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Research article

Certain new applications of Faber polynomial expansion for some new subclasses of υ-fold symmetric bi-univalent functions associated with q-calculus

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Abstract: In this article, we define the q-difference operator and Salagean q-differential operator for υ-fold symmetric functions in open unit disk $\mathcal{U}$ by first applying the concepts of q-calculus operator theory. Then, we considered these operators in order to construct new subclasses for υ-fold symmetric bi-univalent functions. We establish the general coefficient bounds $|a_{\upsilon k+1}|$ for the functions in each of these newly specified subclasses using the Faber polynomial expansion method. Investigations are also performed on Feketo-Sezego problems and initial coefficient bounds for the function h that belong to the newly discovered subclasses. To illustrate the relationship between the new and existing research, certain well-known corollaries of our main findings are also highlighted.

Keywords: quantum (or q-) calculus; analytic functions; q-derivative operator; υ-fold symmetric bi-univalent functions; Faber polynomials expansions
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1. Introduction and definitions

Assume that $\mathfrak{A}$ denotes the set of all analytic functions $\mathfrak{h}(z)$ in the open symmetric unit disk

$$\mathcal{U} = \{z : |z| < 1\},$$

which are normalized by

$$\mathfrak{h}(0) = 0 \ \ and \ \ \mathfrak{h}'(0) = 1.$$

Thus, every function $\mathfrak{h} \in \mathfrak{A}$ can be expressed in the form given in (1.1)

$$\mathfrak{h}(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let an analytic function $\mathfrak{h}$ is said to be univalent if it satisfy the following condition:

$$\mathfrak{h}(z_1) \neq \mathfrak{h}(z_2) \Rightarrow z_1 \neq z_2, \ \forall \, z_1, z_2 \in \mathcal{U}.$$

Furthermore, $\mathcal{S}$ is the subclass of $\mathfrak{A}$ whose members are univalent in $\mathcal{U}$. The idea of subordination was initiated by Lindelof [30] and Little-wood and Rogosinski have further improved this idea, see [31, 35, 36]. For $\mathfrak{h}$, $y \in \mathfrak{A}$, and $\mathfrak{h}$ subordinate to y in $\mathcal{U}$, denoted by

$$\mathfrak{h}(z) \prec y(z), \qquad z \in \mathcal{U},$$

if we have a function u, such that

$$u \in \mathcal{B} = \{u : u \in \mathfrak{A}, |u(z)| < 1, \ and \ u(0) = 0, \ z \in \mathcal{U}\}$$

and

$$\mathfrak{h}(z) = y(u(z)), \ \ z \in \mathcal{U}.$$

According to the Koebe one-quarter theorem (see [13]), the image of $\mathcal{U}$ under $\mathfrak{h} \in \mathcal{S}$ contains a disk of radius one-quarter centered at origin. Thus, every function $\mathfrak{h} \in \mathcal{S}$ has an inverse $\mathfrak{h}^{-1} = g$, defined as:

$$g(\mathfrak{h}(z)) = z, \qquad z \in \mathcal{U}$$

and

$$\mathfrak{h}(g(w)) = w, \ \ |w| < r_0(\mathfrak{h}), \ \ r_0(\mathfrak{h}) \geq \frac{1}{4}.$$

The power series for the inverse function $g(w)$ is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - Q(a)w^4 +, \cdots, \tag{1.2}$$

Where

$$Q(a) = (5a_2^3 - 5a_2a_3 + a_4).$$

An analytic function $\mathfrak{h}$ is called bi-univalent in $\mathcal{U}$ if $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ are univalent in $\mathcal{U}$ and class of all bi-univalent functions are denoted by Σ. In 1967, for $\mathfrak{h} \in \Sigma$, Levin [32] showed that $|a_2| < 1.51$ and after twelve years Branan and Clunie [8] gave the improvement of $|a_2|$ and proved that $|a_2| \leq \sqrt{2}$. Furthermore, for $\mathfrak{h} \in \Sigma$, Netanyahu [34] proved that $\max |a_2| = \frac{4}{3}$ and an intriguing subclass of analytic and bi-univalent functions was proposed and studied by Branan and Taha [9], who also discovered estimates for the coefficients of the functions in this subclass. Recently, the investigation of numerous subclasses of the analytic and bi-univalent function class Σ was basically revitalized by the pioneering work of Srivastava et al. [41]. In 2012, Xu et al. [44] defined a general subclass of class Σ and investigated coefficient estimates for the functions belonging to the new subclass of class Σ. Recently, several different subclasses of class Σ were introduced and investigated by a number of authors (see for details ([23, 29, 38]). In these recent papers only non-sharp estimates on the initial coefficients were obtained.

Faber polynomials was introduced by Faber [15] and first time he used it to determine the general coefficient bounds $|a_k|$ for $k \geq 4$. Gong [16] interpreted significance of Faber polynomials in mathematical sciences, particularly in Geometric Function Theory. In 1913, Hamidi et al. [18] first time used the Faber polynomials expansion technique on meromorphic bi-starlike functions and determined the coefficient estimates. The Faber polynomials expansion method for analytic bi-close-to-convex functions was examined by Hamidi and Jahangiri [21, 22], who also discovered some new coefficient bounds for new subclasses of close-to-convex functions. Furthermore, many authors [3, 4, 7, 11, 12, 14, 20] used the same technique and determined some interesting and useful properties for analytic bi-univalent functions. For $\mathfrak{h} \in \Sigma$, by using the Faber polynomial expansions methods, only a few works have been done so far and we recognized very little over the bounds of Maclaurin's series coefficient $|a_k|$ for $k \geq 4$ in the literature. Recently only a few authors, used the Faber polynomials expansion technique and determined the general coefficient bounds $|a_k|$ for $k \geq 4$, (see for detail [6, 11, 24, 39, 40, 42]).

A domain $\mathcal{U}$ is said to be the υ-fold symmetric if

$$\mathfrak{h}_\upsilon\left(e^{k(\frac{2\pi}{\upsilon})}(z)\right) = e^{k(\frac{2\pi}{\upsilon})}\mathfrak{h}_\upsilon(z), \quad z \in \mathcal{U}, \upsilon \in \mathbb{Z}^+, \mathfrak{h} \in \mathfrak{A}$$

and every $\mathfrak{h}_\upsilon$ has the series of the form

$$\mathfrak{h}_\upsilon = z + \sum_{k=1}^{\infty} a_{\upsilon k+1} z^{\upsilon k+1}. \tag{1.3}$$

The class $\mathcal{S}^\upsilon$ represents the set of all υ-fold symmetric univalent functions. For $\upsilon = 1$, then $\mathcal{S}^\upsilon$ reduce to the class $\mathcal{S}$ of univalent functions. If the inverse g_υ of univalent $\mathfrak{h}$ is univalent then $\mathfrak{h}$ is called υ-fold symmetric bi-univalent functions in $\mathcal{U}$ and denoted by Σ_υ. The series expansion of inverse function g_υ investigated by Srivastava et al. in [43]:

$$\begin{aligned} g_\upsilon(w) = w - a_{\upsilon+1}w^{\upsilon+1} + ((\upsilon+1)a_{\upsilon+1}^2 - a_{2\upsilon+1})w^{2\upsilon+1} - \bigg\{\frac{1}{2}(\upsilon+1)(3\upsilon+2)a_{\upsilon+1}^3 \\ - \frac{1}{2}(\upsilon+1)(\upsilon+2)a_{\upsilon+1}^3 - ((3\upsilon+2)a_{\upsilon+1}a_{2\upsilon+1} + a_{3\upsilon+1})\bigg\}w^{3\upsilon+1}. \end{aligned} \tag{1.4}$$

For $\upsilon = 1$, the series in (1.4) reduces to the (1.2) of the class Σ. In [43] Srivastava et al. defined a subclass of υ-fold symmetric bi-univalent functions and investigated coeffiients problem for υ-fold symmetric bi-univalent functions. Hamidi and Jahangiri [19] defined υ-fold symmetric bi-starlike functions and discussed the unpredictability of the coefficients of υ-fold symmetric bi-starlike functions.

Many researchers have used the q-calculus and fractional q-calculus in the field of Geometric Function Theory (GFT) and they defined and studied several new subclasses of analytic, univalent and bi-univalent functions. In 1909, Jackson ([26, 27]), gave the idea of q-calculus operator and defined the q-difference operator (D_q) while in [25], Ismail et al. was the first who used D_q in order to define a class of q-starlike functions in open unit disk $\mathcal{U}$. The most signifcant usages of q-calculus in the perspective of GFT was basically furnished and the basic (*or* $q-$) hypergeometric functions were first used in GFT in a book chapter by Srivastava (see, for details, [37]). For more study about q-calculus operator theory in GFT, see the following articles [5, 28, 33].

Now we recall, some basic definitions and concepts of the q-calculus which will be used to define some new subclasses of the this paper.

For a non-negative integer t, the q-number $[t, q]$, $(0 < q < 1)$, is defined by

$$[t, q] = \frac{1 - q^t}{1 - q}, \ \textit{and} \ [0, q] = 0$$

and the q-number shift factorial is given by

$$[t, q]! = [1, q][2, q][3, q] \cdots [t, q],$$

$$[0, q]! = 1.$$

For $q \to 1-$, then $[t, q]!$ reduces to $t!$.
The q-generalized Pochhammer symbol is defined by

$$[t, q]_k = \frac{\Gamma_q(t + k)}{\Gamma_q(t)}, \ k \in \mathbb{N}, \ t \in \mathbb{C}.$$

Remark 1.1. *For $q \to 1-$, then $[t, q]_k$ reduces to $(t)_k = \frac{\Gamma(t+k)}{\Gamma(t)}$.*

Definition 1.2. *Jackson [27] defined the q-integral of function $\mathfrak{h}(z)$ as follows:*

$$\int \mathfrak{h}(z) d_q(z) = \sum_{k=0}^{\infty} z(1 - q)\mathfrak{h}(q^k(z))q^k.$$

Jackson [26] introduced the q-difference operator for analytic functions as follows:

Definition 1.3. *[26]. For $\mathfrak{h} \in \mathcal{A}$, the q-difference operator is defined as:*

$$D_q\mathfrak{h}(z) = \frac{\mathfrak{h}(qz) - \mathfrak{h}(z)}{z(q - 1)}, \qquad z \in \mathcal{U}.$$

Note that, for $k \in \mathbb{N}$ and $z \in \mathcal{U}$ and

$$D_q(z^k) = [k, q]z^{k-1}, \ D_q\Big(\sum_{k=1}^{\infty} a_k z^k\Big) = \sum_{k=1}^{\infty} [k, q]a_k z^{k-1}.$$

Here, we introduce the q-difference operator for υ-fold symmetric functions related to the q-calculus as follows:

Definition 1.4. *Let $\mathfrak{h}_\upsilon \in \Sigma_\upsilon$, of the form* (1.3). *Then q-difference operator will be defined as*

$$D_q\mathfrak{h}_\upsilon(z) = \frac{\mathfrak{h}_\upsilon(qz) - \mathfrak{h}_\upsilon(z)}{(q - 1)z}, \qquad z \in \mathcal{U}, \tag{1.5}$$

$$= 1 + \sum_{k=1}^{\infty} [\upsilon k + 1, q]a_{\upsilon k+1} z^{\upsilon k}$$

and

$$D_q\Big(\sum_{k=1}^{\infty} a_{\upsilon k+1} z^{\upsilon k+1}\Big) = \sum_{k=1}^{\infty} [\upsilon k + 1, q]a_{\upsilon k+1} z^{\upsilon k},$$

$$D_q(z)^{\upsilon k+1} = [\upsilon k + 1, q]z^{\upsilon k}.$$

Now we define Salagean q-differential operator for υ-fold symmetric functions as follows:

Definition 1.5. *For $m \in \mathbb{N}$, the Salagean q-differential operator for $\mathfrak{h}_\upsilon \in \Sigma_\upsilon$ is defined by*

$$\nabla_q^0 \mathfrak{h}_\upsilon(z) = \mathfrak{h}_\upsilon(z), \quad \nabla_q^1 \mathfrak{h}_\upsilon(z) = zD_q \mathfrak{h}_\upsilon(z) = \frac{\mathfrak{h}_\upsilon(qz) - \mathfrak{h}_\upsilon(z)}{(q-1)}, \cdots,$$

$$\nabla_q^m \mathfrak{h}_\upsilon(z) = zD_q(\nabla_q^{m-1}\mathfrak{h}_\upsilon(z)) = \left(z + \sum_{k=1}^{\infty}([\upsilon k+1, q])^m z^{\upsilon k+1}\right),$$

$$\nabla_q^m \mathfrak{h}_\upsilon(z) = z + \sum_{k=1}^{\infty}([\upsilon k+1, q])^m a_{\upsilon k+1} z^{\upsilon k+1}. \tag{1.6}$$

Remark 1.6. *For $\upsilon = 1$, we have Salagean q-differential operator for analytic functions proved in [17].*

Motivated by the following articles [1, 10, 25] and using the q-analysis in order to define new subclasses of class Σ_υ, we apply Faber polynomial expansions technique in order to determine the estimates for the general coefficient bounds $|a_{\upsilon k+1}|$. We also derive initial coefficients $|a_{\upsilon+1}|$ and $|a_{2\upsilon+1}|$ and obtain Feketo-Sezego coefficient bounds for the functions belonging to the new subclasses of Σ_υ.

Definition 1.7. *A function $\mathfrak{h}_\upsilon \in \Sigma_\upsilon$ is in the class $\mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi)$ if and only if*

$$1 + \frac{1}{b}\left\{(D_q\mathfrak{h}_\upsilon(z) + \gamma z D_q^2 \mathfrak{h}_\upsilon(z)) - 1\right\} \prec \varphi(z)$$

and

$$1 + \frac{1}{b}\left\{(D_q g_\upsilon(w) + \gamma w D_q^2 g_\upsilon(w)) - 1\right\} \prec \varphi(w),$$

where, $\varphi \in \mathcal{P}$, $\gamma \geq 0$, $b \in \mathbb{C} \setminus \{0\}$, $z, w \in \mathcal{U}$, and $g_\upsilon(w)$ is defined by (1.4).

Remark 1.8. *For $q \to 1-$, $\upsilon = 1$, and $\gamma = 0$, then $\mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi) = \mathcal{R}_b(\varphi)$ introduced in [22].*

Definition 1.9. *A function $\mathfrak{h}_\upsilon \in \Sigma_\upsilon$, is in the class $\mathcal{R}_b^{\upsilon}(b, \alpha, \gamma)$ if and only if*

$$\left|\left(1 + \frac{1}{b}\left\{(D_q\mathfrak{h}_\upsilon(z) + \gamma z D_q^2 \mathfrak{h}_\upsilon(z)) - 1\right\}\right) - \frac{1-\alpha q}{1-q}\right| < \frac{1-\alpha}{1-q}$$

and

$$\left|\left(1 + \frac{1}{b}\left\{(D_q g_\upsilon(w) + \gamma w D_q^2 g_\upsilon(w)) - 1\right\}\right) - \frac{1-\alpha q}{1-q}\right| < \frac{1-\alpha}{1-q}.$$

Or equivalently by using subordination, we can write the above conditions as:

$$1 + \frac{1}{b}\left\{(D_q\mathfrak{h}_\upsilon(z) + \gamma z D_q^2 \mathfrak{h}_\upsilon(z)) - 1\right\} \prec \frac{1 + [1-\alpha(1+q)]z}{1-qz}$$

and

$$1 + \frac{1}{b}\left\{(D_q g_\upsilon(w) + \gamma w D_q^2 g_\upsilon(w)) - 1\right\} \prec \frac{1 + [1-\alpha(1+q)]w}{1-qw},$$

where ,$0 \leq \alpha < 1$, $\gamma \geq 0$, $b \in \mathbb{C} \setminus \{0\}$, $z, w \in \mathcal{U}$, $g_\upsilon(w)$ is defined by (1.4).

Remark 1.10. *For $q \to 1-$, $\upsilon = 1$, $\alpha = 0$ and $\gamma = 0$, then $\mathcal{R}_b^{\upsilon}(b,\alpha,\gamma) = \mathcal{R}_b(\varphi)$ introduced in [22].*

Definition 1.11. *A function $\mathfrak{h}_\upsilon \in \Sigma_\upsilon$, is in the class $\mathcal{R}_{b,q}^{\upsilon,\gamma,m}(\varphi)$ if and only if*

$$1 + \frac{1}{b}\left\{\left(\frac{\nabla_q^m \mathfrak{h}_\upsilon(z)}{z} + \gamma z D_q\left(\frac{\nabla_q^m \mathfrak{h}_\upsilon(z)}{z}\right)\right) - 1\right\} \prec \varphi(z)$$

and

$$1 + \frac{1}{b}\left\{\left(\frac{\nabla_q^m g_\upsilon(w)}{w} + \gamma w D_q\left(\frac{\nabla_q^m g_\upsilon(w)}{w}\right)\right) - 1\right\} \prec \varphi(w),$$

where, $\varphi \in \mathcal{P}$, $\gamma \geq 0, m \in \mathbb{N}$, $b \in \mathbb{C} \setminus \{0\}$, z, $w \in \mathcal{U}$, $g_\upsilon(w)$ is defined by (1.4).

2. The faber polynomial expansion method and application

Using the Faber polynomial technique for the analytic function $\mathfrak{h}$, then the coefficient of its inverse map g can be written as follows (see [2, 4]):

$$g_\upsilon w) = w + \sum_{k=2}^{\infty} \frac{1}{k} \mathfrak{R}_{k-1}^{k}(a_2, a_3, ...)w^k,$$

where

$$\begin{aligned}\mathfrak{R}_{k-1}^{-k} =& \frac{(-k)!}{(-2k+1)!(k-1)!}a_2^{k-1} + \frac{(-k)!}{[2(-k+1)]!(k-3)!}a_2^{k-3}a_3 \\ &+ \frac{(-k)!}{(-2k+3)!(k-4)!}a_2^{k-4}a_4 \\ &+ \frac{(-k)!}{[2(-k+2)]!(k-5)!}a_2^{k-5}[a_5 + (-k+2)a_3^2] \\ &+ \frac{(-k)!}{(-2k+5)!(k-6)!}a_2^{k-6}[a_6 + (-2k+5)a_3a_4] \\ &+ \sum_{i\geq 7} a_2^{k-i}Q_i,\end{aligned}$$

and Q_i is a homogeneous polynomial in the variables $a_2, a_3, ...a_k$, for $7 \leq i \leq k$. Particularly, the first three term of $\mathfrak{R}_{k-1}^{-k}$ are

$$\frac{1}{2}\mathfrak{R}_1^{-2} = -a_2, \frac{1}{3}\mathfrak{R}_2^{-3} = 2a_2^2 - a_3,$$
$$\frac{1}{4}\mathfrak{R}_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for $r \in N$ *and* $k \geq 2$, an expansion of $\mathfrak{R}_k^r$ of the form:

$$\mathfrak{R}_k^r = ra_k + \frac{r(r-1)}{2}E_k^2 + \frac{r!}{(r-3)!3!}E_k^3 + ... + \frac{r!}{(r-k)!k!}E_k^k,$$

where,

$$E_k^r = E_k^r(a_2, a_3, ...)$$

and by [2], we have

$$E_k^{\upsilon}(a_2, a_3, ...a_k) = \sum_{k=1}^{\infty} \frac{\upsilon!(a_2)^{\mu_1}...(a_k)^{\mu_k}}{\mu_{1!}, ..., \mu_{k!}}, \quad for \ \ a_1 = 1 \ \ and \ \ \upsilon \leq k.$$

The sum is taken over all non negative integer $\mu_1, ..., \mu_k$ which is satisfying

$$\mu_1 + \mu_2 + ... + \mu_k = \upsilon,$$
$$\mu_1 + 2\mu_2 + ... + (k)\mu_k = k.$$

Clearly,

$$E_k^k(a_1, ..., a_k) = E_1^k$$

and

$$E_k^k = a_1^k \ \ and \ \ E_k^1 = a_k$$

are first and last polynomials.

Now,using the Faber polynomial expansion for $\mathfrak{h}_{\upsilon}$ of the form (1.3) we have

$$\mathfrak{h}_{\upsilon}(z) = z + \sum_{k=1}^{\infty} a_{\upsilon k+1} z^{\upsilon k+1}.$$

The coefficient of inverse map g_{υ} can be expressed of the form:

$$g_{\upsilon}(z) = w + \sum_{k=1}^{\infty} \frac{1}{(\upsilon k+1)} \mathfrak{K}_k^{-(\upsilon k+1)}(a_{\upsilon+1}, a_{2\upsilon+1}, ...a_{\upsilon k+1}) w^{\upsilon k+1}.$$

Theorem 2.1. *For $b \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{h}_{\upsilon} \in \mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi)$ by given by* (1.3)*. If $a_{\upsilon i+1} = 0, \ \ 1 \leq i \leq k-1$, then*

$$|a_{\upsilon k+1}| \leq \frac{2|b|}{(1 + \gamma[\upsilon k, q])[\upsilon k+1, q]}, \quad for \ \ k \geq 2.$$

Proof. For $\mathfrak{h}_{\upsilon} \in \mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi)$ we have

$$1 + \frac{1}{b}\left\{(D_q \mathfrak{h}_{\upsilon}(z) + \gamma z D_q^2 \mathfrak{h}_{\upsilon}(z)) - 1\right\}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(1 + \gamma[\upsilon k, q])[\upsilon k+1, q]}{b} a_{\upsilon k+1} z^{\upsilon k} \tag{2.1}$$

and

$$1 + \frac{1}{b}\left\{(D_q g_{\upsilon}(w) + \gamma w D_q^2 g_{\upsilon}(w)) - 1\right\}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(1+\gamma[\upsilon k, q])[\upsilon k+1, q]}{b} A_{\upsilon k+1} w^{\upsilon k}, \tag{2.2}$$

where,

$$A_{\upsilon k+1} = \frac{1}{(\upsilon k+1)} \mathfrak{R}_k^{-(\upsilon k+1)}(a_{\upsilon+1}, a_{2\upsilon+1}, ...a_{\upsilon k+1}), \quad for \ \ k \geq 1.$$

Since $\mathfrak{h}_\upsilon \in \mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi)$ and $g_\upsilon \in \mathcal{R}_{b,q}^{\upsilon,\gamma}(\varphi)$ by definition, we have

$$p(z) = \sum_{k=1}^{\infty} c_k z^{\upsilon k} \tag{2.3}$$

and

$$r(w) = \sum_{k=1}^{\infty} d_k w^{\upsilon k} \tag{2.4}$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \varphi_l \mathfrak{R}_k^l(c_1, c_2, ..., c_k) z^{\upsilon k}, \tag{2.5}$$

$$\varphi(r(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \varphi_l \mathfrak{R}_k^l(d_1, d_2, ..., d_k) w^{\upsilon k}. \tag{2.6}$$

Equating the coefficient of (2.1) and (2.5) we obtain

$$\Big(\frac{(1+\gamma[\upsilon k, q])[\upsilon k+1, q]}{b}\Big) a_{\upsilon k+1} = \sum_{l=1}^{k-1} \varphi_l \mathfrak{R}_k^l(c_1, c_2, ..., c_k). \tag{2.7}$$

Similarly, corresponding coefficient of (2.2) and (2.6), we have

$$\Big(\frac{(1+\gamma[\upsilon k, q])[\upsilon k+1, q]}{b}\Big) A_{\upsilon k+1} = \sum_{l=1}^{k-1} \varphi_l \mathfrak{R}_k^l(d_1, d_2 ..., d_k). \tag{2.8}$$

Since, $1 \leq i \leq k-1, \ \ and \ \ a_{\upsilon i+1} = 0$; we have

$$A_{\upsilon k+1} = -a_{\upsilon k+1}$$

and

$$\frac{(1+\gamma[\upsilon k, q])[\upsilon k+1, q]}{b} a_{\upsilon k+1} = \varphi_1 c_k, \tag{2.9}$$

$$\frac{(1+\gamma[\upsilon k, q])[\upsilon k+1, q]}{b} A_{\upsilon k+1} = \varphi_1 d_k. \tag{2.10}$$

Taking the modulus on both sides of (2.9) and (2.10), we have

$$\Big|\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}a_{\upsilon k+1}\Big| = |\varphi_1 c_k|,$$

$$\Big|\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}A_{\upsilon k+1}\Big| = |\varphi_1 d_k|.$$

Now using the fact $|\varphi_1| \leq 2, |c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$|a_{\upsilon k+1}| \leq \frac{|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}|\varphi_1 c_k|$$

$$= \frac{|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]})|\varphi_1 d_k|,$$

$$|a_{\upsilon k+1}| \leq \frac{2|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}.$$

Hence, Theorem 2.1 is completed. □

For $\upsilon = 0, \gamma = 0, q \to 1-, k = n-1$, in Theorem 2.1, we obtain known corollary proved in [22].

Corollary 2.2. *For $b \in \mathbb{C} \setminus \{0\}$, Let $\mathfrak{h}_\upsilon \in \mathcal{R}_b(\varphi)$, If $a_{\upsilon i+1} = 0, 1 \leq i \leq n$. Then*

$$|a_n| \leq \frac{2|b|}{n}, \quad for \;\; n \geq 3.$$

Theorem 2.3. *For $b \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{h}_\upsilon \in \mathcal{R}^{\upsilon,\gamma}_{b,q}(\varphi)$ be given by (1.3). Then*

$$|a_{\upsilon+1}| \leq \begin{cases} \frac{2|b|}{(1+\gamma[\upsilon k,q])[\upsilon+1,q]}, & if \;\; |b| < \psi_1(\upsilon,q), \\ \sqrt{|b|\,\psi_1(\upsilon,q)}, & if \;\; |b| \geq \psi_1(\nu,q), \end{cases}$$

$$|a_{2\upsilon+1}| \leq \begin{cases} |b|\psi_2(\upsilon,q) + \frac{2|b|^2}{(1+\gamma[\upsilon,q])[\upsilon+1,q]}, & if \;\; |b| < \psi_2(\upsilon,q), \\ 2|b|\psi_2(\upsilon,q), & if \;\; |b| \geq \psi_2(\upsilon,q), \end{cases}$$

$$|a_{2\upsilon+1} - (1+\gamma[\upsilon,q])[\upsilon+1,q]a^2_{\upsilon+1}| \leq 2|b|\psi_2(\upsilon,q),$$

$$\Big|a_{2\upsilon+1} - \frac{1}{\psi_2(\upsilon,q)}a^2_{\upsilon+1}\Big| \leq |b|\psi_2(\upsilon,q),$$

where,

$$\psi_1(\upsilon,q) = \frac{8}{((1+\gamma[2\upsilon,q])[2\upsilon+1,q])((1+\gamma[\upsilon,q])[\upsilon+1,q])},$$

$$\psi_2(\upsilon,q) = \frac{2}{((1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Proof. Taking $k = 1$ and $k = 2$ in (2.7) and (2.8), then, we have

$$\frac{(1+\gamma[\upsilon,q])[\upsilon+1,q]}{b}a_{\upsilon+1} = \varphi_1 c_1, \tag{2.11}$$

$$\frac{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{b}a_{2\upsilon+1} = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{2.12}$$

$$-\frac{(1+\gamma[\upsilon,q])[\upsilon+1,q]}{b}a_{\upsilon+1} = \varphi_1 d_1, \tag{2.13}$$

$$\{(1+\gamma[\upsilon,q])[\upsilon+1,q]a_{\upsilon+1}^2 - a_{2\upsilon+1}\} = \frac{b(\varphi_1 d_2 + \varphi_2 d_1^2)}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}. \tag{2.14}$$

From (2.11) and (2.13) and using the fact $|\varphi_1| \leq 2, |c_k| \leq 1$ and $|d_k| \leq 1$, we have

$$\begin{aligned} |a_{\upsilon+1}| &\leq \frac{|b|}{(1+\gamma[\upsilon,q])[\upsilon+1,q]}|\varphi_1 c_1| = \frac{|b|}{(1+\gamma[\upsilon,q])[\upsilon+1,q]}|\varphi_1 d_1| \\ &\leq \frac{2|b|}{1+\gamma[\upsilon,q])[\upsilon+1,q]}. \end{aligned} \tag{2.15}$$

Adding (2.12) and (2.14) we have

$$a_{\upsilon+1}^2 = \frac{b\{\varphi_1(c_2+d_2) + \varphi_2(c_1^2+d_1^2)\}}{((1+\gamma[2\upsilon,q])[2\upsilon+1,q])((1+\gamma[\upsilon,q])[\upsilon+1,q])}. \tag{2.16}$$

Taking absolute value of (2.16), we have

$$|a_{\upsilon+1}| \leq \sqrt{\frac{8|b|}{((1+\gamma[2\upsilon,q])[2\upsilon+1,q])((1+\gamma[\upsilon,q])[\upsilon+1,q])}}.$$

Now the bounds given for $|a_{\upsilon+1}|$ can be justified since

$$|b| < \sqrt{\frac{8}{((1+\gamma[2\upsilon,q])[2\upsilon+1,q])((1+\gamma[\upsilon,q])[\upsilon+1,q])}}$$

for

$$|b| < \frac{8}{((1+\gamma[2\upsilon,q])[2\upsilon+q])((1+\gamma[\upsilon,q])[\upsilon+1,q])}.$$

From (2.12), we get

$$|a_{2\upsilon+1}| = \frac{|b||\varphi_1 c_2 + \varphi_2 c_1^2|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]} \leq \frac{4\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}. \tag{2.17}$$

Subtract (2.14) from (2.12), we have

$$\begin{aligned} &\frac{2(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{b}\left\{a_{2\upsilon+1} - \frac{(1+\gamma[\upsilon,q])[\upsilon+1,q]}{2}a_{\upsilon+1}^2\right\} \\ &= \varphi_1(c_2-d_2) + \varphi_2(c_1^2-d_1^2) = \varphi_1(c_2-d_2), \end{aligned} \tag{2.18}$$

or

$$a_{2\upsilon+1} = \frac{(1+\gamma[\upsilon,q])[\upsilon+1,q]}{2}a_{\upsilon+1}^2 + \frac{\varphi_1 b(c_2-d_2)}{2(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}. \tag{2.19}$$

Taking the absolute, we have

$$| a_{2\upsilon+1}| \leq \frac{|\varphi_1 \|\, b \,\| c_2-d_2|}{2(1+\gamma[2\upsilon,q])[2\upsilon+1,q]} + \frac{(1+\gamma[\upsilon,q])[\upsilon+1,q]}{2} \mid a_{\upsilon+1}^2|. \tag{2.20}$$

Using the assertion (2.15) on (2.20), we have

$$| a_{2\upsilon+1} |\leq \frac{2\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]} + \frac{2\mid b\mid^2}{(1+\gamma[\upsilon,q])[\upsilon+1,q]}. \tag{2.21}$$

Follows from (2.17) and (2.21) upon nothing that

$$\frac{2\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]} + \frac{2\mid b\mid^2}{(1+\gamma[\upsilon,q])[\upsilon+1,q]}$$
$$\leq \frac{2\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]} \quad if \quad \mid b\mid < \frac{2}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Now, rewrite (2.14) as follows:

$$(1+\gamma[\upsilon,q])[\upsilon+1,q]a_{\upsilon+1}^2 - a_{2\upsilon+1} = \frac{b(\varphi_1 d_2 + \varphi_2 d_1^2)}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Using the fact $|\varphi_1| \leq 2$,$|c_k| \leq 1$ and $|d_k| \leq 1$, we have

$$| a_{2\upsilon+1} - (1+\gamma[\upsilon,q])[\upsilon+1,q]a_{\upsilon+1}^2 |\leq \frac{4\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

From (2.18), we have

$$\frac{2(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{b}\Big\{a_{2\upsilon+1} - \frac{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{2}a_{\upsilon+1}^2\Big\} = \varphi_1(c_2-d_2).$$

Again using the fact $|\varphi_1| \leq 2$,$|c_k| \leq 1$ and $|d_k| \leq 1$, we have

$$\left|a_{2\upsilon+1} - \frac{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{2}a_{\upsilon+1}^2\right| \leq \frac{2\mid b\mid}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

□

Take $q \to 1-, \gamma = 0, \upsilon = 1$, and $k = n-1$ in the Theorem 2.3, we get known corollary.

Corollary 2.4. *[22]. For $b \in \mathbb{C} \setminus \{0\}$, let $\mathfrak{h} \in \mathcal{R}_b(\varphi)$ be given by* (1.1)*, then*

$$|a_2| \leq \begin{cases} |b|, & if \ \ |b| < \frac{4}{3}, \\ \sqrt{\frac{4|b|}{3}}, & if \ \ |b| \geq \frac{4}{3}, \end{cases}$$

$$|a_3| \leq \begin{cases} \frac{2|b|}{3} + |b|^2, & if \;\; |b| < \frac{2}{3}, \\ \frac{4|b|}{3}, & if \;\; |b| \geq \frac{2}{3}, \end{cases}$$

$$|a_3 - 2a_2^2| \leq \frac{4|b|}{3},$$

$$|a_3 - a_2^2| \leq \frac{2|b|}{3}.$$

Theorem 2.5. *For $b \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{h}_\upsilon \in \mathcal{R}_q^\upsilon(b, \alpha, \gamma)$ by given by (1.3). If $a_{\upsilon i+1} = 0, 1 \leq i \leq k-1$. Then*

$$|a_{\upsilon k+1}| \leq \frac{(\mathfrak{B}_0 - \mathfrak{B}_1)|b|}{(1 + \gamma[\upsilon k, q])[\upsilon k + 1, q]}, \quad for \;\; k \geq 2.$$

where, $\mathfrak{B}_0 = 1 - \alpha(1+q)$ and $\mathfrak{B}_1 = -q$.

Proof. Let $\mathfrak{h}_\upsilon \in \mathcal{R}_q^\upsilon(b, \alpha, \gamma)$. Then

$$\begin{aligned} & 1 + \frac{1}{b}\{(D_q \mathfrak{h}_\upsilon(z) + \gamma z D_q^2 \mathfrak{h}_\upsilon(z)) - 1\} \\ =& 1 + \sum_{k=1}^{\infty} \frac{(1 + \gamma[\upsilon k, q])[\upsilon k + 1, q]}{b} a_{\upsilon k+1} z^{\upsilon k} \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} & 1 + \frac{1}{b}\{(D_q g_\upsilon(w) + \gamma w D_q^2 g_\upsilon(w)) - 1\} \\ =& 1 + \sum_{k=1}^{\infty} \frac{(1 + \gamma[\upsilon k, q])[\upsilon k + 1, q]}{b} A_{\upsilon k+1} w^{\upsilon k}. \end{aligned} \tag{2.23}$$

where,

$$A_{\upsilon k+1} = \frac{1}{(\upsilon k + 1)} \mathfrak{R}^{-(\upsilon k+1)}(a_{\upsilon+1}, a_{2\upsilon+1}, ..., a_{\upsilon k+1}), \;\; k \geq 1.$$

Since $h_\upsilon \in \mathcal{R}_q^\upsilon(b, \alpha, \gamma)$ and $g_\upsilon \in \mathcal{R}_q^\upsilon(b, \alpha, \gamma)$ by definition, there exist two positive real functions $p(z)$ and $r(w)$ given in (2.3) and (2.4), then we have

$$= \frac{1 + \mathfrak{B}_0(p(z))}{1 + \mathfrak{B}_1(p(z))} = 1 - \sum_{k=1}^{\infty} \sum_{l=1}^{k} (\mathfrak{B}_0 - \mathfrak{B}_1) \mathfrak{R}_k^{-1}(c_1, c_2, ..., c_k, \mathfrak{B}_1) z^{\upsilon k} \tag{2.24}$$

$$= \frac{1 + \mathfrak{B}_0(r(w))}{1 + \mathfrak{B}_1(r(w))} = 1 - \sum_{k=1}^{\infty} \sum_{l=1}^{k} (\mathfrak{B}_0 - \mathfrak{B}_1) \mathfrak{R}_k^{-1}(d_1, d_2, ..., d_k, \mathfrak{B}_1) w^{\upsilon k}. \tag{2.25}$$

Equating the corresponding coefficients of (2.22) and (2.24), we have

$$\frac{(1 + \gamma[\upsilon k, q])[\upsilon k + 1, q]}{b} a_{\upsilon k+1} = (\mathfrak{B}_0 - \mathfrak{B}_1) \mathfrak{R}_k^{-1}(c_1, c_2, ..., c_k, \mathfrak{B}_1) z^{\upsilon k}. \tag{2.26}$$

Similarly, corresponding coefficient of (2.23)and (2.25), we have

$$\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}A_{\upsilon k+1}=(\mathfrak{B}_0-\mathfrak{B}_1)\mathfrak{R}_k^{-1}(d_1,d_2,...,d_k,\mathfrak{B}_1)w^{\upsilon k}. \tag{2.27}$$

For $a_{\upsilon i+1}=0;\ 1\leq i\leq k-1$, we get

$$A_{\upsilon k+1}=-a_{\upsilon k+1}$$

and we have

$$\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}a_{\upsilon k+1}=(\mathfrak{B}_0-\mathfrak{B}_1)c_k, \tag{2.28}$$

and

$$-\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}A_{\upsilon k+1}=(\mathfrak{B}_0-\mathfrak{B}_1)d_k. \tag{2.29}$$

Taking modulus on (2.28) and (2.29), we have

$$\left|\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}a_{\upsilon k+1}\right|=|(\mathfrak{B}_0-\mathfrak{B}_1)c_k|,$$
$$\left|-\frac{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}{b}A_{\upsilon k+1}\right|=|(\mathfrak{B}_0-\mathfrak{B}_1)d_k|.$$

Since

$$|c_k|\leqq 1 \quad \text{and} \quad |d_k|\leqq 1(see[14]),$$

we have

$$\begin{aligned}|a_{\upsilon k+1}|&\leq\frac{|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}|(\mathfrak{B}_0-\mathfrak{B}_1)c_k|\\ &=\frac{|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]}|(\mathfrak{B}_0-\mathfrak{B}_1)d_k,|\\ |a_{\upsilon k+1}|&\leq\frac{(\mathfrak{B}_0-\mathfrak{B}_1)|b|}{(1+\gamma[\upsilon k,q])[\upsilon k+1,q]},\end{aligned}$$

which complete the proof of Theorem.

For $b=1,k=1,\upsilon=n-1,q\to 1-, and\gamma\geqq 0$ in the above Theorem 2.5, we obtain the following result given in [40]. □

Corollary 2.6. *Let* $\mathfrak{h}_\upsilon\in\mathcal{R}(n,\alpha,\gamma)$ *be given by* (1.3)*. If* $a_{n-1}=0$*, and* $1\leq i\leq k-1$*, then*

$$|a_n|\leqq\frac{2(1-\alpha)}{n(1+\gamma(n-1))},\quad n\in\mathbb{N}\setminus\{1,2\}.$$

Theorem 2.7. *For* $b \in \mathbb{C} \setminus \{0\}$, *let* $\mathfrak{h}_\upsilon \in \mathcal{R}_q^\upsilon(b, \alpha, \gamma)$ *be given by* (1.3)*, then*

$$|a_{\upsilon+1}| \leq \begin{cases} \dfrac{(\mathfrak{B}_0 - \mathfrak{B}_1)|b|}{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}, & if \ \ |b| < \psi_3(\upsilon, q), \\ \\ \sqrt{2|b|\psi_3(\upsilon, q)} & if \ \ |b| \geq \psi_3(\upsilon, q), \end{cases}$$

$$|a_{2\upsilon+1}| \leq \begin{cases} |b|\psi_4(\upsilon, q) + \psi_4(\upsilon, q)|(\mathfrak{B}_0 - \mathfrak{B}_1)|\,|b|^2, & if \ \ |b| < \psi_4(\upsilon, q), \\ \\ |b|(|\mathfrak{B}_1| + 1)\psi_4(\upsilon, q) & if \ \ |b| \geq \psi_4(\upsilon, q), \end{cases}$$

$$|a_{2\upsilon+1} - (1 + \gamma[\upsilon, q])[\upsilon + 1, q]a_{\upsilon+1}^2| \leq |b|(|\mathfrak{B}_1| + 1|)\psi_4(\upsilon, q)$$

and

$$\left| a_{2\upsilon+1} - \frac{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}{2} a_{\upsilon+1}^2 \right| \leq |b|\psi_4(\upsilon, q),$$

where

$$\psi_3(\upsilon, q) = \frac{|\mathfrak{B}_0 - \mathfrak{B}_1|\{|\mathfrak{B}_1| + 1\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}$$

$$\psi_4(\upsilon, q) = \frac{|\mathfrak{B}_0 - \mathfrak{B}_1|}{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}.$$

Proof. Take $k = 1$ and $k = 2$ in (2.26) and (2.27). Then we have

$$\frac{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}{b} a_{\upsilon+1} = (\mathfrak{B}_0 - \mathfrak{B}_1)c_1, \tag{2.30}$$

$$\frac{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}{b} a_{2\upsilon+1} = (\mathfrak{B}_0 - \mathfrak{B}_1)(-\mathfrak{B}_1 c_1^2 + c_2), \tag{2.31}$$

$$-\frac{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}{b} a_{\upsilon+1} = -(\mathfrak{B}_0 - \mathfrak{B}_1)d_1, \tag{2.32}$$

$$(1 + \gamma[\upsilon, q])[\upsilon + 1, q]a_{\upsilon+1}^2 - a_{2\upsilon+1} = \frac{b(\mathfrak{B}_0 - \mathfrak{B}_1)(-\mathfrak{B}_1 d_1^2 + d_2)}{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}. \tag{2.33}$$

From (2.30) and (2.32) and using the fact $|\varphi_1| \leq 2, |c_k| \leq 1$ and $|d_k| \leq 1$, we have

$$\begin{aligned} |a_{\upsilon+1}| &\leq \frac{|b|}{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]} |(\mathfrak{B}_0 - \mathfrak{B}_1)c_1| \\ &= \frac{|b|}{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]} |(\mathfrak{B}_0 - \mathfrak{B}_1)d_1| \\ &\leq \frac{(\mathfrak{B}_0 - \mathfrak{B}_1)|b|}{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}. \end{aligned} \tag{2.34}$$

Adding (2.31) and (2.33) we have

$$a_{\upsilon+1}^{2} = \frac{b(\mathfrak{B}_0 - \mathfrak{B}_1)\{(c_2 + d_2) + \mathfrak{B}_1(c_1^2 + d_1^2)\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}$$

and

$$|a_{\upsilon+1}|^2 \leq \frac{2|b|\,|\mathfrak{B}_0 - \mathfrak{B}_1|\{|\mathfrak{B}_1| + 1\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}. \tag{2.35}$$

Taking the square-root of (2.35), we have

$$|a_{\upsilon+1}| \leq \sqrt{\frac{2|b||\mathfrak{B}_0 - \mathfrak{B}_1|\{|\mathfrak{B}_1| + 1\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}}.$$

Now the bounds given for $|a_{\upsilon+1}|$ can be justified since

$$|b| < \sqrt{\frac{2|b||\mathfrak{B}_0 - \mathfrak{B}_1|\{|\mathfrak{B}_1| + 1\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}}$$

$$\text{for} \quad |b| < \frac{2|b||\mathfrak{B}_0 - \mathfrak{B}_1|\{|\mathfrak{B}_1| + 1\}}{((1 + \gamma[2\upsilon, q])[2\upsilon + 1, q])((1 + \gamma[\upsilon, q])[\upsilon + 1, q])}.$$

From (2.31), we have

$$\begin{aligned} |a_{2\upsilon+1}| &= \frac{|b||(\mathfrak{B}_0 - \mathfrak{B}_1)(\mathfrak{B}_1 c_1^2 + c_2)|}{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]} \\ &\leq \frac{|b||\mathfrak{B}_0 - \mathfrak{B}_1|(|\mathfrak{B}_1| + 1)}{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}. \end{aligned} \tag{2.36}$$

Next we subtract (2.33) from (2.31), we get

$$\begin{aligned} &\frac{2(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}{b}\left\{a_{2\upsilon+1} - \frac{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}{2}a_{\upsilon+1}^2\right\} \\ &= (\mathfrak{B}_0 - \mathfrak{B}_1)\{\mathfrak{B}_1(d_1^2 - c_1^2) - (c_2 - d_2)\} = (\mathfrak{B}_0 - \mathfrak{B}_1)(c_2 - d_2), \end{aligned} \tag{2.37}$$

or

$$a_{2\upsilon+1} = \frac{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}{2}a_{\upsilon+1}^2 + \frac{(\mathfrak{B}_0 - \mathfrak{B}_1)b(c_2 - d_2)}{2(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]}. \tag{2.38}$$

Taking the absolute values yield

$$|a_{2\upsilon+1}| \leq \frac{|(\mathfrak{B}_0 - \mathfrak{B}_1)|\,|b|\,|c_2 - d_2|}{2(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]} + \frac{(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}{2}|a_{\upsilon+1}^2|. \tag{2.39}$$

Using the assertion (2.34) on (2.39), we have

$$|a_{2\upsilon+1}| \leq \frac{|(\mathfrak{B}_0 - \mathfrak{B}_1)|\,|b|}{(1 + \gamma[2\upsilon, q])[2\upsilon + 1, q]} + \frac{|(\mathfrak{B}_0 - \mathfrak{B}_1)|^2\,|b|^2}{2(1 + \gamma[\upsilon, q])[\upsilon + 1, q]}. \tag{2.40}$$

It follows from (2.36) and (2.40) upon noting that

$$\frac{|(\mathfrak{B}_0-\mathfrak{B}_1)|\,|b|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}+\frac{|(\mathfrak{B}_0-\mathfrak{B}_1)|^2\,|b|^2}{2(1+\gamma[\upsilon,q])[\upsilon+1,q]}.$$
$$\leq \frac{|(\mathfrak{B}_0-\mathfrak{B}_1)|\,|b|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}\ \text{ if }\ |b|<\frac{|(\mathfrak{B}_0-\mathfrak{B}_1)|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Now, we rewrite (2.33) as follows:

$$\{(1+\gamma[\upsilon,q])[\upsilon+1,q]a_{\upsilon+1}^2-a_{2\upsilon+1}\}=\frac{b(\mathfrak{B}_0-\mathfrak{B}_1)(-\mathfrak{B}_1d_1^2+d_2)}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Taking the modulus and using $|\varphi_1|\leq 2$, $|c_k|\leq 1$ and $|d_k|\leq 1$, we have

$$|a_{2\upsilon+1}-(1+\gamma[\upsilon,q])[\upsilon+1,q]a_{\upsilon+1}^2|\leq\frac{(\mathfrak{B}_0-\mathfrak{B}_1)(|\mathfrak{B}_1|+1)\,|b|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Finally, from (2.37), we have

$$\left\{a_{2\upsilon+1}-\frac{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{2}a_{\upsilon+1}^2\right\}=\frac{b(\mathfrak{B}_0-\mathfrak{B}_1)(c_2-d_2)}{2(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

Taking the modulus and using $|c_k|\leq 1$ and $|d_k|\leq 1$, we have

$$\left|a_{2\upsilon+1}-\frac{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}{2}a_{\upsilon+1}^2\right|\leq\frac{(\mathfrak{B}_0-\mathfrak{B}_1)|b|}{(1+\gamma[2\upsilon,q])[2\upsilon+1,q]}.$$

For $\upsilon=1,\gamma=0,q\to 1-,k=n-1$ in Theorem 2.7, then we obtain result proved in [22]. □

Corollary 2.8. *[22]. For $b\in\mathbb{C}\setminus\{0\}$, let $\mathfrak{h}_\upsilon\in\mathcal{R}_b(\varphi)$ be given by* (1.1)*, then*

$$|a_2|\leq\begin{cases}|b|, & if\ \ |b|<\dfrac{4}{3},\\[2ex] \sqrt{\dfrac{4|b|}{3}}, & if\ \ |b|\geq\dfrac{4}{3},\end{cases}$$

$$|a_3|\leq\begin{cases}\frac{2|b|}{3}+|b|^2, & if\ \ |b|<\dfrac{2}{3},\\[2ex] \frac{4|b|}{3}, & if\ \ |b|\geq\dfrac{2}{3},\end{cases}$$

$$|a_3-2a_2^2|\leq\frac{4|b|}{3},$$

$$|a_3-a_2^2|\leq\frac{2|b|}{3}.$$

2.1. Applications of our main results

Here, in this section, we consider the newly defined Salagean $q-$differential operator for subclass of $R_{b,q}^{\upsilon,\gamma,m}(\varphi)$ of class of $\sum_{\upsilon}$ and investigate some new application in the form of results

Theorem 2.9. *For $b \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{h}_{\upsilon} \in R_{b,q}^{\upsilon,\gamma,m}(\varphi)$ by given by* (1.3). *If $a_{\upsilon i+1} = 0$, and $1 \leq i \leq k-1$, then*

$$|a_{\upsilon k+1}| \leq \frac{2|b|}{(1+\gamma[\upsilon k, q])(\upsilon k+1, q)^{m}}, \quad for \;\; k \geq 2.$$

Proof. We can prove Theorem 2.9 by using the similar method of Theorem 2.1. □

Theorem 2.10. *For $b \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{h}_{\upsilon} \in R_{b,q}^{\upsilon,\gamma,m}(\varphi)$ by given by* (1.3). *Then*

$$|a_{\upsilon+1}| \leq \begin{cases} \frac{2|b|}{(1+\gamma[\upsilon,q])(\upsilon+1,q)^{m}}, & if \;\; |b| < \psi_3(\upsilon, q), \\ \\ \sqrt{|b| \;\; \psi_1(\upsilon, q)}, & if \;\; |b| \geq \psi_3(\upsilon, q), \end{cases}$$

$$|a_{2\upsilon+1}| \leq \begin{cases} |b|\psi_2(\upsilon, q) + \frac{2|b|^2}{(1+\gamma[\upsilon,q])[\upsilon+1,q]^{m}}, & if \, |b| < \psi_4(\upsilon, q), \\ \\ 2|b|\psi_2(\upsilon, q) \;\; if \;\; |b| \geq \psi_4(\upsilon, q), \end{cases}$$

$$|a_{2\upsilon+1} - (1+\gamma[\upsilon, q])[\upsilon+1, q]^{m} a_{\upsilon+1}^{2}| \leq 2|b|\psi_4(\upsilon, q),$$

$$\left| a_{2\upsilon+1} - \frac{1}{\psi_2(\upsilon, q)} a_{\upsilon+1}^{2} \right| \leq |b|\psi_4(\upsilon, q),$$

where

$$\psi_3(\upsilon, q) = \frac{8}{((1+\gamma[2\upsilon, q])[2\upsilon+1, q]^{m})((1+\gamma[\upsilon, q])[\upsilon+1, q]^{m})},$$

$$\psi_4(\upsilon, q) = \frac{2}{(1+\gamma[2\upsilon, q])[2\upsilon+1, q]^{m}}.$$

Proof. We can prove Theorem 2.10 by using the similar method of Theorem 2.3. □

3. Conclusions

In this article, first of all, we used the q-difference operator for υ-fold symmetric functions in order to define some new subclasses of the υ-fold symmetric bi-univalent functions in the open symmetric unit disk $\mathcal{U}$. We also used the basic concepts of q-calculus and defined the Salagean q-differential operator for υ-fold symmetric functions. We considered this operator and investigated a new subclass of υ-fold symmetric bi-univalent functions. Faber Polynomial expansion method and q-analysis are used in order to determined general coefficient bounds $|a_{\upsilon+1}|$ for functions in each of these newly defined υ-fold symmetric bi-univalent functions classes. Feketo-Sezego problems and initial coefficient bounds $|a_{\upsilon+1}|$ and $|a_{2\upsilon+1}|$ for the function belonging to the subclasses of υ-fold symmetric bi-univalent functions in open symmetric unit disk $\mathcal{U}$ are also investigated.

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Conflict of interest

The author declares no conflict of interest.

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