases}
\left|\dfrac{\omega_1(x)}{h_1\left(x,\omega_1(x),\omega_2(g_1(x))\right)} - \dfrac{\omega_1(a)}{h_1\left(a,\omega_1(a),\omega_2(g_1(a))\right)} - I_a^{\tau}\Big(f_1\Big(x,\omega_1(x),\omega_2(g_2(x))\Big)\Big)\right| \leq I_a^{\tau}(\varrho), \\[2ex]
\left|\dfrac{\omega_2(x)}{h_2\left(x,\omega_1(x),\omega_2(g_1(x))\right)} - \dfrac{\omega_2(a)}{h_2\left(a,\omega_1(a),\omega_2(g_1(a))\right)} - I_a^{\tau}\Big(f_2\Big(x,\omega_1(x),\omega_2(g_2(x))\Big)\Big)\right| \leq I_a^{\tau}(\varrho),
\end{cases} \tag{3.21}$$

and using the fact that $\upsilon_1(a) = \omega_1(a) = \omega_2(a) = \lambda_1,\ \upsilon_2(g_1(a)) = \omega_2(g_1(a)) = \lambda_2,\ \omega = (\omega_1, \omega_2)$, we get:

$$\left|\omega_1(x) - T_1\omega_1(x)\right| \leq \varrho(I_a^\tau(1)\left|h_1(x, \omega_1(x), \omega_2(g_1(x)))\right|,$$

$$\left|\omega_2(x) - T_2\omega_2(x)\right| \leq \varrho(I_{a^+}^\tau(1)\left|h_2(x, \omega_1(x), \omega_2(g_1(x)))\right|,$$

$$\left|\omega_1(x) - T_1\omega_1(x)\right| \leq \varrho\frac{1}{\alpha}\left(1 - e^{\frac{\alpha}{\tau}(b-a)}\right)\left|h_1(x, \omega_1(x), \omega_2(g_1(x)))\right|,$$

$$\left|\omega_2(x) - T_2\omega_2(x)\right| \leq \frac{\varrho}{\alpha}\left(1 - e^{\frac{\alpha}{\tau}(b-a)}\right)\left|h_2(x, \omega_1(x), \omega_2(g_1(x)))\right|,$$

$$\left\|\omega_1 - T_1\omega_1\right\|_C \leq \varrho\frac{1}{\alpha}M_1^*, \tag{3.22}$$

and

$$\left\|\omega_2 - T_2\omega_2\right\|_C \leq \frac{\varrho}{\alpha}M_2^*. \tag{3.23}$$

Summing (3.22) and (3.23), we have

$$\left\|\omega - T\omega\right\|_\Lambda \leq \frac{\varrho}{\alpha}(M_1^* + M_2^*). \tag{3.24}$$

On the other hand, since υ is the unique solution of the coupled system (1.1) and T is a contraction, it yields that for ω in Λ satisfying the system of inequalities and for $0 < \beta = \dfrac{2\lambda_1}{\theta} + (X_1 + X_2)\,\Theta < 1$, we have

$$\left\|T\omega - T\upsilon\right\|_\Lambda \leq \beta\left\|\omega - \upsilon\right\|_\Lambda. \tag{3.25}$$

Therefore,

$$\begin{aligned}\left\|\omega - \upsilon\right\|_\Lambda &\leq \left\|\omega - T\omega\right\| + \left\|T\omega - \upsilon\right\|,\\ &\leq \left\|\omega - T\omega\right\| + \left\|T\omega - T\upsilon\right\|,\\ &\leq \frac{\varrho}{\alpha}(M_1^* + M_2^*) + \beta\left\|\omega - \upsilon\right\|_\Lambda\end{aligned}$$

It yields

$$\left(1 - \beta\right)\left\|_\Lambda \omega - \upsilon\right\| \leq \frac{\varrho}{\alpha}(M_1^* + M_2^*), \tag{3.26}$$

hence

$$\left\|\omega - \upsilon\right\|_\Lambda \leq \frac{(M_1^* + M_2^*)}{\alpha\left(1 - \beta\right)}\varrho. \tag{3.27}$$

That is

$$\left\|\omega - \upsilon\right\|_\Lambda \leq \Upsilon_{(f_1,f_2)}(\varrho), \tag{3.28}$$

where $\Upsilon_{(f_1,f_2)}(\varrho) = \dfrac{(M_1^* + M_2^*)}{\alpha\left(1 - \beta\right)}\varrho$. Then the coupled system (1.1) is Ulam-Hyers stable. □

4. An Example

Let us consider the following coupled system

$$\begin{cases} D^{\frac{1}{4}}\left(\dfrac{\upsilon_1}{h_1\left(.,\upsilon_1(.),\upsilon_2(g_1(.))\right)}\right)(x) = f_1\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big), \\ \\ D^{\frac{1}{4}}\left(\dfrac{\upsilon_2}{h_2\left(.,\upsilon_1(.),\upsilon_2(g_1(.))\right)}\right)(x) = f_2\Big(x,\upsilon_1(x),\upsilon_2(g_2(x))\Big), \\ \\ x \in I = [0,1], \\ \\ \upsilon_1(a) = \upsilon_2(a) = \lambda_1,\ \upsilon_2\left(g_1(a)\right) = \lambda_2,\ \upsilon = (\upsilon_1\upsilon_2), \end{cases} \tag{4.1}$$

where $x \in [0\ 1], y, z \in \mathbb{R}$,

$$f_1(x,y,z) = \frac{e^{-3x}}{100}(y+z-0.05)$$

$$f_2(x,y,z) = \frac{\sin(x)}{x^2+100}\left(\frac{y^2}{y^2+1}+\frac{z^2}{z^2+1}-0.02\right)$$

$$h_1(x,y,z) = \frac{y+0.01}{2\ln(x+1)+200}+\frac{ze^{-x}}{x+200}$$

$$h_2(x,y,z) = \frac{1}{x^2+200}(x\sin(y)+z+0.035),$$

$$g_1(x) = \frac{|x|}{|x|+1}$$

$$g_2(x) = e^{-x}.$$

For all $x \in [0\ 1]$ and for all $y, y', z, z' \in \mathbb{R}$, we have:

(P_1)

$$\left|h_1(x,y,z) - f_1\left(x,y',z'\right)\right| \leq \frac{1}{200}\left(|y-y'|+|z-z'|\right),\ \text{ then }\ k_1 = \frac{1}{200},$$

$$\left|h_2(x,y,z) - f_1\left(x,y',z'\right)\right| \leq \frac{1}{200}\left(|y-y'|+|z-z'|\right),\ \text{ then }\ k_2 = \frac{1}{200},$$

(P_2) For $0 < \lambda_1 \leq 0.035$ and $\lambda_2 = 400\lambda_1$, there exists $\theta = 2\lambda_1 + \dfrac{\lambda_1}{200}$ such that

$$\left|h_1(0,\lambda_1,\lambda_2)\right| \geq \theta \geq 2\lambda_1$$

and

$$\left|h_2(0,\lambda_1,\lambda_2)\right| \geq \theta \geq 2\lambda_1$$

(P_3)

$$\left|f_1(x,y,z)-f_1\left(x,y',z'\right)\right| \leq \frac{1}{100}\left(|y-y'|+|z-z'|\right),\quad \text{then}\quad q_1=\frac{1}{100},$$

$$\left|f_2(x,y,z)-f_2\left(x,y',z'\right)\right| \leq \frac{1}{25}\left(|y-y'|+|z-z'|\right),\quad \text{then}\quad q_2=\frac{1}{25},$$

$$\delta_1=\frac{0.01}{200},\ \delta_2=\frac{0.035}{200},$$

(P_4)

$$\kappa_1=\frac{0.05}{100},\ \kappa_2=\frac{0.02}{100}.$$

$$\begin{cases}\tau=\frac{1}{4},\quad \alpha=\frac{3}{4},\quad \Theta=25.20,\quad A_1=0.252,\quad A_2=1.008,\quad \frac{\lambda_1}{\theta}=0.49,\\ B_1=0.502,\quad B_2=0.495,\quad \xi=0.00516,\quad \gamma=0.0063,\quad \nu=0.000111,\\ \sigma=6.99\times10^{-7}<\frac{1}{4},\ 1-2\sqrt{\sigma}=0.998,\ \frac{\theta-2\lambda_1}{\Theta\theta}=9.89\times10^{-5},\ X_1=0.000004,\ X_2=0.00000925.\end{cases}$$

Hence

$$0<\xi\leq 1-2\sqrt{\sigma}$$

and

$$X_1+X_2=1.32\times10^{-5}\leq\frac{\theta-2\lambda_1}{\Theta\theta}.$$

Therefore, all assumptions of Theorem 3.4 are satisfied which implies that the coupled system (1.1) has a unique solution and it is Ulam-Hyers stable.

Remark 4.1. Since for all $x\in[0\ 1]$, we have $f_i(x,0,0)\neq 0$. Then, the unique solution of the coupled system (1.1) is nontrivial.

5. Conclusions

In this paper, we investigate the existence and uniqueness of solution for a particular coupled system, namely, coupled system of two generalized hybrid-type pantograph equations involving deformable. The novelty of the manuscript lies in the fact that it combines three notions in the same problem: A coupled system, generalized hybrid pantograph equation, and deformable derivative. The study of the existence and uniqueness of solutions and Ulam stability for such problems has not been mentioned before. We use the Banach contraction principle to prove our results.

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Conflict of interest

The authors declare that they have no competing interests.

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