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Research article

Study of modified prism networks via fractional metric dimension

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Abstract: For a connected network Γ, the distance between any two vertices is the length of the shortest path between them. A vertex c in a connected network is said to resolve an edge e if the distances of c from its endpoints are unequal. The collection of all the vertices which resolve an edge is called the local resolving neighborhood set of this edge. A local resolving function is a real-valued function is defend as $\eta : V(\Gamma) \to [0,1]$ such that $\eta(R_x(e)) \geq 1$ for each edge $e \in E(\Gamma)$, where $R_x(e)$ represents the local resolving neighborhood set of a connected network. Thus the local fractional metric dimension is defined as $dim_{LF}(\Gamma) = min\ \{|\eta| : \eta\ is\ the\ minimal\ local\ resolving\ function\ of\ \Gamma\}$, where $|\eta| = \sum_{a \in R_x(e)} \eta(a)$. In this manuscript, we have established sharp bounds of the local fractional metric dimension of different types of modified prism networks and it is also proved that local fractional metric dimension remains bounded when the order of these networks approaches to infinity.

Keywords: metric dimension; fractional metric dimension; modified prism networks
Mathematics Subject Classification: 05C12, 05C76, 05C90

1. Introduction

The notion of resolving sets in general networks is introduced by Slater in 1975 and he called the minimum cardinality of a resolving set location number [1]. In next year Harary and Melter also introduced the same concept with different name and they called it the metric dimension (MD) of the connected networks. They provide a characterization of MD of the trees and they also proved that the MD of wheel $W_{1,z}$ and complete network K_z is 2 and $z-1$ respectively [2]. Later on the results of the MD of $W_{1,z}$, were improved by S. Khuller et al. and they also characterized the connected networks that those have MD 1 and 2 [3]. Shanmukha et al. improved the results of Harary and Melter and they computed the MD of wheel-related networks [4]. Chartrand et al. established the bounds on MD of connected networks in terms of the order and diameter of a network [5].

The concept of MD arises in diverse areas including network discovery and verification [6], robot navigation [7], strategies for the Mastermind game [8], combinatorial optimization [9], coin weighting [10], navigation of robots in networks [11] and image processing [12]. There are some new types of MD are discovered in recent times as local MD [13], k- MD [14], edge MD [15], fault tolrent MD [16] and some interesting results of fault-tolerant MD of convex polytope networks have been derived by Raza et al [17].

The idea of MD to find the solution of specific integer programming (IPP) is introduced by Chartrand et al. [5] and Currie and Ollermann introduced the concept of fractional metric dimension (FMD) to find improved solution of IPP [18]. The concept of FMD in the field of networking theory is formally introduced by Arumugam and Mathew, they developed different combinatorial techniques to find the exact value of FMD of different connected networks. Moreover, they also found the FMD of Petersen, cycle, friendship and cartesian product of different connected networks [19, 20]. Feng et al. established a computational technique to find FMD of vertex transitive networks and as an application they computed the FMD of hamming and generalized Johnson networks [21]. Javaid et al. characterize all those connected networks that attain FMD exactly 1 [22, 23] and Zafar et al. computed the exact value of FMD of different connected networks [24].

The notion of latest derived form of FMD known as a local fractional metric dimension (LFMD) is defined by Asiyah et al. and they calculated the exact values of the LFMD of the corona product of connected networks [25]. Javaid et al. purposed a unique methodology to compute the sharp bounds of LFMD for all the connected networks and they also proved that the lower bound of LFMD of non-bipartite networks is greater than 1 [26, 27]. Some interesting results of LFMD of different connected networks can be seen in [28–30].

In this paper, the lower and upper bounds of LFMD of generalized modified prism networks have been computed. It is also proved that all the upper bounds of all these networks is less or equal to 2, when the order of these networks approaches to ∞. The rest of the paper is organized as follows: Section 2 deals with preliminaries, Section 3 consists of the main results of LFMD of generalized modified prism network, Section 4 represents the conclusion and comparison among all the main results.

2. Preliminaries

A network Γ is a pair $(V(\Gamma) \times E(\Gamma))$ with $V(\Gamma)$ is a vertex set and $E(\Gamma) \subseteq (V(\Gamma) \times V(\Gamma))$ an edge set. A walk is a sequence of edges and vertices of a network. A path is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. For any two vertices x , y of $V(\Gamma)$ then the distance $d(x, y)$ between them is the number of edges between the shortest path connecting them. A network is called connected if there exist a path between every pair of vertices of Γ. A vertex $x \in V(\Gamma)$ resolves a pair (a, b) if $d(x, a) \neq d(x, b)$. Let $\mathbb{R} = \{r_1, r_2, r_3, ..., r_z\} \subset V(\Gamma)$ be a ordered set is considered as resolving set of Γ if each pair of vertices of Γ is resolved by some vertex in $\mathbb{R}$. A resolving set with minimum cardinality is called the metric dimension of Γ and it is defined as

$$dim(\Gamma) = min \{ |\mathbb{R}| \ : \ \mathbb{R} \ is \ resolving \ set \ of \ \Gamma\}.$$

For an edge $ab \in E(\Gamma)$ the local resolving neighbourhood set (RLN) $R_x(ab)$ of ab is defined as $R_x(ab) = \{c \in V(\Gamma) : d(a, c) \neq d(b, c)\}$. A local resolving function (LRF) is defined as $\eta : V(\Gamma) \rightarrow [0, 1]$ such that $\eta(R_x(ab)) \geq 1$ for each $R_x(ab)$ of Γ. A local resolving function η is called minimal if there exists a function $\mu : V(\Gamma) \to [0, 1]$ such that $\mu \leq \eta$ and $\mu(a) \neq \eta(a)$ for at least one $a \in \Gamma(V)$ that is not a local resolving function of Γ. If $|\eta| = \sum_{a \in R_x(ab)} \eta(a)$ then LFMD of Γ is donated by $dim_{LF}(\Gamma)$ is defined as

$$dim_{LF}(\Gamma) = min\ \{\ |\eta|\ :\ \eta\ is\ minimal\ local\ resolving\ function\ of\ \Gamma\}.$$

Throughout the paper, we have used the symbol of local resolving neighbourhood set of an edge $ab \in E(\Gamma)$ is $R_x(ab)$. For more details about local resolving neighbourhood set and local resolving function, we refer [25].

Lemma X. [26] Let $\Gamma = (V(\Gamma)\ \times\ E(\Gamma))$ be a connected network. If $|R_x(e) \cap A| \geq \omega$, $\forall e \in E(\Gamma)$ then

$$1 \leq dim_{lf}(\Gamma) \leq \frac{|V(\Gamma)|}{\omega}$$

where $\omega = min\{|R_x(e)| : e \in E(\Gamma)\}$, where $A = \cup\{R_x(e) : |R_x(e) = \omega\}$.

Lemma Y. [27] Let $\Gamma = (V(\Gamma)\ \times\ E(\Gamma))$ be a connected network. Then

$$dim_{lf}(\Gamma) \geq \frac{|V(\Gamma)|}{\sigma}$$

where $\sigma = max\{|R_x(e)| : e \in E(\Gamma)\}$.

2.1. Modified prism networks

For $z \geq 5$ the modified prism network $\mathbb{MP}_{z,1,2}$ with vertex set vertex set $V(\mathbb{MP}_{z,1,2}) = \{a_j, a'_j\ :\ 1 \leq j \leq z\}$ and edge set $E(\mathbb{MP}_{z,1,2}) = \{a_j a_{j+2} : 1 \leq j \leq z-2\} \cup \{a'_j a'_{j+1} : 1 \leq j \leq z\} \cup \{a_j a'_j : 1 \leq j \leq z\} \cup \{a_j a_{j+1} : 1 \leq j \leq z\}$, where $|V(\mathbb{MP}_{z,1,2})| = 2z$ and $|E(\mathbb{MP}_{z,1,2})| = 4z$. Fore more details see Figure 1.

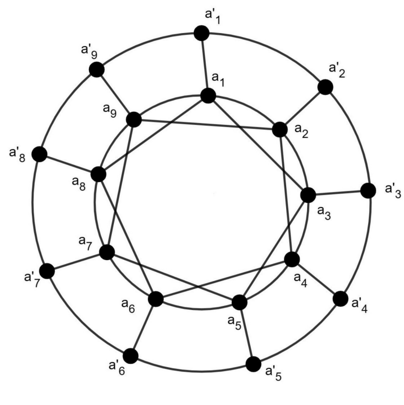

Figure 1. Modified prism network $\mathbb{MP}_{9,1,2}$.

For $z \geq 5$ the modified prism network $\mathbb{MQ}_{z,1,2}$ with vertex set vertex set $V(\mathbb{MP}_{z,1,2}) = \{a_j, a'_j\ :\ 1 \leq j \leq z\}$ and edge set $E(\mathbb{MP}_{z,1,2}) = \{a_j a_{j+2} : 1 \leq j \leq z-2\} \cup \{a'_j a'_{j+1} : 1 \leq j \leq z\} \cup \{a_j a'_j : 1 \leq j \leq z\} \cup \{a_j a_{j+1} : 1 \leq j \leq z\} \cup \{a'_j b_j : 1 \leq j \leq z\} \cup \{a_j a_{j+1} : 1 \leq j \leq z\} \cup \{b_j b_{j+1} : 1 \leq j \leq z\}$, where $|V(\mathbb{MP}_{z,1,2})| = 3z$ and $|E(\mathbb{MP}_{z,1,2})| = 6z$. Fore more details see Figure 2.

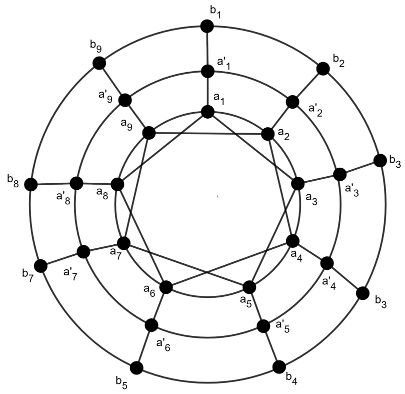

Figure 2. Modified prism network $\mathbb{MQ}_{9,1,2}$.

3. Main results

In this dissertation, our objective is to compute RLN Sets and LFMD of modified prism networks ($\mathbb{MP}_{z,1,2}, \mathbb{MQ}_{z,1,2}$) in the form of sharp upper and lower bounds.

4. RLN sets and LFMD of modified prism network $\mathbb{MP}_{z,1,2}$

In this section, we compute the RLN sets and LFMD of modified prism network ($\mathbb{MP}_{z,1,2}$).

Lemma 4.1. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network , where $z \cong 1 \ (mod\ 4)$. Then*
(a) $|R_x(a_ja_{j+1})| = z - 1$ and $\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$ where $|R_x(y)|$ are the other possible resolving local neighbourhood sets.

Proof. Let a_j inner, a'_j be the outer vertices of modified generalized Prism network, for $1 \leq j \leq z$, where $z + 1 \cong (1\ mod\ z)$, we have following possibilities
(a) $R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}....., a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}, ..., a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\} \cup \{a_{\frac{z+2i+2}{2}}\} \cup \{a'_{\frac{z+2i+2}{2}}\}$ and $|R_x(a_ja_{j+1})| = z - 1$ and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1})| = 3z = |V(\mathbb{MP}_{z,1,2})|$.
(b) $R_x(a_ja'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-3}, a_{z+j-4}\}$, $R_x(a_ja_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}\}$, $R_x(a'_ja'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+j-3}, a'_{i+4}, a'_{i+6}, a'_{i+8}, a'_{i+10}, ..., a'_{z+i-5}\}$. □

The cardinalities among all these RLN sets are classified in Table 1.

Table 1. Cardinality of each RLN set.

RLN Set	Cardinality
$R_x(a_ja'_j)$	$2z - 4 > z - 1$
$R_x(a_ja_{j+2})$	$2z - 2 > z - 1$
$R_x(a'_ja'_{j+1})$	$z + 3 > z - 1$

It is clear from above Table 1 that cardinality of $R_x(a_j a_{j+1})$ is less then all other RLN sets.

Theorem 4.2. *Let* $\mathbb{MP}_{z,1,2}$ *be a modified prism network , where* $z \cong 1$ $(mod\ 4)$. *Then*

$$\frac{z}{z-1} \le dim_{LF}(\mathbb{MP}_{z,1,2}) \le \frac{2z}{z-1}.$$

Proof. **Case 1.** For $z = 5$, we have the following RLN sets
$R_x(a_1a_2) = R_x(a'_1a'_2) = \{a_1, a_2, a'_1, a'_2\}$,
$R_x(a_2a_3) = R_x(a'_2a'_3) = \{a_2, a_3, a'_2, a'_3\}$,
$R_x(a_3a_4) = R_x(a'_3a'_4) = \{a_3, a_4, a'_3, a'_4\}$,
$R_x(a_4a_5) = R_x(a'_4a'_5) = \{a_4, a_5, a'_4, a'_5\}$,
$R_x(a_5a_1) = R_x(a'_5a'_1) = \{a_1, a_5, a'_1, a'_5\}$,
$R_x(a_1a_3) = \{a_1, a_3, a'_1, a'_3\}$,
$R_x(a_1a_4) = \{a_1, a_4, a'_1, a'_4\}$,
$R_x(a_2a_4) = \{a_2, a_4, a'_2, a'_4\}$,
$R_x(a_2a_5) = \{a_2, a_5, a'_2, a'_5\}$,
$R_x(a_3a_5) = \{a_3, a_5, a'_3, a'_5\}$,
$R_x(a_1a'_1) = V(\mathbb{MP}_{5,1,2}) - \{a'_3, a'_4\}$,
$R_x(a_2a'_2) = V(\mathbb{MP}_{5,1,2}) - \{a'_4, a'_5\}$,
$R_x(a_3a'_3) = V(\mathbb{MP}_{5,1,2}) - \{a'_5, a'_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MP}_{5,1,2}) - \{a'_1, a'_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MP}_{5,1,2}) - \{a'_2, a'_3\}$.

For $1 \le j \le 5$ it is clear that $|R_x(a_ja_{j+1})| = 8$ and $|R_x(a_ja_{j+1})| \le |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{5,1,2}$. Therefore, an upper LRF $\eta : V(\mathbb{MP}_{5,1,2}) \to [0, 1]$ is defined as $\eta(y) = \frac{1}{4}$ for each $y \in V(\mathbb{MP}_{5,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta(y)'$: $V(\mathbb{MP}_{5,1,2}) \to [0, 1]$ as $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{5,1,2}$. Therefore, $dim_{LF}(\mathbb{MP}_{5,1,2}) \le \sum_{1}^{10} \frac{1}{4} = \frac{5}{2}$. In the same context, for $1 \le j \le z$ it is clear from the above RLN sets that $|R_x(a_ja'_j)| = 8$ and $|R_x(a_ja'_j)| \ge |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{5,1,2}$. Therefore, a lower LRF $\eta : V(\mathbb{MP}_{5,1,2}) \to [0, 1]$ is defined as $\eta(y) = \frac{1}{21}$ for all $y \in V(\mathbb{MP}_{5,1,2})$ hence $dim_{LF}(\mathbb{MP}_{5,1,2}) \ge \sum_{1}^{10} \frac{1}{8} = \frac{5}{4}$. Consequently,

$$\frac{5}{4} \le dim_{LF}(\mathbb{MP}_{5,1,2}) \le \frac{5}{2}.$$

Case 2. For $1 \le j \le z$ from Lemma 4.1 it is clear that $|R_x(a_ja_{j+1})| = z + 1$ and $|R_x(a_ja_{j+1})| \le |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$. Therefore, an upper LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0, 1]$ is defined as $\eta(y) = \frac{1}{z-1}$ for each $y \in V(\mathbb{MP}_{z,1,2})$. In order to show that η is a minimal RLF, we define another RLF η' : $V(\mathbb{MP}_{z,1,2}) \to [0, 1]$ as $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a RLF of $(\mathbb{MP}_{z,1,2})$. Therefore, by Lemma X $dim_{LF}(\mathbb{MP}_{z,1,2}) \le \sum_{j=1}^{2z} \frac{1}{z-1} = \frac{2z}{z-1}$. In the same way, for $1 \le j \le z$ it is clear from Lemma 4.1 $|R_x(a_ja_{j+1})| = 2z - 2$ and $|R_x(a_ja_{j+2})| \ge |R_x(e)|$,where $R_x(e)$ are the other LRN sets of $\mathbb{MP}_{z,1,2}$. Therefore, a lower RLF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0, 1]$ is defined as $\eta(y) = \frac{1}{2z-4}$ for each $y \in V(\mathbb{MP}_{z,1,2})$ hence by Lemma Y $dim_{LF}(\mathbb{MP}_{z,1,2}) \ge \sum_{j=1}^{2z} \frac{1}{2z-2} = \frac{z}{z-2}$. Consequently,

$$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z-1}.$$

□

Lemma 4.3. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network, where $z \cong 3\ (mod\ 4)$. Then*
(a) $|R_x(a_ja_{j+1})| = z+1$ and $\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$ where $|R_x(e)|$ are the other possible RLN sets.

Proof. Let a_j inner, a'_j be the outer vertices of modified prism network, for $1 \leq j \leq z$, where $z+1 \cong 1(\ mod\ z)$, we have following possibilities,
$(a)R_x(a_ja_{j+1})= V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+i-5}, a_{z+i-3}, a_z\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}, ..., a'_{z+i-5}, a'_{z+i-3}, a'_z\}$
and $|R_x(a_ja_{j+1})| = z+1$ and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1})| = 3z = |V(\mathbb{MP}_{z,1,2})|$.
(b) $R_x(a_ja'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-3}, a_{z+j-4}\}$, $R_x(a_ja_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}\}$, $R_x(a'_ja'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+j-3}, a'_{i+4}, a'_{i+6}, a'_{i+8}, a'_{i+10}, ..., a'_{z+i-5}\}$. □

The RLN sets are classified in Table 2 and it is clear that $|R_x(a_ja_{j+1})|$ is less then the all other RLN sets of $\mathbb{MP}_{z,1,2}$.

Table 2. Cardinality of each RLN set.

RLN Set	Cardinality
$R_x(a_ja'_j)$	$2z-4 > z+1$
$R_x(a_ja_{j+2})$	$2z-2 > z+1$
$R_x(a'_ja'_{j+1})$	$z+3 > z+1$

Theorem 4.4. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network, where $z \cong 3\ (mod\ 4)$. Then*

$$\frac{z}{z-1} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+1}.$$

Proof. **Case 1.** For $z = 7$, we have the following RLN sets
$R_x(a_1a_2) = V(\mathbb{MP}_{7,1,2}) - \{a_3, a_5, a_7, a'_3, a'_5, a'_7\}$,
$R_x(a_2a_3) = V(\mathbb{MP}_{7,1,2}) - \{a_4, a_6, a_1, a'_4, a'_6, a'_1\}$,
$R_x(a_3a_4) = V(\mathbb{MP}_{7,1,2}) - \{a_5, a_7, a_2, a'_5, a'_7, a'_2\}$,
$R_x(a_4a_5) = V(\mathbb{MP}_{7,1,2}) - \{a_6, a_1, a_3, a'_6, a'_1, a'_3\}$,
$R_x(a_5a_6) = V(\mathbb{MP}_{7,1,2}) - \{a_7, a_2, a_4, a'_7, a'_2, a'_4\}$,
$R_x(a_6a_7) = V(\mathbb{MP}_{7,1,2}) - \{a_1, a_3, a_5, a'_1, a'_3, a'_5\}$,
$R_x(a_7a_1) = V(\mathbb{MP}_{7,1,2}) - \{a_2, a_4, a_6, a'_2, a'_4, a'_6\}$,
$R_x(a_1a'_1) = V(\mathbb{MP}_{7,1,2}) - \{a'_3, a'_4, a'_5, a'_6\}$,
$R_x(a_2a'_2) = V(\mathbb{MP}_{7,1,2}) - \{a'_4, a'_5, a'_6, a'_7\}$,

$R_x(a_3a_3') = V(\mathbb{MP}_{7,1,2}) - \{a_5', a_6', a_7', a_1'\}$,
$R_x(a_4a_4') = V(\mathbb{MP}_{7,1,2}) - \{a_6', a_7', a_1', a_2'\}$,
$R_x(a_5a_5') = V(\mathbb{MP}_{7,1,2}) - \{a_7', a_1', a_2', a_3'\}$,
$R_x(a_6a_6') = V(\mathbb{MP}_{7,1,2}) - \{a_1', a_2', a_3', a_4'\}$,
$R_x(a_7a_7') = V(\mathbb{MP}_{7,1,2}) - \{a_2', a_3', a_4', a_5', \}$,
$R_x(a_1a_3) = V(\mathbb{MP}_{7,1,2}) - \{a_2, a_2'\}$,
$R_x(a_2a_4) = V(\mathbb{MP}_{7,1,2}) - \{a_3, a_3'\}$,
$R_x(a_3a_5) = V(\mathbb{MP}_{7,1,2}) - \{a_4, a_4'\}$,
$R_x(a_4a_6) = V(\mathbb{MP}_{7,1,2}) - \{a_5, a_5'\}$,
$R_x(a_5a_7) = V(\mathbb{MP}_{7,1,2}) - \{a_6, a_6'\}$,
$R_x(a_6a_1) = V(\mathbb{MP}_{7,1,2}) - \{a_7, a_7'\}$,
$R_x(a_7a_2) = V(\mathbb{MP}_{7,1,2}) - \{a_1, a_1'\}$,
$R_x(a_1'a_2') = V(\mathbb{MP}_{7,1,2}) - \{a_3, a_5, a_7, a_5'\}$,
$R_x(a_2'a_3') = V(\mathbb{MP}_{7,1,2}) - \{a_4, a_6, a_1, a_6'\}$,
$R_x(a_3'a_4') = V(\mathbb{MP}_{7,1,2}) - \{a_5, a_7, a_2, a_7'\}$,
$R_x(a_4'a_5') = V(\mathbb{MP}_{7,1,2}) - \{a_6, a_1, a_3, a_1'\}$,
$R_x(a_5'a_6') = V(\mathbb{MP}_{7,1,2}) - \{a_7, a_2, a_4, a_2'\}$,
$R_x(a_6'a_7') = V(\mathbb{MP}_{7,1,2}) - \{a_1, a_3, a_5, a_3'\}$,
$R_x(a_7'a_1') = V(\mathbb{MP}_{7,1,2}) - \{a_2, a_4, a_6, a_4'\}$.

For $1 \leq j \leq 7$ it is clear that $|R_x(a_ja_{j+1})| = 8$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{7,1,2})$. Therefore, an upper LRF $\eta : V(\mathbb{MP}_{7,1,2}) \to [0,1]$ is defined as $\eta(y) = \frac{1}{8}$ for each $y \in V(\mathbb{MP}_{7,1,2})$. In order to show that $\eta(y)$ is a minimal upper LRF, we define another LRF $\eta(y)' : V(\mathbb{MP}_{7,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a local resolving function of $\mathbb{P}_{7,1,2})$. Therefore, $dim_{LF}(\mathbb{MP}_{7,1,2}) \leq \sum_{1}^{14} \frac{1}{8} = \frac{7}{4}$. In the same context, for $1 \leq j \leq z$ it is clear from the above RLN sets that $|R_x(a_ja_{j+2})| = 12$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{7,1,2})$. Therefore, a lower LRF $\eta : V(\mathbb{MP}_{7,1,2}) \to [0,1]$ is defined as $\eta(y) = \frac{1}{21}$ for each $y \in V(\mathbb{MP}_{7,1,2})$ hence $dim_{LF}(\mathbb{MP}_{7,1,2}) \geq \sum_{1}^{14} \frac{1}{12} = \frac{7}{6}$. Since $\mathbb{MP}_{7,1,2}$ is a non-bipartite network so its lower bound must be greater then 1. Consequently,

$$\frac{7}{6} \leq dim_{LF}(\mathbb{MP}_{7,1,2}) \leq \frac{7}{4}.$$

Case 2. For $1 \leq j \leq z$ from Lemma 4.3, it is clear that $|R_x(a_ja_{j+1})| = z + 1$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$. Therefore, an upper LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ is defined as $\eta(y) = \frac{2}{3n+6}$ for each $y \in V(\mathbb{MP}_{z,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta' : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ as $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{7,1,2}$ hence by Lemma X $dim_{LF} \leq \sum_{j=1}^{2z} \frac{1}{z+1} = \frac{2z}{z+1}$. In the same way, for $1 \leq j \leq z$ it is clear from Lemma 4.3 $|R_x(a_ja_{j+1})| = 2z - 2$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN of $\mathbb{MP}_{z,1,2}$. Therefore, a maximal lower LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ is defined as $\eta(y) = \frac{1}{2z-2}$ for each $y \in V(\mathbb{MP}_{z,1,2})$ hence by Lemma Y $dim_{LF}(\mathbb{MP}_{z,1,2}) \geq \sum_{j=1}^{2z} \frac{1}{2z-2} = \frac{z}{z-1}$. Consequently,

$$\frac{z}{z-1} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+1}.$$

□

Lemma 4.5. *Let $\mathbb{MP}_{z,1,2}$ be a modified generalized prism network, where $z \cong 0\ (mod\ 4)$. Then*
(a) $|R_x(a_ja_{j+1})| = z$ and $\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$, where $|R_x(y)|$ are the other possible RLN sets.

Proof. Let a_j inner, a'_j be the outer vertices of modified generalized Prism network, for $1 \leq j \leq z$, where $z + 1 \cong (1\ mod\ z)$, we have following possibilities
(a) $R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}....., a_{\frac{z+2j}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, a_{\frac{z+2j+10}{2}}, ...a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}....., a'_{\frac{z+2j}{2}}, a'_{\frac{z+2j+2}{2}}, a'_{\frac{z+2j+6}{2}}, a'_{\frac{z+2j+10}{2}}, ...a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\}$ and $|R_x(a_ja_{j+1})| = z$ and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1})| = 2z = |V(\mathbb{MP}_{z,1,2})|$.
(b)$R_x(a_ja'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-2}, a_{z+j-3}\}$, $R_x(a_ja_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}, a_{\frac{n+2j+2}{2}}, a'_{\frac{n+2j+2}{2}}\}$, $R_x(a'_ja'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{\frac{z+2j}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, ..., a_{z+j-6}, a_{z+j-3}, a_{z+j-1}\}$. □

The RLN sets are classified in Table 3 and it is clear that cardinality of $R_x(a_ja_{j+1})$ is less then all other RLN sets of $\mathbb{MP}_{z,1,2}$.

Table 3. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_ja'_j)$	$2z - 4 > z$
$R_x(a_ja_{j+2})$	$2z - 2 > z$
$R_x(a'_ja'_{j+1})$	$z + 3 > z$

Theorem 4.6. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network , where $z \cong 0\ (mod\ 4)$. Then*

$$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2.$$

Proof. **Case 1.** For $z = 8$, we have the following RLN sets;
$R_x(a_1a_2) = V(\mathbb{MP}_{8,1,2}) - \{a_3, a_5, a_6, a_8, a'_3, a'_5, a'_6, a'_8\}$,
$R_x(a_2a_3) = V(\mathbb{MP}_{8,1,2}) - \{a_4, a_6, a_1, a_2, a'_4, a'_6, a'_7, a'_1\}$,
$R_x(a_3a_4) = V(\mathbb{MP}_{8,1,2}) - \{a_5, a_7, a_8, a_3, a'_5, a'_7, a'_8, a'_2\}$,
$R_x(a_4a_5) = V(\mathbb{MP}_{8,1,2}) - \{a_6, a_8, a_1, a_4, a'_6, a'_8, a'_1, a'_3\}$,
$R_x(a_5a_6) = V(\mathbb{MP}_{8,1,2}) - \{a_7, a_1, a_2, a_5, a'_7, a'_1, a'_2, a'_4\}$,
$R_x(a_6a_7) = V(\mathbb{MP}_{8,1,2}) - \{a_8, a_2, a_3, a_6, a'_8, a'_2, a'_3, a'_5\}$,
$R_x(a_7a_8) = V(\mathbb{MP}_{8,1,2}) - \{a_1, a_3, a_4, a_7, a'_1, a'_3, a'_4, a'_6\}$,
$R_x(a_1a_8) = V(\mathbb{MP}_{8,1,2}) - \{a_2, a_4, a_5, a_8, a'_2, a'_4, a'_5, a'_7\}$,
$R_x(a_1a'_1) = V(\mathbb{MP}_{8,1,2}) - \{a'_3, a'_4, a'_6, a'_7\}$,
$R_x(a_2a'_2) = V(\mathbb{MP}_{8,1,2}) - \{a'_4, a'_5, a'_7, a'_8\}$,
$R_x(a_3a'_3) = V(\mathbb{MP}_{8,1,2}) - \{a'_5, a'_6, a'_8, a'_1\}$,

$R_x(a_4a_4') = V(\mathbb{MP}_{8,1,2}) - \{a_6', a_7', a_1', a_2'\}$,
$R_x(a_5a_5') = V(\mathbb{MP}_{8,1,2}) - \{a_7', a_8', a_2', a_3'\}$,
$R_x(a_6a_6') = V(\mathbb{MP}_{8,1,2}) - \{a_8', a_1', a_3', a_4'\}$,
$R_x(a_7a_7') = V(\mathbb{MP}_{8,1,2}) - \{a_1', a_2', a_4', a_5'\}$,
$R_x(a_8a_8') = V(\mathbb{MP}_{8,1,2}) - \{a_2', a_3', a_5', a_6'\}$,
$R_x(a_1'a_2') = V(\mathbb{MP}_{8,1,2}) - \{a_3, a_5, a_6, a_8\}$,
$R_x(a_2'a_3') = V(\mathbb{MP}_{8,1,2}) - \{a_4, a_6, a_7, a_1\}$,
$R_x(a_3'a_4') = V(\mathbb{MP}_{8,1,2}) - \{a_5, a_7, a_8, a_2\}$,
$R_x(a_4'a_5') = V(\mathbb{MP}_{8,1,2}) - \{a_6, a_8, a_1, a_3\}$,
$R_x(a_5'a_6') = V(\mathbb{MP}_{8,1,2}) - \{a_7, a_1, a_2, a_4\}$,
$R_x(a_6'a_7') = V(\mathbb{MP}_{8,1,2}) - \{a_8, a_2, a_3, a_5\}$,
$R_x(a_7'a_8') = V(\mathbb{MP}_{8,1,2}) - \{a_1, a_3, a_4, a_6\}$,
$R_x(a_8'a_1') = V(\mathbb{MP}_{8,1,2}) - \{a_2, a_4, a_5, a_7\}$,
$R_x(a_1a_3) = V(\mathbb{MP}_{8,1,2}) - \{a_2, a_6, a_2', a_6'\}$,
$R_x(a_2a_4) = V(\mathbb{MP}_{8,1,2}) - \{a_3, a_7, a_3', a_7'\}$,
$R_x(a_3a_5) = V(\mathbb{MP}_{8,1,2}) - \{a_4, a_8, a_4', a_8'\}$,
$R_x(a_4a_6) = V(\mathbb{MP}_{8,1,2}) - \{a_5, a_1, a_5', a_1'\}$,
$R_x(a_5a_7) = V(\mathbb{MP}_{8,1,2}) - \{a_6, a_2, a_6', a_2'\}$,
$R_x(a_6a_8) = V(\mathbb{MP}_{8,1,2}) - \{a_3, a_7, a_3', a_7'\}$,
$R_x(a_7a_1) = V(\mathbb{MP}_{8,1,2}) - \{a_8, a_1, a_8', a_1'\}$,
$R_x(a_8a_2) = V(\mathbb{MP}_{8,1,2}) - \{a_1, a_5, a_1', a_5'\}$.

For $1 \leq j \leq 8$ it is clear that $|R_x(a_ja_{j+1})| = 8$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the RLN sets of $\mathbb{MP}_{8,1,2}$. Then there exits an upper LRF $\eta : V(\mathbb{MP}_{8,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{8}$ for each $y \in V(\mathbb{MP}_{8,1,2})$. In order to show that $\eta(y)$ is a minimal LRF, we define another LRF $\eta'(y) : V(\mathbb{MP}_{8,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{8,1,2}$. Therefore, $dim_{LF}(\mathbb{MP}_{8,1,2}) \leq \sum_{1}^{16} \frac{1}{8} = 2$. In the same context, for $1 \leq j \leq z$ it is clear from RLN sets that $|R_x(a_ja_{j+2})| = 12$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{8,1,2}$). Then there exist a lower LRF $\eta : V(\mathbb{MP}_{8,1,2}) \to [0,1]$ and it is defined $\eta(y) = \frac{1}{21}$ for each $y \in V(\mathbb{MP}_{7,1,2})$ hence $dim_{LF}(\mathbb{MP}_{8,1,2}) \geq \sum_{1}^{16} \frac{1}{12} = \frac{4}{3}$. Since $\mathbb{MP}_{8,1,2}$ is a non-bipartite network so its lower bound must be greater then 1. Consequently,

$$\frac{4}{3} \leq dim_{LF}(\mathbb{MP}_{8,1,2}) \leq 2.$$

Case 2. For $1 \leq j \leq z$, it is clear from Lemma 4.5 it is that $|R_x(a_ja_{j+1})| = z$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$. Then there exits an upper LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ an it is defined as $\eta(y) = \frac{1}{z}$ for each $y \in V(\mathbb{MP}_{z,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta' : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{8,1,2}$ hence by Lemma X $dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \sum_{j=1}^{2z} \frac{1}{z} = 2$. In the same way, For $1 \leq j \leq z$ it is clear from Lemma 4.5 $|R_x(a_ja_{j+1})| = 2z - 4$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$). Then there exits a maximal lower LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{z-1}$ for each $y \in V(\mathbb{MP}_{z,1,2})$ hence by Lemma Y $dim_{LF}(\mathbb{MP}_{z,1,2}) \geq \sum_{j=1}^{2z} \frac{1}{2z-4} = \frac{z}{z-2}$. Consequently,

$$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2.$$

□

Lemma 4.7. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network, where $z \cong 2\ (mod\ 4)$. Then*
(a) $|R_x(a_j a_{j+1})| = z+2$ and $\bigcup_{j=1}^{z} R_x(a_j a_{j+1}) = V(\mathbb{MP}_{z,1,2})$.
(b) $|R_x(a_j a_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{z} R_x(a_j a_{j+1}) \cap R_x(y)| > |R_x(a_j a_{j+1})|$ where $|R_x(y)|$ are the other possible resolving local neighbourhood sets.

Proof. Let a_j inner, a'_j be the outer vertices of modified generalized Prism network, for $1 \leq j \leq z$, where $z+1 \cong (1\ mod\ z)$, we have following possibilities
(a) $R_x(a_j a_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}....., a_{\frac{z+j-1}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, a_{\frac{z+2j+10}{2}}, ...a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}....., a'_{\frac{z+j-1}{2}}, a'_{\frac{z+2j+2}{2}}, a'_{\frac{z+2j+6}{2}}, a'_{\frac{z+2j+10}{2}}, ...a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\}$ and $|R_x(a_j a_{j+1})| = z$ and $|\bigcup_{j=1}^{z} R_x(a_j a_{j+1})| = 2z = |V(\mathbb{MP}_{z,1,2})|$.
(b) $R_x(a_j a'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-2}, a_{z+j-3}\}$, $R_x(a_j a_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}, a_{\frac{n+2j+2}{2}}, a'_{\frac{n+2j+2}{2}}\}$, $R_x(a'_j a'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{\frac{z+j-1}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, ..., a_{z+j-6}, a_{z+j-3}, a_{z+j-1}\}$. □

The RLN sets are classified in Table 4 and it is clear that $|R_x(a_j a_{j+1})|$ is less then all other RLN sets of $\mathbb{MP}_{z,1,2}$.

Table 4. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_j a'_j)$	$2z-4 > z+2$
$R_x(a_j a_{j+2})$	$2z-4 > z+2$
$R_x(a'_j a'_{j+1})$	$2z-4 > z+2$

Theorem 4.8. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network, where $z \cong 2\ (mod\ 4)$. Then*

$$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+2}.$$

Proof. **Case 1.** For $z=6$, we have the following RLN sets;
$R_x(a_1a_2) = V(\mathbb{MP}_{6,1,2}) - \{a_3, a_6, a'_3, a'_5, a'_6\}$,
$R_x(a_2a_3) = V(\mathbb{MP}_{6,1,2}) - \{a_4, a_1, a'_4, a'_6, a'_1\}$,
$R_x(a_3a_4) = V(\mathbb{MP}_{6,1,2}) - \{a_5, a_2, a'_5, a'_1, a'_2\}$,
$R_x(a_4a_5) = V(\mathbb{MP}_{6,1,2}) - \{a_6, a_3, a'_6, a'_2, a'_3\}$,
$R_x(a_5a_6) = V(\mathbb{MP}_{6,1,2}) - \{a_1, a_4, a'_1, a'_3, a'_4\}$,
$R_x(a_6a_1) = V(\mathbb{MP}_{6,1,2}) - \{a_2, a_5, a'_2, a'_4, a'_5\}$,
$R_x(a_1a'_1) = V(\mathbb{MP}_{6,1,2}) - \{a'_3, a'_4, a'_5\}$,
$R_x(a_2a'_2) = V(\mathbb{MP}_{6,1,2}) - \{a'_4, a'_5, a'_6\}$,
$R_x(a_3a'_3) = V(\mathbb{MP}_{6,1,2}) - \{a'_5, a'_6, a'_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MP}_{6,1,2}) - \{a'_6, a'_1, a'_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MP}_{6,1,2}) - \{a'_1, a'_2, a'_3\}$,

$R_x(a_6a_6') = V(\mathbb{MP}_{6,1,2}) - \{a_2', a_3', a_4'\}$,
$R_x(a_1a_3) = V(\mathbb{MP}_{6,1,2}) - \{a_2, a_5, a_2', a_5'\}$,
$R_x(a_2a_4) = V(\mathbb{MP}_{6,1,2}) - \{a_3, a_6, a_3', a_6'\}$,
$R_x(a_3a_5) = V(\mathbb{MP}_{6,1,2}) - \{a_4, a_1, a_4', a_1'\}$,
$R_x(a_4a_6) = V(\mathbb{MP}_{6,1,2}) - \{a_5, a_2, a_5', a_2'\}$,
$R_x(a_5a_1) = V(\mathbb{MP}_{6,1,2}) - \{a_6, a_3, a_6', a_3'\}$,
$R_x(a_6a_2) = V(\mathbb{MP}_{6,1,2}) - \{a_1, a_4, a_1', a_4'\}$,
$R_x(a_1'a_2') = V(\mathbb{MP}_{6,1,2}) - \{a_3, a_6\}$,
$R_x(a_2'a_3') = V(\mathbb{MP}_{6,1,2}) - \{a_4, a_1\}$,
$R_x(a_3'a_4') = V(\mathbb{MP}_{6,1,2}) - \{a_5, a_2\}$,
$R_x(a_4'a_5') = V(\mathbb{MP}_{6,1,2}) - \{a_6, a_3\}$,
$R_x(a_5'a_6') = V(\mathbb{MP}_{6,1,2}) - \{a_1, a_4\}$,
$R_x(a_1'a_6') = V(\mathbb{MP}_{6,1,2}) - \{a_2, a_5\}$.

For $1 \leq j \leq 6$ it is clear that $|R_x(a_ja_{j+1})| = 7$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{6,1,2}$). Then there exits an upper LRF $\eta : V(\mathbb{MP}_{6,1,2}) \rightarrow [0,1]$ is defined as $\eta(y) = \frac{1}{7}$ for each $y \in V(\mathbb{MP}_{6,1,2})$. In order to show that $\eta(y)$ is a minimal LRF, we define another LRF $\eta'(y) : V(\mathbb{MP}_{6,1,2}) \rightarrow [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{6,1,2}$ hence $dim_{LF}(\mathbb{MP}_{6,1,2}) \leq \sum_{1}^{12} \frac{1}{8} = \frac{3}{2}$. In the same context, for $1 \leq j \leq z$ it is clear that $|R_x(a_ja_{j+2})| = 12$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$, where $R_x(e)$ are the other resolving local neighbour sets of $\mathbb{MP}_{6,1,2}$). Then there exits a lower LRF $\eta : V(\mathbb{MP}_{6,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{21}$ for each $y \in V(\mathbb{MP}_{6,1,2})$ hence $dim_{LF}(\mathbb{MP}_{6,1,2}) \geq \sum_{1}^{12} \frac{1}{10} = \frac{6}{5}$. Since $\mathbb{MP}_{6,1,2}$ is a non bipartite network so its lower bound must be greater then 1. Consequently,

$$\frac{6}{5} < dim_{LF}(\mathbb{MP}_{6,1,2}) \leq \frac{3}{2}.$$

Case 2. For $1 \leq j \leq z$ from Lemma 4.7 that $|R_x(a_ja_{j+1})| = z$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$. Then there exits an upper LRF $\eta : V(\mathbb{MP}_{z,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{z+2}$ for each $y \in V(\mathbb{MP}_{z,1,2})$. In order to show that η is a minimal upper LRF, we define another LRF $\eta' : V(\mathbb{MP}_{z,1,2}) \rightarrow [0,1]$ as $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{6,1,2}$ hence by Lemma X $dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \sum_{j=1}^{2z} \frac{1}{z+2} = \frac{2z}{z+2}$. In the same way, for $1 \leq j \leq z$ it is clear from Lemma 4.7 $|R_x(a_ja_{j+1})| = 2z-4$ and $|R_x(a_ja_{j+2})| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MP}_{z,1,2}$). Then there exits a lower LRF $\eta : V(\mathbb{MP}_{z,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{z-1}$ for each $y \in V(\mathbb{MP}_{z,1,2})$ hence by Lemma Y$dim_{LF}(\mathbb{MP}_{z,1,2}) \geq \sum_{j=1}^{2z} \frac{1}{2z-4} = \frac{z}{z-2}$. Consequently,

$$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+2}.$$

□

5. RLN sets LFMD of modified prism network $\mathbb{MQ}_{z,1,2}$

In this section, we compute RLN sets and LFMD of modified prism network $\mathbb{MQ}_{z,1,2}$ in the form of bounds.

Lemma 5.1. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 2$ (mod 4) . Then*
(a) $|R_x(a_ja_{j+1})| = \frac{3z+6}{2}$ *and* $\bigcup_{j=1}^{3z} R_x(a_ja_{j+1}) = V(\mathbb{MQ}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, *and* $|\bigcup_{j=1}^{3z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$ *where* $|R_x(e)|$ *are the other possible RLN sets.*

Proof. Let a_i inner, a'_i middle and b_i be the outer vertices of modified generalized Prism network, for $1 \leq j \leq z$, where $z+1 \cong 1($ mod $z)$, we have the following possibilities
(a) $R_x(a_ja_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}....., a_{\frac{z+j-1}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, a_{\frac{z+2j+10}{2}}, ...a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}....., a'_{\frac{z+j-1}{2}}, a'_{\frac{z+2j+2}{2}}, a'_{\frac{z+2j+6}{2}}, a'_{\frac{z+2j+10}{2}}, ...a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\} \cup \{b_{j+2}, b_{j+4}, b_{j+6}....., b_{\frac{z+j-1}{2}}, b_{\frac{z+2j+2}{2}}, b_{\frac{z+2j+6}{2}}, b_{\frac{z+2j+10}{2}}, ...b_{z+i-5}, b_{z+i-3}, b_{z+i-1}\}$ and $|R_x(a_ja_{j+1})| = \frac{3z+6}{2}$ and $|\bigcup_{j=1}^{3z} R_x(a_ja_{j+1})| = 3z = |V(\mathbb{MQ}_{z,1,2})|$.
(b) $R_x(a_ja'_j) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}, a_{i+3}, a_{z+j-3}, a_{z+j-2}, b_{j+2}, b_{j+3}, b_{z+j-3}, b_{z+j-2}\}$, $R_x(a_ja_{j+2}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+1}, a_{\frac{z+2j+2}{2}}, a'_{j+1}, a'_{\frac{z+2j+2}{2}}, b_{j+1}, b_{\frac{z+2j+2}{2}}\}$, $R_x(a'_ja'_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{z+j-1}\}$, $R_x(b_jb_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}\}$, $R_x(a'_jb_j) = V(\mathbb{MQ}_{z,1,2})$. □

The RLN sets classified in Table 5 and it is clear that $|R_x(a_ja_{j+1}|$ is less then all other RLN sets of $\mathbb{MQ}_{z,1,2}$.

Table 5. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_ja'_j)$	$3z-4 > \frac{3z+6}{2}$
$R_x(a_ja_{j+2})$	$3z-4 > \frac{3z+6}{2}$
$R_x(a'_jb_j)$	$3z > \frac{3z+6}{2}$
$R_x(a'_ja'_{j+1})$	$3z-1 > \frac{3z+6}{2}$
$R_x(b_jb_{j+1})$	$3z-1 > \frac{3z+6}{2}$

Theorem 5.2. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 2$ (mod 4). Then*

$$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+2}.$$

Proof. **Case 1.** For $z = 6$, we have the following RLN sets
$R_x(a_1a_2) = V(\mathbb{MQ}_{6,1,2}) - \{a_3, a_6, a'_3, a'_6, b_3, b_6\}$,
$R_x(a_2a_3) = V(\mathbb{MQ}_{6,1,2}) - \{a_4, a_1, a'_4, a'_1, b_4, b_1\}$,
$R_x(a_3a_4) = V(\mathbb{MQ}_{6,1,2}) - \{a_5, a_6, a'_5, a'_2, b_5, b_2\}$,
$R_x(a_4a_5) = V(\mathbb{MQ}_{6,1,2}) - \{a_6, a_1, a'_6, a'_3, b_6, b_3\}$,
$R_x(a_5a_6) = V(\mathbb{MQ}_{6,1,2}) - \{a_1, a_2, a'_1, a'_2, b_1, b_4\}$,
$R_x(a_6a_1) = V(\mathbb{MQ}_{6,1,2}) - \{a_2, a_3, a'_2, a'_3, b_2, b_5\}$,
$R_x(a_1a'_1) = V(\mathbb{MQ}_{6,1,2}) - \{a'_3, a'_4, a'_5, b_3, b_4, b_5\}$,
$R_x(a_2a'_2) = V(\mathbb{MQ}_{6,1,2}) - \{a'_4, a'_5, a'_6, b_4, b_5, b_6\}$,

$R_x(a_3a'_4) = V(\mathbb{MQ}_{6,1,2}) - \{a'_5, a'_6, a'_1, b_5, b_6, b_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MQ}_{6,1,2}) - \{a'_6, a'_1, a'_2, b_6, b_1, b_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MQ}_{6,1,2}) - \{a'_1, a'_2, a'_3, b_1, b_2, b_3\}$,
$R_x(a_6a'_6) = V(\mathbb{MQ}_{6,1,2}) - \{a'_2, a'_3, a'_4, b_2, b_3, b_4\}$,
$R_x(a_1a_3) = V(\mathbb{MQ}_{6,1,2}) - \{a_2, a_5, a'_2, a'_5, b_2, b_5\}$,
$R_x(a_2a_4) = V(\mathbb{MQ}_{6,1,2}) - \{a_3, a_6, a'_3, a'_6, b_3, b_6\}$,
$R_x(a_3a_5) = V(\mathbb{MQ}_{6,1,2}) - \{a_4, a_1, a'_4, a'_1, b_4, b_1\}$,
$R_x(a_4a_6) = V(\mathbb{MQ}_{6,1,2}) - \{a_5, a_2, a'_5, a'_2, b_5, b_2\}$,
$R_x(a_5a_1) = V(\mathbb{MQ}_{6,1,2}) - \{a_6, a_3, a'_6, a'_3, b_6, b_3\}$,
$R_x(a_6a_2) = V(\mathbb{MQ}_{6,1,2}) - \{a_1, a_4, a'_6, a'_4, b_1, b_4\}$,
$R_x(a'_1a'_2) = V(\mathbb{MQ}_{6,1,2}) - \{a_3, a_6\}$,
$R_x(a'_2a'_3) = V(\mathbb{MQ}_{6,1,2}) - \{a_4, a_5\}$,
$R_x(a'_3a'_4) = V(\mathbb{MQ}_{6,1,2}) - \{a_5, a_6\}$,
$R_x(a'_4a'_5) = V(\mathbb{MQ}_{6,1,2}) - \{a_6, a_1\}$,
$R_x(a'_5a'_6) = V(\mathbb{MQ}_{6,1,2}) - \{a_1, a_2\}$,
$R_x(a'_6a'_1) = V(\mathbb{MQ}_{6,1,2}) - \{a_2, a_3\}$,
$R_x(b_1b_2) = V(\mathbb{MQ}_{6,1,2}) - \{a_3, a_6\}$,
$R_x(b_2b_3) = V(\mathbb{MQ}_{6,1,2}) - \{a_4, a_1\}$,
$R_x(b_3b_4) = V(\mathbb{MQ}_{6,1,2}) - \{a_5, a_2\}$,
$R_x(b_4b_5) = V(\mathbb{MQ}_{6,1,2}) - \{a_6, a_1\}$,
$R_x(b_5b_6) = V(\mathbb{MQ}_{6,1,2}) - \{a_1, a_2\}$,
$R_x(b_1b_6) = V(\mathbb{MQ}_{6,1,2}) - \{a_2, a_1\}$,
$R_x(a'_1b_1) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_2b_2) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_3b_3) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_4b_4) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_5b_5) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_6b_6) = V(\mathbb{MQ}_{6,1,2})$,
$R_x(a'_6b_6) = V(\mathbb{MP}_{6,1,2})$.

For $1 \leq j \leq 6$ it is clear that $|R_x(a_ja_{j+1})| = 12$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{6,1,2}$. Then there exits an upper LRF $\eta : V(\mathbb{MP}_{6,1,2}) \rightarrow [0,1]$ and is defined as $\eta(y) = \frac{1}{12}$ for each $y \in V(\mathbb{MQ}_{6,1,2})$. In order to show that $\eta(y)$ is a minimal LRF, we define another LRF $\eta(y)' : V(\mathbb{MP}_{6,1,2}) \rightarrow [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not LRF. Therefore, $dim_{LF}(\mathbb{MQ}_{6,1,2}) \leq \sum_{1}^{18} \frac{1}{12} = \frac{3}{2}$. For $1 \leq j \leq 6$ it is clear from the above RLN sets that $|R_x(b_jb_{j+1})| = 18$ and $|R_x(b_jb_{j+1})| \geq |R_x(e)|$, where $R_x(e)$ are other RLN sets of $\mathbb{MQ}_{6,1,2})$. Then there exits a lower LRF $\eta : V(\mathbb{MQ}_{6,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{18}$ for each $y \in V(\mathbb{MQ}_{6,1,2})$ hence $dim_{LF}(\mathbb{MQ}_{6,1,2}) \geq \sum_{1}^{18} \frac{1}{18} = 1$. Since $\mathbb{MQ}_{6,1,2}$ is a non-bipartite network so its lower bound must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{6,1,2}) \leq \frac{3}{2}.$$

Case 2. For $1 \leq j \leq z$ from Lemma 5.1 it is clear from the above resolving local neighbourhood sets that $|R_x(a_ja_{j+1})| = \frac{2}{3z+6}$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2})$. Then there exits an upper LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{2}{3n+6}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta' : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{6,1,2}$. Therefore by Lemma X $dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \sum_{j=1}^{3z} \frac{2}{3z+6} = \frac{2z}{z+2}$.

For $1 \leq j \leq z$ it is clear from Lemma 5.1 $|R_x(a'_j b'_j)| = 3z$ and $|R_x(b_j b_{j+1})| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2}$. Then there exits a maximal lower LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{3z}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. Hence by Lemma Y $dim_{LF}(\mathbb{MQ}_{z,1,2}) \geq \sum_{j=1}^{3z} \frac{1}{3z} = 1$. Since $\mathbb{MQ}_{z,1,2}$ is a non-bipartite network so its lower of LFMD bound must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z+2}.$$

□

Lemma 5.3. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 0\ (mod\ 4)$. Then*
(a) $|R_x(a_j a_{j+1})| = \frac{3z}{2}$ and $\bigcup_{j=1}^{3z} R_x(a_j a_{j+1}) = V(\mathbb{MQ}_{z,1,2})$.
(b) $|R_x(a_j a_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{3z} R_x(a_j a_{j+1}) \cap R_x(y)| > |R_x(a_j a_{j+1})|$, where $|R_x(y)|$ are the other possible RLN sets.

Proof. Let a_i inner, a'_i middle and b_i be the outer vertices of modified generalized Prism network, for $1 \leq j \leq z$, where $z + 1 \cong 1(\ mod\ z)$, we have following possibilities
(a) $R_x(a_j a_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6} \ldots.., a_{\frac{z+j-1}{2}}, a_{\frac{z+2j+2}{2}}, a_{\frac{z+2j+6}{2}}, a_{\frac{z+2j+10}{2}}, \ldots..a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6} \ldots.., a'_{\frac{z+j-1}{2}}, a'_{\frac{z+2j+2}{2}}, a'_{\frac{z+2j+6}{2}}, a'_{\frac{z+2j+10}{2}}, \ldots..a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\}\{b_{j+2}, b_{j+4}, b_{j+6} \ldots.., b_{\frac{z+j-1}{2}}, b_{\frac{z+2j+2}{2}}, b_{\frac{z+2j+6}{2}}, b_{\frac{z+2j+10}{2}}, \ldots..b_{z+i-5}, b_{z+i-3}, b_{z+i-1}\}$ and $|R_x(a_j a_{j+1})| = \frac{3z}{2}$ and $|\bigcup_{j=1}^{3z} R_x(a_j a_{j+1})| = 3z = |V(\mathbb{MQ}_{z,1,2})|$.
(b) $R_x(a_j a'_j) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}, a_{i+3}, a_{z+j-3}, a_{z+j-2}, b_{j+2}, b_{j+3}, b_{z+j-3}, b_{z+j-2}\}$, $R_x(a_j a_{j+2}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+1}, a_{\frac{z+2j+2}{2}}, a'_{j+1}, a'_{\frac{z+2j+2}{2}}, b_{j+1}, b_{\frac{z+2j+2}{2}}\}$, $R_x(a'_j a'_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{z+j-1}\}$, $R_x(b_j b_{j+1}) = V(\mathbb{MQ}_{z,1,2}) - \{a_{j+2}\}$, $R_x(a'_j b_j) = V(\mathbb{MQ}_{z,1,2})$. □

The RLN sets are classified in Table 6 and it is clear that $|R_x(a_j a_{j+1}|$ is less then all other RLN sets of $\mathbb{MQ}_{z,1,2}$.

Table 6. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_j a'_j)$	$3z - 4 > \frac{3z}{2}$
$R_x(a_j a_{j+2})$	$3z - 4 > \frac{3z}{2}$
$R_x(a'_j b_j)$	$3z > \frac{3z}{2}$
$R_x(a'_j a'_{j+1})$	$3z - 1 > \frac{3z}{2}$
$R_x(b_j b_{j+1})$	$3z - 1 > \frac{3z}{2}$

Theorem 5.4. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 0\ (mod\ 4)$. Then*

$$1 \leq dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2.$$

Proof. **Case 1.** For $z = 8$, we have the following RLN sets;
$R_x(a_1a_2) = V(\mathbb{MQ}_{8,1,2}) - \{a_3, a_5, a_6, a_8, a'_3, a'_5, a'_6, a'_8, b_3, b_5, b_7, b_8\}$,
$R_x(a_2a_3) = V(\mathbb{MQ}_{8,1,2}) - \{a_4, a_6, a_1, a_2, a'_4, a'_6, a'_7, a'_1, b_4, b_6, b_8, b_1\}$,
$R_x(a_3a_4) = V(\mathbb{MQ}_{8,1,2}) - \{a_5, a_7, a_8, a_3, a'_5, a'_7, a'_8, a'_2, b_5, b_7, b_1, b_2\}$,
$R_x(a_4a_5) = V(\mathbb{MQ}_{8,1,2}) - \{a_6, a_8, a_1, a_4, a'_6, a'_8, a'_1, a'_3, b_6, b_8, b_2, b_3\}$,
$R_x(a_5a_6) = V(\mathbb{MQ}_{8,1,2}) - \{a_7, a_1, a_2, a_5, a'_7, a'_1, a'_2, a'_4, b_7, b_1, b_3, b_4\}$,
$R_x(a_6a_7) = V(\mathbb{MQ}_{8,1,2}) - \{a_8, a_2, a_3, a_6, a'_8, a'_2, a'_3, a'_5, b_8, b_2, b_4, b_5\}$,
$R_x(a_7a_8) = V(\mathbb{MQ}_{8,1,2}) - \{a_1, a_3, a_4, a_7, a'_1, a'_3, a'_4, a'_6, b_1, b_3, b_5, b_6\}$,
$R_x(a_1a_8) = V(\mathbb{MQ}_{8,1,2}) - \{a_2, a_4, a_5, a_8, a'_2, a'_4, a'_5, a'_7, b_2, b_4, b_6, b_7\}$,
$R_x(a_1a'_1) = V(\mathbb{MQ}_{8,1,2}) - \{a'_3, a'_4, a'_6, a'_7, b_3, b_4, b_6, b_7\}$,
$R_x(a_2a'_2) = V(\mathbb{MQ}_{8,1,2}) - \{a'_4, a'_5, a'_7, a'_8, b_4, b_5, b_7, b_8\}$,
$R_x(a_3a'_3) = V(\mathbb{MQ}_{8,1,2}) - \{a'_5, a'_6, a'_8, a'_1, b_5, b_6, b_8, b_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MQ}_{8,1,2}) - \{a'_6, a'_7, a'_1, a'_2, b_6, b_7, b_1, b_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MQ}_{8,1,2}) - \{a'_7, a'_8, a'_2, a'_3, b_7, b_8, b_2, b_3\}$,
$R_x(a_6a'_6) = V(\mathbb{MQ}_{8,1,2}) - \{a'_8, a'_1, a'_3, a'_4, b_8, b_1, b_3, b_4\}$,
$R_x(a_7a'_7) = V(\mathbb{MQ}_{8,1,2}) - \{a'_1, a'_2, a'_4, a'_5, b_1, b_2, b_4, b_5\}$,
$R_x(a_8a'_8) = V(\mathbb{MQ}_{8,1,2}) - \{a'_2, a'_3, a'_5, a'_6, b_2, b_3, b_5, b_6\}$,
$R_x(a'_1a'_2) = V(\mathbb{MQ}_{8,1,2}) - \{a_3, a_5, a_6, a_8, \}$,
$R_x(a'_2a'_3) = V(\mathbb{MQ}_{8,1,2}) - \{a_4, a_6, a_7, a_1\}$,
$R_x(a'_3a'_4) = V(\mathbb{MQ}_{8,1,2}) - \{a_5, a_7, a_8, a_2\}$,
$R_x(a'_4a'_5) = V(\mathbb{MQ}_{8,1,2}) - \{a_6, a_8, a_1, a_3\}$,
$R_x(a'_5a'_6) = V(\mathbb{MQ}_{8,1,2}) - \{a_7, a_1, a_2, a_4\}$,
$R_x(a'_6a'_7) = V(\mathbb{MQ}_{8,1,2}) - \{a_8, a_2, a_3, a_5\}$,
$R_x(a'_7a'_8) = V(\mathbb{MQ}_{8,1,2}) - \{a_1, a_3, a_4, a_6\}$,
$R_x(a'_8a'_1) = V(\mathbb{MQ}_{8,1,2}) - \{a_2, a_4, a_5, a_7\}$,
$R_x(a_1a_3) = V(\mathbb{MQ}_{8,1,2}) - \{a_2, a_6, a'_2, a'_6, b_2, b_6\}$,
$R_x(a_2a_4) = V(\mathbb{MQ}_{8,1,2}) - \{a_3, a_7, a'_3, a'_7, b_3, b_7\}$,
$R_x(a_3a_5) = V(\mathbb{MQ}_{8,1,2}) - \{a_4, a_8, a'_4, a'_8, b_4, b_8\}$,
$R_x(a_4a_6) = V(\mathbb{MQ}_{8,1,2}) - \{a_5, a_1, a'_5, a'_1, b_5, b_1\}$,
$R_x(a_5a_7) = V(\mathbb{MQ}_{8,1,2}) - \{a_6, a_2, a'_6, a'_2, b_6, b_2\}$,
$R_x(a_6a_8) = V(\mathbb{MQ}_{8,1,2}) - \{a_3, a_7, a'_3, a'_7, b_7, b_3\}$,
$R_x(a_7a_1) = V(\mathbb{MQ}_{8,1,2}) - \{a_8, a_1, a'_8, a'_1, b_8, b_4\}$,
$R_x(a_8a_2) = V(\mathbb{MQ}_{8,1,2}) - \{a_1, a_5, a'_1, a'_5, b_1, b_5\}$,
$R_x(b_1b_2) = V(\mathbb{MQ}_{8,1,2}) - \{a_3, a_5, a_6, a_8\}$,
$R_x(b_2b_3) = V(\mathbb{MQ}_{8,1,2}) - \{a_4, a_6, a_7, a_1\}$,
$R_x(b_3b_4) = V(\mathbb{MQ}_{8,1,2}) - \{a_5, a_7, a_8, a_2\}$,
$R_x(b_4b_5) = V(\mathbb{MQ}_{8,1,2}) - \{a_6, a_8, a_1, a_3\}$,
$R_x(b_5b_6) = V(\mathbb{MQ}_{8,1,2}) - \{a_7, a_1, a_2, a_4\}$,
$R_x(b_6b_7) = V(\mathbb{MQ}_{8,1,2}) - \{a_8, a_2, a_3, a_5\}$,
$R_x(b_7b_8) = V(\mathbb{MQ}_{8,1,2}) - \{a_1, a_3, a_4, a_6\}$,
$R_x(b_8b_1) = V(\mathbb{MQ}_{8,1,2}) - \{a_2, a_4, a_5, a_7\}$,
$R_x(a'_1b_1) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'_2b_2) = V(\mathbb{MQ}_{8,1,2})$,

$R_x(a'_3b_3) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'_4b_4) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'_5b_5) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'6b_6) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'_7b_7) = V(\mathbb{MQ}_{8,1,2})$,
$R_x(a'_8b_8) = V(\mathbb{MQ}_{8,1,2})$.

For $1 \leq j \leq 8$ it is clear that $|R_x(a_ja_{j+1})| = 12$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{8,1,2}$). Then there exits an upper LRF $\eta : V(\mathbb{MQ}_{8,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{8}$ for each $y \in V(\mathbb{MQ}_{8,1,2})$. In order to show that $\eta(y)$ is a minimal LRF, we define another resolving function $\eta(y)' : V(\mathbb{MQ}_{8,1,2}) \rightarrow [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{8,1,2}$ hence by Lemma X$dim_{LF}(\mathbb{MQ}_{8,1,2}) \leq \sum_{1}^{24} \frac{1}{12} = 2$. In the same context, for $1 \leq j \leq z$ it is clear from the above RLN sets that $|R_x(a_ja_{j+2})| = 12$ and $|R_x(a'_ja'_{j+1})| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{8,1,2}$. Then there exits a lower LRF $\eta : V(\mathbb{MQ}_{8,1,2}) \rightarrow [0,1]$ such that $\eta(y) = \frac{1}{24}$ for each $y \in V(\mathbb{MQ}_{8,1,2})$ hence $dim_{LF}(\mathbb{MQ}_{8,1,2}) \geq \sum_{1}^{24} \frac{1}{24} = 1$. Since $\mathbb{MQ}_{8,1,2}$ is non bipartite network so its lower bound of LFMD must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{8,1,2}) \leq 2.$$

Case 2. For $1 \leq j \leq z$ it is clear that $|R_x(a_ja_{j+1})| = z$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2}$. Then there exits an upper LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \rightarrow [0,1]$ is defined as $\eta(y) = \frac{1}{z}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta' : V(\mathbb{MQ}_{z,1,2}) \rightarrow [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{8,1,2}$ hence by Lemma X $dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \sum_{j=1}^{3z} \frac{2}{3z} = 2$. In the same way, for $1 \leq j \leq z$ it is clear from Lemma 5.3 that $|R_x(a'_jb_j)| = 3z$ and $|R_x(a'_jb_j)| \geq |R_x(e)|$,where $R_x(e)$ are the other resolving local neighbour sets of $\mathbb{MQ}_{z,1,2}$). Then there exits a maximal lower LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \rightarrow [0,1]$ and it is defined as $\eta(y) = \frac{1}{3z}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$ hence by Lemma Y$dim_{LF}(\mathbb{MQ}_{z,1,2}) \geq \sum_{j=1}^{3z} \frac{1}{3z} = 1$. Since $\mathbb{MQ}_{z,1,2}$ is a non-bipartite network so its lower bound of LFMD must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2.$$

□

Lemma 5.5. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 1\ (mod\ 4)$. Then*
(a) $|R_x(a_ja_{j+1})| = \frac{3z-3}{2}$ and $\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) = V(\mathbb{MQ}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$ where $|R_x(y)|$ are the other possible RLN sets.

Proof. Let a_j inner, a'_j middle and b_j are be the outer vertices of modified prism network, for $1 \leq j \leq z$, where $z + 1 \cong (1\ mod\ z)$, we have following possibilities
(a) $R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}....., a_{z+i-5}, a_{z+i-3}, a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}, ..., a'_{z+i-5}, a'_{z+i-3}, a'_{z+i-1}\} \cup \cup\{b_{j+2}, b_{j+4}, b_{j+6}, ..., b_{z+i-5}, b_{z+i-3}, b_{z+i-1}\} \cup \{a_{\frac{z+2i+2}{2}}\} \cup \{a'_{\frac{z+2i+2}{2}}\}$

and $|R_x(a_j a_{j+1})| = z - 1$ and $|\bigcup_{j=1}^{3z} R_x(a_j a_{j+1})| = 3z = |V(\mathbb{MP}_{z,1,2})|$.
(b) $R_x(a_j a'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-3}, a_{z+j-4}, b_{j+2}, b_{j+3}, b_{z+j-3}, b_{z+j-4}\}$, $R_x(a_j a_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}, b_{j+1}, a_{\frac{z+2j+1}{2}}, a'_{\frac{z+2j+1}{2}}, b_{\frac{z+2j+1}{2}}\}$,$R_x(b_j b_{j+1}) = R_x(a'_j a'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+j-3}, a_{z+j-1}\} \cup \{a_{\frac{z+2j+1}{2}}\} \cup \{a'_{\frac{z+2j+1}{2}}, b_{\frac{z+2j+1}{2}}\}$. $R_x(a'_j b_j) = V(\mathbb{MP}_{z,1,2})$. □

The RLN sets are classified in Table 7 and it is clear $|R_x(a_j a_{j+1})|$ is less then all other RLN sets of $\mathbb{MQ}_{z,1,2}$.

Table 7. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_j a'_j)$	$3z - 8 > \frac{3z-3}{2}$
$R_x(a_j a_{j+2})$	$3z - 6 > \frac{3z-3}{2}$
$R_x(a'_j a'_{j+1})$	$5x - 25 > \frac{3z-3}{2}$
$R_x(a'_j b_j)$	$3z > \frac{3z-3}{2}$
$R_x(b_j a_{j+1})$	$5z - 25 > \frac{3z-3}{2}$

Theorem 5.6. *Let $\mathbb{MP}_{z,1,2}$ be a modified prism network , where $z \cong 1 \ (mod\ 4)$. Then*

$$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z-1}.$$

Proof. **Case 1.** For $z = 5$, we have the following RLN sets
$R_x(a_1a_2) = \{a_1, a_2, a'_1, a'_2, b_1, b_2\}$,
$R_x(a_2a_3) = \{a_2, a_3, a'_2, a'_3, b_2, b_3\}$,
$R_x(a_3a_4) = \{a_3, a_4, a'_3, a'_4, b_3, b_4\}$,
$R_x(a_4a_5) = \{a_4, a_5, a'_4, a'_5, b_4, b_5\}$,
$R_x(a_5a_1) = \{a_1, a_5, a'_1, a'_5, b_5, b_1\}$,
$R_x(a_1a_3) = \{a_1, a_3, a'_1, a'_3, b_1, b_3\}$,
$R_x(a_1a_4) = \{a_1, a_4, a'_1, a'_4, b_2, b_4\}$,
$R_x(a_2a_4) = \{a_2, a_4, a'_2, a'_4, b_3, b_5\}$,
$R_x(a_2a_5) = \{a_2, a_5, a'_2, a'_5, b_4, b_1\}$,
$R_x(a_3a_5) = \{a_3, a_5, a'_3, a'_5, b_5, b_2\}$,
$R_x(a_1a'_1) = V(\mathbb{MP}_{5,1,2}) - \{a'_3, a'_4, b_3, b_4\}$,
$R_x(a_2a'_2) = V(\mathbb{MP}_{5,1,2}) - \{a'_4, a'_5, b_4, b_5\}$,
$R_x(a_3a'_3) = V(\mathbb{MP}_{5,1,2}) - \{a'_5, a'_1, b_5, b_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MP}_{5,1,2}) - \{a'_1, a'_2, b_1, b_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MP}_{5,1,2}) - \{a'_2, a'_3, b_2, b_3\}$,
$R_x(a'_1a'_2) = V(\mathbb{MP}_{5,1,2}) - \{a_3, a_4, a_5, a'_4, b_4\}$,
$R_x(a'_2a'_3) = V(\mathbb{MP}_{5,1,2}) - \{a_4, a_5, a_1, a'_5, b_5\}$,
$R_x(a'_3a'_4) = V(\mathbb{MP}_{5,1,2}) - \{a_5, a_1, a_2, a'_1, b_1\}$,
$R_x(a'_4a'_5) = V(\mathbb{MP}_{5,1,2}) - \{a_1, a_2, a_3, a'_2, b_5\}$,
$R_x(a'_5a'_1) = V(\mathbb{MP}_{5,1,2}) - \{a_2, a_3, a_4, a'_3, b_1\}$,
$R_x(b_1b_2) = V(\mathbb{MP}_{5,1,2}) - \{a_3, a_4, a_5, a'_4, b_4\}$,
$R_x(b_2b_3) = V(\mathbb{MP}_{5,1,2}) - \{a_4, a_5, a_1, a'_5, b_5\}$,

$R_x(b_3b_4) = V(\mathbb{MP}_{5,1,2}) - \{a_5, a_1, a_2, a'_1, b_1\}$,
$R_x(b_4b_5) = V(\mathbb{MP}_{5,1,2}) - \{a_1, a_2, a_3, a'_2, b_2\}$,
$R_x(b_5b_1) = V(\mathbb{MP}_{5,1,2}) - \{a_2, a_3, a_4, a'_3, b_3\}$,
$R_x(a'_1b_1) = V(\mathbb{MP}_{5,1,2})$,
$R_x(a'_2b_2) = V(\mathbb{MP}_{5,1,2})$,
$R_x(a'_3b_3) = V(\mathbb{MP}_{5,1,2})$,
$R_x(a'_4b_4) = V(\mathbb{MP}_{5,1,2})$,
$R_x(a'_5b_5) = V(\mathbb{MP}_{5,1,2})$.

For $1 \leq j \leq 5$ it is clear that $|R_x(a_ja_{j+1})| = 8$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{5,1,2}$. Then there exists an upper LRF $\eta : V(\mathbb{MQ}_{5,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{6}$ for each $y \in V(\mathbb{MQ}_{5,1,2}$. In order to show that $\eta(y)$ is a minimal resolving local function, we define another resolving function $\eta'(y) : V(\mathbb{MQ}_{5,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MP}_{5,1,2}$ hence $dim_{LF}(\mathbb{MQ}_{5,1,2}) \leq \sum_{1}^{15} \frac{1}{6} = \frac{5}{2}$. In the same context, for $1 \leq j \leq z$ it is clear that $|R_x(a_ja'_j)| = 8$ and $|R_x(a'_jb_j)| \geq |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{5,1,2}$. Then there exits a maximal lower LRF $\eta : V(\mathbb{MQ}_{5,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{15}$ for each $y \in V(\mathbb{MQ}_{5,1,2})$ hence $dim_{LF}(\mathbb{MQ}_{5,1,2}) \geq \sum_{1}^{15} \frac{1}{15} = 1$. Since $\mathbb{MQ}_{5,1,2})$ is a non bipartite network so its lower bound must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{5,1,2}) \leq \frac{5}{2}.$$

Case 2. For $1 \leq j \leq z$ from Lemma 5.5 it is clear that $|R_x(a_ja_{j+1})| = z + 1$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2})$. Then there exists an upper LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{z-1}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. In order to show that η is a minimal LRF, we define another LRF $\eta' : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{5,1,2}$. Therefore, by Lemma X $dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \sum_{j=1}^{3z} \frac{2}{3z-3} = \frac{2z}{z-1}$. In the same context, for $1 \leq j \leq z$ it is clear from Lemma 5.5 that $|R_x(a'_jb_j)| = 3z$ and $|R_x(a'_jb_j)| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2}$. Then there exists an upper LRF $\eta : V(\mathbb{MP}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{3z}$ for each $y \in V(\mathbb{MQ})$ hence by Lemma Y $dim_{LF}(\mathbb{MQ}_{z,1,2}) \geq \sum_{j=1}^{3z} \frac{1}{3z} = 1$. Since $\mathbb{MQ}_{z,1,2}$ is a non-bipartite network so its lower bound of LFMD must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z-1}.$$

□

Lemma 5.7. *Let $\mathbb{MQ}_{z,1,2}$ be a modified prism network, where $z \cong 3(\ mod\ 4)$. Then*
(a) $|R_x(a_ja_{j+1})| = \frac{3z+3}{2}$ and $\bigcup_{j=1}^{3z} R_x(a_ja_{j+1}) = V(\mathbb{MQ}_{z,1,2})$.
(b) $|R_x(a_ja_{j+1})| < |R_x(y)|$, and $|\bigcup_{j=1}^{3z} R_x(a_ja_{j+1}) \cap R_x(y)| > |R_x(a_ja_{j+1})|$ where $|R_x(y)|$ are the other possible resolving local neighbourhood sets.

Proof. Let a_i inner, a'_i middle and b_i be the outer vertices of modified generalized prism network, for $1 \leq j \leq z$, where $z + 1 \cong (1\ mod\ z)$, we have following possibilities

(a) $R_x(a_ja_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+i-1}\} \cup \{a'_{j+2}, a'_{j+4}, a'_{j+6}, ..., a'_{z+i-1}\} \cup \{b_{j+2}, b_{j+4}, b_{j+6}, ..., b_{z+i-1}\}$ and $|R_x(a_ja_{j+1})| = \frac{3z+3}{2}$ and $|\bigcup_{j=1}^{3z} R_x(a_ja_{j+1})| = 3z = |V(\mathbb{MP}_{z,1,2})|$.
(b) $R_x(a_ja'_j) = V(\mathbb{MP}_{z,1,2}) - \{a'_{j+2}, a'_{j+3}, a'_{z+j-3}, a_{z+j-2}, b_{j+2}, b_{j+3}, b_{z+j-3}, b_{z+j-2}\}$, $R_x(a_ja_{j+2}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+1}, a'_{j+1}, b_{j+1}\}$, $R_x(a'_ja'_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+j-1}, a'_{\frac{z+2j+1}{2}}, b_{\frac{z+2j+1}{2}}\}$, $R_x(b_jb_{j+1}) = V(\mathbb{MP}_{z,1,2}) - \{a_{j+2}, a_{j+4}, a_{j+6}, ..., a_{z+i-1}, a'_{\frac{z+2j+1}{2}}, b_{\frac{z+2j+1}{2}}\}$, $R_x(a'_jb_j) = V(\mathbb{MP}_{z,1,2})$. □

The RLN sets are classified in Table 8 and it is clear that $|R_x(a_ja_{j+1})|$ is less then all other RLN sets $\mathbb{MQ}_{z,1,2}$.

Table 8. Cardinality of each LRN set.

RLN Set	Cardinality
$R_x(a_ja'_j)$	$3z - 4 > \frac{3z+3}{2}$
$R_x(a_ja_{j+2})$	$3z - 4 > \frac{3z+3}{2}$
$R_x(a'_jb_j)$	$3z > \frac{3z+3}{2}$
$R_x(a'_ja'_{j+1})$	$3z - 1 > \frac{3z+3}{2}$
$R_x(b_jb_{j+1})$	$3z - 1 > \frac{3z+3}{2}$

Theorem 5.8. *Let $\mathbb{MQ}_{z,1,2}$ be a generalized modified prism network, where $z \cong 3\ (mod\ 4)$. Then*

$$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z+2}.$$

Proof. **Case 1.** For $z = 7$, we have the following RLN sets
$R_x(a_1a_2) = V(\mathbb{MQ}_{7,1,2}) - \{a_3, a_5, a_7, a'_3, a'_5, a'_7, b_3, b_5, b_7\}$,
$R_x(a_2a_3) = V(\mathbb{MQ}_{7,1,2}) - \{a_4, a_6, a_1, a'_4, a'_6, a'_1, b_4, b_6, b_1\}$,
$R_x(a_3a_4) = V(\mathbb{MQ}_{7,1,2}) - \{a_5, a_7, a_2, a'_5, a'_7, a'_2, b_5, b_7, b_2\}$,
$R_x(a_4a_5) = V(\mathbb{MQ}_{7,1,2}) - \{a_6, a_1, a_3, a'_6, a'_1, a'_3, b_6, b_1, b_3\}$,
$R_x(a_5a_6) = V(\mathbb{MQ}_{7,1,2}) - \{a_7, a_2, a_4, a'_7, a'_2, a'_4, b_7, b_2, b_4\}$,
$R_x(a_6a_7) = V(\mathbb{MQ}_{7,1,2}) - \{a_1, a_3, a_5, a'_1, a'_3, a'_5, b_1, b_3, b_5\}$,
$R_x(a_7a_1) = V(\mathbb{MQ}_{7,1,2}) - \{a_2, a_4, a_6, a'_2, a'_4, a'_6, b_2, b_4, b_6\}$,
$R_x(a_1a'_1) = V(\mathbb{MQ}_{7,1,2}) - \{a'_3, a'_4, a'_5, a'_6, b_3, b_4, b_5, b_6\}$,
$R_x(a_2a'_2) = V(\mathbb{MQ}_{7,1,2}) - \{a'_4, a'_5, a'_6, a'_7, b_4, b_5, b_6, b_7\}$,
$R_x(a_3a'_3) = V(\mathbb{MQ}_{7,1,2}) - \{a'_5, a'_6, a'_7, a'_1, b_5, b_6, b_7, b_1\}$,
$R_x(a_4a'_4) = V(\mathbb{MQ}_{7,1,2}) - \{a'_6, a'_7, a'_1, a'_2, b_5, b_7, b_1, b_2\}$,
$R_x(a_5a'_5) = V(\mathbb{MQ}_{7,1,2}) - \{a'_7, a'_1, a'_2, a'_3b_6, b_1, b_2, b_3\}$,
$R_x(a_6a'_6) = V(\mathbb{MQ}_{7,1,2}) - \{a'_1, a'_2, a'_3, a'_4, b_7, b_2, b_3, b_4\}$,
$R_x(a_7a'_7) = V(\mathbb{MQ}_{7,1,2}) - \{a'_2, a'_3, a'_4, a'_5, b_1, b_3, b_4, b_5\}$,
$R_x(a_1a_3) = V(\mathbb{MQ}_{7,1,2}) - \{a_2, a'_2, b_2\}$,
$R_x(a_2a_4) = V(\mathbb{MQ}_{7,1,2}) - \{a_3, a'_3, b_3\}$,
$R_x(a_3a_5) = V(\mathbb{MQ}_{7,1,2}) - \{a_4, a'_4, b_4\}$,
$R_x(a_4a_6) = V(\mathbb{MQ}_{7,1,2}) - \{a_5, a'_5, b_5\}$,
$R_x(a_5a_7) = V(\mathbb{MQ}_{7,1,2}) - \{a_6, a'_6, b_6\}$,

$R_x(a_6a_1) = V(\mathbb{MQ}_{7,1,2}) - \{a_7, a'_7, b_7\}$,
$R_x(a_7a_2) = V(\mathbb{MQ}_{7,1,2}) - \{a_1, a'_1, b_1\}$,
$R_x(b_1b_2) = V(\mathbb{MQ}_{7,1,2}) - \{a_3, a_5, a_7, b_5\}$,
$R_x(b_2b_3) = V(\mathbb{MQ}_{7,1,2}) - \{a_4, a_6, a_1, b_6\}$,
$R_x(b_3b_4) = V(\mathbb{MQ}_{7,1,2}) - \{a_5, a_7, a_2, b_7\}$,
$R_x(b_4b_5) = V(\mathbb{MQ}_{7,1,2}) - \{a_6, a_1, a_3, b_1\}$,
$R_x(b_5b_6) = V(\mathbb{MQ}_{7,1,2}) - \{a_7, a_2, a_4, b_2\}$,
$R_x(b_6b_7) = V(\mathbb{MQ}_{7,1,2}) - \{a_1, a_3, a_5, b_3\}$,
$R_x(b_7b_1) = V(\mathbb{MQ}_{7,1,2}) - \{a_2, a_4, a_6, b_4\}$,
$R_x(a'_1a'_2) = V(\mathbb{MQ}_{7,1,2}) - \{a_3, a_5, a_7, a'_5, b_5\}$,
$R_x(a'_2a'_3) = V(\mathbb{MQ}_{7,1,2}) - \{a_4, a_6, a_1, a'_6, b_6\}$,
$R_x(a'_3a'_4) = V(\mathbb{MQ}_{7,1,2}) - \{a_5, a_7, a_2, a'_7, b_7\}$,
$R_x(a'_4a'_5) = V(\mathbb{MQ}_{7,1,2}) - \{a_6, a_1, a_3, a'_1, b_1\}$,
$R_x(a'_5a'_6) = V(\mathbb{MQ}_{7,1,2}) - \{a_7, a_2, a_4, a'_2, b_2\}$,
$R_x(a'_6a'_7) = V(\mathbb{MQ}_{7,1,2}) - \{a_1, a_3, a_5, a'_3, b_3\}$,
$R_x(a'_7a'_1) = V(\mathbb{MQ}_{7,1,2}) - \{a_2, a_4, a_6, a'_4, b_4\}$,
$R_x(a'_1b_1) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_2b_2) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_3b_3) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_4b_4) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_5b_5) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_6b_6) = V(\mathbb{MQ}_{7,1,2})$,
$R_x(a'_7b_7) = V(\mathbb{MQ}_{7,1,2})$.

For $1 \leq j \leq 7$ $|R_x(a_ja'_{j+1})| = 13$ and $|R_x(a_ja_{j+1})| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{7,1,2})$. Then there exists an upper LRF $\eta : V(\mathbb{MQ}_{7,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{13}$ for each $y \in V(\mathbb{MQ}_{7,1,2})$. In order to show that $\eta(y)$ is a minimal LRF, we define another LRF $\eta(y)' : V(\mathbb{MQ}_{7,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{7,1,2}$ hence $dim_{LF} \leq \sum_{1}^{21} \frac{1}{12} = \frac{7}{12}$. In the same context,for $1 \leq j \leq 7$ it is clear that $|R_x(a_ja_{j+1})| = 21$ and $|R_x(a_ja_{j+1})| \geq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{7,1,2})$. Then there exists a maximal LLRF $\eta : V(\mathbb{MQ}_{7,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{1}{21}$ for each $y \in V(\mathbb{MQ}_{7,1,2})$ hence $dim_{LF} = \sum_{1}^{21} \frac{1}{21} = 1$. Since $\mathbb{MQ}_{7,1,2})$ is non-bipartite network so its lower bound must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{7,1,2}) \leq \frac{7}{12}.$$

Case 2. For $1 \leq j \leq z$ from Lemma 5.7 it is clear that $|R_x(a_ja'_j)| = \frac{2}{3z+6}$ and $|R_x(a_ja'_j)| \leq |R_x(e)|$, where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2})$. Then there exits an upper LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$ and it is defined as $\eta(y) = \frac{2}{3n+6}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. In order to show that η is a minimal LRF of $\mathbb{MQ}_{z,1,2})$, we define another LRF $\eta' : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$ such that $|\eta'(y)| < |\eta(y)|$ then $\eta(R_x(e)) < 1$ which shows that η' is not a LRF of $\mathbb{MQ}_{z,1,2})$ hence by Lemma X $dim_{LF} \leq \sum_{j=1}^{3z} \frac{2}{3z+3} = \frac{2z}{z+1}$. In the same context for $1 \leq j \leq z$ it is clear from Lemma 5.7 that $|R_x(a'_jb_j)| = 3z$ and $|R_x(b_jb_{j+1})| \geq |R_x(e)|$,where $R_x(e)$ are the other RLN sets of $\mathbb{MQ}_{z,1,2}$. Then there exists a maximal lower LRF $\eta : V(\mathbb{MQ}_{z,1,2}) \to [0,1]$

and it is defined as $\eta(y) = \frac{1}{3z}$ for each $y \in V(\mathbb{MQ}_{z,1,2})$. Therefore, by Lemma Y $dim_{LF} \sum_{j=1}^{3z} \frac{1}{3z} = 1$. Since $\mathbb{MQ}_{z,1,2}$ is a non-bipartite network so its lower bound of LFMD must be greater then 1. Consequently,

$$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z+1}.$$

□

6. Conclusions

In this paper, we have computed the local fractional metric dimension of generalized modified prism networks ($\mathbb{MP}_{z,1,2}, \mathbb{MQ}_{z,1,2}$) in the form of lower and upper bounds. The lower bounds of all the modified prism networks $\mathbb{MQ}_{z,1,2}$ is strictly greater than 1 in all cases. Moreover, all of these modified prism networks remain bounded when $z \to \infty$ as shown in Table 9.

Table 9. Limiting values of LFMDs of modified prism networks.

$z \cong$	LFMDs	Limiting LFMDs as $z \to \infty$	Comment
$1(mod4)$	$\frac{z}{z-1} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z-1}$	$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2$	Bounded
$3(mod4)$	$\frac{z}{z-1} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+1}$	$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2$	Bounded
$0(mod4)$	$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2$	$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2$	Bounded
$2(mod4)$	$\frac{z}{z-2} \leq dim_{LF}(\mathbb{MP}_{z,1,2}) \leq \frac{2z}{z+2}$	$1 < dim_{LF}(\mathbb{MP}_{z,1,2}) \leq 2$	Bounded
$2(mod4)$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z+2}$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2$	Bounded
$0(mod4)$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2$	Bounded
$1(mod4)$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z-1}$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2$	Bounded
$3(mod4)$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq \frac{2z}{z+2}$	$1 < dim_{LF}(\mathbb{MQ}_{z,1,2}) \leq 2$	Bounded

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Conflict of interest

The authors declare that they have no conflicts of interest.

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