



Research article

Some fixed point results based on contractions of new types for extended b -metric spaces

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Abstract: The construction of contraction conditions plays an important role in science for formulating new findings in fixed point theories of mappings under a set of specific conditions. The aim of this work is to take advantage of the idea of extended b -metric spaces in the sense introduced by Kamran et al. [A generalization of b -metric space and some fixed point theorems, *Mathematics*, **5** (2017), 1–7] to construct new contraction conditions to obtain new results related to fixed points. Our results enrich and extend some known results from b -metric spaces to extended b -metric spaces. We construct some examples to show the usefulness of our results. Also, we provide some applications to support our results.

Keywords: fixed point; extended metric space of type (γ, β) ; extended b -metric space

Mathematics Subject Classification: 37C25, 47H10, 54H25

1. Introduction

Nowadays, the subject of fixed point theory is one of the most beautiful and attractive subjects in science because this subject has many applications in all aspects of science. Many authors have extracted many fixed point results, and applied their results to give a set of conditions for such integral equations, and ordinary differential equations to guarantee existence solutions to such equations, see [1–4] and the references cited there. Ameer et al. [5] employed the directed graph to introduce new mappings, called “hybrid Ćirić type graphic Υ, Λ -contraction mappings”, and then applied their new results to study some applications to electric circuit and fractional differential equations.

The thought of metric spaces has been extended in many directions, such as G -metric spaces, cone metric spaces and b -metric spaces in order to expand Banach’s contraction [6] to more beneficial

forms. Recently, Kamran et al. [7] expanded the thought of b -metric spaces in the sense introduced by Baktain [8] and Czerwik [9] into a new concept called “extended b -metric spaces”. The benefit of extending metric spaces to new spaces is to enrich the sciences with new findings relevant to fixed points for mappings that satisfy a set of suitable conditions to ensure the existence of fixed points for some findings in b -metric and extended b -spaces, see [10–17].

Abdeljawad et al. [18] extended some results in fixed points to partial b -metric spaces. Shatanawi et al. [19] utilized the ordered relation to present a new extension of Banach’s contraction theorem. Roshan et al. [20] presented some common fixed points in ordered b -metric spaces for functions that satisfy contraction condition based on two different functions. Recently, Mlaiki et al. [21] launched a new space, called “controlled metric type space”, and they gave a new version of the Banach contraction theorem. Then, some authors obtained good results on this new topic, see [22–25]. Farhan et al. [26] studied some results of Reich-type and (α, F) -contractions in partially ordered, double-controlled metric-type spaces. Then, they applied their results to obtain some applications to non-linear fractional differential equations.

Henceforth, Q stands for a non-empty set.

Definition 1.1. [7] For a set Q , let $\theta : Q \times Q \rightarrow [1, \infty)$ be a function. Then, the function $\nu : Q \times Q \rightarrow [0, \infty)$ is called an extended b -metric, if $\forall \zeta, \varphi, \varrho \in Q$, we have

- (1) $\nu(\zeta, \varphi) = 0 \iff \zeta = \varphi$,
- (2) $\nu(\zeta, \varphi) = \nu(\varphi, \zeta)$,
- (3) $\nu(\zeta, \varphi) \leq \theta(\zeta, \varphi)[\nu(\zeta, \varrho) + \nu(\varrho, \varphi)]$.

The pair (Q, ν) is referred to as an extended b -metric space.

Some examples for (Q, ν) are stated here:

Example 1.1. For $Q = [0, \infty)$, set $\theta : Q \times Q \rightarrow [1, \infty)$ and $\nu : Q \times Q \rightarrow [1, \infty)$ via $\theta(\zeta_1, \zeta_2) = 1 + \zeta_1 + \zeta_2$, $\forall \zeta_1, \zeta_2 \in Q$, and

$$\nu(\zeta_1, \zeta_2) = \begin{cases} \zeta_1 + \zeta_2, & \text{for all } \zeta_1, \zeta_2 \in Q; \zeta_1 \neq \zeta_2, \\ 0, & \zeta_1 = \zeta_2. \end{cases}$$

Example 1.2. For $Q = [0, \infty)$, set $\theta : Q \times Q \rightarrow [1, \infty)$ and $\nu : Q \times Q \rightarrow [0, \infty)$ via $\theta(\zeta, \nu) = \frac{3+\zeta+\nu}{2}$, for all $\zeta, \nu \in Q$, and

- (1) $\nu(\zeta, \nu) = 0$, for all $\zeta, \nu \in Q$, $\zeta = \nu$,
- (2) $\nu(\zeta, \nu) = \nu(\nu, \zeta) = 5$, for all $\zeta, \nu \in Q - \{0\}$, $\zeta \neq \nu$,
- (3) $\nu(\zeta, 0) = \nu(0, \zeta) = 2$, for all $\zeta \in Q - \{0\}$.

In this paper, we take advantage of the notion of extended b -metric to present new contraction conditions, and we make use of our new contractions to formulate new results related to a fixed point of a mapping that satisfies a set of conditions. More precisely, we will prove six new fixed point theorems in the context of extended b -metric spaces. Also, we construct two examples to show the validity and usefulness of our findings. Furthermore, we add an application to an integral equation to support our results.

2. Mains results

From now on, FP stands for a fixed point.

Theorem 2.1. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(\nu, Tv), \quad (2.1)$$

for all $s, \nu \in Q$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m)\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1-h}{r}, \quad (2.2)$$

where $s_i = T^i s_0$ for $s_0 \in Q$. Also, suppose that for any $\nu, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(\nu, T^i s) \text{ exists and is finite.}$$

Then, T has a FP in Q .

Proof. Let $s_0 \in Q$. Then, set up a sequence (s_t) in Q by putting $s_t = Q^t s_0$. For $t \in \mathbf{N}$, condition (2.1) gives

$$\nu(s_t, s_{t+1}) = \nu(Qs_{t-1}, Qs_t) \leq r\theta(s_{t-1}, s_t)\nu(s_{t-1}, s_t) + h\nu(s_t, s_{t+1}). \quad (2.3)$$

Simplifying inequality (2.3) to have

$$\nu(s_t, s_{t+1}) \leq \frac{r}{1-h}\theta(s_{t-1}, s_t)\nu(s_{t-1}, s_t). \quad (2.4)$$

For $t \in \mathbf{N}$, inequality (2.4) yields

$$\nu(s_t, s_{t+1}) \leq \left(\frac{r}{1-h}\right)^t \prod_{s=1}^t \theta(s_{s-1}, s_s)\nu(s_0, s_1). \quad (2.5)$$

For $t, m \in \mathbf{N}$ with $m > t$, we choose $k \in \mathbf{N}$ with $m = t + k$. The triangular inequality of the definition ν produces

$$\begin{aligned} \nu(s_t, s_{t+k}) &\leq \theta(s_t, s_{t+k})\nu(s_t, s_{t+1}) + \theta(s_t, s_{t+k})\nu(s_{t+1}, s_{t+k}) \\ &\leq \theta(s_t, s_{t+k})\nu(s_t, s_{t+1}) + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\nu(s_{t+1}, s_{t+2}) \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\nu(s_{t+2}, s_{t+k}) \\ &\leq \theta(s_t, s_{t+k})\nu(s_t, s_{t+1}) + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\nu(s_{t+1}, s_{t+2}) \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\theta(s_{t+2}, s_{t+k})\nu(s_{t+2}, s_{t+3}) \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\theta(s_{t+2}, s_{t+k})\nu(s_{t+3}, s_{t+k}) \\ &\leq \dots \\ &\leq \theta(s_t, s_{t+k})\nu(s_t, s_{t+1}) + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\nu(s_{t+1}, s_{t+2}) \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\theta(s_{t+2}, s_{t+k})\nu(s_{t+2}, s_{t+3}) \\ &\quad + \\ &\quad \vdots \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k}) \dots \theta(s_{t+k-2}, s_{t+k-1})\nu(s_{t+k-2}, s_{t+k-1}) \\ &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k}) \dots \theta(s_{t+k-2}, s_{t+k-1})\nu(s_{t+k-1}, s_{t+k}). \end{aligned}$$

In light of the values of θ greater than or equal to 1, the above inequalities imply

$$\begin{aligned}
 \nu(s_t, s_{t+k}) &\leq \theta(s_t, s_{t+k})\nu(s_t, s_{t+1}) + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\nu(s_{t+1}, s_{t+2}) \\
 &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k})\theta(s_{t+2}, s_{t+k})\nu(s_{t+2}, s_{t+3}) \\
 &\quad + \\
 &\quad \vdots \\
 &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k}) \dots \theta(s_{t+k-2}, s_{t+k-1})\nu(s_{t+k-2}, s_{t+k-1}) \\
 &\quad + \theta(s_t, s_{t+k})\theta(s_{t+1}, s_{t+k}) \dots \theta(s_{t+k-2}, s_{t+k-1})\theta(s_{t+k-1}, s_{t+k})\nu(s_{t+k-1}, s_{t+k}) \\
 &= \sum_{j=t}^{t+k-1} \prod_{i=t}^j \theta(s_i, s_{t+k})\nu(s_j, s_{j+1}). \tag{2.6}
 \end{aligned}$$

By using inequalities (2.5) and (2.6), it becomes

$$\nu(s_t, s_m) \leq \sum_{j=t}^{t+k-1} \prod_{i=t}^j \theta(s_i, s_{t+k}) \left(\frac{r}{1-h}\right)^j \prod_{y=1}^j \theta(s_{y-1}, s_y)\nu(s_0, s_1). \tag{2.7}$$

Define

$$\prod_{i=t}^j \theta(s_i, s_{t+k}) \left(\frac{r}{1-h}\right)^j \prod_{y=1}^j \theta(s_{y-1}, s_y)\nu(s_0, s_1) := I_j. \tag{2.8}$$

Then,

$$\lim_{j \rightarrow +\infty} \frac{I_{j+1}}{I_j} = \lim_{j \rightarrow +\infty} \theta(s_{j+1}, s_{t+k})\theta(s_j, s_{j+1}) \frac{r}{1-h} < 1.$$

The ratio test makes certain

$$\left(\sum_{i=t}^{t+k-1} \prod_{i=t}^j \theta(s_i, s_{t+k}) \left(\frac{r}{1-h}\right)^j \prod_{y=1}^j \theta(s_{y-1}, s_y)\nu(s_0, s_1) \right)$$

is Cauchy, accordingly the sequence (s_t) is Cauchy in (Q, ν) . So, $\exists s' \in Q$ as an output of the completeness of (Q, ν) , such that

$$\lim_{t \rightarrow \infty} \nu(s_t, s') = 0. \tag{2.9}$$

Our mission is to verify $Ts' = s'$. Before that, we need to verify

$$\lim_{t \rightarrow +\infty} \nu(s_t, s_{t+1}) = 0. \tag{2.10}$$

The triangular inequality with addition to inequality (2.2) emphasize

$$\limsup_{t \rightarrow +\infty} \nu(s_t, s_{t+1}) \leq \frac{1-h}{r} \lim_{t \rightarrow +\infty} (\nu(s_t, s') + \nu(s', s_{t+1})) = 0. \tag{2.11}$$

Thus, (2.10) has been achieved. Again, the triangular inequality and (2.1) yield

$$\begin{aligned}
 \nu(s', Ts') &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(Ts', s_{t+1}) \\
 &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')(r\theta(s', s_t)\nu(s', s_t) + h\nu(s_t, s_{t+1})). \tag{2.12}
 \end{aligned}$$

On letting $t \rightarrow +\infty$ in (2.12) and benefiting from (2.9) and (2.10), we arrive at $\nu(s', Ts') = 0$. Accordingly, $Ts' = s'$.

In the following result, we assume that θ is continuous in its variables.

Theorem 2.2. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, \infty)$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(\nu, Tv),$$

for all $s, \nu \in Q$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m)\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1-h}{r},$$

where $s_i = T^i s_0$ for $s_0 \in Q$. If θ is continuous in its variables, then T has a FP in Q .

Proof. We proceed in the same way as in proof of Theorem 2.1, to generate a sequence $(s_t = T^t s_0)$ in Q , such that $s_t \rightarrow s' \in Q$ and

$$\lim_{t \rightarrow +\infty} \nu(s_t, s_{t+1}) = \lim_{t \rightarrow +\infty} \nu(s_t, s') = \lim_{t \rightarrow +\infty} \nu(s, s_t) = 0.$$

Also, the continuity of θ in its variables implies that

$$\lim_{t \rightarrow +\infty} \theta(s', s_t) = \theta(s', s'). \quad (2.13)$$

Now,

$$\begin{aligned} \nu(s', Ts') &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(Ts', s_{t+1}) \\ &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')(r\theta(s', s_t)\nu(s', s_t) + h\nu(s_t, s_{t+1})). \end{aligned} \quad (2.14)$$

Allow $t \rightarrow +\infty$ in the above inequalities, and make use of (2.13) and (2.14) to obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} \nu(s', Ts') &\leq \theta(s', Ts') \lim_{t \rightarrow +\infty} \nu(s', s_{t+1}) + \theta(s', Ts')(r \lim_{t \rightarrow +\infty} \theta(s', s_t)\nu(s', s_t) + h \lim_{t \rightarrow +\infty} \nu(s_t, s_{t+1})) \\ &= 0. \end{aligned}$$

This means that $Ts' = s'$. Thus, the desired result is obtained.

The uniqueness of the FP in Theorem 2.1 or Theorem 2.2 can be obtained if an appropriate condition is added.

Theorem 2.3. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(\nu, Tv),$$

for all $s, \nu \in Q$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m)\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1-h}{r},$$

where $s_i = T^i s_0$ for $s_0 \in Q$. Moreover, assume that for any $\nu, s_0 \in Q$,

$$\limsup_{i \rightarrow +\infty} \theta(\nu, T^i s_0) \text{ exists and is finite or } \theta \text{ is continuous.}$$

Also, suppose that $\forall v, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(T^i v, T^i s) \text{ exists and less than } \frac{1}{r}.$$

Then, T has only one FP in Q .

Proof. If for any $v, s_0 \in Q$, we have $\limsup_{i \rightarrow +\infty} \theta(v, T^i s_0)$ exists and less than $\frac{1-h}{r}$, then Theorem 2.1 ensures that $\exists s' \in Q$, such that $T s' = s'$.

If θ is continuous in its variables, then Theorem 2.2 ensures that $\exists s' \in Q$, such that $T s' = s'$.

To verify that T achieves only one FP, let $v' \in Q$, such that $T v' = v'$. Now,

$$\begin{aligned} \nu(v', s') = \nu(T v', T s') &\leq r\theta(v', s')\nu(v', s') + k\nu(s', T s') \\ &= r\theta(T^t v', T^t s')\nu(v', s') + h\nu(s', T s') \\ &= r\theta(T^t v', T^t s')\nu(v', s'). \end{aligned}$$

On taking the limit of supremum as $t \rightarrow +\infty$ in the above inequality, we reach to

$$\nu(v', s') < \nu(v', s'),$$

a contradiction. Thus, $v' = s'$ and we conclude that T has only one FP.

Corollary 2.1. Suppose (Q, ν) is complete. Assume there exists $r \in (0, 1]$, such that $T : Q \rightarrow Q$ satisfies

$$\nu(T s, T v) \leq r\theta(s, v)\nu(s, v)$$

for all $s, v \in Q$. For $s_0 \in Q$, let $s_n = T^n s_0$. Assume for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m)\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1}{r}.$$

Also, suppose that for any $v, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(v, T^i s) \text{ exists and is finite.}$$

Then, T has a FP in Q .

Proof. By choosing $h = 0$ in Theorem 2.1, we obtain the result as desired.

Corollary 2.2. Suppose (Q, ν) is complete. Assume there exists $r \in (0, 1]$, such that $T : Q \rightarrow Q$ satisfies

$$\nu(T s, T v) \leq r\theta(s, v)\nu(s, v)$$

for all $s, v \in Q$. For $s_0 \in Q$, let $s_n = T^n s_0$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_{i+m})\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1}{r}.$$

If θ is continuous in its variables, then, T has a FP in Q .

Proof. By choosing $h = 0$ in Theorem 2.2, we obtain the result as desired.

Corollary 2.3. *Suppose (Q, ν) is complete. Assume there exists $r \in (0, 1]$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu),$$

for all $s, \nu \in Q$. For $s_0 \in Q$, let $s_n = T^n s_0$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m)\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1}{r}.$$

Moreover, assume that for any $\nu, s_0 \in Q$,

$$\limsup_{i \rightarrow +\infty} \theta(\nu, T^i s_0) \text{ exists and is finite or } \theta \text{ is continuous.}$$

Suppose for any $\nu, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(T^i \nu, T^i s) \text{ exists and less than } \frac{1}{r}.$$

Then, T has only one FP in Q .

Proof. By taking $h = 0$ in Theorem 2.3, we get the desired result.

Corollary 2.4. *Suppose (Q, ν) is a complete b -metric space with constant $b \geq 1$. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$ with $b^2 r + h < 1$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq rb\nu(s, \nu) + h\nu(\nu, Tv), \quad (2.15)$$

for all $s, \nu \in Q$. Then, T has only one FP in Q .

Proof. Define $\theta : Q \times Q \rightarrow [0, +\infty)$ via $\theta(s, p) = b, \forall s, \nu \in Q$. Now, for $s_0 \in Q$, we have

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_{i+m})\theta(s_i, s_{i+1}) = b^2 < \frac{1-h}{r}.$$

Also, from $br \leq b^2 r + h < 1$, we arrive at

$$\limsup_{i \rightarrow +\infty} \theta(\nu, T^i s_0) = b < \frac{1}{r}.$$

So, all conditions of Theorem 2.3 are met. So, the result also follows.

Corollary 2.5. *Suppose (Q, ν) is complete. Assume there exists $r \in (0, 1]$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\nu(s, \nu), \quad (2.16)$$

for all $s, \nu \in Q$. Assume that for any $m \in \mathbf{N}$,

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_{i+m})\theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1-h}{r},$$

where $s_i = T^i s_0$ for $s_0 \in Q$. Suppose that for all $\nu, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(T^i \nu, T^i s) \text{ exists and less than } \frac{1}{r}.$$

Then, T has only one FP in Q .

Proof. Let $(s_{t+1} = Ts_t)$ be a sequence in Q constructed as in the proof of Theorem 2.1. So,

$$\nu(s_t, s_{t+1}) \leq r^t \nu(s_0, s_1).$$

Thus,

$$\lim_{t \rightarrow +\infty} \nu(s_t, s_{t+1}) = 0$$

has been obtained. Take $m \in \mathbf{N}$, as in the proof of Theorem 2.1, we obtain

$$\lim_{t \rightarrow +\infty} \nu(s_t, s_{t+m}) = 0,$$

and, hence, (s_t) is Cauchy in Q . Then, one can show that T has a FP, say $t \in Q$. Since $r < 1$, then the uniqueness of t follows from inequality 2.16.

Theorem 2.4. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(s, Tv), \quad (2.17)$$

for all $s, \nu \in Q$. Also, suppose that for any $m \in \mathbf{N}$,

$$\limsup_{j \rightarrow +\infty} \theta(s_{j+1}, s_m) \left(\frac{r\theta(s_j, s_{j+1}) + h\theta(s_j, s_{j+2})}{1 - h\theta(s_j, s_{j+2})} \right) < 1, \quad (2.18)$$

where $s_i = T^i s_0$ for $s_0 \in Q$. Moreover, assume that for any $\nu \in Q$, we have $\limsup_{i \rightarrow +\infty} \nu(\nu, s_i)$ exists and is finite. Then, T has a FP in Q .

Proof. Presume $s_0 \in Q$. Then, set up a sequence (s_t) in Q , such that $s_t = Q^t s_0$ owns condition (2.18). For $t \in \mathbf{N}$, condition (2.17) gives

$$\begin{aligned} \nu(s_t, s_{t+1}) &= \nu(Ts_{t-1}, Ts_t) \\ &\leq r\theta(s_{t-1}, s_t)\nu(s_{t-1}, s_t) + h\nu(s_{t-1}, s_{t+1}) \\ &\leq r\theta(s_{t-1}, s_t)\nu(s_{t-1}, s_t) + h\theta(s_{t-1}, s_{t+1})\nu(s_{t-1}, s_t) + h\theta(s_{t-1}, s_{t+1})\nu(s_t, s_{t+1}). \end{aligned} \quad (2.19)$$

Simplifying inequality (2.19) to have

$$\nu(s_t, s_{t+1}) \leq \frac{r\theta(s_{t-1}, s_t) + h\theta(s_{t-1}, s_{t+1})}{1 - h\theta(s_{t-1}, s_{t+1})} \nu(s_{t-1}, s_t). \quad (2.20)$$

For $n \in \mathbf{N}$, inequality (2.20) yields

$$\nu(s_t, s_{t+1}) \leq \prod_{y=1}^t \frac{r\theta(s_{y-1}, s_y) + h\theta(s_{y-1}, s_{y+1})}{1 - h\theta(s_{y-1}, s_{y+1})} \nu(s_0, s_1). \quad (2.21)$$

Choose $t, m \in \mathbf{N}$ in such a way that $m > t$. Select $k \in \mathbf{N}$, such that $m = t + k$. By helping with triangular inequality of the definition ν and imitation of the procedure in the proof of Theorem 2.1, at the end of the day, we obtain

$$\nu(s_t, s_{t+k}) \leq \sum_{j=t}^{t+k-1} \prod_{i=t}^j \theta(s_i, s_{i+k}) \nu(s_j, s_{j+1}). \quad (2.22)$$

By employing inequality (2.21), inequality (2.22) can be written as

$$\nu(s_t, s_m) \leq \sum_{j=t}^{t+k-1} \prod_{i=t}^j \theta(s_i, s_{t+k}) \prod_{y=1}^j \frac{r\theta(s_{y-1}, s_y) + h\theta(s_{y-1}, s_{y+1})}{1 - h\theta(s_{y-1}, s_{y+1})} \nu(s_0, s_1). \quad (2.23)$$

Define

$$\prod_{i=t}^j \theta(s_i, s_{t+k}) \prod_{y=1}^j \frac{r\theta(s_{y-1}, s_y) + h\theta(s_{y-1}, s_{y+1})}{1 - h\theta(s_{y-1}, s_{y+1})} \nu(s_0, s_1) := I_j. \quad (2.24)$$

Then,

$$\lim_{j \rightarrow +\infty} \frac{I_{j+1}}{I_j} = \lim_{j \rightarrow +\infty} \theta(s_{j+1}, s_{t+k}) \left(\frac{r\theta(s_j, s_{j+1}) + h\theta(s_j, s_{j+2})}{1 - h\theta(s_j, s_{j+2})} \right) < 1.$$

Ratio test implies that

$$\begin{aligned} & \sum_{j=1}^{+\infty} \prod_{i=t}^j \theta(s_i, s_{t+k}) \prod_{y=1}^j \frac{r\theta(s_{y-1}, s_y) + h\theta(s_{y-1}, s_{y+1})}{1 - h\theta(s_{y-1}, s_{y+1})} \nu(s_0, s_1) \\ \rightarrow s &= \sum_{j=1}^{+\infty} \prod_{i=t}^j \theta(s_i, s_{t+k}) \prod_{y=1}^j \frac{r\theta(s_{y-1}, s_y) + h\theta(s_{y-1}, s_{y+1})}{1 - h\theta(s_{y-1}, s_{y+1})} \nu(s_0, s_1). \end{aligned}$$

By moving towards infinity in (2.23), the following will be achieved:

$$\lim_{t, m \rightarrow +\infty} \nu(s_t, s_m) = 0,$$

and, hence, (s_t) is Cauchy in (Q, ν) . As an output of the completeness of (Q, ν) , we find $s' \in Q$, such that $s_t \rightarrow s'$; that is,

$$\lim_{t \rightarrow \infty} \nu(s_t, s') = \lim_{t \rightarrow \infty} \nu(s', s_t) = 0. \quad (2.25)$$

Our task is to verify $Ts' = s'$. Now, (2.25) and (2.17) lead us to

$$\lim_{t \rightarrow +\infty} \nu(s_{t+1}, Ts') = \lim_{t \rightarrow +\infty} \nu(Ts', Ts_t) \leq \lim_{t \rightarrow +\infty} (r\theta(s', s_t)\nu(s', s_t) + h\nu(s', s_{t+1})) = 0. \quad (2.26)$$

By using the triangular inequality, and then moving towards infinity to obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} \nu(s', Ts') &\leq \lim_{t \rightarrow +\infty} (\theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(s_{t+1}, Ts')) \\ &= 0, \end{aligned}$$

and, hence, $\nu(s', Ts') = 0$. Accordingly, $Ts' = s'$.

In our next result, we assume that θ is continuous in its variables.

Theorem 2.5. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(s, Tv), \quad (2.27)$$

for all $s, \nu \in Q$. Also, suppose that for any $m \in \mathbf{N}$,

$$\limsup_{j \rightarrow +\infty} \theta(s_{j+1}, s_m) \left(\frac{r\theta(s_j, s_{j+1}) + h\theta(s_j, s_{j+2})}{1 - h\theta(s_j, s_{j+2})} \right) < 1, \quad (2.28)$$

where $s_j = T^j s_0$ for $s_0 \in Q$. If T is continuous, then T has a FP in Q .

Proof. By starting with $s_0 \in Q$, we launch a sequence (s_t) as in the proof of Theorem 2.4, such that there exists $s' \in Q$ with

$$\lim_{t \rightarrow +\infty} \nu(s_t, s') = \lim_{t \rightarrow +\infty} \nu(s', s_t) = \lim_{t \rightarrow +\infty} \nu(s_t, s_{t+1}) = 0.$$

Now, we show that $Ts' = s'$. The triangular inequality and inequality 2.27 imply that

$$\begin{aligned} \nu(s', Ts') &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(s_{t+1}, Ts') \\ &= \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(Ts_t, Ts') \\ &= \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')\nu(Ts', Ts_t) \\ &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')r\theta(s', s_t)\nu(s', s_t) + h\theta(s', Ts')\nu(s', Ts_t) \\ &\leq \theta(s', Ts')\nu(s', s_{t+1}) + \theta(s', Ts')r\theta(s', s_t)\nu(s', s_t) + h\theta(s', Ts')\nu(s', s_{t+1}). \end{aligned}$$

By permitting $t \rightarrow +\infty$ in the above inequalities, we have $\nu(s', Ts') = 0$ and, hence, $s' = Ts'$.

The uniqueness of the FP can be achieved in Theorem 2.4 or Theorem 2.5 if a suitable condition is added.

Theorem 2.6. *Suppose (Q, ν) is complete. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq r\theta(s, \nu)\nu(s, \nu) + h\nu(s, Tv),$$

for all $s, \nu \in Q$. Also, suppose that for any $m \in \mathbf{N}$,

$$\limsup_{j \rightarrow +\infty} \theta(s_{j+1}, s_{j+1+m}) \left(\frac{r\theta(s_j, s_{j+1}) + h\theta(s_j, s_{j+2})}{1 - h\theta(s_j, s_{j+2})} \right) < 1,$$

where $s_j = T^j s_0$ for $s_0 \in Q$. Moreover, assume that for any $\nu \in Q$, we have $\limsup_{t \rightarrow +\infty} \theta(\nu, s_t)$ exists and is finite. Also, suppose for any $\nu, s \in Q$, $\limsup_{t \rightarrow +\infty} \theta(T^t \nu, T^t s)$ exists and less than $\frac{1-h}{r}$. Then, T has only one FP in Q .

Proof. Theorem 2.4 ensures that there exists $s' \in Q$ with $Ts' = s'$. To verify that T achieves only one fixed point, we suppose there exists $\nu' \in Q$ with $s' \neq \nu'$, such that $T\nu' = \nu'$. Now,

$$\begin{aligned} \nu(s', \nu') &= \nu(Ts', T\nu') \leq r\theta(s', \nu')\nu(s', \nu') + h\nu(s', T\nu') \\ &= r\theta(T^t s', T^t \nu')\nu(s', \nu') + h\nu(s', \nu'). \end{aligned}$$

Rewrite the above inequality in a proper form, then we have

$$\begin{aligned} \nu(s', \nu') &\leq \frac{r}{1-h}\theta(s', \nu')\nu(s', \nu') \\ &= \frac{r}{1-h}\theta(T^t s', T^t \nu')\nu(s', \nu'). \end{aligned}$$

By permitting t tends to infinity in the above inequality, we get $\nu(s', \nu') < \nu(s', \nu')$, a contradiction. Thus, $s' = \nu'$ and, hence, T has only FP in Q .

Corollary 2.6. *Suppose (Q, ν) is a complete b -metric space with constant $b \geq 1$. Assume there exist $r \in (0, 1]$ and $h \in [0, 1)$ with $b^2 r + h(b^2 + b) < 1$, such that $T : Q \rightarrow Q$ satisfies*

$$\nu(Ts, Tv) \leq rb\nu(s, \nu) + h\nu(s, Tv), \quad (2.29)$$

for all $s, \nu \in Q$. Then, T has only one FP in Q .

Proof. Define $\theta : Q \times Q \rightarrow [1, +\infty)$ via $\theta(s, v) = b$, $\forall s, v \in Q$. Now, for $s_0 \in Q$ and $m \in \mathbf{N}$, we have

$$\limsup_{j \rightarrow +\infty} \theta(s_{j+1}, s_{j+1+m}) \left(\frac{r\theta(s_j, s_{j+1}) + h\theta(s_j, s_{j+2})}{1 - h\theta(s_j, s_{j+2})} \right) = b \left(\frac{rb + hb}{1 - hb} \right) < 1.$$

Also, from $br + h \leq b^2r + h(b^2 + b) < 1$, we arrive at

$$\lim_{t \rightarrow +\infty} \theta(v, T^t s_0) = b < \frac{1-h}{r}.$$

So, all conditions of Theorem 2.6 are met. So, the result also follows.

Now, we present some examples of our results.

Example 2.1. Let $Q = [0, +\infty)$. Define $T : Q \rightarrow Q$ via $Tv = \frac{1}{4}v$ and $\theta : Q \times Q \rightarrow [1, \infty)$ by $\theta(v, s) = v + s + 1$. Also, define $\nu : Q \times Q \rightarrow [0, +\infty)$ via

$$\nu(v, s) = \begin{cases} 0, & \text{if } v = s, \\ \frac{v}{1+v}, & \text{if } v \neq 0, s = 0, \\ \frac{s}{1+s}, & \text{if } v = 0, s \neq 0, \\ \max\{v, s\}, & \text{if } 0 \neq v \neq s \neq 0. \end{cases}$$

Then:

- (1) ν is extended b -metric, which is not b -metric.
- (2) (Q, ν) is complete.
- (3) Let $v_0 \in Q$, take $v_n = T^n v_0$. Then, for $m \in \mathbf{N}$, we have

$$\limsup_{i \rightarrow +\infty} \theta(v_i, v_m) \theta(v_i, v_{i+1}) = 1 < 2 = \frac{1-h}{r}.$$

- (4) For any $v, s_0 \in Q$, we have

$$\limsup_{n \rightarrow +\infty} \theta(v, T^n s_0) = v + 1 \text{ exists and is finite.}$$

- (5) For any $v, s \in Q$, we have

$$\limsup_{t \rightarrow +\infty} \theta(T^t v, T^t s) = 1 \leq 4 = \frac{1}{r}.$$

- (6) For $s, v \in Q$, we have

$$\nu(Ts, Tv) \leq \frac{1}{4}\theta(s, v)\nu(s, v) + \frac{1}{2}\nu(v, Ts).$$

We note that the hypotheses of Theorem 2.3 have been fulfilled for $r = \frac{1}{4}$ and $h = \frac{1}{2}$.

Example 2.2. Let $Q = \{0, 1, 2, 3, \dots\}$. Define $T : Q \rightarrow Q$ via

$$Tv = \begin{cases} \frac{v}{2}, & \text{if } v \text{ is even,} \\ \frac{v-1}{2}, & \text{if } v \text{ is odd,} \end{cases}$$

and $\theta : Q \times Q \rightarrow [1, \infty)$ by

$$\theta(v, s) = \begin{cases} v + s, & \text{if } (v, s) \neq (0, 0), \\ 1, & \text{if } (v, s) = (0, 0). \end{cases}$$

Also, define $\nu : Q \times Q \rightarrow [0, +\infty)$ via

$$\nu(v, s) = \begin{cases} 0, & \text{if } v = s, \\ 1, & \text{if one of } v \text{ or } s \text{ is even and the other is odd,} \\ \min\{v, s\}, & \text{if both of } v \text{ and } s \text{ are even or both are odd, provided that } v \neq s. \end{cases}$$

Then:

- (1) ν is extended b -metric, which is not b -metric.
- (2) (Q, ν) is complete.
- (3) Let $s_0 \in Q$, take $(s_t) = (T^t s_0)$. Then, for $m \in \mathbf{N}$, we have

$$\limsup_{j \rightarrow +\infty} \theta(s_{j+1}, s_m) \left(\frac{\frac{1}{4}\theta(s_j, s_{j+1}) + \frac{1}{4}\theta(s_j, s_{j+2})}{1 - \frac{1}{4}\theta(s_j, s_{j+2})} \right) = \frac{2}{3} < 1.$$

- (4) For any $v, q \in Q$, we have

$$\limsup_{n \rightarrow +\infty} \theta(v, T^n q_0) = v \text{ or } 0 \text{ exists and is finite.}$$

- (5) For any $v, s \in Q$, we have

$$\limsup_{n \rightarrow +\infty} \theta(T^n v, T^n s) = 1 \leq 3 = \frac{1-h}{r}.$$

- (6) For $v, s \in Q$, we have

$$\nu(Tv, Ts) \leq \frac{1}{4}\theta(v, s)\nu(v, s) + \frac{1}{4}\nu(v, Ts).$$

We note that the hypotheses of Theorem 2.6 have been fulfilled for $r = \frac{1}{4}$ and $h = \frac{1}{4}$.

3. Application

In this section, our goal is to present some applications of our findings.

We start this section by giving a solution to the following integral equation:

$$f(t) = \int_0^1 K(t, f(s))ds. \quad (3.1)$$

Now, let $Q = C([0, 1])$ be the set of all continuous functions on $[0, 1]$. Define $\|\cdot\|_\infty : Q \rightarrow [0, +\infty)$ by

$$\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|.$$

Also, define $\nu : Q \times Q \rightarrow [0, +\infty)$ via

$$\nu(f, g) = \begin{cases} 0, & \text{if } f = g, \\ \max\{\|f\|_\infty, \|g\|_\infty\}, & \text{if } f \neq g, \end{cases}$$

and $\theta : Q \times Q \rightarrow [1, +\infty)$ via

$$\theta(f, g) = \max \{1 + \|f\|_\infty, 1 + \|g\|_\infty\}.$$

Then, (Q, ν) is an extended b -metric space.

Now, let

$$A := \{f : f \in Q, \|f\|_\infty \leq 1\}.$$

Then, one can show that A is a closed subspace of Q . So, (A, ν) is complete.

Theorem 3.1. *Suppose the following conditions:*

- (1) $K : [0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$ is continuous, and
- (2) There exists $r \in [0, \frac{1}{4})$, such that

$$\sup_{t \in [0, 1]} \int_0^1 |K(t, f(s))| ds \leq r(1 + \|f\|_\infty)\|f\|_\infty$$

for all $f \in A$.

Then, the integral equation (3.1) has a unique solution.

Proof. Define $T : A \rightarrow A$ by

$$(Tf)(t) = \int_0^1 K(t, f(s)) ds, \quad t \in [0, 1].$$

Now, for $f \in A$, we have

$$\begin{aligned} \|Tf\|_\infty &= \sup_{t \in [0, 1]} |(Tf)(t)| \\ &= \sup_{[0, 1]} \left| \int_0^1 K(t, f(s)) ds \right| \\ &\leq \sup_{[0, 1]} \int_0^1 |K(t, f(s))| ds \\ &\leq r(1 + \|f\|_\infty)\|f\|_\infty \\ &\leq \frac{1}{2}. \end{aligned}$$

So, we conclude that

$$\|T\| \leq \frac{1}{2}. \quad (3.2)$$

For $f, g \in Q$, we have

$$\begin{aligned} \nu(Tf, Tg) &= \max \{\|Tf\|_\infty, \|Tg\|_\infty\} \\ &\leq r \max \{(1 + \|f\|_\infty)\|f\|_\infty, (1 + \|g\|_\infty)\|g\|_\infty\} \\ &\leq r \max \{1 + \|f\|_\infty, 1 + \|g\|_\infty\} \max \{\|f\|_\infty, \|g\|_\infty\} \\ &= r\theta(f, g)\nu(f, g). \end{aligned}$$

Let $f_0 \in A$. Then, for $t \in [0, 1]$ and $i \in \mathbf{N}$, we have $|f_i(t)| = |T^i(f_0)(t)| \leq \|T\|_\infty^i \|f_0\|_\infty |t|$. So, $\|f_i\|_\infty \leq \|T\|_\infty^i$. Therefore,

$$\begin{aligned} \limsup_{i \rightarrow +\infty} \theta(f_i, f_{i+1}) \theta(f_i, f_m) &= \limsup_{i \rightarrow +\infty} \max \{1 + \|f_i\|_\infty, 1 + \|f_{i+1}\|_\infty\} \max \{1 + \|f_i\|_\infty, 1 + \|f_m\|_\infty\} \\ &\leq \limsup_{i \rightarrow +\infty} \max \{1 + \|T\|_\infty^i, 1 + \|T\|_\infty^{i+1}\} \max \{1 + \|T\|_\infty^i, 1 + \|T\|_\infty^m\} \\ &\leq \limsup_{i \rightarrow +\infty} \max \left\{ 1 + \left(\frac{1}{2}\right)^i, 1 + \left(\frac{1}{2}\right)^{i+1} \right\} \max \left\{ 1 + \left(\frac{1}{2}\right)^i, 1 + \left(\frac{1}{2}\right)^{i+m} \right\} \\ &= 1 \leq 4 < \frac{1}{r}. \end{aligned}$$

Let $v, f \in Q$. Then,

$$\begin{aligned} \limsup_{i \rightarrow +\infty} \theta(v, T^i f) &= \limsup_{i \rightarrow +\infty} \max \{1 + \|v\|_\infty, 1 + \|T^i f\|_\infty\} \\ &\leq \limsup_{i \rightarrow +\infty} \max \{1 + \|v\|_\infty, 1 + \|T\|_\infty^i \|f\|_\infty\} \\ &\leq 2. \end{aligned}$$

So, $\limsup_{i \rightarrow +\infty} \theta(v, T^i f)$ exists and is finite. So, all conditions of Theorem 2.1 have been achieved. Accordingly, T has a fixed point in $A \subseteq Q$. So, the integral equation (3.1) has a solution in Q .

Our second application is to give a solution to equation of the form $f(s) = 0$, where $s \in Q = [0, +\infty)$.

Theorem 3.2. For $t \geq 1$ and two integers m and k with $4m^2 < (1+t)^2$ and $(1+t) \leq k$, the equation

$$(s-1)(s+t)^{2m} + (k^{2m} + 1)s - k^{2m} = 0$$

has a unique real solution s' in $[0, +\infty)$.

Proof. Let $Q = [0, +\infty)$. Define $T : Q \rightarrow Q$ by

$$Ts = \frac{(s+t)^{2m} + k^{2m}}{(s+t)^{2m} + k^{2m} + 1}.$$

Also, define $\theta : Q \times Q \rightarrow [1, +\infty)$ by

$$\theta(s, v) = \sum_{i=1}^{2m} (s+t)^{2m-i} (v+t)^{i-1}.$$

On $Q \times Q$, we define the complete extended b -metric ν by

$$\nu(s, v) = |s - v|.$$

Then:

(1) For $s, v \in Q$, we have

$$\nu(Ts, Tv) \leq \frac{1}{k^{4m}} \theta(s, v) \nu(s, v).$$

Indeed,

$$\begin{aligned}
 \nu(Ts, Tv) &= |Ts - Tv| \\
 &= \left| \frac{(s+t)^{2m} + k^{2m}}{(s+t)^{2m} + k^{2m} + 1} - \frac{(v+t)^{2m} + k^{2m}}{(v+t)^{2m} + k^{2m} + 1} \right| \\
 &= \left| \frac{(s+t)^{2m} - (v+t)^{2m}}{((s+t)^{2m} + k^{2m} + 1)((v+t)^{2m} + k^{2m} + 1)} \right| \\
 &\leq \frac{1}{k^{4m}} \left| (s+t)^{2m} - (v+t)^{2m} \right| \\
 &\leq \frac{1}{k^{4m}} \sum_{i=1}^{2m} (s+t)^{2m-i} (v+t)^{i-1} |s-v| \\
 &= \frac{1}{k^{4m}} \theta(v, l) \nu(v, l).
 \end{aligned}$$

(2) For $s_0 \in Q$, put $s_i = T^i s_0$. Then, one can show that

$$\limsup_{i \rightarrow \infty} \theta(s_i, s_m) \theta(s_i, s_{i+1}) \text{ exists and less than } \frac{1}{r} = k^{4m}.$$

(3) θ is continuous in its variables.

(4) For $v, s \in Q$, we have

$$\limsup_{i \rightarrow +\infty} \theta(T^i v, T^i s) \text{ exists and less than } \frac{1}{r} = k^{4m}.$$

So, all the conditions of Corollary 2.3 have been fulfilled. Therefore, T has a unique FP in $[0, +\infty)$.

Corollary 3.1. For $k \geq 3$, the equation

$$s^3 + (k^2 + 2k - 3)s^2 + (-4k + 3)s - k^2 + 2k - 1 = 0$$

has a unique real solution s' in $[0, +\infty)$.

Proof. We can show that the equation

$$s^3 + (k^2 + 2k - 3)s^2 + (-4k + 3)s - k^2 + 2k - 1 = 0$$

is equivalent to

$$(s-1)(s+k-1)^2 + (k^2+1)s - k^2 = 0.$$

The result follows from Theorem 3.2 by taking $m = 1$, $t = k - 1$ and noting that $4m^2 < (1+t)^2$ and $1+t \leq k$.

Example 3.1. The equation

$$s^3 + 117s^2 - 37s - 81 = 0$$

has a unique real solution s' in $[0, +\infty)$.

Proof. The result follows from Corollary 3.1 by taking $k = 10$.

4. Conclusions

In this work, we have taken advantage of the notion of extended b -metric to present new contraction conditions. Next, we proved several new fixed point theorems in the context of extended b -metric spaces. Two examples are provided to show the validity and usefulness of our findings. Furthermore, two applications were added to support our findings.

Acknowledgments

The first author would like to thank the Prince Sultan University for facilitating publishing this paper through TAS lab.

Conflict of interest

The authors declare no conflicts of interest.

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