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Research article

N-dimensional fractional Hardy operators with rough kernels on central Morrey spaces with variable exponents

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Abstract: In this paper, we obtain some boundedness of the n-dimensional fractional Hardy operators with rough kernels and their commutators on central Morrey spaces with variable exponents.

Keywords: n-dimensional fractional Hardy operator; commutator; rough kernel; central Morrey space; variable exponent; functional spaces
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1. Introduction

The development of Morrey-type spaces has gradually become the mainstream of modern harmonic analysis [1, 9], since the work of Torchinsky [21]. Some scholars established the λ-central BMO estimates of commutators of some operators on central Morrey spaces in [3, 6, 20], including some classical operators such as singular integral operators, fractional integral operators and Hardy operators. With the development of science and technology and the deepening of research content, people gradually find that the function spaces with variable exponents are also very important in harmonic analysis. As early as 1931, Orlicz [17] began to express and study the theory of variable Lebesgue space. Mizuta et al. [15] solved the boundedness of Hardy-Littlewood maximal operators on non-homogeneous central Morrey spaces with variable exponents. In [22], Wang et al. introduced the central BMO spaces with variable exponents and estimated the boundedness of commutators of Hardy operators and their adjoint operators on variable Lebesgue spaces. In particular, Fu et al. [7] gave the definitions of central Morrey spaces and λ-central BMO spaces with variable exponent. Moreover, the estimates of some integral operators and their commutators are given in [23, 24].

In recent years, more and more people studied and developed the theory of Hardy operators. In [6], Fu gave the λ-central BMO estimates for commutators of n-dimensional Hardy operators on central Morrey spaces. And the boundedness of other Hardy-type operators and their commutators has also been discussed on different function spaces [2, 8, 10, 11, 13, 18, 19]. Inspired by the above references, in this paper, we aim to study the boundedness of n-dimensional fractional Hardy operators with rough kernels and their commutators on central Morrey spaces with variable exponents.

Let f be a locally integrable function in $\mathbb{R}^n$ and $0 < \beta < n$. Suppose that $\mathcal{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. The n-dimensional fractional Hardy operator with rough kernels and its adjoint operator can be defined by

$$\begin{aligned} \mathcal{H}_{\Omega,\beta} f(x) &= \frac{1}{|x|^{n-\beta}} \int_{|t| \leq |x|} \Omega(x-t) f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \\ \mathcal{H}^*_{\Omega,\beta} f(x) &= \int_{|t| > |x|} \frac{1}{|t|^{n-\beta}} \Omega(x-t) f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{1.1}$$

where $\Omega \in L^s(\mathcal{S}^{n-1})$, $1 \leq s < \infty$, is homogenous of degree zero.

The commutators of $\mathcal{H}_{\Omega,\beta}$ and $\mathcal{H}^*_{\Omega,\beta}$ are defined by

$$\begin{aligned} \mathcal{H}^{b}_{\Omega,\beta} f &= b\mathcal{H}_{\Omega,\beta} f - \mathcal{H}_{\Omega,\beta}(bf), \\ \mathcal{H}^{b,*}_{\Omega,\beta} f &= b\mathcal{H}^*_{\Omega,\beta} f - \mathcal{H}^*_{\Omega,\beta}(bf), \end{aligned} \tag{1.2}$$

with locally functions b on $\mathbb{R}^n$.

Next, let us explain the outline of this article. In Section 2, we first briefly review some standard notations and lemmas in variable Lebesgue spaces and give the definitions of λ-central BMO spaces and central Morrey spaces with variable exponents. In Section 3, we will establish the boundedness for n-dimensional fractional Hardy operator with rough kernels and its adjoint operator on central Morrey spaces with variable exponents. In Section 4, we will demonstrate the boundedness for the commutators of n-dimensional fractional Hardy operator with rough kernels and its adjoint operator on central Morrey spaces with variable exponents.

2. Function spaces with variable exponents

In this section, we are going to introduce some basic properties of variable Lebesgue spaces and definitions related to the variable exponent function spaces. Throughout this article, we denote by $|B|$ and χ_B the Lebesgue measure and characteristic function of a measurable set $B \subset \mathbb{R}^n$, respectively, where

$$B = \{x \in \mathbb{R}^n : |x| \leq R\}.$$

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot)$: $E \longrightarrow [1, \infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces.

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) := \{f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}(E)$ to be the set of $p(\cdot)$: $E \longrightarrow [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty,$$

and $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ which satisfies

$$\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1.$$

Let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ defining

$$\mathcal{M}f = \sup_{R>0} \frac{1}{|B_r|} \int_{B_r \cap E} |f| dy$$

is bounded on $L^{p(\cdot)}(E)$, where

$$B_r = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

In variable L^p spaces there are some important lemmas as follows.

Lemma 2.1. [4] Given an open set $E \subset \mathbb{R}^n$. If $p(\cdot) \in \mathcal{P}(E)$ and satisfies

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}, \quad |x - y| \leq 1/2, \tag{2.1}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|, \tag{2.2}$$

then $p(\cdot) \in \mathcal{B}(E)$, that is the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(E)$.

Lemma 2.2. [14] (Generalized Hölder's inequality) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then f, g are integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

Lemma 2.3. [12] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $0 < \delta < 1$ and a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|}, \tag{2.3}$$

and

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta}. \tag{2.4}$$

Lemma 2.4. [12] Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{2.5}$$

Lemma 2.5. [5] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1. Then

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & if\ |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}} & if\ |Q| \ge 1, \end{cases}$$

for every cube (or ball) $Q \in \mathbb{R}^n$, where

$$p(\infty) = \lim_{x\to\infty} p(x).$$

Lemma 2.6. [5] Let $p(\cdot),\ q(\cdot),\ s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},$$

for almost every $x \in \mathbb{R}^n$. Then

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.7. [16] If $f \in L^s(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$, and

$$\frac{1}{s} + \frac{1}{q(\cdot)} = \frac{1}{p(\cdot)},$$

then we have

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^s(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of f and g.

Now we recall that the λ-central BMO space with variable exponent and the central Morrey space with variable exponent in [7] are defined as follows.

Definition 2.1. [7] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda < 1/n$. The λ-central BMO space with variable exponent $CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$CBMO^{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Definition 2.2. [7] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The central Morrey space with variable exponent $\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Remark 2.1. *Denote by $CMO^{p(\cdot),\lambda}(\mathbb{R}^n)$ and $\mathcal{B}^{p(\cdot),\lambda}(\mathbb{R}^n)$ the inhomogeneous versions of the λ-central BMO spaces and central Morrey spaces with variable exponents, which are defined, respectively, by taking the supremum over $R \geq 1$ in Definitions 2.1 and 2.2 instead of $R > 0$ there.*

Remark 2.2. *Our results in this paper remain true for the inhomogeneous versions of λ-central BMO spaces and central Morrey spaces with variable exponents.*

3. *N*-dimensional fractional Hardy operators with rough kernels

We begin this section by illustrating and proving Theorem 3.1, which present the boundedness of n-dimensional fractional Hardy operator with rough kernels and its adjoint operator on central Morrey spaces with variable exponents. Here and subsequently, for simplicity, we write 2^kB with the same center as B and 2^k times of its radius,

$$C_k = 2^kB\backslash 2^{k-1}B,$$

for $k \in \mathbb{Z}$.

Theorem 3.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1. Suppose that $\Omega \in L^s(\mathcal{S}^{n-1})$ with $s > p'(\cdot)$.

(1) If $\lambda_2 = \lambda_1 + \frac{\beta}{n}$ and $\lambda_2 + \delta > 0$, then

$$\|\mathcal{H}_{\Omega,\beta} f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

(2) If $\lambda_2 = \lambda_1 + \frac{\beta}{n} < 0$ and $\lambda_2 + \delta > 0$, then

$$\|\mathcal{H}^*_{\Omega,\beta} f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

To start the proof of Theorem 3.1, we need the following lemma.

Lemma 3.1. [8] For $x \in 2^kB, t \in 2^iB$, we have

$$0 \leq |x-t| \leq |x| + |2^iB|^{\frac{1}{n}} \leq 2\cdot|2^kB|^{\frac{1}{n}},$$

and

$$\int_{B_i} |\Omega(x-t)|^s dt \leq \int_0^{2\cdot|2^kB|^{\frac{1}{n}}} \int_{\mathcal{S}^{n-1}} |\Omega(x')|^s d\sigma(x') r^{n-1} dr \leq C|2^kB|.$$

Proof of Theorem 3.1. Through the definition of $\mathcal{H}_{\Omega,\beta}$ and generalized Hölder's inequality to $L^{p(\cdot)}$ and $L^{p'(\cdot)}$ we have

$$\begin{aligned}
|\mathcal{H}_{\Omega,\beta} f(x)\chi_B(x)| &\leq \frac{1}{|x|^{n-\beta}}\left|\int_{|t|\leq|x|}\Omega(x-t)f(t)dt\right|\cdot\chi_B(x)\\
&\leq C\sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\beta}}\left|\int_{|t|\leq|x|}\Omega(x-t)f(t)dt\right|\cdot\chi_{C_k}(x)\\
&\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}\left|\int_{2^kB}\Omega(x-t)f(t)dt\right|\cdot\chi_{C_k}(x)\\
&\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}\|\Omega(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\cdot\chi_{C_k}(x).
\end{aligned} \tag{3.1}$$

In view of the condition $s > p'(\cdot)$ and $\Omega \in L^s(\mathcal{S}^{n-1})$, set

$$\frac{1}{p'(\cdot)} = \frac{1}{s} + \frac{1}{q(\cdot)},$$

then by Lemmas 2.5, 2.7 and 3.1 we have

$$\begin{aligned}\|\Omega(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)} &\leq C\|\Omega(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leq C|2^kB|^{\frac{1}{s}}\|\chi_{2^kB}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\approx C|2^kB|^{\frac{1}{s}}|2^kB|^{\frac{1}{q(\cdot)}}\\ &= C|2^kB|^{\frac{1}{p'(\cdot)}}\\ &\approx C\|\chi_{2^kB}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{3.2}$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (3.1) both sides and using Minkowski inequality, then by Lemma 2.4, we write

$$\begin{aligned}\|\mathcal{H}_{\Omega,\beta}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta}{n}+\lambda_1}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\frac{1}{|2^kB|}\|\chi_{2^kB}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}|2^kB|^{\lambda_2}.\end{aligned}$$

Finally, by inequality (2.4), we can obtain

$$\begin{aligned}\|\mathcal{H}_{\Omega,\beta}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\frac{|2^kB|^{\lambda_2}}{|B|^{\lambda_2}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}2^{nk(\lambda_2+\delta)}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

where the fact $\lambda_2 + \delta > 0$ has been used in the last inequality.

Thus we have

$$\|\mathcal{H}_{\Omega,\beta}f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Next we prove the boundedness of adjoint n-dimensional fractional Hardy operator $\mathcal{H}^*_{\Omega,\beta}$.

It can also be seen from the definition of $\mathcal{H}^*_{\Omega,\beta}$ and generalized Hölder's inequality to $L^{p(\cdot)}$ and $L^{p'(\cdot)}$, we have

$$
\begin{aligned}
|\mathcal{H}^*_{\Omega,\beta}f(x)\chi_B(x)| &\le \left|\int_{|t|>|x|}\frac{1}{|t|^{n-\beta}}\Omega(x-t)f(t)dt\right|\cdot\chi_B(x)\\
&\le \sum_{k=-\infty}^{0}\left|\int_{|t|>|x|}\frac{1}{|t|^{n-\beta}}\Omega(x-t)f(t)dt\right|\cdot\chi_{C_k}(x)\\
&\le \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}\frac{1}{|t|^{n-\beta}}\Omega(x-t)f(t)dt\right|\cdot\chi_{C_k}(x) \qquad (3.3)\\
&\le \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta-n}{n}}\left|\int_{2^jB}\Omega(x-t)f(t)dt\right|\cdot\chi_{C_k}(x)\\
&\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta-n}{n}}\|\Omega(x-\cdot)\chi_{2^jB}(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^jB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\cdot\chi_{C_k}(x).
\end{aligned}
$$

Using (3.2) and taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (3.3) both sides, we can see that

$$
\begin{aligned}
\|\mathcal{H}^*_{\Omega,\beta}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta}{n}}\|f\chi_{2^jB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{1}{|2^jB|}\|\chi_{2^jB}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta}{n}}\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}|2^jB|^{\lambda_1}\frac{1}{|2^jB|}\|\chi_{2^jB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{2^jB}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\lambda_1+\frac{\beta}{n}}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\lambda_2}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, through inequality (2.4), $\lambda_2 < 0$ and $\lambda_2 + \delta > 0$, we can obtain

$$
\begin{aligned}
\|\mathcal{H}^*_{\Omega,\beta}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\frac{|2^jB|^{\lambda_2}}{|2^kB|^{\lambda_2}}\frac{|2^kB|^{\lambda_2}}{|B|^{\lambda_2}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&= C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}2^{n(j-k)\lambda_2}2^{kn(\lambda_2+\delta)}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we have

$$
\|\mathcal{H}^*_{\Omega,\beta}f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_2}(\mathbb{R}^n)} \le C\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbb{R}^n)}.
$$

The proof of Theorem 3.1 is completed.

4. Commutators of n-dimensional fractional Hardy operators with rough kernels

In Section 3 we have proved the boundedness of n-dimensional fractional Hardy operators with rough kernels on central Morrey spaces with variable exponents. In this section, we mainly prove the boundedness of their commutators on central Morrey spaces with variable exponents.

Theorem 4.1. Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 with

$$\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}.$$

Suppose that $\Omega \in L^s(\mathcal{S}^{n-1})$, $s > p'(\cdot)$. If

$$b \in CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n),\quad \lambda = \lambda_1 + \lambda_2 + \frac{\beta}{n},$$

and $\lambda + \delta > 0$, then

$$\|\mathcal{H}^b_{\Omega,\beta} f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Proof. We decompose the integral as

$$\begin{aligned}
|\mathcal{H}^b_{\Omega,\beta} f(x)\chi_B(x)| &\leq \left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(x)-b(t))f(t)dt\right|\cdot\chi_B(x)\\
&\leq \sum_{k=-\infty}^{0}\left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(x)-b(t))f(t)dt\right|\cdot\chi_{C_k}(x)\\
&\leq \sum_{k=-\infty}^{0}\left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(x)-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right|\\
&\quad+\sum_{k=-\infty}^{0}\left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(t)-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right|\\
&=: D+E,
\end{aligned}\tag{4.1}$$

where

$$D = \sum_{k=-\infty}^{0}\left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(x)-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right|,$$

and

$$E = \sum_{k=-\infty}^{0}\left|\frac{1}{|x|^{n-\beta}}\int_{|t|\leq|x|}\Omega(x-t)(b(t)-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right|.$$

Estimate D, firstly. By using generalized Hölder's inequality to $L^{p_1(\cdot)}$ and $L^{p_1'(\cdot)}$ we have

$$\begin{aligned}
D &\leq \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\beta}}\left|\int_{|t|\leq|x|}\Omega(x-t)f(t)dt\cdot(b(x)-b_{2^kB})\chi_{C_k}(x)\right|\\
&\leq \sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}|(b(x)-b_{2^kB})\chi_{C_k}(x)|\cdot\left|\int_{2^kB}\Omega(x-t)f(t)dt\right|\\
&\leq C\sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta-n}{n}}|(b(x)-b_{2^kB})\chi_{C_k}(x)|\cdot\|\Omega(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

In view of the condition $s > p'(\cdot) > p_1'(\cdot)$, we can take

$$\frac{1}{p_1'(\cdot)} = \frac{1}{s} + \frac{1}{q_1(\cdot)}.$$

We note that $\Omega \in L^s(\mathcal{S}^{n-1})$, and using Lemma 2.4 and (3.2) to the term D, we can produce

$$\begin{aligned} D &\le C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta-n}{n}} |(b(x) - b_{2^k B})\chi_{C_k}(x)| \cdot \|\chi_{2^k B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\le C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta-n}{n}} |(b(x) - b_{2^k B})\chi_{C_k}(x)| \cdot \|\chi_{2^k B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \\ &\times \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} |2^k B|^{\lambda_1} \\ &\le C \sum_{k=-\infty}^{0} |2^k B|^{\lambda_1 + \frac{\beta}{n}} |(b(x) - b_{2^k B})\chi_{C_k}(x)| \cdot \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}. \end{aligned} \tag{4.2}$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (4.2) both sides. Noting $\lambda + \delta > 0$ and

$$\frac{1}{p(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_1(\cdot)},$$

by using Lemma 2.6 and inequality (2.4) we can obtain

$$\begin{aligned} \|D\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C \sum_{k=-\infty}^{0} \|(b - b_{2^k B})\chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2^k B|^{\lambda_1 + \frac{\beta}{n}} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} |2^k B|^{\lambda_2 + \lambda_1 + \frac{\beta}{n}} \|\chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\ &= C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} \frac{|2^k B|^{\lambda}}{|B|^{\lambda}} \frac{\|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} 2^{nk(\lambda+\delta)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Next, we simply estimate E, because its proof method is similar to D. In view of the condition $s > p'(\cdot)$, by

$$\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)},$$

we can get

$$\frac{1}{p_1'(\cdot)} = \frac{1}{p'(\cdot)} + \frac{1}{p_2(\cdot)} > \frac{1}{s} + \frac{1}{p_2(\cdot)}.$$

So we can take

$$\frac{1}{p_1'(\cdot)} = \frac{1}{s} + \frac{1}{p_2(\cdot)} + \frac{1}{q_2(\cdot)}.$$

Using generalized Hölder's inequality, Lemmas 2.4, 2.6 and 2.7 we have

$$
\begin{aligned}
E &= \sum_{k=-\infty}^{0} \left| \frac{1}{|x|^{n-\beta}} \int_{|t|\le|x|} \Omega(x-t)(b(t)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x) \right| \\
&\le \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} \left| \int_{|t|\le|x|} \Omega(x-t)(b(t)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x) \right| \\
&\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} \|\Omega(x-\cdot)(b(\cdot)-b_{2^kB})\chi_{C_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_{C_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} \|\Omega(x-\cdot)\chi_{2^kB}\|_{L^s(\mathbb{R}^n)} \|(b(\cdot)-b_{2^kB})\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\times \|f\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} |2^kB|^{\frac{1}{s}} \|(b(\cdot)-b_{2^kB})\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|f\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2^kB|^{\lambda_1+\lambda_2} |2^kB|^{\frac{1}{s}} \\
&\times \|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta-n}{n}} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |2^kB|^{\lambda_1+\lambda_2} \|\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\le C \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}+\lambda_1+\lambda_2} \cdot \chi_{C_k}(x).
\end{aligned}
$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on E both sides and using inequality (2.4),

$$\lambda = \lambda_1 + \lambda_2 + \frac{\beta}{n},$$

and $\lambda + \delta > 0$, we can obtain

$$
\begin{aligned}
\|E\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} |2^kB|^{\lambda_1+\lambda_2+\frac{\beta}{n}} \|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} \frac{|2^kB|^{\lambda}}{|B|^{\lambda}} \frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} 2^{nk(\lambda+\delta)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

In summary, taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (4.1) both sides and using Minkowski's inequality, we thus have established the following inequality if we combine the above estimates for D and E,

$$
\begin{aligned}
\|\mathcal{H}^b_{\Omega,\beta} f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le \|D\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|E\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we have

$$\|\mathcal{H}^{b}_{\Omega,\beta}f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

The proof of Theorem 4.1 is completed.

Theorem 4.2. Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 with

$$\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}.$$

Suppose that $\Omega \in L^s(\mathcal{S}^{n-1})$, $s > p'(\cdot)$. If

$$b \in CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n),\quad \lambda = \lambda_1 + \lambda_2 + \frac{\beta}{n} < 0,\quad \lambda + \delta > 0,\quad \text{and}\quad 0 \leq \lambda_2 < 1/n,$$

then

$$\|\mathcal{H}^{b,*}_{\Omega,\beta}f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Proof. We decompose the integral as follows

$$\begin{aligned}
|\mathcal{H}^{b,*}_{\Omega,\beta}f(x)\chi_B(x)| &= \left|\int_{|t|>|x|} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(x)-b(t))f(t)dt \cdot \chi_B(x)\right| \\
&\leq \sum_{k=-\infty}^{0} \left|\int_{|t|>|x|} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(x)-b(t))f(t)dt \cdot \chi_{C_k}(x)\right| \\
&\leq \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty} \left|\int_{C_j} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(x)-b(t))f(t)dt \cdot \chi_{C_k}(x)\right| \\
&\leq \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty} \left|\int_{C_j} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(x)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x)\right| \\
&\quad + \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty} \left|\int_{C_j} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(t)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x)\right| \\
&=: F + G,
\end{aligned} \tag{4.3}$$

where

$$F = \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty} \left|\int_{C_j} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(x)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x)\right|,$$

and

$$G = \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty} \left|\int_{C_j} \frac{1}{|t|^{n-\beta}}\Omega(x-t)(b(t)-b_{2^kB})f(t)dt \cdot \chi_{C_k}(x)\right|.$$

For F, we follow a procedure similar to the estimate of Theorem 3.1.

$$
\begin{aligned}
F &\leq \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta-n}{n}} \left| \int_{C_j} \Omega(x-t) f(t) dt \right| \cdot |(b(x) - b_{2^k B}) \chi_{C_k}(x)| \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta-n}{n}} \|\Omega(x-\cdot)\chi_{2^j B}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_{2^j B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot |(b(x) - b_{2^k B}) \chi_{C_k}(x)| \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta-n}{n}} \|\chi_{2^j B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_{2^j B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot |(b(x) - b_{2^k B}) \chi_{C_k}(x)| \qquad (4.4) \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta}{n}+\lambda_1} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \frac{1}{|2^j B|} \|\chi_{2^j B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2^j B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} |(b(x) - b_{2^k B}) \chi_{C_k}(x)| \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta}{n}+\lambda_1} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} |(b(x) - b_{2^k B}) \chi_{C_k}(x)|.
\end{aligned}
$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (4.4) both sides. Noting

$$
\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)},
$$

$$
\lambda = \lambda_1 + \lambda_2 + \frac{\beta}{n} < 0,
$$

and

$$
\lambda + \delta > 0,
$$

by using Lemma 2.6 and inequality (2.4), we can obtain

$$
\begin{aligned}
\|F\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta}{n}+\lambda_1} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|(b - b_{2^k B}) \chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta}{n}+\lambda_1} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|(b - b_{2^k B}) \chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} |2^j B|^{\frac{\beta}{n}+\lambda_1+\lambda_2} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \\
&\times \|\chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&= C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} \frac{|2^j B|^{\lambda}}{|2^k B|^{\lambda}} \frac{|2^k B|^{\lambda}}{|B|^{\lambda}} \frac{\|\chi_{2^k B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\
&\times \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&= C \sum_{k=-\infty}^{0} \sum_{j=k}^{\infty} 2^{n(j-k)\lambda} 2^{nk(\lambda+\delta)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For G, we decompose the integral appearing as follows

$$\begin{aligned} G &\le \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)(b(t)-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right| \\ &\le \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)(b(t)-b_{2^jB})f(t)dt\cdot\chi_{C_k}(x)\right| \\ &+ \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)(b_{2^jB}-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right| \\ &=: G_1+G_2, \end{aligned}$$

where

$$G_1 = \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)(b(t)-b_{2^jB})f(t)dt\cdot\chi_{C_k}(x)\right|,$$

and

$$G_2 = \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)(b_{2^jB}-b_{2^kB})f(t)dt\cdot\chi_{C_k}(x)\right|.$$

We estimate G_1 and G_2 respectively. Before estimate G_2, the following boundedness for $|b_{2^jB}-b_{2^kB}|$ can be shown

$$\begin{aligned} |b_{2^jB}-b_{2^kB}| &\le \sum_{m=k}^{j-1}|b_{2^{m+1}B}-b_{2^mB}| \\ &\le C\sum_{m=k}^{j-1}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^{m+1}B|^{\lambda_2}\frac{1}{|2^{m+1}B|}\|\chi_{2^{m+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{m+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \\ &\le C\sum_{m=k}^{j-1}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^{m+1}B|^{\lambda_2} \\ &\le C(j-k)\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^jB|^{\lambda_2}. \end{aligned}$$

By generalized Hölder's inequality to $p_2(\cdot)$ and $p_2'(\cdot)$ and applying Lemma 2.4, we can found

$$\begin{aligned} G_2 &= \sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\left|\int_{C_j}|2^jB|^{\frac{\beta-n}{n}}\Omega(x-t)f(t)dt\right|\cdot|b_{2^jB}-b_{2^kB}|\cdot\chi_{C_k}(x) \\ &\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta-n}{n}}\left|\int_{C_j}\Omega(x-t)f(t)dt\right|\cdot\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^jB|^{\lambda_2}\cdot\chi_{C_k}(x) \\ &\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta}{n}+\lambda_2-1}\|\Omega(x-\cdot)\chi_{2^jB}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^jB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\cdot\chi_{C_k}(x) \\ &\le C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta}{n}+\lambda_2-1}\|\chi_{2^jB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{2^jB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\cdot\chi_{C_k}(x) \end{aligned}$$

$$\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta}{n}+\lambda_1+\lambda_2}\cdot\chi_{C_k}(x).$$

Following the steps taken to estimate F and using inequality (2.4), $\lambda<0$ and $\lambda+\delta>0$, we can obtain

$$\begin{aligned}
\|G_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta}{n}+\lambda_1+\lambda_2}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)|2^jB|^{\frac{\beta}{n}+\lambda_1+\lambda_2}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&= C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)\frac{|2^jB|^{\lambda}}{|2^kB|^{\lambda}}\frac{|2^kB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&= C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}(j-k)2^{n(j-k)\lambda}2^{nk(\lambda+\delta)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, similar to the proof method of E in Theorem 4.1, we have

$$\begin{aligned}
G_1 &\leq C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\|\Omega(x-\cdot)(b(\cdot)-b_{2^jB})\chi_{C_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_{C_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\cdot\chi_{C_k}(x)\\
&\leq C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta}{n}+\lambda_1+\lambda_2}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\cdot\chi_{C_k}(x).
\end{aligned}$$

So we can take the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on G_1 both sides. We write

$$\begin{aligned}
\|G_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}|2^jB|^{\frac{\beta}{n}+\lambda_1+\lambda_2}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}\frac{|2^jB|^{\lambda}}{|2^kB|^{\lambda}}\frac{|2^kB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&= C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k}^{\infty}2^{n(j-k)\lambda}2^{nk(\lambda+\delta)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Finally, taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on (4.3) both sides and using Minkowski's inequality, we have established the following inequality if we combine the above estimates for F and G,

$$\begin{aligned}
\|\mathcal{H}^{b,*}_{\Omega,\beta}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq \|F\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\|G\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus we have

$$\|\mathcal{H}^{b,*}_{\Omega,\beta}f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leq C\|b\|_{CBMO^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

The proof of Theorem 4.2 is completed.

5. Conclusions

In this article, we first establish the boundedness for n-dimensional fractional Hardy operator with rough kernels $\mathcal{H}_{\Omega,\beta}$ and its adjoint operator $\mathcal{H}^*_{\Omega,\beta}$ on central Morrey spaces with variable exponents. Furthermore, we prove that their commutators $\mathcal{H}^b_{\Omega,\beta}$ and $\mathcal{H}^{b,*}_{\Omega,\beta}$ are bounded on central Morrey spaces with variable exponents.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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