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Research article

Some algebraic invariants of the edge ideals of perfect $[h, d]$-ary trees and some unicyclic graphs

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Abstract: This article is mainly concerned with computations of some algebraic invariants of quotient rings of edge ideals of perfect $[h, d]$-ary trees and unicyclic graphs. We compute exact values of depth and Stanley depth and consequently projective dimension for above mentioned quotient rings, except for the one special case of unicyclic graph for which best possible bounds of Stanley depth are given.

Keywords: depth; edge ideal; Stanley depth; projective dimension; perfect $[h, d]$-ary tree; unicyclic graph
Mathematics Subject Classification: 05C38, 05E99, 13C15, 13F20

1. Introduction

Let K be a field and $S = K[x_1, x_2, \dots, x_d]$ be the polynomial ring over K with standard grading, that is, $\deg(x_i) = 1$, for all i. Let M be a finitely generated graded S-module. Suppose that M admits the following minimal free resolution:

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p,j}(M)} \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p-1,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

Let pdim(M) denotes the *projective dimension* of M. Then

$$\operatorname{pdim}(M) = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}.$$

Depth is an algebraic invariant of module M denoted by depth(M) and is defined as the common length of maximal regular sequences on M in the ideal $\mathbf{m} = (x_1, x_2, \dots, x_d)$, where $\mathbf{m}$ is the unique graded maximal ideal of S. If M is a finitely generated $\mathbb{Z}^d$ graded module over the $\mathbb{Z}^d$ graded ring S, then for a homogeneous element $u \in M$ and a subset $W \subset \{x_1, x_2, \dots, x_d\}$, $uK[W]$ denotes the K-subspace of

M generated by all homogeneous elements of the form uv, where v is a monomial in $K[W]$. Such a linear K-subspace $uK[W]$ is called a Stanley space of dimension $|W|$ if it is a free $K[W]$-module, where $|W|$ denotes the number of indeterminates in W. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{s} a_i K[W_i].$$

The Stanley depth of decomposition $\mathcal{D}$ is $\mathrm{sdepth}(\mathcal{D}) = \min\{|W_i| : i = 1, 2, \dots, s\}$. The Stanley depth of M is

$$\mathrm{sdepth}(M) = \max\{\mathrm{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

We refer the readers to [1] for a detailed introduction to Stanley depth. In 1982 Stanley conjectured in [2] that, if M is a finitely generated $\mathbb{Z}^d$-graded S-module, then $\mathrm{sdepth}(M) \geq \mathrm{depth}(M)$. Ichim et al. gave an algorithm for computing Stanley depth in [3]. Let $I \subset J$ be monomial ideal of S, then Ichim et al. [4] reduced the Stanley's conjecture for the module of the type J/I to the case when I and J are square free monomial ideals. This conjecture was proved for some special classes of modules; see for instance [5–7]. However, this conjecture was later disproved by Duval et al. in [8]. For some other interesting results on Stanley depth we refer the readers to [9–12].

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{x_1, x_2, \dots, x_d\}$ and edge set $E(G)$. All graphs considered in this paper are simple and undirected. The *degree* of a vertex in a graph G is the number of edges incident to that vertex. A vertex of a graph is a *pendant* vertex if it is of degree one. An edge ideal of a graph G is a square free monomial ideal of polynomial ring S, that is, $I(G) = (x_i x_j : \{x_i, x_j\} \in E(G))$. Several papers have been written on depth and Stanley depth of $S/I(G)$; see for instance [13–15]. A graph is known as a path if it is a sequence of vertices such that each vertex in the sequence is adjacent to the vertex next to it. A *tree* is a graph in which there is a unique path between any two vertices of it. A tree in which one vertex is selected as a root vertex and all other vertices are directed away from it, is known as a *rooted tree*. A vertex of tree is known as *parent*, if at least one edge is directed from it to another vertex of a tree. A vertex to which edge is directed from the parent is known as a *child*. A tree is called an *h-ary tree* if each of its vertex has at most h children. An *h-ary tree* is called a *perfect h-ary tree* of height t if each of its parent vertex has h children, and all non parent vertices are at distance t from the root vertex. B. Shaukat et al. [16] computed formulas for the values of depth, Stanley depth and projective dimension of residue class rings of the edge ideals of perfect h-ary trees.

In this article we define a tree which we call a perfect $[h, d]$-ary tree and a unicyclic graph closely related to the perfect $[h, d]$-ary tree. We compute exact values of depth, Stanley depth and projective dimension for quotient rings associated to the edge ideals of perfect $[h, d]$-ary trees and quotient rings associated to the edge ideals of considered unicyclic graphs, except for the one special case of unicyclic graphs in which we compute tight bounds for the Stanley depth. For the values of depth see Theorem 3.4, 3.7, 4.2 and 4.5, and for the results related to Stanley depth see Theorem 3.5, 3.8, 4.3 and 4.6.

2. Preliminaries

Here we discuss some of the terminologies of Graph Theory. A graph is called *simple* if it is loopless and contains no multiple edges. Let $n \geq 1$, if P_d represents a path graph on d vertices say $x_1, x_2, \ldots, x_d$, then $E(P_d) = \bigcup_{i=1}^{d-1}\{\{x_i, x_{i+1}\}\}$ (if $d = 1$, then $E(P_1) = \emptyset$ and $I(P_1) = (0)$). A graph denoted by C_d with edge set $E(C_d) = \bigcup_{i=1}^{d-1}\{\{x_i, x_{i+1}\}\} \bigcup \{x_1, x_d\}$ is called a *cycle* on d vertices. The vertices x_1 and x_2 in a graph G are said to be *fused* or *merged* or *identified* if x_1 and x_2 are replaced by a single new vertex x, such that, every edge that was adjacent to either x_1 or x_2 or both, is adjacent to x. A path with end vertices x_i and x_j is known as $x_i x_j$*-path*. A graph G is *connected* if it has $x_i x_j$-path for each $x_i, x_j \in V(G)$. A subgraph of a graph G is *maximal connected* if it is connected and is not contained in any other connected subgraph of G. A maximal connected subgraph of a graph G is called its *component*. For vertices x_i and x_j of a graph G, the length of a shortest path from x_i to x_j is called the distance between x_i and x_j denoted by $d_G(x_i, x_j)$. If no such path exists between x_i and x_j, then $d_G(x_i, x_j) = \infty$. As defined earlier a tree is a graph in which there is a unique path between any two vertices of it. A *forest* is a graph whose all components are trees. A *caterpillar* is a tree in which all the vertices are at distance at most one from the central path. A *lobster* graph is a tree in which all the vertices are at distance at most two from the central path.

If $h = 1$, then a perfect h-ary tree is a path, and we designate one of its pendant vertex as a root. If $h \geq 2$, then there is only one vertex of degree h in a perfect h-ary tree, and we designate that unique vertex as a root. Let $h \geq 2$, $d \geq 1$ and $H_1, H_2, \ldots, H_d$ be perfect $(h-1)$-ary trees such that $H_i \cong H_j$ for all i and j. Let $x_1, x_2, \ldots, x_d$ be the root vertices of $H_1, H_2, \ldots, H_d$, respectively, and $y_1, y_2, \ldots, y_d$ be the vertices of the path P_d. Let $\mathscr{H}$ be a forest with $d + 1$ components $H_1, H_2, \ldots, H_d$ and P_d. If we fuse the vertex x_i with y_i for all $i \in \{1, 2, \ldots, d\}$ in $\mathscr{H}$, then we get a tree which we call a *perfect* $[h, d]$*-ary tree*. If t is the common height of all perfect $(h-1)$-ary trees in $\mathscr{H}$, then we denote such a perfect $[h, d]$-ary tree by $\mathscr{P}_{h,t,d}$. See Figures 1 and 2 for an example of $\mathscr{P}_{h,t,d}$. Now let $d \geq 3$. If we add an edge between the vertices x_1 and x_d of $\mathscr{P}_{h,t,d}$, then we get a unicyclic graph and we denote this unicyclic graph by $\mathscr{C}_{h,t,d}$.

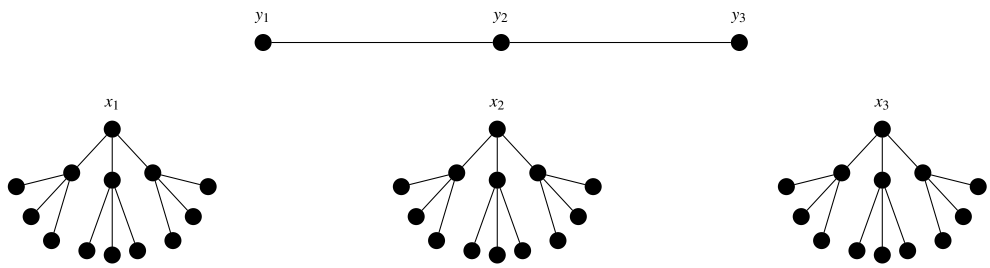

Figure 1. A forest consisting of three perfect 3-ary trees and P_3.

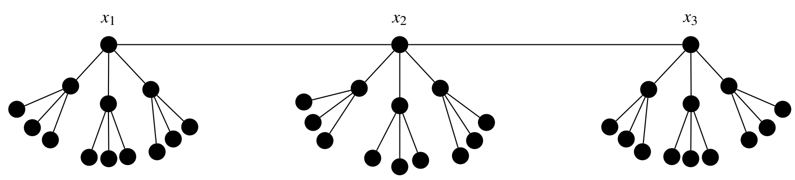

Figure 2. A perfect $[4, 3]$-ary tree.

See Figures 3 and 4 for examples and labeling of vertices of $\mathscr{P}_{h,t,d}$ and $\mathscr{C}_{h,t,d}$.

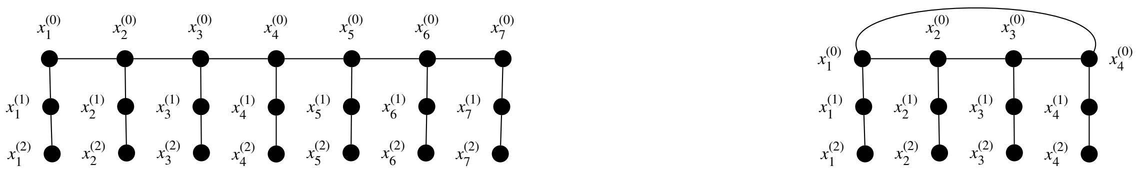

Figure 3. $\mathscr{P}_{2,2,7}$ and $\mathscr{C}_{2,2,4}$, respectively.

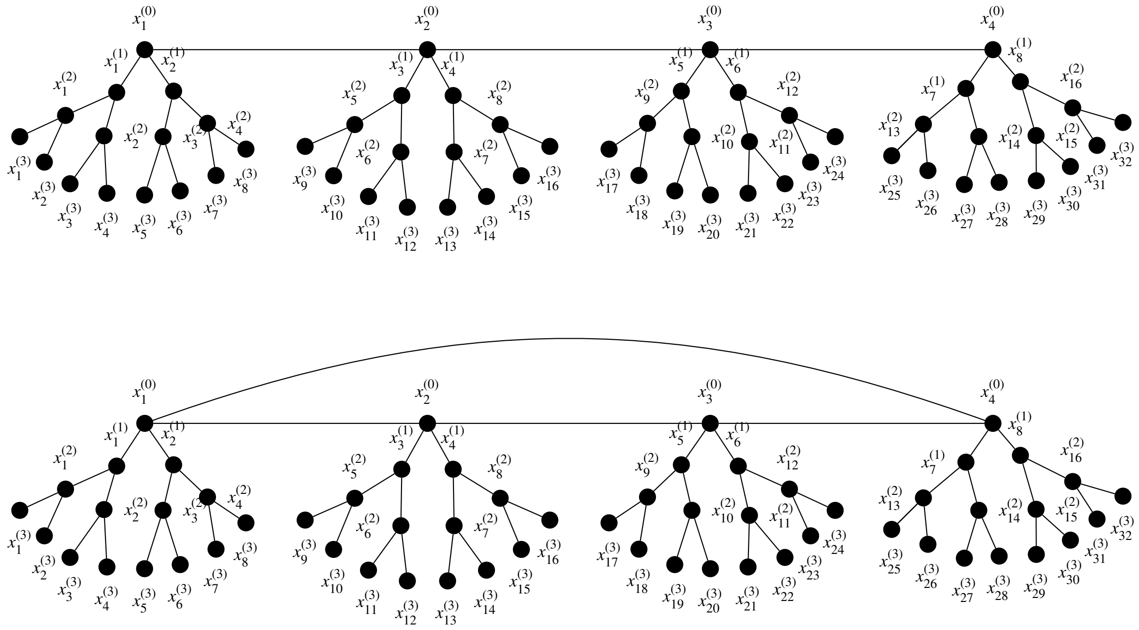

Figure 4. $\mathscr{P}_{3,3,4}$ and $\mathscr{C}_{3,3,4}$, respectively.

Remark 2.1. *Note that*

(1) If $h \geq 2$, $t \geq 1$, *and* $d = 1$, *then* $\mathscr{P}_{h,t,1}$ *is a perfect* $(h-1)$*-ary tree.*

(2) If $h \geq 2$, $t = 1$ *and* $d \geq 1$, *then* $\mathscr{P}_{h,1,d}$ *belongs to the class of caterpillar trees.*

(3) If $h \geq 2$, $t = 2$ *and* $d \geq 1$, *then* $\mathscr{P}_{h,2,d}$ *is a special class of lobster trees.*

Now we state some results that will be used frequently throughout this article.

Lemma 2.2. *Let* $I = I(P_d) \subseteq S = K[x_1, x_2, \dots, x_d]$ *be an ideal of* S, *then*

(1) $\operatorname{depth}(S/I) = \lceil \frac{d}{3} \rceil$ *([17, Lemma 2.8]).*

(2) $\operatorname{sdepth}(S/I) = \lceil \frac{d}{3} \rceil$ *([18, Lemma 4]).*

Proposition 2.3. [19, Proposition 1.3 and 1.8] *Let* $I = I(C_d) \subseteq S = K[x_1, x_2, \dots, x_d]$ *be an ideal of* S, *then* $\operatorname{depth}(S/I) = \lceil \frac{d-1}{3} \rceil$ *and* $\operatorname{sdepth}(S/I) \geq \lceil \frac{d-1}{3} \rceil$.

Lemma 2.4. [20, Theorem 2.6] *Let $h \geq 2$, $t = 1$ and $d = 1$. If $S = K[V(\mathscr{P}_{h,1,1})]$, then* $\operatorname{depth}(S/I(\mathscr{P}_{h,1,1})) = \operatorname{sdepth}(S/I(\mathscr{P}_{h,1,1})) = 1$.

Theorem 2.5. [21, Theorem 3.5 and 3.6] *Let $S = K[V(\mathscr{P}_{h,2,d})]$, $h \geq 2$, $t = 2$ and $d \geq 1$, then* $\operatorname{depth}(S/I(\mathscr{P}_{h,2,d})) = \operatorname{sdepth}(S/I(\mathscr{P}_{h,2,d})) = (h-1)d$.

Theorem 2.6. [21, Theorem 4.3 and 4.4] *Let $S = K[V(\mathscr{C}_{h,2,d})]$, $h \geq 2$, $t = 2$ and $d \geq 3$, then* $\operatorname{depth}(S/I(\mathscr{C}_{h,2,d})) = \operatorname{sdepth}(S/I(\mathscr{C}_{h,2,d})) = (h-1)d$.

Proposition 2.7. [16, Proposition 2.2] *Let $h \geq 3$ and $t \geq 1$. If $S = K[V(\mathscr{P}_{h,t,1})]$, then*

$$\operatorname{depth}(S/I(\mathscr{P}_{h,t,1})) = \operatorname{sdepth}(S/I(\mathscr{P}_{h,t,1})) = \begin{cases} \frac{(h-1)^2((h-1)^t-1)}{(h-1)^3-1} + 1, & if\ \ t \equiv 0 (\mathrm{mod}\, 3); \\ \frac{(h-1)^{t+2}-1}{(h-1)^3-1}, & if\ \ t \equiv 1 (\mathrm{mod}\, 3); \\ \frac{(h-1)^{t+2}-h+1}{(h-1)^3-1}, & if\ \ t \equiv 2 (\mathrm{mod}\, 3). \end{cases}$$

Theorem 2.8. [22, Theorem 1.3.3] *(Auslander–Buchsbaum formula) Let R be a commutative Noetherian local ring and M be a non-zero finitely generated R-module. If* $\operatorname{pdim}(M) < \infty$*, then*

$$\operatorname{pdim}(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Now we give some results that play important role while proving our main results of this article.

Lemma 2.9. [23, Lemma 2.2] *Let $0 \to R_1 \to R_2 \to R_3 \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^d$-graded $S-$modules, then* $\operatorname{sdepth}(R_2) \geq \min\{\operatorname{sdepth}(R_1), \operatorname{sdepth}(R_3)\}$.

Proposition 2.10. [24, Proposition 2.7] *Let $J \subset S$ be a monomial ideal and w be a monomial such that $w \notin J$, then* $\operatorname{sdepth}(S/(J : w)) \geq \operatorname{sdepth}(S/J)$.

Lemma 2.11. [22, Proposition 1.2.9] *(Depth Lemma) Let $0 \to R_1 \to R_2 \to R_3 \to 0$ be a short exact sequence of $\mathbb{Z}^d$-graded S-modules, then*

(1) $\operatorname{depth}(R_2) \geq \min\{\operatorname{depth}(R_1), \operatorname{depth}(R_3)\}$.

(2) $\operatorname{depth}(R_1) \geq \min\{\operatorname{depth}(R_2), \operatorname{depth}(R_3) + 1\}$.

(3) $\operatorname{depth}(R_3) \geq \min\{\operatorname{depth}(R_1) - 1, \operatorname{depth}(R_2)\}$.

Remark 2.12. Let $J \subset S$ be a monomial ideal. Then for $x \notin J$, the short exact sequence

$$0 \longrightarrow S/(J : x) \xrightarrow{\cdot x} S/J \longrightarrow S/(J, x) \longrightarrow 0$$

implies that

(1) $\operatorname{depth}(S/J) \geq \min\{\operatorname{depth}(S/(J : x)), \operatorname{depth}(S/(J, x))\}$,

(2) $\operatorname{sdepth}(S/J) \geq \min\{\operatorname{sdepth}(S/(J : x)), \operatorname{sdepth}(S/(J, x))\}$.

Moreover, if $\operatorname{depth}(S/(J : x)) \leq \operatorname{depth}(S/(J, x))$, then by Depth Lemma $\operatorname{depth}(S/J) = \operatorname{depth}(S/(J : x))$.

The following results will be used frequently in our proofs and we will not be referring it again and again.

Lemma 2.13. [25, Lemma 3.6] *Let* $I \subset S = K[x_1, x_2, \dots, x_d]$ *be a monomial ideal. If* $S' = S \otimes_K K[x_{d+1}] \cong S[x_{d+1}]$, *then* $\operatorname{depth}(S'/IS') = \operatorname{depth}(S/I) + 1$ *and* $\operatorname{sdepth}(S'/IS') = \operatorname{sdepth}(S/I) + 1$.

Lemma 2.14. [26, Lemma 2.12 and 2.13] *Let* $J_1 \subset S_1 = K[x_1, x_2, \dots, x_t]$ *and* $J_2 \subset S_2 = K[x_{t+1}, x_{t+2}, \dots, x_d]$ *be monomial ideals where* $1 \le t < d$. *If* $S = S_1 \otimes_K S_2$, *then*

(1) $\operatorname{depth}_S(S_1/J_1 \otimes_K S_2/J_2) = \operatorname{depth}_S(S/(J_1S + J_2S)) = \operatorname{depth}_{S_1}(S_1/J_1) + \operatorname{depth}_{S_2}(S_2/J_2)$.

(2) $\operatorname{sdepth}_S(S_1/J_1 \otimes_K S_2/J_2) \ge \operatorname{sdepth}_{S_1}(S_1/J_1) + \operatorname{sdepth}_{S_2}(S_2/J_2)$.

The following lemma is proved by B. Shaukat et al. [27]. Since the paper is not yet published, we present here a short proof for the sake of completeness.

Lemma 2.15. [27] *Let* $h \ge 2$, $t = 1$.

(1) If $d \ge 1$ *and* $S = K[V(\mathscr{P}_{h,1,d})]$, *then*

$$\operatorname{depth}(S/I(\mathscr{P}_{h,1,d})) = \operatorname{sdepth}(S/I(\mathscr{P}_{h,1,d})) = d + (h-2)\lceil \frac{d-1}{2} \rceil.$$

(2) If $d \ge 3$ *and* $S = K[V(\mathscr{C}_{h,1,d})]$, *then*

$$\operatorname{depth}(S/I(\mathscr{C}_{h,1,d})) = \operatorname{sdepth}(S/I(\mathscr{C}_{h,1,d})) = d + (h-2)\lceil \frac{d}{2} \rceil.$$

Proof. (1) Result is proved by induction on d. If $d = 1$, then the result follows from Lemma 2.4. Now let $d \ge 2$. We have

$$S/(I(\mathscr{P}_{h,1,d}) : x_d^{(0)}) \cong K[V(\mathscr{P}_{h,1,d-2})]/I(\mathscr{P}_{h,1,d-2}) \otimes_k K[x^{(1)}_{(d-2)(h-1)+1}, \dots, x^{(1)}_{(d-1)(h-1)}, x_d^{(0)}]$$

(if $d = 2$, $K[V(\mathscr{P}_{h,1,d-2})]/I(\mathscr{P}_{h,1,d-2}) \cong K$) and

$$S/(I(\mathscr{P}_{h,1,d}), x_d^{(0)}) \cong K[V(\mathscr{P}_{h,1,d-1})]/I(\mathscr{P}_{h,1,d-1}) \otimes_k K[x^{(1)}_{(d-1)(h-1)+1}, \dots, x^{(1)}_{(d)(h-1)}].$$

By Lemma 2.13 we have,

$$\begin{aligned}\operatorname{depth}(S/(I(\mathscr{P}_{h,1,d}) : x_d^{(0)})) = &\operatorname{depth}(K[V(\mathscr{P}_{h,1,d-2})]/I(\mathscr{P}_{h,1,d-2})) \\ &+ \operatorname{depth}(K[x^{(1)}_{(d-2)(h-1)+1}, \dots, x^{(1)}_{(d-1)(h-1)}, x_d^{(0)}])\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(S/(I(\mathscr{P}_{h,1,d}), x_d^{(0)})) = &\operatorname{depth}(K[V(\mathscr{P}_{h,1,d-1})]/I(\mathscr{P}_{h,1,d-1})) \\ &+ \operatorname{depth}(K[x^{(1)}_{(d-1)(h-1)+1}, \dots, x^{(1)}_{(d)(h-1)}]).\end{aligned}$$

Then again by Lemma 2.13 and induction, we have

$$\operatorname{depth}(S/(I(\mathscr{P}_{h,1,d}) : x_d^{(0)})) = (d-2) + (h-2)\lceil \frac{d-3}{2} \rceil + h = d + (h-2)\lceil \frac{d-1}{2} \rceil$$

and

$$\operatorname{depth}(S/(I(\mathscr{P}_{h,1,d}), x_d^{(0)})) = (d-1) + (h-2)\lceil \frac{d-2}{2} \rceil + h - 1 = d + (h-2)\lceil \frac{d}{2} \rceil.$$

Hence by Remark 2.12 we have $\operatorname{depth}(S/I(\mathscr{P}_{h,1,d})) = d + (h-2)\lceil \frac{d-1}{2} \rceil$. Now we discuss the proof for Stanley depth. For lower bound we use the same arguments as used in the proof of depth and we use Lemma 2.9 instead of Lemma 2.11. And we compute the desired upper bound by using Proposition 2.10.

(2) The proof of this result is similar to the proof given above and it involves the use of values of depth and Stanley depth of $S/I(\mathscr{P}_{h,1,d})$ which are computed above. □

3. Depth, Stanley depth and projective dimension of cyclic modules associated to perfect $[h,d]$-ary trees

In this section we compute depth, Stanley depth and projective dimension of the quotient ring $K[V(\mathscr{P}_{h,t,d})]/I(\mathscr{P}_{h,t,d})$. We also prove that the depth and Stanley depth are equal.

Remark 3.1. In this remark we introduce some terms that appear in some special cases of our proofs when we use induction.

(1) If $d = 0$, we define $K[V(\mathscr{P}_{h,t,0})]/I(\mathscr{P}_{h,t,0}) := K$, hence

$$\operatorname{depth}(K[V(\mathscr{P}_{h,t,0})]/I(\mathscr{P}_{h,t,0})) = \operatorname{sdepth}(K[V(\mathscr{P}_{h,t,0})]/I(\mathscr{P}_{h,t,0})) = 0.$$

(2) If $d \geq 2$, we define $K[V(\mathscr{P}_{h,0,d})]/I(\mathscr{P}_{h,0,d}) := K[V(P_d)]/I(P_d)$, hence by Lemma 2.2

$$\operatorname{depth}(K[V(\mathscr{P}_{h,0,d})]/I(\mathscr{P}_{h,0,d})) = \operatorname{sdepth}(K[V(\mathscr{P}_{h,0,d})]/I(\mathscr{P}_{h,0,d})) = \lceil \frac{d}{3} \rceil.$$

Remark 3.2. While doing the computations of depth and Stanley depth for the quotient ring associated to edge ideals of both perfect $[h,d]$-ary trees and a unicyclic graph, it was observed that the patterns of computed values were different for $h = 2$ and $h \geq 3$. So our results are based on two cases of h, that is for $h = 2$ and $h \geq 3$. Further, these two cases are classified on the basis of variable t.

Remark 3.3. Let $I \subset S$ be a squarefree monomial ideal whose minimal generating set comprises of monomials of degree at most two. We associate a graph G_I to an ideal I with vertex set and edge set defined as $V(G_I) = \operatorname{supp}(I)$ and $E(G_I) = \{\{x_a, x_b\} : x_a x_b \in G(I)\}$, respectively. We know that for $x_i \notin I$, $(I : x_i)$ and (I, x_i) are also squarefree monomial ideals minimally generated by monomials of degree at most two. The graphs $G_{(I:x_i)}$ and $G_{(I,x_i)}$ are subgraphs of graph G_I. See Figure 5 for examples of $G_{(I(\mathscr{P}_{3,3,3}):x_3^{(0)})}$ and $G_{(I(\mathscr{P}_{3,3,3}),x_3^{(0)})}$ that are subgraphs of $G_{I(\mathscr{P}_{3,3,3})}$. By using the structures of these subgraphs we have the following isomorphisms:

$$S/(I(\mathscr{P}_{3,3,3}) : x_3^{(0)}) \cong K[V(\mathscr{P}_{3,3,1})]/I(\mathscr{P}_{3,3,1}) \otimes_K \bigotimes_{i=1}^{2}{}_K K[V(\mathscr{P}_{3,2,1})]/I(\mathscr{P}_{3,2,1}) \otimes_K \bigotimes_{i=1}^{4}{}_K K[V(\mathscr{P}_{3,1,1})]/I(\mathscr{P}_{3,1,1}) \otimes_K K[x_3^{(0)}]$$

and

$$S/(I(\mathscr{P}_{3,3,3}), x_3^{(0)}) \cong K[V(\mathscr{P}_{3,3,2})]/I(\mathscr{P}_{3,3,2}) \otimes_K \bigotimes_{i=1}^{2}{}_K K[V(\mathscr{P}_{3,2,1})]/I(\mathscr{P}_{3,2,1}).$$

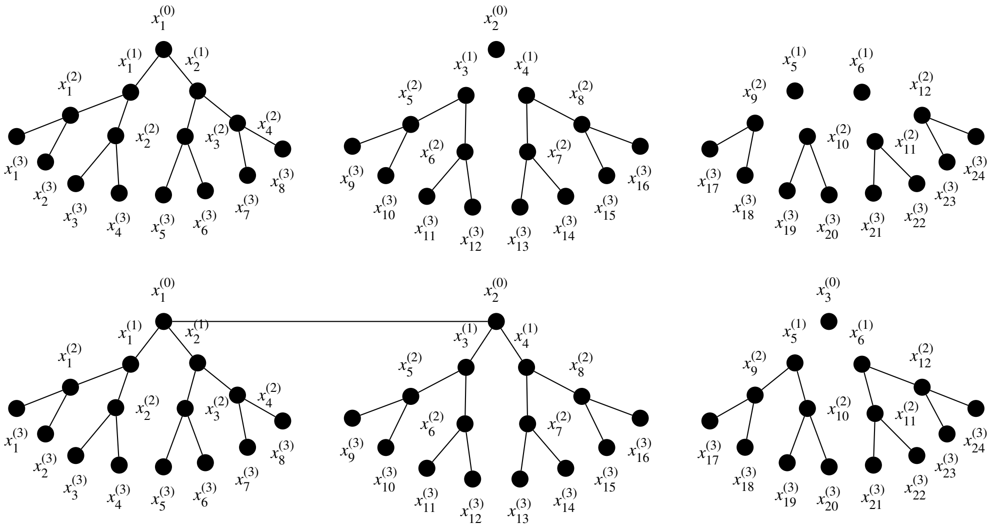

Figure 5. $\mathscr{G}_{(I(\mathscr{P}_{3,3,3}):x_3^{(0)})}$ and $\mathscr{G}_{(I(\mathscr{P}_{3,3,3}),x_3^{(0)})}$, respectively.

Theorem 3.4. *Let $t \geq 1$ and $d \geq 1$. If $S = K[V(\mathscr{P}_{2,t,d})]$, then*

$$\operatorname{depth}(S/I(\mathscr{P}_{2,t,d})) = \begin{cases} \frac{dt}{3} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\operatorname{mod} 3); \\ \lceil \frac{t}{3} \rceil \cdot d, & t \equiv 1, 2 (\operatorname{mod} 3). \end{cases}$$

Proof. We will prove this result by induction on d. For $d = 1$ and $d = 2$, $K[V(\mathscr{P}_{2,t,1})]/I(\mathscr{P}_{2,t,1}) \cong K[V(P_{t+1})]/I(P_{t+1})$ and $K[V(\mathscr{P}_{2,t,2})]/I(\mathscr{P}_{2,t,2}) \cong K[V(P_{2(t+1)})]/I(P_{2(t+1)})$, respectively. Then by Lemma 2.2, $\operatorname{depth}(S/I(\mathscr{P}_{2,t,1})) = \lceil \frac{t+1}{3} \rceil$ and $\operatorname{depth}(S/I(\mathscr{P}_{2,t,2})) = \lceil \frac{2(t+1)}{3} \rceil$, as desired. Now let $d \geq 3$. For $t = 1$ the result follows from Lemma 2.15 and for $t = 2$ the result follows from Theorem 2.5. Now let $t \geq 3$, we have the following cases:

(1) Let $t \equiv 0 (\operatorname{mod} 3)$. We have

$$\begin{aligned} S/(I(\mathscr{P}_{2,t,d}) : x_{d-1}^{(0)}) \cong & K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3}) \otimes_K \bigotimes_{i=1}^{2}{}_K\, K[V(P_t)]/I(P_t) \\ & \otimes_K K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[x_{d-1}^{(0)}] \end{aligned}$$

and

$$\begin{aligned} S/(I(\mathscr{P}_{2,t,d}), x_{d-1}^{(0)}) \cong & K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2}) \otimes_K K[V(P_{t+1})]/I(P_{t+1}) \\ & \otimes_K K[V(P_t)]/I(P_t). \end{aligned}$$

Then by induction, Lemma 2.14 and Lemma 2.2 we have

$$\begin{aligned} \operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_{d-1}^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3})) + 1 \\ &\quad + \sum_{i=1}^{2} \operatorname{depth}(K[V(P_t)]/I(P_t)) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) \\ &= \frac{t}{3} \cdot (d-3) + \lceil \frac{d-3}{3} \rceil + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t-1}{3} \rceil + 1 \end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}), x_{d-1}^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2})) \\ &\quad + \operatorname{depth}(K[V(P_{t+1})]/I(P_{t+1})) + \operatorname{depth}(K[V(P_t)]/I(P_t)) \\ &= \frac{t}{3}\cdot(d-2) + \lceil\frac{d-2}{3}\rceil + \lceil\frac{t+1}{3}\rceil + \lceil\frac{t}{3}\rceil.\end{aligned}$$

Since $t \equiv 0(\bmod 3)$ implies $\lceil\frac{t-1}{3}\rceil = \lceil\frac{t}{3}\rceil = (\frac{t}{3})$ and $\lceil\frac{t+1}{3}\rceil = \lceil\frac{t}{3}\rceil + 1$. Thus we have $\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_{d-1}^{(0)})\right) = \frac{dt}{3} + \lceil\frac{d}{3}\rceil$ and $\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}), x_{d-1}^{(0)})\right) = \frac{dt}{3} + \lceil\frac{d-2}{3}\rceil + 1 = \frac{dt}{3} + \lceil\frac{d+1}{3}\rceil$. Hence by Remark 2.12 $\operatorname{depth}(S/I(\mathscr{P}_{2,t,d})) = \frac{dt}{3} + \lceil\frac{d}{3}\rceil$.

(2) Let $t \equiv 1(\bmod 3)$. We have

$$\begin{aligned}S/(I(\mathscr{P}_{2,t,d}) : x_d^{(0)}) \cong& K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2}) \otimes_K K[V(P_t)]/I(P_t) \\ &\otimes_K K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[x_d^{(0)}]\end{aligned}$$

and

$$S/(I(\mathscr{P}_{2,t,d}), x_d^{(0)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_t)]/I(P_t).$$

By induction, Lemma 2.14 and Lemma 2.2 we have

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2})) + 1 \\ &\quad + \operatorname{depth}(K[V(P_t)]/I(P_t)) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) \\ &= \lceil\frac{t}{3}\rceil\cdot(d-2) + \lceil\frac{t}{3}\rceil + \lceil\frac{t-1}{3}\rceil + 1,\end{aligned}$$

since $t \equiv 1(\bmod 3)$ implies $\lceil\frac{t-1}{3} + 1\rceil = \lceil\frac{t}{3}\rceil$ so $\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_d^{(0)})\right) = \lceil\frac{t}{3}\rceil\cdot d$ and

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}), x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) + \operatorname{depth}(K[V(P_t)]/I(P_t)) \\ &= \lceil\frac{t}{3}\rceil\cdot(d-1) + \lceil\frac{t}{3}\rceil \\ &= \lceil\frac{t}{3}\rceil\cdot d.\end{aligned}$$

Hence by Remark 2.12 $\operatorname{depth}(S/I(\mathscr{P}_{2,t,d})) = \lceil\frac{t}{3}\rceil\cdot d$.

(3) Let $t \equiv 2(\bmod 3)$. We have

$$S/(I(\mathscr{P}_{2,t,d}) : x_d^{(1)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_{t-2})]/I(P_{t-2}) \otimes_K K[x_d^{(1)}].$$

Now let $J = (I(\mathscr{P}_{2,t,d}), x_d^{(1)})$. We have the following isomorphisms:

$$\begin{aligned}S/(J : x_d^{(0)}) \cong& K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2}) \otimes_K K[V(P_t)]/I(P_t) \\ &\otimes_K K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[x_d^{(0)}]\end{aligned}$$

and

$$S/(J, x_d^{(0)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_{t-1})]/I(P_{t-1}).$$

Then by induction, Lemma 2.14 and Lemma 2.2 we have

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_d^{(1)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}))\\ &\quad + \operatorname{depth}(K[V(P_{t-2})]/I(P_{t-2})) + 1\\ &= \lceil \frac{t}{3} \rceil \cdot (d-1) + \lceil \frac{t-2}{3} \rceil + 1\\ &= \lceil \frac{t}{3} \rceil \cdot d - \lceil \frac{t}{3} \rceil + \lceil \frac{t+1}{3} \rceil,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}\left(S/(J : x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-2})]/I(\mathscr{P}_{2,t,d-2})) + \operatorname{depth}(K[V(P_t)]/I(P_t))\\ &\quad + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) + 1\\ &= \lceil \frac{t}{3} \rceil \cdot (d-2) + \lceil \frac{t}{3} \rceil + \lceil \frac{t-1}{3} \rceil + 1\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(S/(J, x_d^{(0)})) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1}))\\ &= \lceil \frac{t}{3} \rceil \cdot (d-1) + \lceil \frac{t-1}{3} \rceil\\ &= \lceil \frac{t}{3} \rceil \cdot d - \lceil \frac{t}{3} \rceil + \lceil \frac{t+1}{3} - \frac{2}{3} \rceil.\end{aligned}$$

Since $t \equiv 2 (\operatorname{mod} 3)$ implies $\lceil \frac{t+1}{3} \rceil = \lceil \frac{t}{3} \rceil$, $\lceil \frac{t-1}{3} + 1 \rceil = \lceil \frac{t}{3} \rceil + 1$ and $\lceil \frac{t+1}{3} - \frac{2}{3} \rceil = \lceil \frac{t}{3} \rceil$. So we have $\operatorname{depth}\left(S/(I(\mathscr{P}_{2,t,d}) : x_d^{(1)})\right) = \lceil \frac{t}{3} \rceil \cdot d$, $\operatorname{depth}\left(S/(J : x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d + 1$ and $\operatorname{depth}\left(S/(J, x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d$. Thus by Remark 2.12 $\operatorname{depth}(S/J) \geq \lceil \frac{t}{3} \rceil \cdot d$ and also $J = (I(\mathscr{P}_{2,t,d}), x_d^{(1)})$ implies $\operatorname{depth}(S/(I(\mathscr{P}_{2,t,d}), x_d^{(1)})) \geq \lceil \frac{t}{3} \rceil \cdot d$. Hence again by Remark 2.12 $\operatorname{depth}(S/I(\mathscr{P}_{2,t,d})) = \lceil \frac{t}{3} \rceil \cdot d$.

□

Theorem 3.5. *Let $t \geq 1$ and $d \geq 1$. If $S = K[V(\mathscr{P}_{2,t,d})]$, then*

$$\operatorname{sdepth}(S/I(\mathscr{P}_{2,t,d})) = \begin{cases} \frac{dt}{3} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\operatorname{mod} 3);\\ \lceil \frac{t}{3} \rceil \cdot d, & t \equiv 1, 2 (\operatorname{mod} 3).\end{cases}$$

Proof. For $d = 1, 2$ result holds and proof is similar to as done in Theorem 3.4 by using Lemma 2.2. Now let $d \geq 3$. With similar arguments as in Theorem 3.4 and using Lemma 2.14 and Remark 2.12 we get the lower bounds

$$\operatorname{sdepth}(S/I(\mathscr{P}_{2,t,d})) \geq \begin{cases} \frac{dt}{3} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\operatorname{mod} 3);\\ \lceil \frac{t}{3} \rceil \cdot d, & t \equiv 1, 2 (\operatorname{mod} 3).\end{cases}$$

Now we will compute upper bound by induction on t. If $t = 1$, by Lemma 2.15 we have $\mathrm{sdepth}(S/I(\mathscr{P}_{2,t,d})) = d$, as desired. For $t = 2$, by Theorem 2.5 $\mathrm{sdepth}(S/I(\mathscr{P}_{2,t,d})) = d$ satisfies the result. Now let $t \geq 3$ and u be a monomial such that $u := x_1^{(t-1)} x_2^{(t-1)} \dots x_d^{(t-1)}$. We have the following isomorphism:
$S/(I(\mathscr{P}_{2,t,d}) : u) \cong K[V(\mathscr{P}_{2,t-3,d})]/I(\mathscr{P}_{2,t-3,d}) \otimes_K K[x_1^{(t-1)}, x_2^{(t-1)}, \dots, x_d^{(t-1)}]$, then we have

$$\mathrm{sdepth}(S/(I(\mathscr{P}_{2,t,d}) : u)) = \mathrm{sdepth}(K[V(\mathscr{P}_{2,t-3,d})]/I(\mathscr{P}_{2,t-3,d})) + d. \tag{3.1}$$

By Proposition 2.10 we have $\mathrm{sdepth}(S/I(\mathscr{P}_{2,t,d})) \leq \mathrm{sdepth}(S/(I(\mathscr{P}_{2,t,d}) : u))$.

(1) Let $t \equiv 0 (\mathrm{mod}\, 3)$. Since $t - 3 \equiv 0 (\mathrm{mod}\, 3)$, then from Eq (3.1) and by induction on t we have

$$\mathrm{sdepth}(K[V(\mathscr{P}_{2,t-3,d})]/I(\mathscr{P}_{2,t-3,d})) + d = \frac{t-3}{3} \cdot d + \lceil \frac{d}{3} \rceil + d = \frac{dt}{3} + \lceil \frac{d}{3} \rceil.$$

(2) Let $t \equiv 1, 2 (\mathrm{mod}\, 3)$. Since $t - 3 \equiv 1, 2 (\mathrm{mod}\, 3)$, then from Eq (3.1) and by induction on t we have

$$\mathrm{sdepth}(K[V(\mathscr{P}_{2,t-3,d})]/I(\mathscr{P}_{2,t-3,d})) + d = \lceil \frac{t-3}{3} \rceil \cdot d + d = \lceil \frac{t}{3} \rceil \cdot d.$$

□

Corollary 3.6. *Let $h = 2$, $t \geq 1$ and $d \geq 1$. If $S = K[V(\mathscr{P}_{2,t,d})]$, then*

$$\mathrm{pdim}\,(S/I(\mathscr{P}_{2,t,d})) = \begin{cases} (\frac{2t}{3} + 1)d - \lceil \frac{d}{3} \rceil, & t \equiv 0 (\mathrm{mod}\, 3); \\ d(t+1) - \lceil \frac{t}{3} \rceil \cdot d, & t \equiv 1, 2 (\mathrm{mod}\, 3). \end{cases}$$

Proof. We have $|V(\mathscr{P}_{2,t,d})| = d(t+1)$, therefore, $\mathrm{depth}(S) = d(t+1)$. Hence we get the required result by using Theorem 3.4 and Theorem 2.8. □

Theorem 3.7. *Let $h \geq 3$, $t \geq 1$ and $d \geq 1$. If $S = K[V(\mathscr{P}_{h,t,d})]$, then*

$$\mathrm{depth}\,(S/I(\mathscr{P}_{h,t,d})) = \begin{cases} \frac{d\left((h-1)^{t+2} - (h-1)^2\right)}{(h-1)^3 - 1} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\mathrm{mod}\, 3); \\ \frac{d\left((h-1)^{t+2} - 1\right)}{(h-1)^3 - 1} + (h-2)\lceil \frac{d-1}{2} \rceil, & t \equiv 1 (\mathrm{mod}\, 3); \\ \frac{d\left((h-1)^{t+2} - (h-1)\right)}{(h-1)^3 - 1}, & t \equiv 2 (\mathrm{mod}\, 3). \end{cases}$$

Proof. We will prove this result by induction on d. If $d = 1$, the result follows from Proposition 2.7. Now let $d \geq 2$. For $t = 1$ the result follows from Lemma 2.15 and for $t = 2$ the result follows from Theorem 2.5. Now let $t \geq 3$, we consider the following three cases:

(1) Let $t \equiv 0 (\mathrm{mod}\, 3)$. We consider two subcases

(i) $d = 2$.

(ii) $d \geq 3$.

If $d = 2$, then we have the following isomorphisms:

$$S/(I(\mathscr{P}_{h,t,2}) : x_1^{(0)}) \cong \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[x_1^{(0)}]$$

and

$$S/(I(\mathscr{P}_{h,t,2}), x_1^{(0)}) \cong K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}).$$

Now if $d \geq 3$, then we have

$$\begin{aligned} S/(I(\mathscr{P}_{h,t,d}) : x_{d-1}^{(0)}) \cong{}& K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \\ & \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \\ & \otimes_K K[x_{d-1}^{(0)}] \end{aligned}$$

and

$$\begin{aligned} S/(I(\mathscr{P}_{h,t,d}), x_{d-1}^{(0)}) \cong{}& K[V(\mathscr{P}_{h,t,d-2})]/I(\mathscr{P}_{h,t,d-2}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \\ & \otimes_K K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}). \end{aligned}$$

First we prove the result for subcase (ii) and the proof for subcase (i) is similar and will be discussed later. Since $t-1 \equiv 2 (\bmod 3)$ and $t-2 \equiv 1 (\bmod 3)$, then by Lemma 2.14, Proposition 2.7 and induction we have

$$\begin{aligned} &\operatorname{depth}(S/(I(\mathscr{P}_{h,t,d}) : x_{d-1}^{(0)})) \\ &= \operatorname{depth}(K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3})) + \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) \\ &\quad + \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})) + \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) \\ &\quad + \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) + 1 \\ &= \frac{(h-1)^{t+2} - (h-1)^2}{(h-1)^3 - 1} \cdot (d-3) + \lceil \frac{d-3}{3} \rceil + 2(h-1) \cdot \frac{(h-1)^{t-1+2} - (h-1)}{(h-1)^3 - 1} \\ &\quad + (h-1)^2 \cdot \frac{(h-1)^{t-2+2} - 1}{(h-1)^3 - 1} + 1 \\ &= \frac{d((h-1)^{t+2} - (h-1)^2)}{(h-1)^3 - 1} + \lceil \frac{d}{3} \rceil \end{aligned}$$

and

$$
\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{P}_{h,t,d}), x_{d-1}^{(0)}))\\
&\quad= \operatorname{depth}(K[V(\mathscr{P}_{h,t,d-2})]/I(\mathscr{P}_{h,t,d-2})) + \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&\qquad+ \operatorname{depth}(K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}))\\
&\quad= \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\cdot(d-2) + \lceil\frac{d-2}{3}\rceil + (h-1)\cdot\frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1}\\
&\qquad+ \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1} + 1\\
&\quad= \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil\frac{d+1}{3}\rceil.
\end{aligned}
$$

Hence by Remark 2.12 depth $(S/(I(\mathscr{P}_{h,t,d})) = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil\frac{d}{3}\rceil$.

For subcase (i) the proof is similar, therefore we omit the detailed proof. We have the values $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}) : x_1^{(0)})) = \frac{2((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + 1$ and $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}), x_1^{(0)})) = \frac{2((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + 1$. The required result follows by using Remark 2.12.

(2) Let $t \equiv 1(\bmod 3)$. We again consider two subcases that is $d = 2$ and $d \geq 3$.
(i) Let $d = 2$. We have the following isomorphims:

$$
\begin{aligned}
S/(I(\mathscr{P}_{h,t,2}) : x_2^{(0)}) \cong &\bigotimes_{K\, i=1}^{(h-1)} K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{K\, i=1}^{(h-1)^2} K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})\\
&\otimes_K K[x_2^{(0)}]
\end{aligned}
$$

and

$$
S/(I(\mathscr{P}_{h,t,2}), x_2^{(0)}) \cong K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}) \otimes_K \bigotimes_{K\, i=1}^{(h-1)} K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}).
$$

Since $t-1 \equiv 0(\bmod 3)$ and $t-2 \equiv 2(\bmod 3)$, then by Lemma 2.14 and Proposition 2.7

$$
\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}) : x_2^{(0)}))\\
&\quad= \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) + \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})) + 1\\
&\quad= (h-1)\cdot\Big\{\frac{(h-1)^{t-1+2}-(h-1)^2}{(h-1)^3-1} + 1\Big\} + (h-1)^2\cdot\frac{(h-1)^{t-2+2}-(h-1)}{(h-1)^3-1} + 1\\
&\quad= \frac{2((h-2)^{t+2}-1)}{(h-1)^3-1} + (h-2)
\end{aligned}
$$

and

$$\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}), x_2^{(0)})) \\
&= \operatorname{depth}(K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1})) + \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) \\
&= \frac{(h-1)^{t+2}-1}{(h-1)^3-1} + (h-1)\cdot\Big\{\frac{(h-1)^{t-1+2}-(h-1)^2}{(h-1)^3-1}+1\Big\} \\
&= \frac{2((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2).
\end{aligned}$$

Hence by Remark 2.12, depth $(S/(I(\mathscr{P}_{h,t,2})) = \frac{2((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)$, as desired.
(ii) Now let $d \geq 3$. We have the following isomorphisms:

$$\begin{aligned}
S/(I(\mathscr{P}_{h,t,d}) : x_d^{(0)}) \cong & K[V(\mathscr{P}_{h,t,d-2})]/I(\mathscr{P}_{h,t,d-2}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \\
& \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[x_d^{(0)}],
\end{aligned}$$

$$S/(I(\mathscr{P}_{h,t,d}), x_d^{(0)}) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}).$$

Then similarly as done for subcase (i) by using same arguments for each terms of above isomorphisms except for first terms on which induction on d is applied and we have the values $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,d}) : x_d^{(0)})) = \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\cdot\lceil\frac{d-1}{2}\rceil$ and $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,d}), x_d^{(0)})) = \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\cdot\lceil\frac{d}{2}\rceil$. Hence by Remark 2.12 we have depth $(S/I(\mathscr{P}_{h,t,d})) = \frac{d((h-1)^{t+2}-1)}{(h-1)^3} + (h-2)\cdot\lceil\frac{d-1}{2}\rceil$.

(3) Let $t \equiv 2(\operatorname{mod} 3)$.
(i) Let $d = 2$. We have the following isomorphisms:

$$\begin{aligned}
S/(I(\mathscr{P}_{h,t,2}) : x_{2(h-1)}^{(1)}) \cong & K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \\
& \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K [x_{2(h-1)}^{(1)}],
\end{aligned}$$

Now let $J = (I(\mathscr{P}_{h,t,2}), x_{2(h-1)}^{(1)})$. We have

$$S/(J : x_2^{(0)}) \cong \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K [x_2^{(0)}],$$

$$\begin{aligned}
S/(J, x_2^{(0)}) \cong & K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \\
& \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}).
\end{aligned}$$

Since $t-1\equiv 1(\bmod 3)$ and $t-3\equiv 2(\bmod 3)$, then by Lemma 2.14 and Proposition 2.7 we have

$$\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}) : x^{(1)}_{2(h-1)}))\\
&\quad= \operatorname{depth}(K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1})) + \sum_{i=1}^{(h-2)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&\qquad+ \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1})) + 1\\
&\quad= \frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1} + (h-2)\cdot\frac{(h-1)^{t-1+2}-1}{(h-1)^3-1}\\
&\qquad+ (h-1)^2\cdot\frac{(h-1)^{t-3+2}-(h-1)}{(h-1)^3-1} + 1\\
&\quad= \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1},
\end{aligned}$$

$$\begin{aligned}
&\operatorname{depth}(S/(J : x^{(0)}_2))\\
&\quad= \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&\qquad+ \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})) + 1\\
&\quad= (h-1)\cdot\frac{(h-1)^{t-1+2}-1}{(h-1)^3-1} + (h-1)^2\cdot\Big\{\frac{(h-1)^{t-2+2}-(h-1)^2}{(h-1)^3-1}+1\Big\} + 1\\
&\quad= \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1} + (h-1)(h-2) + 1
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{depth}(S/(J, x^{(0)}_2))\\
&\quad= \operatorname{depth}(K[V(\mathscr{P}_{h,t,1})]/I(\mathscr{P}_{h,t,1})) + \sum_{i=1}^{(h-2)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&\qquad+ \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}))\\
&\quad= \frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1} + (h-2)\cdot\frac{(h-1)^{t-1+2}-1}{(h-1)^3-1}\\
&\qquad+ (h-1)\cdot\Big\{\frac{(h-1)^{t-2+2}-(h-1)^2}{(h-1)^3-1}+1\Big\}\\
&\quad= \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1} + (h-2).
\end{aligned}$$

Thus by Remark 2.12 we have $\operatorname{depth}(S/J) \geq \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1})+(h-2)$ and also $J = (I(\mathscr{P}_{h,t,2}), x^{(1)}_{2(h-1)})$ implies $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,2}), x^{(1)}_{2(h-1)})) \geq \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1} + (h-2)$. Hence again by Remark 2.12 we

have, $\operatorname{depth}(S/I(\mathscr{P}_{h,t,2})) = \frac{2((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$.
(ii) Now let $d \geq 3$.
Let $J = (I(\mathscr{P}_{h,t,d}), x_{(h-1)d}^{(1)})$. We have the following isomorphisms:

$$S/(I(\mathscr{P}_{h,t,d}) : x_{(h-1)d}^{(1)}) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K K[x_{(h-1)d}^{(1)}],$$

$$S/(J : x_d^{(0)}) \cong K[V(\mathscr{P}_{h,t,d-2})]/I(\mathscr{P}_{h,t,d-2}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})) \otimes_K K[x_d^{(0)}]$$

and

$$S/(J, x_d^{(0)}) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}).$$

Then similarly as done for $d = 2$ by using the same arguments for each terms of above isomorphisms except for the first terms on which induction on d is applied we get the values $\operatorname{depth}(S/(I(\mathscr{P}_{h,t,d}) : x_{(h-1)d}^{(1)})) = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$, $\operatorname{depth}(S/(J : x_d^{(0)})) = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1} + (h-1)(h-2) + 1$ and $\operatorname{depth}(S/(J, x_d^{(0)})) = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1} + (h-2)$. And so we have $\operatorname{depth}(S/I(\mathscr{P}_{h,t,d})) = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$.

□

Theorem 3.8. *Let $d \geq 1$, $t \geq 1$ and $h \geq 3$. If $S = K[V(\mathscr{P}_{h,t,d})]$, then*

$$\operatorname{sdepth}(S/I(\mathscr{P}_{h,t,d})) = \begin{cases} \frac{d\left((h-1)^{t+2}-(h-1)^2\right)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\bmod 3); \\ \frac{d\left((h-1)^{t+2}-1\right)}{(h-1)^3-1} + (h-2)\lceil \frac{d-1}{2} \rceil, & t \equiv 1 (\bmod 3); \\ \frac{d\left((h-1)^{t+2}-(h-1)\right)}{(h-1)^3-1}, & t \equiv 2 (\bmod 3). \end{cases}$$

Proof. If $d = 1$ the result follows from Proposition 2.7. Now let $d \geq 2$. With similar arguments as in Theorem 3.7 and using Lemma 2.14 and Remark 2.12 we get lower bounds that is

$$\operatorname{sdepth}(S/I(\mathscr{P}_{h,t,d})) \geq \begin{cases} \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil, & t \equiv 0 (\bmod 3); \\ \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d-1}{2} \rceil, & t \equiv 1 (\bmod 3); \\ \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}, & t \equiv 2 (\bmod 3). \end{cases}$$

Now we will compute upper bound by induction on t. If $t = 1$, by Lemma 2.15

$$\operatorname{sdepth}(S/I(\mathscr{P}_{h,t,d})) = d + (h-2)\lceil \frac{d-1}{2} \rceil.$$

If $t = 2$, by Theorem 2.5 $\operatorname{sdepth}(S/I(\mathscr{P}_{h,t,d})) = (h-1)(d)$, as desired. Now let $t \geq 3$ and u be a monomial such that $u := x_1^{(t-1)}x_2^{(t-1)}\dots x_{d(h-1)^{t-1}}^{(t-1)}$. We have the following isomorphism:

$$S/(I(\mathscr{P}_{h,t,d}):u) \cong K[V(\mathscr{P}_{h,t-3,d})]/I(\mathscr{P}_{h,t-3,d}) \otimes_K K[x_1^{(t-1)}, x_2^{(t-1)}, \dots, x_{d(h-1)^{t-1}}^{(t-1)}],$$

then we have

$$\operatorname{sdepth}(S/(I(\mathscr{P}_{h,t,d}):u)) = \operatorname{sdepth}(K[V(\mathscr{P}_{h,t-3,d})]/I(\mathscr{P}_{h,t-3,d})) + d(h-1)^{t-1}. \tag{3.2}$$

By Proposition 2.10 we have $\operatorname{sdepth}(S/I(\mathscr{P}_{h,t,d})) \leq \operatorname{sdepth}(S/(I(\mathscr{P}_{h,t,d}):u))$.

(1) Let $t \equiv 0 (\bmod 3)$. Since $t-3 \equiv 0 (\bmod 3)$ thus by Eq (3.2) and induction on t, $\operatorname{sdepth}(K[V(\mathscr{P}_{h,t-3,d})]/I(\mathscr{P}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2-3}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil$.

(2) Let $t \equiv 1 (\bmod 3)$. Since $t-3 \equiv 1 (\bmod 3)$ thus by Eq (3.2) and induction on t we have, $\operatorname{sdepth}(K[V(\mathscr{P}_{h,t-3,d})]/I(\mathscr{P}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2-3}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d-1}{2} \rceil + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d-1}{2} \rceil$.

(3) Let $t \equiv 2 (\bmod 3)$. Since $t-3 \equiv 2 (\bmod 3)$ thus by Eq (3.2) and induction on t we have, $\operatorname{sdepth}(K[V(\mathscr{P}_{h,t-3,d})]/I(\mathscr{P}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2-3}-(h-1))}{(h-1)^3-1} + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$.

□

Corollary 3.9. *Let $h \geq 3$, $t \geq 1$ and $d \geq 3$. If $S = K[V(\mathscr{P}_{h,t,d})]$, then*

$$\operatorname{pdim}(S/I(\mathscr{P}_{h,t,d})) = \begin{cases} \left(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\right)(d) - \lceil \frac{d}{3} \rceil, & t \equiv 0 (\bmod 3); \\ \left(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-1}{(h-1)^3-1}\right)(d) - (h-2)\lceil \frac{d-1}{2} \rceil, & t \equiv 1 (\bmod 3); \\ \left(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1}\right)(d), & t \equiv 2 (\bmod 3). \end{cases}$$

Proof. We have $|V(\mathscr{P}_{h,t,d})| = \left(\frac{(h-1)^{t+1}-1}{(h-2)}\right)(d)$, therefore, $\operatorname{depth}(S) = \left(\frac{(h-1)^{t+1}-1}{(h-2)}\right)(d)$. Hence we get the required result by using Theorem 3.7 and Theorem 2.8. □

4. Depth, Stanley depth and projective dimension of cyclic modules associated to some unicyclic graphs

In this section we compute depth, Stanley depth and projective dimension of quotient ring $K[V(\mathscr{C}_{h,t,d})]/I(\mathscr{C}_{h,t,d})$.

Remark 4.1. In this remark we introduce some terms that appear in special cases of our proofs.

(1) If $d = 0$ we define $K[V(\mathscr{C}_{h,t,0})]/I(\mathscr{C}_{h,t,0}) := K$, hence

$$\operatorname{depth}(K[V(\mathscr{C}_{h,t,0})]/I(\mathscr{C}_{h,t,0})) = \operatorname{sdepth}(K[V(\mathscr{C}_{h,t,0})]/I(\mathscr{C}_{h,t,0})) = 0.$$

(2) If $d \geq 3$ we define $K[V(\mathscr{C}_{h,0,d})]/I(\mathscr{C}_{h,0,d}) := K[V(C_d)]/I(C_d)$, hence by Proposition 2.3

$$\operatorname{depth}(K[V(\mathscr{C}_{h,0,d})]/I(\mathscr{C}_{h,0,d})) = \lceil \frac{d-1}{3} \rceil \text{ and } \operatorname{sdepth}(K[V(\mathscr{C}_{h,0,d})]/I(\mathscr{C}_{h,0,d})) \geq \lceil \frac{d-1}{3} \rceil.$$

Theorem 4.2. *Let $t \geq 1$ and $d \geq 3$. If $S = K[V(\mathscr{C}_{2,t,d})]$, then*

$$\operatorname{depth}(S/I(\mathscr{C}_{2,t,d})) = \begin{cases} \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil, & t \equiv 0 (\bmod 3); \\ \lceil \frac{t}{3} \rceil \cdot d, & t \equiv 1, 2 (\bmod 3). \end{cases}$$

Proof. Let $d \geq 3$. For $t = 1$ the result follows from Lemma 2.15 and for $t = 2$ the result follows from Theorem 2.6. Now let $t \geq 3$.

(1) Let $t \equiv 0(\bmod 3)$. We have

$$S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)}) \cong K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3}) \otimes_K \bigotimes_{i=1}^{2}{}_K\, K[V(P_t)]/I(P_t) \\ \otimes_K K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[x_d^{(0)}]$$

and

$$S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_t)]/I(P_t).$$

Then by Lemma 2.14, Lemma 2.2 and Theorem 3.4

$$\begin{aligned} \operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3})) \\ &\quad + \sum_{i=1}^{2} \operatorname{depth}(K[V(P_t)]/I(P_t)) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) + 1 \\ &= \frac{dt}{3} - \frac{t}{3} \cdot 3 + \lceil \frac{d-3}{3} \rceil + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t-1}{3} \rceil + 1 \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) + \operatorname{depth}(K[V(P_t)]/I(P_t)) \\ &= \frac{t}{3} \cdot (d-1) + \lceil \frac{d-1}{3} \rceil + \lceil \frac{t}{3} \rceil. \end{aligned}$$

Since $t \equiv 0(\bmod 3)$ implies $\lceil \frac{t-1}{3} \rceil = \lceil \frac{t}{3} \rceil = \frac{t}{3}$. Hence $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})\right) = \frac{dt}{3} + \lceil \frac{d-3}{3} + 1 \rceil = \frac{dt}{3} + \lceil \frac{d}{3} \rceil$ and $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)})\right) = \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil$. Since for $d \equiv 0, 2(\bmod 3)$ $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})\right) = \operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)})\right)$. Thus by Remark 2.12 $\operatorname{depth}(S/I(\mathscr{C}_{2,t,d})) = \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil$.

Now let $d \equiv 1(\bmod 3)$. We have the following isomorphism:

$$\frac{(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})}{I(\mathscr{C}_{2,t,d})} \cong x_1^{(0)} A_0 \oplus x_{d-1}^{(0)} A_1 \oplus x_d^{(1)} A_2, \tag{4.1}$$

where

$$A_0 = \Big(K[x_3^{(0)}, x_4^{(0)}, \ldots, x_{d-1}^{(0)}, x_2^{(1)}, \ldots, x_d^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots, x_d^{(2)}, \ldots, \\ x_1^{(t)}, x_2^{(t)}, \ldots, x_d^{(t)}]/(G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(0)}x_2^{(0)}, x_2^{(0)}x_2^{(1)}, \\ x_2^{(0)}x_3^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_1^{(0)}x_d^{(0)}, x_d^{(0)}x_d^{(1)}\})\Big)[x_1^{(0)}],$$

$$A_1 = \Big(K[x_2^{(0)}, x_3^{(0)}, \dots, x_{d-3}^{(0)}, x_1^{(1)}, x_2^{(1)}, \dots, x_{d-2}^{(1)}, x_d^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_d^{(2)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_d^{(t)}]/(G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(0)}x_2^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-2}^{(0)}x_{d-3}^{(0)}, x_{d-2}^{(0)}x_{d-2}^{(1)}, x_{d-2}^{(0)}x_{d-1}^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-1}^{(1)}, x_d^{(0)}x_d^{(1)}, x_{d-1}^{(1)}x_{d-1}^{(2)}\})\Big)[x_{d-1}^{(0)}],$$

and

$$A_2 = \Big(K[x_2^{(0)}, x_3^{(0)}, \dots, x_{d-2}^{(0)}, x_1^{(1)}, x_2^{(1)}, \dots, x_{d-1}^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{d-1}^{(2)}, x_1^{(3)}, x_2^{(3)}, \dots, x_d^{(3)}, \dots x_1^{(t)}, x_2^{(t)}, \dots, x_d^{(t)}]/(G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(0)}x_2^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-2}^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-1}^{(1)}, x_d^{(0)}x_d^{(1)}, x_d^{(1)}x_d^{(2)}, x_d^{(2)}x_d^{(3)}\})\Big)[x_d^{(1)}].$$

Indeed, if w is a monomial such that $w \in (I(\mathscr{C}_{2,t,d}) : x_d^{(0)})$ but $w \notin I(\mathscr{C}_{2,t,d})$, then w is divisible by at least one variable from the set $\{x_1^{(0)}, x_{d-1}^{(0)}, x_d^{(1)}\}$. If w is not divisible by any variable form the set $\{x_1^{(0)}, x_{d-1}^{(0)}, x_d^{(1)}\}$, then $w \in I(\mathscr{C}_{2,t,d})$, a contradiction. Let w be a monomial such that $w \in \frac{(I(\mathscr{C}_{2,t,d}):x_d^{(0)})}{I(\mathscr{C}_{2,t,d})}$. In order to establish the isomorphism as given in Eq (4.1) one has to consider the following cases.

Case 1. If $x_1^{(0)}|w$, then $w = (x_1^{(0)})^{\alpha}u$, where $\alpha \geq 1$, $u \in K[x_3^{(0)}, x_4^{(0)}, \dots, x_{d-1}^{(0)}, x_2^{(1)}, x_2^{(1)}, \dots, x_d^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_d^{(2)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_d^{(t)}]$ and $u \notin (G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(0)}x_2^{(0)}, x_2^{(0)}x_2^{(1)}, x_2^{(0)}x_3^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_1^{(0)}x_d^{(0)}, x_d^{(0)}x_d^{(1)}\})$. Thus $w \in x_1^{(0)}A_0$ and it is easy to see that

$$A_0 \cong K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3}) \otimes_K \bigotimes_{i=1}^{2}{}_K K[V(P_t)]/I(P_t) \otimes_K K[x_1^{(0)}].$$

Case 2. If $x_{d-1}^{(0)}|w$ and $x_1^{(0)} \nmid w$, then $w = (x_{d-1}^{(0)})^{\beta}v$, where $\beta \geq 1$, $v \in K[x_2^{(0)}, x_3^{(0)}, \dots, x_{d-3}^{(0)}, x_1^{(1)}, x_2^{(1)}, \dots, x_{d-2}^{(1)}, x_d^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_d^{(2)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_d^{(t)}]$ and $v \notin (G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(0)}x_2^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-2}^{(0)}x_{d-3}^{(0)}, x_{d-2}^{(0)}x_{d-2}^{(1)}, x_{d-2}^{(0)}x_{d-1}^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-1}^{(1)}, x_d^{(0)}x_d^{(1)}, x_{d-1}^{(1)}x_{d-1}^{(2)}\})\big)$. Thus $w \in x_{d-1}^{(0)}A_1$ and we have

$$A_1 \cong K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[V(\mathscr{P}_{2,t,d-4})]/I(\mathscr{P}_{2,t,d-4}) \otimes_K \bigotimes_{i=1}^{3}{}_K K[V(P_t)]/I(P_t) \otimes_K K[x_{d-1}^{(0)}].$$

Case 3. If $x_d^{(1)}|w$, $x_1^{(0)} \nmid w$ and $x_{d-1}^{(0)} \nmid w$, then $w = (x_d^{(1)})^{\gamma}z$, where $\gamma \geq 1$, $z \in K[x_2^{(0)}, x_3^{(0)}, \dots, x_{d-2}^{(0)}, x_1^{(1)}, x_2^{(1)}, \dots, x_{d-1}^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{d-1}^{(2)}, x_1^{(3)}, x_2^{(3)}, \dots, x_d^{(3)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_d^{(t)}]$ and $z \notin (G(I(\mathscr{C}_{2,t,d})) \setminus \{x_1^{(0)}x_1^{(1)}, x_1^{(0)}x_2^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-2}^{(0)}, x_{d-1}^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_{d-1}^{(1)}, x_d^{(0)}x_d^{(1)}, x_d^{(1)}x_d^{(2)}, x_d^{(2)}x_d^{(3)}\})$. Hence $w \in x_d^{(1)}A_2$ and

$$A_2 \cong K[V(P_{t-2})]/I(P_{t-2}) \otimes_K K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3}) \otimes_K \bigotimes_{i=1}^{2}{}_K K[V(P_t)]/I(P_t) \otimes_K K[x_d^{(1)}].$$

It is also easy to see that $x_1^{(0)}A_0 \cong A_0$, $x_{d-1}^{(0)}A_1 \cong A_1$ and $x_d^{(1)}A_2 \cong A_2$. Thus by Lemma 2.2, Lemma 2.14 and Theorem 3.4

$$\begin{aligned}\operatorname{depth}(A_0) &= \lceil\frac{t-1}{3}\rceil + \frac{t}{3}\cdot(d-3) + \lceil\frac{d-3}{3}\rceil + 2\lceil\frac{t}{3}\rceil + 1 \\ &= \frac{dt}{3} + \lceil\frac{d}{3}\rceil,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(A_1) &= \lceil \frac{t-1}{3} \rceil + \frac{t}{3} \cdot (d-4) + \lceil \frac{d-4}{3} \rceil + 3\lceil \frac{t}{3} \rceil + 1 \\ &= \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil, \\ \operatorname{depth}(A_2) &= \lceil \frac{t-2}{3} \rceil + \frac{t}{3} \cdot (d-3) + \lceil \frac{d-3}{3} \rceil + 2\lceil \frac{t}{3} \rceil + 1 \\ &= \frac{dt}{3} + \lceil \frac{d}{3} \rceil.\end{aligned}$$

Hence, depth $\left(\frac{(I(\mathscr{C}_{2,t,d}):x_d^{(0)})}{I(\mathscr{C}_{2,t,d})}\right) = \min\{\operatorname{depth}(A_0), \operatorname{depth}(A_1), \operatorname{depth}(A_2)\} = \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil$. Now by applying Depth Lemma on following short exact sequence

$$0 \longrightarrow \frac{(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})}{I(\mathscr{C}_{2,t,d})} \longrightarrow S/I(\mathscr{C}_{2,t,d}) \longrightarrow S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)}) \longrightarrow 0,$$

we get $\operatorname{depth}(S/I(\mathscr{C}_{2,t,d})) = \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil$.

(2) Let $t \equiv 1 (\mathrm{mod}\, 3)$. In a similar way as done in Case 1 we have

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3})) \\ &+ \sum_{i=1}^{2} \operatorname{depth}(K[V(P_t)]/I(P_t)) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot (d-3) + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t-1}{3} \rceil + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot d - 3\lceil \frac{t}{3} \rceil + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t+2}{3} - 1 \rceil + 1\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) + \operatorname{depth}(K[V(P_t)]/I(P_t)) \\ &= \lceil \frac{t}{3} \rceil \cdot (d-1) + \lceil \frac{t}{3} \rceil.\end{aligned}$$

Since $t + 2 \equiv 0 (\mathrm{mod}\, 3)$ implies $\lceil \frac{t+2}{3} \rceil = \lceil \frac{t}{3} \rceil$. Hence $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d$ and $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}), x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d$. Hence by Remark 2.12 we have $\operatorname{depth}(S/I(\mathscr{C}_{2,t,d})) = \lceil \frac{t}{3} \rceil \cdot d$.

(3) Let $t \equiv 2 (\mathrm{mod}\, 3)$

We have

$$S/(I(\mathscr{C}_{2,t,d}) : x_d^{(1)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_{t-2})]/I(P_{t-2}) \otimes_K K[x_d^{(1)}].$$

Now let $J = (I(\mathscr{C}_{2,t,d}), x_d^{(1)})$. We have the following isomorphisms:

$$\begin{aligned}S/(J : x_d^{(0)}) \cong & K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3}) \otimes_K \bigotimes_{K, i=1}^{2} K[V(P_t)]/I(P_t) \\ & \otimes_K K[V(P_{t-1})]/I(P_{t-1}) \otimes_K K[x_d^{(0)}]\end{aligned}$$

and

$$S/(J, x_d^{(0)}) \cong K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1}) \otimes_K K[V(P_{t-1})]/I(P_{t-1}).$$

Then by Lemma 2.14, Lemma 2.2 and Theorem 3.4 we have

$$\begin{aligned}\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(1)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) \\ &\quad + \operatorname{depth}(K[V(P_{t-2})]/I(P_{t-2})) + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot (d-1) + \lceil \frac{t-2}{3} \rceil + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot d - \lceil \frac{t}{3} \rceil + \lceil \frac{t+1}{3} \rceil,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}\left(S/(J : x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-3})]/I(\mathscr{P}_{2,t,d-3})) + \sum_{i=1}^{2} \operatorname{depth}(K[V(P_t)]/I(P_t)) \\ &\quad + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot (d-3) + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t-1}{3} \rceil + 1 \\ &= \lceil \frac{t}{3} \rceil \cdot d - 3\lceil \frac{t}{3} \rceil + 2\lceil \frac{t}{3} \rceil + \lceil \frac{t+1}{3} - \frac{2}{3} \rceil + 1\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}\left(S/(J, x_d^{(0)})\right) &= \operatorname{depth}(K[V(\mathscr{P}_{2,t,d-1})]/I(\mathscr{P}_{2,t,d-1})) + \operatorname{depth}(K[V(P_{t-1})]/I(P_{t-1})) \\ &= \lceil \frac{t}{3} \rceil \cdot (d-1) + \lceil \frac{t-1}{3} \rceil \\ &= \lceil \frac{t}{3} \rceil \cdot d - \lceil \frac{t}{3} \rceil + \lceil \frac{t+1}{3} - \frac{2}{3} \rceil.\end{aligned}$$

Since $t + 1 \equiv 0 (\operatorname{mod} 3)$ implies $\lceil \frac{t+1}{3} \rceil = \lceil \frac{t}{3} \rceil$. Hence $\operatorname{depth}\left(S/(I(\mathscr{C}_{2,t,d}) : x_d^{(1)})\right) = \lceil \frac{t}{3} \rceil \cdot d$, $\operatorname{depth}\left(S/(J : x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d + 1$ and $\operatorname{depth}\left(S/(J, x_d^{(0)})\right) = \lceil \frac{t}{3} \rceil \cdot d$. Thus by Remark 2.12 $\operatorname{depth}(S/J) \geq \lceil \frac{t}{3} \rceil \cdot d$ and also $J = (I(\mathscr{C}_{2,t,d}), x_d^{(1)})$ implies $\operatorname{depth}(S/(I(\mathscr{C}_{2,t,d}), x_d^{(1)})) \geq \lceil \frac{t}{3} \rceil \cdot d$. Hence again by Remark 2.12 $\operatorname{depth}(S/I(\mathscr{C}_{2,t,d})) = \lceil \frac{t}{3} \rceil \cdot d$.

□

Theorem 4.3. *Let $d \geq 3$ and $t \geq 1$. If $S = K[V(\mathscr{C}_{2,t,d})]$, then* $\operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) = \lceil \frac{t}{3} \rceil \cdot d$ *for $t \equiv 1, 2 (\operatorname{mod} 3)$. Otherwise,* $\frac{dt}{3} + \lceil \frac{d-1}{3} \rceil \leq \operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) \leq \frac{dt}{3} + \lceil \frac{d}{3} \rceil$.

Proof. We compute lower bound by using similar arguments as in Theorem 4.2 and using Lemma 2.14 and Remark 2.12 that is $\operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) \geq \lceil \frac{t}{3} \rceil \cdot d$ for $t \equiv 1, 2 (\operatorname{mod} 3)$ otherwise, $\operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) \geq \frac{dt}{3} + \lceil \frac{d-1}{3} \rceil$. Now we compute upper bound by induction on t. If $t = 1$, then by Lemma 2.15 $\operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) = d$, as desired. For $t = 2$, by Theorem 2.6 $\operatorname{sdepth}(S/I(\mathscr{C}_{2,t,d})) = d$, as desired. For $t = 3$ we have the following isomorphism:

$$S/(I(\mathscr{C}_{2,3,d}) : x_1^{(0)} x_1^{(2)} x_2^{(2)} \dots x_d^{(2)}) \cong K[V(P_{d-3})]/I(P_{d-3}) \otimes_K K[x_1^{(0)}, x_1^{(2)}, x_2^{(2)}, \dots, x_d^{(2)}].$$

Then we have

$$\begin{aligned}\mathrm{sdepth}(S/(I(\mathscr{C}_{2,3,d}) : x_1^{(0)}x_1^{(2)}x_2^{(2)}\dots x_d^{(2)})) &= \mathrm{sdepth}(K[V(P_{d-3})]/I(P_{d-3})) + d + 1\\ &= \lceil\frac{d-3}{3}\rceil + d + 1\\ &= \lceil\frac{d}{3}\rceil + d.\end{aligned}$$

Also by Proposition 2.10 we have $\mathrm{sdepth}(S/I(\mathscr{C}_{2,3,d})) \leq d + \lceil\frac{d}{3}\rceil$, and as already we have computed lower bound that is $\mathrm{sdepth}(S/I(\mathscr{C}_{2,3,d})) \geq d + \lceil\frac{d-1}{3}\rceil$. Hence

$$d + \lceil\frac{d-1}{3}\rceil \leq \mathrm{sdepth}(S/I(\mathscr{C}_{2,3,d})) \leq d + \lceil\frac{d}{3}\rceil.$$

Now let $t \geq 4$ and u be a monomial such that $u := x_1^{(t-1)}x_2^{(t-1)}\dots x_d^{(t-1)}$. We have the following isomorphism:

$$S/(I(\mathscr{C}_{2,t,d}) : u) \cong K[V(\mathscr{C}_{2,t-3,d})]/I(\mathscr{C}_{2,t-3,d})\otimes_K K[x_1^{(t-1)}, x_2^{(t-1)}, \dots, x_d^{(t-1)}].$$

Then we have

$$\mathrm{sdepth}(S/(I(\mathscr{C}_{2,t,d}) : x_1^{(t-1)}x_2^{(t-1)}\dots x_d^{(t-1)})) = \mathrm{sdepth}(K[V(\mathscr{C}_{2,t-3,d})]/I(\mathscr{C}_{2,t-3,d})) + d. \quad (4.2)$$

By Proposition 2.10, $\mathrm{sdepth}(S/I(\mathscr{C}_{2,t,d})) \leq \mathrm{sdepth}(S/(I(\mathscr{C}_{2,t,d}) : u))$.

(1) Let $t \equiv 1, 2 (\mathrm{mod}\, 3)$. Since $t - 3 \equiv 1, 2(\mathrm{mod}\, 3)$, then by using induction on Eq (4.2) we have $\mathrm{sdepth}(K[V(\mathscr{C}_{2,t-3,d})]/I(\mathscr{C}_{2,t-3,d})) + d = \lceil\frac{t-3}{3}\rceil \cdot d + d = \lceil\frac{t}{3}\rceil \cdot d$.

(2) Let $t \equiv 0(\mathrm{mod}\, 3)$. Since $t - 3 \equiv 0(\mathrm{mod}\, 3)$, then by using induction on Eq 4.2 we have $\mathrm{sdepth}(K[V(\mathscr{C}_{2,t-3,d})]/I(\mathscr{C}_{2,t-3,d})) + d = \frac{dt}{3} + \lceil\frac{d}{3}\rceil$.

□

Corollary 4.4. *Let $h = 2$, $t \geq 1$ and $d \geq 3$. If $S = K[V(\mathscr{C}_{2,t,d})]$, then*

$$\mathrm{pdim}\,(S/I(\mathscr{C}_{2,t,d})) = \begin{cases}(\frac{2t}{3} + 1)d - \lceil\frac{d-1}{3}\rceil, & t \equiv 0(\mathrm{mod}\, 3);\\ d(t + 1) - \lceil\frac{t}{3}\rceil \cdot d, & t \equiv 1, 2(\mathrm{mod}\, 3).\end{cases}$$

Proof. We have $|V(\mathscr{C}_{2,t,d})| = d(t + 1)$, therefore, $\mathrm{depth}(S) = d(t + 1)$. Hence we get the required result by using Theorem 4.2 and Theorem 2.8. □

Theorem 4.5. *Let $d \geq 3$, $t \geq 1$ and $h \geq 3$. If $S = K[V(\mathscr{C}_{h,t,d})]$, then*

$$\mathrm{depth}\,(S/I(\mathscr{C}_{h,t,d})) = \begin{cases}\frac{d\left((h-1)^{t+2}-(h-1)^2\right)}{(h-1)^3-1} + \lceil\frac{d-1}{3}\rceil, & t \equiv 0(\mathrm{mod}\, 3);\\ \frac{d\left((h-1)^{t+2}-1\right)}{(h-1)^3-1} + (h - 2)\lceil\frac{d}{2}\rceil, & t \equiv 1(\mathrm{mod}\, 3);\\ \frac{d\left((h-1)^{t+2}-(h-1)\right)}{(h-1)^3-1}, & t \equiv 2(\mathrm{mod}\, 3).\end{cases}$$

Proof. If $t = 1$ the result follows from the Lemma 2.15. If $t = 2$ the result follows from Theorem 2.6. Now let $t \geq 3$.

(1) Let $t \equiv 0(\bmod 3)$. We have the following isomorphisms:

$$S/(I(\mathscr{C}_{h,t,d}):x_d^{(0)}) \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[x_d^{(0)}]$$

and

$$S/(I(\mathscr{C}_{h,t,d}), x_d^{(0)}) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}).$$

Since $t-1 \equiv 2(\bmod 3)$ and $t-2 \equiv 1(\bmod 3)$, then by Lemma 2.14, Theorem 3.7 and Proposition 2.7 we have

$$\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{C}_{h,t,d}):x_d^{(0)}))\\
&= \sum_{i=1}^{2(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) + \operatorname{depth}(K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}))\\
&\quad + \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1})) + 1\\
&= 2(h-1)\cdot \frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1} + \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\cdot(d-3)\\
&\quad + \lceil \frac{d-3}{3} \rceil + (h-1)^2 \cdot \frac{(h-1)^{t-2+2}-1}{(h-1)^3-1} + 1\\
&= \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{C}_{h,t,d}), x_d^{(0)}))\\
&= \operatorname{depth}(K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1})) + \sum_{i=1}^{(h-1)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&= \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\cdot(d-1) + \lceil \frac{d-1}{3} \rceil + (d-1)\cdot \frac{(d-1)^{t-1+2}-(h-1)}{(h-1)^3-1}\\
&= \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d-1}{3} \rceil.
\end{aligned}$$

For $d \equiv 0, 2(\bmod 3)$, $\lceil \frac{d-1}{3} \rceil = \lceil \frac{d}{3} \rceil$. Then by Remark 2.12 we have

$$\operatorname{depth}(S/I(\mathscr{C}_{h,t,d})) = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d-1}{3} \rceil.$$

Now let $d \equiv 1(\bmod 3)$. We have the following isomorphism:

$$\frac{(I(\mathscr{C}_{h,t,d}):x_d^{(0)})}{I(\mathscr{C}_{h,t,d})} \cong x_1^{(0)}A_0 \oplus x_{d-1}^{(0)}A_1 \oplus x_{(h-1)d-(h-2)}^{(1)}B_0 \oplus x_{(h-1)d-(h-3)}^{(1)}B_1 \oplus, \dots, \oplus x_{(h-1)d}^{(1)}B_{h-2} \quad (4.3)$$

where

$$A_0 = \Big(K[x_3^{(0)}, x_4^{(0)}, \dots, x_{d-1}^{(0)}, x_h^{(1)}, x_{h+1}^{(1)}, \dots, x_{(h-1)d}^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{(h-1)^2d}^{(2)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_{(h-1)^td}^{(t)}]/(G(I(\mathscr{C}_{h,t,d})) \setminus \bigcup_{l=0}^{1}\bigcup_{p=1}^{(h-1)^l}\bigcup_{q=(h-1)p-(h-2)}^{(h-1)p}\{x_p^{(l)}x_q^{(l+1)}\} \cup \bigcup_{q=h}^{2(h-1)}\{x_2^{(0)}x_q^{(1)}\} \cup \bigcup_{q=(h-1)d-(h-2)}^{(h-1)d}\{x_d^{(0)}x_q^{(1)}\} \cup \{x_1^{(0)}x_2^{(0)}, x_2^{(0)}x_3^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_d^{(0)}\})\Big)[x_1^{(0)}].$$

Indeed if w is a monomial such that $w \in (I(\mathscr{C}_{h,t,d}) : x_d^{(0)})$ but $w \notin I(\mathscr{C}_{h,t,d})$, then w is divisible by at least one variable from the set $\{x_1^{(0)}, x_{d-1}^{(0)}, x_{(h-1)d-(h-2)}^{(1)}, x_{(h-1)d-(h-3)}^{(1)}, \dots, x_{(h-1)d}^{(1)}\}$. If w is not divisible by any variable from the set $\{x_1^{(0)}, x_{d-1}^{(0)}, x_{(h-1)d-(h-2)}^{(1)}, x_{(h-1)d-(h-3)}^{(1)}, \dots, x_{(h-1)d}^{(1)}\}$, then $w \in I(\mathscr{C}_{h,t,d})$, a contradiction. Let w be a monomial such that $w \in \frac{(I(\mathscr{C}_{h,t,d}):x_d^{(0)})}{I(\mathscr{C}_{h,t,d})}$. In order to establish the isomorphism as given in Eq (4.3) we adopt the similar strategy of Theorem 4.2. If $x_1^{(0)}|w$ then $w = (x_1^{(0)})^{\alpha}u$, where $\alpha \geq 1$, $u \notin (G(I(\mathscr{C}_{h,t,d})) \setminus \bigcup_{l=0}^{1}\bigcup_{p=1}^{(h-1)^l}\bigcup_{q=(h-1)p-(h-2)}^{(h-1)p}\{x_p^{(l)}x_q^{(l+1)}\} \cup \bigcup_{q=h}^{2(h-1)}\{x_2^{(0)}x_q^{(1)}\} \cup \bigcup_{q=(h-1)d-(h-2)}^{(h-1)d}\{x_d^{(0)}x_q^{(1)}\} \cup \{x_1^{(0)}x_2^{(0)}, x_2^{(0)}x_3^{(0)}, x_1^{(0)}x_d^{(0)}, x_{d-1}^{(0)}x_d^{(0)}\})$ and $u \in K[x_3^{(0)}, x_4^{(0)}, \dots, x_{d-1}^{(0)}, x_h^{(1)}, x_{h+1}^{(1)}, \dots, x_{(h-1)d}^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{(h-1)^2d}^{(2)}, \dots, x_1^{(t)}, x_2^{(t)}, \dots, x_{(h-1)^td}^{(t)}]$. Thus $w \in x_1^{(0)}A_0$ where

$$A_0 \cong \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[x_1^{(0)}].$$

Now proceeding in a similar way if $x_{d-1}^{(0)}|w$ and $x_1^{(0)} \nmid w$ then we have $w \in x_{d-1}^{(0)}A_1$. Now let $x_{(h-1)d-(h-2)}^{(0)}|w$, $x_1^{(0)} \nmid w$ and $x_{d-1}^{(0)} \nmid w$ we have $w \in x_{(h-1)d-(h-2)}^{(1)}B_0$. Continuing in the same fashion we get the required isomorphism and

$$A_1 \cong \bigotimes_{i=1}^{3(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-4})]/I(\mathscr{P}_{h,t,d-4}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[x_{d-1}^{(0)}],$$

$$B_0 \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[x_{(h-1)d-(h-2)}^{(1)}],$$

$$B_1 \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K$$
$$\bigotimes_{i=1}^{h-1}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K$$
$$\bigotimes_{i=1}^{(h-3)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[x^{(1)}_{(h-1)d-(h-3)}],$$

$$B_2 \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K$$
$$\bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K$$
$$\bigotimes_{i=1}^{h-4}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[x^{(1)}_{(h-1)d-(h-4)}],$$

$$\vdots$$

$$B_{h-3} \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K$$
$$\bigotimes_{i=1}^{(h-3)(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1})$$
$$\otimes_K K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[x^{(1)}_{(h-1)d-1}],$$

$$B_{h-2} \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K$$
$$\bigotimes_{i=1}^{(h-2)(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1})$$
$$\otimes_K K[x^{(1)}_{(h-1)d}].$$

Now by Lemma 2.14, Theorem 3.7 and Proposition 2.7 we have

$$\begin{aligned}\operatorname{depth}(A_0) &= (h-1)^2 \cdot \frac{(h-1)^{t-2+2}-1}{(h-1)^3-1} + \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1} \cdot (d-3) + \lceil \frac{d-3}{3} \rceil \\ &\quad + 2(h-1) \cdot \frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1} + 1 \\ &= \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(A_1) &= 3(h-1) \cdot \frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1} + \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1} \cdot (d-4) \\ &\quad + \lceil \frac{d-4}{3} \rceil + (h-1)^2 \cdot \frac{(h-1)^{t-2+2}-1}{(h-1)^3-1} + 1 \\ &= \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d-1}{3} \rceil,\end{aligned}$$

$$\begin{aligned}\text{depth}(B_0) &= 2(h-1)\cdot\frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1}+\frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\cdot(d-3)\\ &\quad+\lceil\frac{d-3}{3}\rceil+(h-1)^2\cdot(\frac{(h-1)^{t-3+2}-(h-1)^2}{(h-1)^3-1}+1)\\ &\quad+(h-2)\cdot\frac{(h-1)^{t-1+2}-(h-1)}{(h-1)^3-1}+1\\ &=\frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d}{3}\rceil+(h-1)(h-2).\end{aligned}$$

Similarly

$$\begin{aligned}\text{depth}(B_1) = \text{depth}(B_2) = \ldots = \text{depth}(B_{h-3}) &= \text{depth}(B_{h-2})\\ &=\frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d}{3}\rceil+(h-1)(h-2).\end{aligned}$$

Thus depth $(\frac{(I(\mathscr{C}_{h,t,d}):x_d^{(0)})}{I(\mathscr{C}_{h,t,d})}) = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d-1}{3}\rceil$. Now by applying Lemma 2.11 on the following short exact sequence

$$0\longrightarrow\frac{(I(\mathscr{C}_{h,t,d}):x_d^{(0)})}{I(\mathscr{C}_{h,t,d})}\longrightarrow S/I(\mathscr{C}_{h,t,d})\longrightarrow S/(I(\mathscr{C}_{h,t,d}):x_d^{(0)})\longrightarrow 0,$$

we have the required result that is $\text{depth}(S/I(\mathscr{C}_{h,t,d})) = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d-1}{3}\rceil$.

(2) Let $t\equiv 1(\text{mod}\,3)$. In a similar way as done in Case 1 we have

$$\begin{aligned}&\text{depth}(S/(I(\mathscr{C}_{h,t,d}):x_d^{(0)}))\\ &=\sum_{i=1}^{2(h-1)}\text{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))+\text{depth}(K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}))\\ &\quad+\sum_{i=1}^{(h-1)^2}\text{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}))+1\\ &=2(h-1)\cdot\Big\{\frac{(h-1)^{t-1+2}-(h-1)^2}{(h-1)^3-1}+1\Big\}+\frac{(h-1)^{t+2}-1}{(h-1)^3-1}\cdot(d-3)\\ &\quad+(h-2)\lceil\frac{d-3-1}{2}\rceil+(h-1)^2\cdot\frac{(h-1)^{t-2+2}-(h-1)}{(h-1)^3-1}+1\\ &=\frac{d((h-1)^{t+2}-1)}{(h-1)^3-1}+(h-2)\lceil\frac{d}{2}\rceil\end{aligned}$$

and

$$\begin{aligned}&\text{depth}(S/(I(\mathscr{C}_{h,t,d}),x_d^{(0)}))\\ &=\text{depth}(K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}))+\sum_{i=1}^{(h-1)}\text{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\ &=\frac{(h-1)^{t+2}-1}{(h-1)^3-1}\cdot(d-1)+(h-2)\lceil\frac{d-2}{2}\rceil+(h-1)\cdot\Big\{\frac{(h-1)^{t-1+2}-(h-1)^2}{(h-1)^3-1}+1\Big\}\\ &=\frac{d((h-1)^{t+2}-1)}{(h-1)^3-1}+(h-2)\lceil\frac{d}{2}\rceil.\end{aligned}$$

Hence by Remark 2.12 we have depth$(S/I(\mathscr{C}_{h,t,d}) = \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d}{2} \rceil$.

(3) Let $t \equiv 2 \pmod 3$. We have the following isomorphism:

$$S/(I(\mathscr{C}_{h,t,d}) : x^{(1)}_{(h-1)d}) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1}) \otimes_K K[x^{(1)}_{(h-1)d}].$$

Now let $J = (I(\mathscr{C}_{h,t,d}), x^{(1)}_{(h-1)d})$, then we have

$$S/(J : x^{(0)}_d) \cong \bigotimes_{i=1}^{2(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}) \otimes_K \bigotimes_{i=1}^{(h-1)^2}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}) \otimes_K K[x^{(0)}_d]$$

and

$$S/(J, x^{(0)}_d) \cong K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}) \otimes_K \bigotimes_{i=1}^{(h-2)}{}_K\, K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}) \otimes_K \bigotimes_{i=1}^{(h-1)}{}_K\, K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}).$$

Since $t-1 \equiv 1 \pmod 3$, $t-2 \equiv 0 \pmod 3$ and $t-3 \equiv 2 \pmod 3$, then by Lemma 2.14, Theorem 3.7 and Proposition 2.7 we have

$$\begin{aligned}
&\operatorname{depth}(S/(I(\mathscr{C}_{h,t,d}) : x^{(1)}_{(h-1)d})) \\
&\quad = \operatorname{depth}(K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1})) + \sum_{i=1}^{(h-2)} \operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1})) \\
&\quad\quad + \sum_{i=1}^{(h-1)^2} \operatorname{depth}(K[V(\mathscr{P}_{h,t-3,1})]/I(\mathscr{P}_{h,t-3,1})) + 1 \\
&\quad = \frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1} \cdot (d-1) + (h-2) \cdot \frac{(h-1)^{t-1+2}-1}{(h-1)^3-1} \\
&\quad\quad + (h-1)^2 \cdot \frac{(h-1)^{t-3+2}-(h-1)}{(h-1)^3-1} + 1 \\
&\quad = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1},
\end{aligned}$$

$$\begin{aligned}
&\operatorname{depth}(S/(J:x_d^{(0)}))\\
&\quad=\sum_{i=1}^{2(h-1)}\operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))+\operatorname{depth}(K[V(\mathscr{P}_{h,t,d-3})]/I(\mathscr{P}_{h,t,d-3}))\\
&\qquad+\sum_{i=1}^{(h-1)^2}\operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}))+1\\
&\quad=2(h-1)\cdot\frac{(h-1)^{t-1+2}-1}{(h-1)^3-1}+\frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1}\cdot(d-3)\\
&\qquad+(h-1)^2\cdot\Big\{\frac{(h-1)^{t-2+2}-(h-1)^2}{(h-1)^3-1}+1\Big\}+1\\
&\quad=\frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}+(h-1)(h-2)+1
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{depth}(S/(J,x_d^{(0)}))\\
&\quad=\operatorname{depth}(K[V(\mathscr{P}_{h,t,d-1})]/I(\mathscr{P}_{h,t,d-1}))+\sum_{i=1}^{(h-2)}\operatorname{depth}(K[V(\mathscr{P}_{h,t-1,1})]/I(\mathscr{P}_{h,t-1,1}))\\
&\qquad+\sum_{i=1}^{(h-1)}\operatorname{depth}(K[V(\mathscr{P}_{h,t-2,1})]/I(\mathscr{P}_{h,t-2,1}))\\
&\quad=\frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1}\cdot(d-1)+(h-2)\cdot\frac{(h-1)^{t-1+2}-1}{(h-1)^3-1}\\
&\qquad+(h-1)\cdot\Big\{\frac{(h-1)^{t-2+2}-(h-1)^2}{(h-1)^3-1}+1\Big\}\\
&\quad=\frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}+(h-2).
\end{aligned}$$

Hence by Remark 2.12 we have $\operatorname{depth}(S/J)\geq\frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}+(h-2)$ and also $J=(I(\mathscr{C}_{h,t,d}),x_{(h-1)d}^{(1)})$ implies $\operatorname{depth}(S/(I(\mathscr{C}_{h,t,d}),x_{(h-1)d}^{(1)}))\geq\frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}+(h-2)$. Thus again by Remark 2.12 we have $\operatorname{depth}(S/I(\mathscr{C}_{h,t,d}))=\frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$.

□

Theorem 4.6. *Let $d\geq 3$, $t\geq 0$ and $h\geq 3$. If $S=K[V(\mathscr{C}_{h,t,d})]$, then*

$$\operatorname{sdepth}(S/I(\mathscr{C}_{h,t,d}))=\begin{cases}\frac{d\left((h-1)^{t+2}-1\right)}{(h-1)^3-1}+(h-2)\lceil\frac{d}{2}\rceil, & t\equiv 1(\bmod 3);\\ \frac{d\left((h-1)^{t+2}-(h-1)\right)}{(h-1)^3-1}, & t\equiv 2(\bmod 3).\end{cases}$$

Otherwise, $\frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d-1}{3}\rceil\leq\operatorname{sdepth}(S/I(\mathscr{C}_{h,t,d}))\leq\frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1}+\lceil\frac{d}{3}\rceil$.

Proof. We compute lower bound by using similar arguments as in Theorem 4.5 and using Lemma 2.14 and Remark 2.12 we get.

$$\operatorname{sdepth}(S/I(\mathscr{C}_{h,t,d}))\geq\begin{cases}\frac{d((h-1)^{t+2}-1)}{(h-1)^3-1}+(h-2)\lceil\frac{d}{2}\rceil, & t\equiv 1(\bmod 3);\\ \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}, & t\equiv 2(\bmod 3),\end{cases}$$

otherwise sdepth $(S/I(\mathscr{C}_{h,t,d})) \geq \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d-1}{3} \rceil$.

Now we compute upper bound by induction on t. If $t = 1$ result follows from Lemma 2.15 and for $t = 2$ result follows from Theorem 2.6. For $t = 3$ we have the following isomorphism:

$$S/(I(\mathscr{C}_{h,3,d}) : x_1^{(0)} x_1^{(2)} x_2^{(2)} \dots x_{d(h-1)^2}^{(2)}) \cong K[V(P_{d-3})]/I(P_{d-3}) \otimes_K K[x_1^{(0)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{d(h-1)^2}^{(2)}].$$

Then we have

$$\operatorname{sdepth}(S/(I(\mathscr{C}_{h,3,d}) : x_1^{(0)} x_1^{(2)} x_2^{(2)} \dots x_{d(h-1)^2}^{(2)})) = \operatorname{sdepth}(K[V(P_{d-3})]/I(P_{d-3})) + d(h-1)^2 + 1,$$

then by Lemma 2.2 we have

$$\begin{aligned} \operatorname{sdepth}(S/(I(\mathscr{C}_{h,3,d}) : x_1^{(0)} x_1^{(2)} x_2^{(2)} \dots x_{d(h-1)^2}^{(2)})) &= \lceil \frac{d-3}{3} \rceil + (h-1)^2 d + 1 \\ &= \lceil \frac{d}{3} \rceil + (h-1)^2 d. \end{aligned}$$

Also by Proposition 2.10 we have $\operatorname{sdepth}(S/I(\mathscr{C}_{h,3,d})) \leq (h-1)^2 d + \lceil \frac{d}{3} \rceil$ and already we have computed lower bound that is $\operatorname{sdepth}(S/I(\mathscr{C}_{h,3,d})) \geq (h-1)^2 d + \lceil \frac{d-1}{3} \rceil$. Now let $t \geq 4$ and u be a monomial such that $u := x_1^{(t-1)} x_2^{(t-1)} \dots x_{d(h-1)^{t-1}}^{(t-1)}$. We have

$$S/(I(\mathscr{C}_{h,t,d}) : u) \cong K[V(\mathscr{C}_{h,t-3,d})]/I(\mathscr{C}_{h,t-3,d}) \otimes_K K[x_1^{(t-1)}, x_2^{(t-1)}, \dots, x_{d(h-1)^{t-1}}^{(t-1)}].$$

Then we have

$$\operatorname{sdepth}(S/(I(\mathscr{C}_{h,t,d}) : u)) = \operatorname{sdepth}(K[V(\mathscr{C}_{h,t-3,d})]/I(\mathscr{C}_{h,t-3,d})) + d(h-1)^{t-1}. \tag{4.4}$$

By Proposition 2.10 we have $\operatorname{sdepth}(S/I(\mathscr{C}_{h,t,d})) \leq \operatorname{sdepth}(S/(I(\mathscr{C}_{h,t,d}) : u))$.

(1) Let $t \equiv 1 \pmod 3$ implies $t-3 \equiv 1 \pmod 3$ then from Eq (4.4) and by induction we have $\operatorname{sdepth}(K[V(\mathscr{C}_{h,t-3,d})]/I(\mathscr{C}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2-3}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d}{2} \rceil + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-1)}{(h-1)^3-1} + (h-2)\lceil \frac{d}{2} \rceil$, as desired.

(2) Let $t \equiv 2 \pmod 3$ implies $t-3 \equiv 2 \pmod 3$ then from Eq (4.4) and by induction we have $\operatorname{sdepth}(K[V(\mathscr{C}_{h,t-3,d})]/I(\mathscr{C}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2-3}-(h-1))}{(h-1)^3-1} + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-(h-1))}{(h-1)^3-1}$, as desired.

(3) Let $t \equiv 0 \pmod 3$ implies $t-3 \equiv 0 \pmod 3$ then from Eq (4.4) and by induction we have $\operatorname{sdepth}(K[V(\mathscr{C}_{h,t-3,d})]/I(\mathscr{C}_{h,t-3,d})) + d(h-1)^{t-1} = \frac{d((h-1)^{t+2}-(h-1)^2)}{(h-1)^3-1} + \lceil \frac{d}{3} \rceil$, as desired.

□

Corollary 4.7. *Let $h \geq 3$, $t \geq 1$ and $d \geq 3$. If $S = K[V(\mathscr{C}_{h,t,d})]$, then*

$$\operatorname{pdim}(S/I(\mathscr{C}_{h,t,d})) = \begin{cases} \Big(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-(h-1)^2}{(h-1)^3-1}\Big)(d) - \lceil \frac{d-1}{3} \rceil, & t \equiv 0 \pmod 3; \\ \Big(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-1}{(h-1)^3-1}\Big)(d) - (h-2)\lceil \frac{d}{2} \rceil, & t \equiv 1 \pmod 3; \\ \Big(\frac{(h-1)^{t+1}-1}{(h-2)} - \frac{(h-1)^{t+2}-(h-1)}{(h-1)^3-1}\Big)(d), & t \equiv 2 \pmod 3. \end{cases}$$

Proof. We have $|V(\mathscr{C}_{h,t,d})| = \Big(\frac{(h-1)^{t+1}-1}{(h-2)}\Big)(d)$, therefore, $\operatorname{depth}(S) = \Big(\frac{(h-1)^{t+1}-1}{(h-2)}\Big)(d)$. Hence we get the required result by using Theorem 4.5 and Theorem 2.8. □

Conflict of interest

It is declared by the authors that there is no conflict of interest in this paper.

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