Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(5): 10228–10248.
DOI: 10.3934/math.2023518
Received: 22 November 2022
Revised: 15 February 2023
Accepted: 16 February 2023
Published: 27 February 2023

*Research article*

# Quasi $M$-metric spaces

**Irshad Ayoob**[1,2], **Ng Zhen Chuan**[2] **and Nabil Mlaiki**[1,*]

[1] Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

[2] School of Mathematics, Universiti Sains Malaysia, 11800 Gelugor, Penang, Malaysia

* **Correspondence:** Email: nmlaiki@psu.edu.sa.

**Abstract:** In this paper, we introduce quasi $M$-metric spaces as a generalization of $M$-metric spaces. We establish some fixed point results along with the examples and application of our results to integral equations and system of linear equations.

**Keywords:** quasi $M$-metric spaces; partial metric spaces; $M$-metric spaces; fixed point; integral equations
**Mathematics Subject Classification:** Primary 47H10, Secondary 54H25

## 1. Introduction

Fixed point theory has become the focus of many researchers in the last couple decades and that is due to its applications in the existence and uniqueness of solutions of differential and integral equations, engineering, mathematical economics,dynamical systems, neural networks and many other fields. The classic result of the fixed point, widely studied by researchers, is that of Banach [1], Banach's fixed theorem has been generalized by expanding the underlying metric space or by changing the contraction condition. A few examples of existing concepts where the underlying metric has been extended include cone metric space [8, 29, 30], partial symmetric space [9], partial JS-metric space [10], $M$-metric space [3], $M_b$-metric space [4], extended$M_b$-metric space [5], rectangular $M$-metric space [11], and others. Various types of mappings for which the fixed points have been investigated in the extended metric spaces include Banach contraction mapping, Kannan contraction mapping, Ciric contraction mapping, Riech contraction mapping, Hardy-Roger contraction mapping, Caristi contraction mapping and several others [31–33].

## 2. Preliminaries

In [7], Wilson initiated the concept of *quasi*-metric space (also known as asymmetric metric space) as an extension of the metric space. This is defined as a metric space $(\mathcal{X}, \eta)$, but $\eta$ is not required to be symmetric. The intuitive example of a *quasi*-metric space is a circle, where the distance $\eta$ between two points $A$ and $B$ on the circle is defined as the length of the shortest arc measured in the clockwise direction. Clearly, $\eta(A, B) \neq \eta(B, A)$ unless $A$ and $B$ are diametrically opposite on the circle. Recent developments in applied mathematics have seen a wide range of applications for quasi-metric spaces, including shape-memory alloys [12], questions about the existence and uniqueness of Hamilton-Jacobi equations [13], automated taxonomy construction [14], models for material failure [15] and several others.

We recall the definition of *quasi*-metric space.

**Definition 2.1.** *[7] Let $\mathcal{X}$ be a nonempty set. A quasi-metric on $\mathcal{X}$ is a function $\eta : \mathcal{X}^2 \to [0, +\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$,*

*(1) $\eta(\mu, \omega) = 0$ if and only if $\mu = \omega$,*
*(2) $\eta(\mu, \omega) \leq \eta(\mu, w) + \eta(w, \omega)$.*

*A pair $(\mathcal{X}, \eta)$ is called a quasi-metric space.*

Every metric space is a *quasi*-metric space, while the converse is not true, in general. The topological concepts in *quasi*-metric space such as convergence, Cauchyness, completeness and continuity, differ from those in metric spaces. For these concepts in *quasi*-metric spaces, the reader may refer to [6].

As a further generalization of metric spaces, Matthews [2] introduced the concept of partial metric spaces and established the Banach type fixed point theorem in the same space. Several researchers such as O'Neill [16], Bukatin and Scott [17, 18], Escardo [19], Romaguera and Schellekens [20, 21] and Waszkiewicz [22, 23] have studied the connection between domain theory and partial metrics.

We state the definition of partial metric space.

**Definition 2.2.** *[2] Let $\mathcal{X}$ be a nonempty set. A partial metric on $\mathcal{X}$ is a function $\mathcal{J} : \mathcal{X}^2 \to [0, +\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$,*

*(1) $\mathcal{J}(\mu, \mu) = \mathcal{J}(\omega, \omega) = \mathcal{J}(\mu, \omega)$ if and only if $\mu = \omega$,*
*(2) $\mathcal{J}(\mu, \mu) \leq \mathcal{J}(\mu, \omega)$,*
*(3) $\mathcal{J}(\mu, \omega) = \mathcal{J}(\omega, \mu)$,*
*(4) $\mathcal{J}(\mu, w) \leq \mathcal{J}(\mu, \omega) + \mathcal{J}(\omega, w) - \mathcal{J}(\omega, \omega)$.*

*A pair $(\mathcal{X}, \mathcal{J})$ is called a partial metric space.*

In [3], Asadi et al. extended the definition of partial metric space to $M$-metric space. The authors in [3] also established that every partial metric space is an $M$-metric space, however, every $M$-metric space need not to be a partial metric space.

We need the following notations to state the definition of $M$-metric space.

**Notation 2.1.** *[3]*

*(1) $m_{\mu,\omega} := \min\{\mathcal{N}(\mu, \mu), \mathcal{N}(\omega, \omega)\}$.*

(2) $\mathcal{N}_{\mu,\omega} := \max\{\mathcal{N}(\mu,\mu), \mathcal{N}(\omega,\omega)\}$.

**Definition 2.3.** *[3] Let $\mathcal{X}$ be a nonempty set. An M-metric on $\mathcal{X}$ is a function $\mathcal{N} : \mathcal{X}^2 \to [0,+\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$,*

(1) $\mathcal{N}(\mu,\mu) = \mathcal{N}(\omega,\omega) = \mathcal{N}(\mu,\omega)$ *if and only if* $\mu = \omega$,
(2) $m_{\mu,\omega} \leq \mathcal{N}(\mu,\omega)$,
(3) $\mathcal{N}(\mu,\omega) = \mathcal{N}(\omega,\mu)$,
(4) $\left(\mathcal{N}(\mu,\omega) - m_{\mu,\omega}\right) \leq \left(\mathcal{N}(\mu,w) - m_{\mu,w}\right) + (\mathcal{N}(w,\omega) - m_{w,\omega})$.

*A pair $(\mathcal{X}, \mathcal{N})$ is called an M-metric space.*

**Example 2.1.** Let $\mathcal{X} = [0,\infty)$. Then, $\mathcal{N} : \mathcal{X}^2 \to [0,+\infty)$ defined by $\mathcal{N}(\mu,\omega) = \frac{\mu+\omega}{2}$ is an $M$-metric on $\mathcal{X}$.

**Example 2.2.** [3] Let $\mathcal{X} = \{a,b,c\}$. Define

$$\mathcal{N}(a,a) = 1,\ \mathcal{N}(b,b) = 9,\ \mathcal{N}(c,c) = 5,$$

$$\mathcal{N}(a,b) = \ \mathcal{N}(b,a) = 10,\ \mathcal{N}(a,c) = \mathcal{N}(c,a) = 7,\ \mathcal{N}(b,c) = \mathcal{N}(3,c) = 7,$$

Then, $\mathcal{N}$ is an $M$-metric on $\mathcal{X}$ but not a partial metric.

$M$-metric spaces have been extensively studied by several researchers [4,5,11,24–27]. In this study, we extend the $M$-metric spaces to quasi $M$-metric spaces, and prove the related fixed point results along with the examples and applications. We shall use the following notations:

**Notation 2.2.** *[3]*

(1) $z_{\mu,\omega} := \min\{\zeta(\mu,\mu), \zeta(\omega,\omega)\}$.
(2) $R_{\mu,\omega} := \max\{\zeta(\mu,\mu), \zeta(\omega,\omega)\}$.

**Definition 2.4.** *Let $\mathcal{X}$ be a nonempty set. A quasi M-metric on $\mathcal{X}$ is a function $\zeta : \mathcal{X}^2 \to [0,+\infty)$ such that for all $\mu, \omega, w \in \mathcal{X}$,*

(1) $\zeta(\mu,\mu) = \zeta(\omega,\omega) = \zeta(\mu,\omega) = \zeta(\omega,\mu)$ *if and only if* $\mu = \omega$,
(2) $z_{\mu,\omega} \leq \zeta(\mu,\omega)$,
(3) $\left(\zeta(\mu,\omega) - z_{\mu,\omega}\right) \leq \left(\zeta(\mu,w) - z_{\mu,w}\right) + (\zeta(w,\omega) - z_{w,\omega})$.

*A pair $(\mathcal{X}, \zeta)$ is called a quasi M-metric space.*

Every $M$-metric space is a quasi $M$-metric space, however, the converse is not true in general. We note that in quasi $M$-metric space that self-distance is not necessarily zero, and that symmetry is not necessarily preserved.

**Example 2.3.** Let $\mathcal{X} = \{a,b,c\}$ and $\zeta : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ be defined by

$$\zeta(a,a) = 1, \quad \zeta(b,b) = 9, \quad \zeta(c,c) = 5, \quad \zeta(a,c) = 7 = \zeta(c,a),$$

$$\zeta(b,c) = 8 = \zeta(c,b), \quad \zeta(a,b) = 10, \quad \zeta(b,a) = 11.$$

It is not difficult to verify that $(\mathcal{X}, \zeta)$ is a quasi $M$-metric space. Since $\zeta(a,b) \neq \zeta(b,a)$, we see that $(\mathcal{X}, \zeta)$ is not an $M$-metric space.

**Example 2.4.** Let $\mathcal{X} = [0, 1]$ and $\zeta : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be defined by $\zeta(\mu, \omega) = 2\mu + \omega$. Then $(\mathcal{X}, \zeta)$ is a quasi $M$-metric space.

*Proof.* First suppose that $\mu = \omega$, then we have

$$\zeta(\mu,\mu) = 3\mu,\ \zeta(\omega,\omega) = 3\mu,\ \zeta(\mu,\omega) = 3\mu,\ \zeta(\omega,\mu) = 3\mu,$$

that is,

$$\zeta(\mu,\mu) = \zeta(\omega,\omega) = \zeta(\mu,\omega) = \zeta(\omega,\mu).$$

Conversely, assume that $\zeta(\mu,\mu) = \zeta(\omega,\omega) = \zeta(\mu,\omega) = \zeta(\omega,\mu)$. These imply, $3\mu = 3\omega = 2\mu + \omega = 2\omega + \mu$, which further implies that $\mu = \omega$. Hence the condition (1) of Definition 2.4 is satisfied.

Without loss of generality, we may assume that $\mu < \omega, \mu, \omega \in [0, 1]$, we have

$$\begin{aligned} \zeta_{\mu,\omega} &= \min\{\zeta(\mu,\mu), \zeta(\omega,\omega)\} = \min\{3\mu, 3\omega\} = 3\mu \\ &\leq 2\mu + \omega = \zeta(\mu,\omega), \end{aligned}$$

which proves the condition (2) of Definition 2.4.

Let $\mu, \omega, w \in [0, 1]$. We prove the triangular inequality for the case $\mu < w < \omega$. Other cases can be treated in a similar way.

$$\begin{aligned} \zeta(\mu,\omega) - \zeta_{\mu,\omega} &= 2\mu + \omega - \min\{\zeta(\mu,\mu), \zeta(\omega,\omega)\} \\ &= 2\mu + \omega - \min\{3\mu, 3\omega\} \\ &= 2\mu + \omega - 3\mu \\ &= (2\mu + w - 3\mu) + (2w + \omega - 3w) \\ &= (2\mu + w - \min\{\zeta(\mu,\mu), \zeta(w,w)\}) + (2w + \omega - \min\{\zeta(w,w), \zeta(\omega,\omega)\}) \\ &= (\zeta(\mu,w) - \zeta_{\mu,w}) + (\zeta(w,\omega) - \zeta_{w,\omega}). \end{aligned}$$

□

Similar to the Remark 1.1 in [3], and Proposition 2.4 in [5], it is not difficult to see that the following holds in a quasi $M$-metric space:

**Proposition 2.1.** *Let $(\mathcal{X}, \zeta)$ be a quasi $M$-metric space, then for $\mu, \omega, w \in \mathcal{X}$, we have*

*(1)* $0 \leq R_{\mu\omega} + z_{\mu\omega} = \zeta(\mu,\mu) + \zeta(\omega,\omega)$,
*(2)* $0 \leq R_{\mu\omega} - z_{\mu\omega} = |\zeta(\mu,\mu) - \zeta(\omega,\omega)|$,
*(3)* $R_{\mu\omega} - z_{\mu\omega} \leq \left(R_{\mu w} - z_{\mu w}\right) + (R_{w\omega} - z_{w\omega})$.

**Proposition 2.2.** *Let $(\mathcal{X}, \zeta)$ be a quasi $M$-metric space and $K : \mathcal{X}^2 \to [0, +\infty)$ be defined by*

$$K(\mu,\omega) = \zeta(\mu,\omega) + \zeta(\omega,\mu) - 2z_{\mu,\omega}$$

*for all $\mu, \omega \in \mathcal{X}$. Then $K$ is a Euclidean metric, and the pair $(\mathcal{X}, K)$ is a usual metric space.*

*Proof.* The first two conditions of $K$ being a Euclidean metric follow easily. We establish the triangle inequality.

Let $\mu, \omega, z \in \mathcal{X}$, we have

$$
\begin{aligned}
K(\mu, \omega) &= \zeta(\mu, \omega) + \zeta(\omega, \mu) - 2z_{\mu,\omega} \\
&= (\zeta(\mu, \omega) - z_{\mu,\omega}) + (\zeta(\omega, \mu) - z_{\mu,\omega}) \\
&= (\zeta(\mu, \omega) - z_{\mu,\omega}) + (\zeta(\omega, \mu) - z_{\omega,\mu}) \\
&\leq \left[ \left(\zeta(\mu, w) - z_{\mu,w}\right) + (\zeta(w, \omega) - z_{w,\omega}) \right] + \left[ (\zeta(\omega, w) - z_{\omega,w}) + \left(\zeta(w, \mu) - z_{w,\mu}\right) \right] \\
&\leq \left[ \left(\zeta(\mu, w) - z_{\mu,w}\right) + \left(\zeta(w, \mu) - z_{w,\mu}\right) \right] + \left[ (\zeta(w, \omega) - z_{w,\omega}) + (\zeta(w, \omega) - z_{w,\omega}) \right] \\
&= [\zeta(\mu, w) + \left(\zeta(w, \mu - 2z_{\mu,w}\right))] + [\zeta(w, \omega) + (\zeta(w, \omega) - 2z_{w,\omega})] \\
&= K(\mu, w) + K(w, \omega).
\end{aligned}
$$

□

The proof of the following proposition is easy.

**Proposition 2.3.** *Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space and $H : \mathcal{X}^2 \to [0, +\infty)$ be defined by*

$$h(\mu, \omega) = \frac{\zeta(\mu, \omega) + \zeta(\omega, \mu)}{2}$$

*for all $\mu, \omega \in \mathcal{X}$. Then H is an M-metric, and the pair $(\mathcal{X}, H)$ is an M-metric space.*

## 3. Topology of quasi $M$-metric space

**Definition 3.1.** *Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space. Let $g \in \mathcal{X}$ and $\epsilon > 0$. Then:*

*(1) The forward open ball $B^+$ centered at g is defined as*

$$B^+(g, \epsilon) = \{h \in \mathcal{X} | \zeta(g, h) - z_{g,h} < \epsilon\}.$$

*(2) The backward open ball $B^-$ centered at g is defined as*

$$B^-(g, \epsilon) = \{h \in \mathcal{X} | \zeta(h, g) - z_{h,g} < \epsilon\}.$$

**Remark 3.1.** *A topological space $\mathcal{X}$ is called $T_0$ if there exists an open set that contains one of any two distinct points x and y but not the other. In Definition 3.1, we have defined forward open ball and backward open ball in $(\mathcal{X}, \zeta)$. Both of these open balls give rise to two types of $T_0$ topologies on $\mathcal{X}$, which we call forward topology $\tau^+$ and backward topology $\tau^-$. More precisely, the collection of forward open balls $\{B^+(g, \epsilon) : g \in \mathcal{X}, \epsilon > 0\}$ and backward open balls $\{B^-(g, \epsilon) : g \in \mathcal{X}, \epsilon > 0\}$ constitute the bases for the forward topology $\tau^+$ and backward topology $\tau^-$, respectively.*

In this paper, we shall work with forward topology $\tau^+$.

**Remark 3.2.** *We observe that the collection of forward open balls $\{B^+(g, \frac{1}{n}) : g \in \mathcal{X}\}$ constitutes a countable neighbourhood base for each point $g \in \mathcal{X}$. Therefore, $(\mathcal{X}, \zeta)$ is a first countable space under forward topology $\tau^+$.*

Now we give some topological definitions in $(\mathcal{X}, \zeta)$.

**Definition 3.2.** *(Compactness) Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space. The space $\mathcal{X}$ is called compact with respect to forward topology $\tau^+$ generated by the quasi M-metric $\zeta$ if each of its open covers has a finite subcover.*

**Definition 3.3.** *(Connectedness) Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space. The space $\mathcal{X}$ is called connected with respect to forward topology $\tau^+$ generated by the quasi M-metric $\zeta$ if it cannot be divided into two disjoint non-empty open sets.*

The different sorts of countability and separability axioms, as well as many other characteristics like sequential compactness and path-connectedness for the space $(\mathcal{X}, \zeta)$, are fascinating to investigate in depth in future.

**Definition 3.4.** *Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space, and $\{\theta_n\}$ be a sequence in $\mathcal{X}$.*

*(1) Then, the sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ from the left if and only if*

$$\lim_{n\to+\infty} \left(\zeta(\theta_n, g) - z_{\theta_n,g}\right) = 0.$$

*(2) Then, the sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ from the right if and only if*

$$\lim_{n\to+\infty} \left(\zeta(g, \theta_n) - z_{g,\theta_n}\right) = 0.$$

*(3) The sequence $\{\theta_n\}$ converges to a point $g \in \mathcal{X}$ if and only if it converges to g from the left, and from the right.*

**Definition 3.5.** *Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space, and $\{\theta_n\}$ be a sequence in $\mathcal{X}$. We say that:*

*(1) The sequence $\{\theta_n\}$ is left $\zeta$-Cauchy if and only if*

$$\lim_{n,m\to+\infty} (\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m})$$

*and*

$$\lim_{n,m\to+\infty} (R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m})$$

*exist finitely.*

*(2) The sequence $\{\theta_n\}$ is right $\zeta$-Cauchy if and only if*

$$\lim_{n,m\to+\infty} (\zeta(\theta_m, \theta_n) - z_{\theta_m,\theta_n})$$

*and*

$$\lim_{n,m\to+\infty} (R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n})$$

*exist finitely.*

*(3) The sequence $\{\theta_n\}$ is $\zeta$-Cauchy if and only if it is both left $\zeta$-Cauchy and right $\zeta$-Cauchy.*

Note that the definitions of left $\zeta$-Cauchy and right $\zeta$-Cauchy are essentially same, however, for the sake of interest and completeness, we have included both the definitions.

**Definition 3.6.** *Let $(\mathcal{X}, \zeta)$ be a quasi M-metric space, and $\{\theta_n\}$ be a $\zeta$-Cauchy in $\mathcal{X}$. We say that:*

*(1)* $(\mathcal{X}, \zeta)$ *is left* $\zeta$*-complete, with respect to forward topology* $\tau^+$*, if every left* $\zeta$*-Cauchy sequence converges to a point* $g \in \mathcal{X}$ *such that*

$$\lim_{n\to+\infty}\left(\zeta(\theta_n, g) - z_{\theta_n,g}\right) = 0$$

*and*

$$\lim_{n\to+\infty}\left(R_{\theta_n,g} - z_{\theta_n,g}\right) = 0.$$

*(2)* $(\mathcal{X}, \zeta)$ *is right* $\zeta$*-complete, with respect to forward topology* $\tau^+$*, if every right* $\zeta$*-Cauchy sequence converges to a point* $g \in \mathcal{X}$ *such that*

$$\lim_{n\to+\infty}\left(\zeta(g, \theta_n) - z_{g,\theta_n}\right) = 0$$

*and*

$$\lim_{n,m\to+\infty}\left(R_{g,\theta_n} - z_{g,\theta_n}\right) = 0.$$

*(3)* $(\mathcal{X}, \zeta)$ *is* $\zeta$*-complete, with respect to forward topology* $\tau^+$*, if and only if* $(\mathcal{X}, \zeta)$ *is both left* $\zeta$*-complete and right* $\zeta$*-complete.*

**Definition 3.7.** *Let* $(\mathcal{X}, \zeta)$ *be a quasi M-metric space, and a map* $F : \mathcal{X} \longrightarrow \mathcal{X}$*. We say that:*

*(1)* $F$ *is left* $\zeta$*-continuous if and only for each sequence* $\{\theta_n\}$ *in* $\mathcal{X}$ *converging to* $g \in \mathcal{X}$ *from the left implies that* $\{F\theta_n\}$ *converges to* $Fg$ *from the left, that is, we have*

$$\lim_{n\to+\infty}\left(\zeta(\theta_n, g) - z_{\theta_n,g}\right) = 0 \quad \Longrightarrow \lim_{n\to+\infty}\left(\zeta(F\theta_n, Fg) - z_{F\theta_n,Fg}\right) = 0.$$

*(2)* $F$ *is right* $\zeta$*-continuous if and only for each sequence* $\{\theta_n\}$ *in* $\mathcal{X}$ *converging to* $g \in \mathcal{X}$ *from the right implies that* $\{F\theta_n\}$ *converges to* $Fg$ *from the right, that is, we have*

$$\lim_{n\to+\infty}\left(\zeta(g, \theta_n) - z_{g,\theta_n}\right) = 0 \quad \Longrightarrow \lim_{n\to+\infty}\left(\zeta(Fg, F\theta_n) - z_{Fg,F\theta_n}\right) = 0.$$

*(3)* $F$ *is* $\zeta$*-continuous if it is both left and right* $\zeta$*-continuous.*

Analogous to Lemma 2.1 in [3], we have the following result.

**Proposition 3.1.** *Let* $(\mathcal{X}, \zeta)$ *be a quasi M-metric space, and* $(\mathcal{X}, K)$ *be the corresponding usual metric space given in Proposition* 1.2*. Let* $\{\theta_n\}$ *be a sequence in* $\mathcal{X}$*, then we have:*

*(1) The sequence* $\{\theta_n\}$ *is Cauchy in the usual metric space* $(\mathcal{X}, K)$ *if and only if it is* $\zeta$*-Cauchy in* $(\mathcal{X}, \zeta)$*.*
*(2) The space* $(\mathcal{X}, K)$ *is complete if and only if* $(\mathcal{X}, \zeta)$ *is* $\zeta$*-complete.*

*Proof.* The proof easily follows from the definition of $K$. □

The proof of the following result is similar to Lemma 3.5 in [5].

**Lemma 3.1.** *Let* $(\mathcal{X}, \zeta)$ *be a quasi M-metric space where* $\zeta$ *is continuous in the usual Euclidean metric. Suppose the self mapping* $F : \mathcal{X} \longrightarrow \mathcal{X}$ *satisfies*

$$\zeta(Fg, Fh) \le k\zeta(g, h)$$

*for some $k \in [0,1)$. Define a sequence $\{\theta_n\} \in \mathcal{X}$ by $\theta_n = F\theta_{n-1}$. If $\{\theta_n\}$ converges to a point $s \in \mathcal{X}$ from the left (or right), then $\{F\theta_n\}$ converges to $Fs \in \mathcal{X}$ from the left (or right), in the sense of Definition 2.2. That is,*

$$\lim_{n\to+\infty} \zeta(\theta_n, s) - z_{\theta_n,s} = 0,$$

*implies*

$$\lim_{n\to+\infty} (\zeta(F\theta_n, Fs) - z_{F\theta_n,Fs}) = 0.$$

## 4. Main result

The following result is analogous to the classical Banach contraction principle.

**Theorem 4.1.** *Let $(\mathcal{X}, \zeta)$ be a complete quasi M-metric space. Suppose that $F\colon \mathcal{X} \to \mathcal{X}$ is a self map satisfying*

$$\zeta(Fg, Fh) \leq k\zeta(g,h), \tag{4.1}$$

*for all $g, h \in \mathcal{X}$, where $k \in (0,1)$. Then $F$ has a unique fixed point in $\mathcal{X}$.*

*Proof.* Fix $\theta_0 \in \mathcal{X}$ and define a sequence $\{\theta_n\}$ in $\mathcal{X}$ inductively by taking $\theta_n = F\theta_{n-1},\ n \geq 0$.

$$\begin{aligned} \zeta(\theta_n, \theta_{n+1}) &= \zeta(F\theta_{n-1}, F\theta_n) \\ &\leq k\zeta(\theta_{n-1}, \theta_n) \\ &= k\zeta(F\theta_{n-2}, F\theta_{n-1}) \\ &\leq k^2\zeta(\theta_{n-2}, \theta_{n-1}) \\ &\ \vdots \\ &\leq k^n\zeta(\theta_0, \theta_1). \end{aligned} \tag{4.2}$$

That is,

$$\zeta(\theta_n, \theta_{n+1}) \leq k^n\zeta(\theta_0, \theta_1). \tag{4.3}$$

Similarly, we have

$$\zeta(\theta_{n+1}, \theta_n) \leq k^n\zeta(\theta_1, \theta_0). \tag{4.4}$$

Now, consider $n, m \in \mathbb{N}$ where $n > m$. Then using triangular inequality repeatedly, we have

$$\begin{aligned} \zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m} &\leq \sum_{i=m}^{n-1} (\zeta(\theta_i, \theta_{i+1}) - z_{\theta_i,\theta_{i+1}}) \leq \sum_{i=m}^{n-1} \zeta(\theta_i, \theta_{i+1}) \\ &\leq \sum_{i=m}^{n-1} k^i\zeta(\theta_0, \theta_1) \leq \frac{k^n - k^m}{k-1}\zeta(\theta_0, \theta_1). \end{aligned} \tag{4.5}$$

Since $k \in [0,1)$, letting $n, m \to \infty$, we conclude that

$$\lim_{n,m\to+\infty} (\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m}) = 0. \tag{4.6}$$

Similarly, using (4.4), we can establish that

$$\lim_{n,m\to+\infty} (\zeta(\theta_m, \theta_n) - z_{\theta_m,\theta_n}) = 0. \tag{4.7}$$

For $n > m$, we have

$$\begin{aligned}\zeta(\theta_n, \theta_n) &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\zeta(\theta_{n-1}, \theta_{n-1}) \\ &\ \vdots \\ &\leq k^{n-m}\zeta(\theta_m, \theta_m).\end{aligned} \tag{4.8}$$

The inequality (4.8) implies that

$$R_{\theta_n,\theta_m} = \max\{\zeta(\theta_n, \theta_n), \zeta(\theta_m, \theta_m)\} = \zeta(\theta_n, \theta_n).$$

Hence, we get

$$\begin{aligned}R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m} &\leq R_{\theta_n,\theta_m} \\ &= \zeta(\theta_n, \theta_n) \\ &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\zeta(\theta_{n-1}, \theta_{n-1}) \\ &\ \vdots \\ &\leq k^n\zeta(\theta_0, \theta_0).\end{aligned} \tag{4.9}$$

Letting $n \to \infty$, we deduce that

$$\lim_{n,m\to+\infty} (R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m}) = 0. \tag{4.10}$$

Similarly, we can establish that

$$\lim_{n,m\to+\infty} (R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n}) = 0. \tag{4.11}$$

By (4.6), (4.8), (4.10) and (4.11), we conclude that $\{\theta_n\}$ is $\zeta$-Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is $\zeta$-complete, $\{\theta_n\}$ converges to a point $\theta \in \mathcal{X}$ so that we have

$$\lim_{n\to+\infty} \zeta(\theta_n, \theta) - z_{\theta_n,\theta} = 0 \tag{4.12}$$

and

$$\lim_{n\to+\infty} \zeta(\theta, \theta_n) - z_{\theta,\theta_n} = 0. \tag{4.13}$$

Next, we prove that $F\theta = \theta$.

By the Lemma 3.1, $F$ is $\zeta$-continuous. The Definition 3.7 and Eq (4.12) imply that

$$\lim_{n\to+\infty} (\zeta(\theta_{n-1}, F\theta) - z_{\theta_{n-1},F\theta}) = \lim_{n\to+\infty} (\zeta(F\theta_n, F\theta) - z_{F\theta_n,F\theta}) = 0. \tag{4.14}$$

By the triangular inequality, we have

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq \left(\zeta(\theta, \theta_n) - z_{\theta,\theta_n}\right) + \left(\zeta(\theta_n, F\theta) - z_{\theta_n,F\theta}\right). \tag{4.15}$$

Taking the limit in the above inequality, and using (4.13) and (4.14), we obtain

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq 0. \tag{4.16}$$

By the definition of quasi $M$-metric space, we have

$$z_{\theta,F\theta} - \zeta(\theta, F\theta) \leq 0. \tag{4.17}$$

The inequalities (4.16) and (4.17) imply

$$\zeta(\theta, F\theta) = z_{\theta,F\theta}. \tag{4.18}$$

Now, by condition (4.1), we have $\zeta(F\theta, F\theta) \leq k\zeta(\theta, \theta) < \zeta(\theta, \theta)$. This implies

$$R_{\theta,F\theta} = \max\{\zeta(\theta, \theta), \zeta(F\theta, F\theta)\} = \zeta(\theta, \theta) \tag{4.19}$$

and

$$z_{\theta,F\theta} = \min\{\zeta(\theta, \theta), \zeta(F\theta, F\theta)\} = \zeta(F\theta, F\theta). \tag{4.20}$$

By (4.18) and (4.20), we obtain

$$\zeta(\theta, F\theta) = \zeta(F\theta, F\theta). \tag{4.21}$$

Now,

$$\begin{aligned} \zeta(\theta_n, \theta_n) &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k\zeta(\theta_{n-1}, \theta_{n-1}) \\ &\vdots \\ &\leq k^n\zeta(\theta_0, \theta_0). \end{aligned} \tag{4.22}$$

This implies

$$\lim_{n\to+\infty} \zeta(\theta_n, \theta_n) = 0. \tag{4.23}$$

By Eq (4.23), we get

$$\begin{aligned} \lim_{n\to+\infty} z_{\theta_n,\theta} &= \lim_{n\to+\infty} \min\{\zeta(\theta_n, \theta_n), \zeta(\theta, \theta)\} \\ &= \min\{0, \zeta(\theta, \theta)\} \\ &= 0. \end{aligned} \tag{4.24}$$

By Proposition 2.1, we have

$$\zeta(\theta_n, \theta_n) + \zeta(\theta, \theta) = R_{\theta_n,\theta} + z_{\theta_n,\theta}$$

or

$$\begin{aligned} \zeta(\theta, \theta) &= R_{\theta_n,\theta} + z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n) \\ &= (R_{\theta_n,\theta} - z_{\theta_n,\theta}) + 2z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n). \end{aligned} \tag{4.25}$$

Since $(\mathcal{X}, \zeta)$ is $\zeta$-Complete, by Definition 3.6,

$$\lim_{n\to+\infty} (R_{\theta_n,\theta} - z_{\theta_n,\theta}) = 0. \tag{4.26}$$

Using (4.23), (4.24) and (4.26) in (4.25), we obtain

$$\zeta(\theta, \theta) = 0. \tag{4.27}$$

By Eqs (4.18), (4.19) and (4.27), we have

$$\zeta(\theta, F\theta) = z_{\theta,F\theta} \leq R_{\theta,F\theta} = 0. \tag{4.28}$$

Since $\zeta(\theta, F\theta) \geq 0$, this implies

$$\zeta(\theta, F\theta) = 0. \tag{4.29}$$

Similarly, we may prove

$$\zeta(F\theta, \theta) = 0.$$

The Eqs (4.21), (4.27) and (4.29) imply

$$\zeta(\theta, \theta) = \zeta(F\theta, F\theta) = \zeta(\theta, F\theta) = \zeta(\theta, F\theta) = 0, \tag{4.30}$$

which further implies $\theta = F\theta$ so that $\theta$ is a fixed point of $F$.

Next, we show the uniqueness of the fixed point. Suppose that $F$ has two distinct fixed points $\theta$ and $\delta$, such that $F\theta = \theta$ and $F\delta = \delta$. Thus, $\zeta(\theta, \delta) = \zeta(F\theta, F\delta) \leq k\zeta(\theta, \delta) < \zeta(\theta, \delta)$. This implies, $\zeta(\theta, \delta) = 0$. Also, $\zeta(\theta, \theta) = \zeta(F\theta, F\theta) \leq k\zeta(\theta, \theta) < \zeta(\theta, \theta)$, which implies $\zeta(\theta, \theta) = 0$. Similarly, $\zeta(\delta, \delta) = 0$. Thus we have

$$\zeta(\theta, \delta) = \zeta(\theta, \theta) = \zeta(\delta, \delta) = 0,$$

which by the Definition 2.3 implies $\delta = \theta$. □

Analogous to Shukla fixed point theorem [28], we have the following fixed point theorem in quasi $M$-metric space.

**Theorem 4.2.** *Let $(\mathcal{X}, \zeta)$ be a complete quasi $M$-metric space with $r > 2$ and $F: \mathcal{X} \to \mathcal{X}$ be a self $\zeta$-continuous mapping on $\mathcal{X}$ satisfying*

$$\zeta(Fg, Fh) \leq k[\zeta(g, Fg) + \zeta(h, Fh)] \tag{4.31}$$

*for all $g, h \in \mathcal{X}$, where $k \in [0, \frac{1}{r}]$. Then $F$ has a unique fixed point $\theta$ in $\mathcal{X}$ such that $\zeta(\theta, \theta) = 0$.*

*Proof.* Let $\theta_0 \in \mathcal{X}$ and define a sequence $\{\theta_n\}$ in $\mathcal{X}$ inductively by taking $\theta_n = F\theta_{n-1}$, $n \geq 0$. Set $d_n = \zeta(\theta_n, \theta_{n+1})$ and $D_n = \zeta(\theta_{n+1}, \theta_n)$. Then we have

$$\begin{aligned} d_n = \zeta(\theta_n, \theta_{n+1}) &= \zeta(F\theta_{n-1}, F\theta_n) \\ &\leq k[\zeta(\theta_{n-1}, F\theta_{n-1}) + \zeta(\theta_n, F\theta_n)] \\ &= k[\zeta(\theta_{n-1}, \theta_n) + \zeta(\theta_n, \theta_{n+1})] \\ &\leq k[d_{n-1} + d_n)], \end{aligned} \tag{4.32}$$

which implies

$$d_n \leq \beta d_{n-1}, \tag{4.33}$$

where $\beta = \frac{k}{1-k} < 1$ as $k \in [0, \frac{1}{r}], r > 2$. Thus we have

$$d_n \leq \beta d_{n-1} \leq \beta^2 d_{n-2} \leq ... \leq \beta^n \zeta(\theta_0, \theta_1). \tag{4.34}$$

Similarly we have

$$D_n \leq \beta^n \zeta(\theta_1, \theta_0). \tag{4.35}$$

Now, consider $n, m \in \mathbb{N}$ where $n > m$. Then using the triangular inequality repeatedly, we have

$$\begin{aligned}\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m} &\leq \sum_{i=m}^{n-1} (\zeta(\theta_i, \theta_{i+1}) - z_{\theta_i,\theta_{i+1}}) \leq \sum_{i=m}^{n-1} \zeta(\theta_i, \theta_{i+1}) \\ &= \sum_{i=m}^{n-1} d_i \leq \sum_{i=m}^{n-1} \beta^i \zeta(\theta_0, g_1) \leq \frac{\beta^n - \beta^m}{\beta - 1} \zeta(\theta_0, g_1).\end{aligned} \tag{4.36}$$

Since $\beta \in [0, 1)$, letting $n, m \to \infty$, we conclude that

$$\lim_{n,m\to+\infty} (\zeta(\theta_n, \theta_m) - z_{\theta_n,\theta_m}) = 0. \tag{4.37}$$

Similarly using (4.35), we can prove that

$$\lim_{n,m\to+\infty} (\zeta(\theta_m, \theta_n) - z_{\theta_m,\theta_n}) = 0. \tag{4.38}$$

Without loss of generality, we may assume that

$$R_{\theta_n,\theta_m} = \max\{\zeta(\theta_n, \theta_n), \zeta(\theta_m, \theta_m)\} = \zeta(\theta_n, \theta_n).$$

Hence, we get

$$\begin{aligned}R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m} &\leq R_{\theta_n,\theta_m} = \zeta(\theta_n, \theta_n) = \zeta(F\theta_{n-1}, F\theta_{n-1}) \\ &\leq k[\zeta(\theta_{n-1}, F\theta_{n-1}) + \zeta(\theta_{n-1}, F\theta_{n-1})] \\ &\leq k[\zeta(\theta_{n-1}, \theta_n) + \zeta(\theta_{n-1}, \theta_n)] \\ &= 2k[\zeta(\theta_{n-1}, \theta_n)] \\ &= 2kd_{n-1}.\end{aligned} \tag{4.39}$$

By the inequality (4.34), $\lim_{n\to+\infty} d_n = 0$.

Letting $n \to \infty$ in the above inequality, we deduce that

$$\lim_{n,m\to+\infty} (R_{\theta_n,\theta_m} - z_{\theta_n,\theta_m}) = 0. \tag{4.40}$$

Similarly,we can establish that

$$\lim_{n,m\to+\infty} (R_{\theta_m,\theta_n} - z_{\theta_m,\theta_n}) = 0. \tag{4.41}$$

By (4.37), (4.38), (4.40) and (4.41), we conclude that $\{\theta_n\}$ is $\zeta$-Cauchy in $\mathcal{X}$. Since $\mathcal{X}$ is $\zeta$-complete, $\{\theta_n\}$ converges to a point $\theta \in \mathcal{X}$ so that we have

$$\lim_{n\to+\infty} \zeta(\theta_n, \theta) - z_{\theta_n,\theta} = 0 \tag{4.42}$$

and

$$\lim_{n\to+\infty} \zeta(\theta, \theta_n) - z_{\theta,\theta_n} = 0. \tag{4.43}$$

Now, we prove that $\theta$ is a fixed point of $F$.

Since $F$ is $\zeta$-continuous, the Definition 3.7 and the Eq (4.42) implies

$$\lim_{n\to+\infty}\left(\zeta(\theta_{n-1}, F\theta) - z_{\theta_{n-1},F\theta}\right) = \lim_{n\to+\infty}\left(\zeta(F\theta_n, F\theta) - z_{F\theta_n,F\theta}\right) = 0. \tag{4.44}$$

By the triangular inequality, we have

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq \left(\zeta(\theta, \theta_n) - z_{\theta,\theta_n}\right) + \left(\zeta(\theta_n, F\theta) - z_{\theta_n,F\theta}\right). \tag{4.45}$$

Taking the limit in the above inequality, and using (4.43) and (4.44), we obtain

$$\zeta(\theta, F\theta) - z_{\theta,F\theta} \leq 0. \tag{4.46}$$

By the definition of quasi $M$-metric space, we have

$$z_{\theta,F\theta} - \zeta(\theta, F\theta) \leq 0. \tag{4.47}$$

The inequalities (4.46) and (4.47) imply

$$\zeta(\theta, F\theta) = z_{\theta,F\theta}. \tag{4.48}$$

Now,

$$\begin{aligned}
\zeta(\theta_n, \theta_n) &= \zeta(F\theta_{n-1}, F\theta_{n-1}) \\
&\leq k\left[\zeta(\theta_{n-1}, F\theta_{n-1}) + \zeta(\theta_{n-1}, F\theta_{n-1})\right] \\
&\leq k\left[\zeta(\theta_{n-1}, \theta_n) + \zeta(\theta_{n-1}, \theta_n)\right] \\
&= 2k\left[\zeta(\theta_{n-1}, \theta_n)\right] \\
&= 2kd_{n-1}.
\end{aligned} \tag{4.49}$$

This implies

$$\lim_{n\to+\infty} \zeta(\theta_n, \theta_n) = 0. \tag{4.50}$$

By Eq (4.50), we get

$$\begin{aligned}
\lim_{n\to+\infty} z_{\theta_n,\theta} &= \lim_{n\to+\infty} \min\{\zeta(\theta_n, \theta_n), \zeta(\theta, \theta)\} \\
&= \min\{0, \zeta(\theta, \theta)\} \\
&= 0.
\end{aligned} \tag{4.51}$$

By Proposition 2.1, we have

$$\zeta(\theta_n, \theta_n) + \zeta(\theta, \theta) = R_{\theta_n,\theta} + z_{\theta_n,\theta}$$

or

$$\begin{aligned}
\zeta(\theta, \theta) &= R_{\theta_n,\theta} + z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n) \\
&= (R_{\theta_n,\theta} - z_{\theta_n,\theta}) + 2z_{\theta_n,\theta} - \zeta(\theta_n, \theta_n).
\end{aligned} \tag{4.52}$$

Since $(\mathcal{X}, \zeta)$ is $\zeta$-Complete, by Definition 3.6,

$$\lim_{n\to+\infty} (R_{\theta_n,\theta} - z_{\theta_n,\theta}) = 0. \tag{4.53}$$

Using (4.50), (4.51) and (4.53) in (4.52), we obtain

$$\zeta(\theta,\theta) = 0. \tag{4.54}$$

By Eqs (4.48) and (4.54), we have

$$\begin{aligned}\zeta(\theta, F\theta) &= z_{\theta,F\theta}\\ &= \min\{\zeta(\theta,\theta), \zeta(F\theta, F\theta)\}\\ &= \min\{0, \zeta(F\theta, F\theta)\}\\ &= 0.\end{aligned} \tag{4.55}$$

Similarly, we may prove

$$\zeta(F\theta,\theta) = 0.$$

Using (3.55), we obtain

$$\begin{aligned}\zeta(F\theta, F\theta) &\leq k[\zeta(\theta, F\theta) + \zeta(\theta, F\theta)]\\ &\leq 2k[\zeta(\theta, F\theta)]\\ &= 0.\end{aligned} \tag{4.56}$$

This implies

$$\zeta(F\theta, F\theta) = 0. \tag{4.57}$$

Therefore, by Eqs (4.54), (4.55) and (4.57), we obtain

$$\zeta(\theta,\theta) = \zeta(\theta, F\theta) = \zeta(F\theta,\theta) = \zeta(F\theta, F\theta) = 0,$$

which implies that $F\theta = \theta$.

Finally, we establish the uniqueness of the fixed point. Suppose that $F$ has two distinct fixed points $\theta$ and $\delta$, that $F\theta = \theta$ and $F\delta = \delta$. We have

$$\zeta(\theta,\delta) = \zeta(F\theta, F\delta) \leq k[\zeta(\theta, F\theta) + \zeta(\delta, F\delta)] = k[\zeta(\theta,\theta) + \zeta(\delta,\delta)] = 0,$$

which implies that $\zeta(\theta,\delta) = 0$. Since $\theta$ and $\delta$ are fixed points, by Eq (4.56), we have $\zeta(\theta,\theta) = 0$ and $\zeta(\delta,\delta) = 0$. Therefore,

$$\zeta(\theta,\delta) = \zeta(\theta,\theta) = \zeta(\delta,\delta) = 0,$$

which implies that $\theta = \delta$. □

The authors are intending to prove a Banach type result for Riech contraction mapping along with the applications in their future work.

## 5. Examples and applications

In this section, we illustrate our results with examples and applications. In Example 2.4, we showed that $(\mathcal{X},\zeta)$ with $\mathcal{X} = [0,1]$ and $\zeta(\mu,\omega) = 2\mu + \omega$ is a quasi $M$-metric space. We have the following result.

**Example 5.1.** Let $\mathcal{X} = [0,1]$ and $\zeta(\mu,\omega) = 2\mu + \omega$, then $(\mathcal{X},\zeta)$ is a complete quasi $M$-metric space.

*Proof.* Without loss of generality, assume that $\mu < \omega,\ \mu, \omega \in \mathcal{X}$. Consider the metric $K(\mu, \omega)$ given in Proposition 2.2, we have

$$\begin{aligned} K(\mu,\omega) &= \zeta(\mu,\omega) + \zeta(\omega,\mu) - 2z_{\mu,\omega} \\ &= 2\mu + \omega + 2\omega + \mu - 2\min\{\zeta(\mu,\mu), \zeta(\omega,\omega)\} \\ &= 2\mu + \omega + 2\omega + \mu - 2\min\{3\mu, 3\omega)\} \\ &= 2\mu + \omega + 2\omega + \mu - 6\mu \\ &= 3(\omega - \mu) \\ &= 3|\mu - \omega|. \end{aligned}$$

Since $\mathcal{X}$ is complete with respect to the metric $K(\mu, \omega) = 3|\mu - \omega|$, by Proposition 3.1 we conclude that $(\mathcal{X}, \zeta)$ is a $\zeta$-complete quasi $M$-metric space. □

**Example 5.2.** Let $(\mathcal{X}, \zeta)$ be a complete quasi $M$-metric space with $\mathcal{X} = [2, 3]$ and $\zeta(\mu, \omega) = 2\mu + \omega$. Define a self-map $F : [2, 3] \longrightarrow [2, 3]$ defined by $F(\mu) = 3 - \mu$. Then $F$ has a unique fixed point given by $\mu = \frac{3}{2}$.

*Proof.* Let $\mathcal{X} = [2, 3]$, $\zeta(\mu, \omega) = 2\mu + \omega$ and $F(\mu) = 3 - \mu$. Then, for all $\mu, \omega \in [2, 3]$, we have $\mu \geq 2,\ \omega \geq 2$. This implies that $22 \leq 7\mu + 4\omega$, which further implies

$$21 \leq 7\mu + 4\omega.$$

The above inequality can be further rearranged as

$$\begin{aligned} 2(3-\mu) + (3-\omega) &\leq \frac{1}{3}[2\mu + (3-\mu) + 2\omega + (3-\omega)], \\ \zeta(F\mu, F\omega) &\leq \frac{1}{3}[\zeta(\mu, F\mu) + \zeta(\omega, F\omega)]. \end{aligned}$$

Since $F$ satisfies the conditions of Theorem 4.2 with $k = \frac{1}{3}$, we conclude that $F$ has a unique fixed point. □

**Example 5.3.** Consider the space of continuous real valued functions $\mathcal{X} = C[0, 1]$, and $\zeta(r(\mu), h(\mu)) : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be defined as

$$\zeta(r(\mu), h(\mu)) = \sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)|.$$

It is not difficult to see that $(\mathcal{X}, \zeta)$ is a complete quasi $M$-metric space.

*Proof.* We prove the triangular inequality of Definition 2.4. Other two conditions follow easily.

Let $r(\mu), h(\mu), w(\mu) \in C[0, 1]$, and assume that $\sup_{\mu\in[0,1]} |r(\mu)| \leq \sup_{\mu\in[0,1]} |w(\mu)| \leq \sup_{\mu\in[0,1]} |h(\mu)|$. The other cases may be treated similarly.

We have

$$
\begin{aligned}
&\zeta(r(\mu), h(\mu)) - z_{r(\mu),h(\mu)} \\
= &\sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| - z_{r(\mu),h(\mu)} \\
= &\sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| \\
&- \min\{\sup_{\mu\in[0,1]} |r(\mu) - r(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)|, \sup_{\mu\in[0,1]} |h(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |h(\mu)|\} \\
= &\sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| - \min\{\sup_{\mu\in[0,1]} |r(\mu)|, \sup_{\mu\in[0,1]} |h(\mu)|\} \\
= &\sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| - \sup_{\mu\in[0,1]} |r(\mu)| \\
= &\sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| \\
= &\sup_{\mu\in[0,1]} |r(\mu) - w(\mu) + w(\mu) - h(\mu)| \\
\leq &\sup_{\mu\in[0,1]} |r(\mu) - w(\mu)| + \sup_{\mu\in[0,1]} |w(\mu) - h(\mu)| \\
\leq &\sup_{\mu\in[0,1]} |r(\mu) - w(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| - \sup_{\mu\in[0,1]} |r(\mu)| + \sup_{\mu\in[0,1]} |w(\mu) - h(\mu)| \\
&+ \sup_{\mu\in[0,1]} |w(\mu)| - \sup_{\mu\in[0,1]} |w(\mu)| \\
\leq &\sup_{\mu\in[0,1]} |r(\mu) - w(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)| - \sup_{\mu\in[0,1]} |r(\mu)| \\
&+ \sup_{\mu\in[0,1]} |w(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |w(\mu)| - \sup_{\mu\in[0,1]} |w(\mu)| \\
\leq &\zeta(r(\mu), w(\mu)) - \sup_{\mu\in[0,1]} |r(\mu)| + \zeta(w(\mu), h(\mu)) - \sup_{\mu\in[0,1]} |w(\mu)| \\
\leq &\zeta(r(\mu), w(\mu)) - \min\{\sup_{\mu\in[0,1]} |r(\mu)|, \sup_{\mu\in[0,1]} |w(\mu)|\} \\
&+ \zeta(w(\mu), h(\mu)) - \min\{\sup_{\mu\in[0,1]} |w(\mu)|, \sup_{\mu\in[0,1]} |h(\mu)|\} \\
\leq &\zeta(r(\mu), w(\mu)) - z_{r(\mu),w(\mu)} + \zeta(w(\mu), h(\mu)) - z_{w(\mu),h(\mu)}.
\end{aligned}
\tag{5.1}
$$

Using Proposition 2.2, we can easily prove that $(\mathcal{X}, \zeta)$ is a $\zeta$-complete quasi $M$-metric space. □

**Theorem 5.1.** *Let $\mathcal{X} = C[0, 1]$ be the complete quasi $M$-metric space given in Example* 5.3. *Consider the following integral equation with parameter $\lambda$:*

$$r(\mu) = \lambda \int_0^1 l(\mu, \omega) r(\omega) d\omega, \tag{5.2}$$

*where $l(u, \omega) : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ is a given continuous function with $|l(\mu, \omega)| \leq c$ for all $(\mu, \omega) \in [0, 1] \times [0, 1]$. If $c|\lambda| < 1$, then the integral equation* (4.2) *has a unique solution.*

*Proof.* Let $F : C[0,1] \longrightarrow C[0,1]$ be defined by $F(r(\mu)) = \lambda \int_0^1 l(\mu,\omega)r(\omega)d\omega$, then

$$\begin{aligned}
\zeta(Fr(\mu), Fh(\mu)) &= \sup_{\mu\in[0,1]} |Fr(\mu) - Fh(\mu)| + \sup_{\mu\in[0,1]} |Fr(\mu)| \\
&= \sup_{\mu\in[0,1]} |\lambda \int_0^1 l(\mu,\omega)r(\omega)d\omega - \lambda \int_0^1 l(\mu,\omega)h(\omega)d\omega| \\
&\quad + \sup_{\mu\in[0,1]} |\lambda \int_0^1 l(\mu,\omega)r(\omega)d\omega| \\
&\leq \sup_{\mu\in[0,1]} | \int_0^1 |\lambda||l(\mu,\omega)||r(\omega) - h(\omega)|d\omega + \sup_{\mu\in[0,1]} \int_0^1 |\lambda||l(\mu,\omega)||r(\omega)|d\omega \\
&\leq c|\lambda|( \sup_{\mu\in[0,1]} \int_0^1 |r(\omega) - h(\omega)|d\omega + \sup_{\mu\in[0,1]} \int_0^1 |r(\omega)|d\omega) \\
&\leq c|\lambda|( \sup_{\mu\in[0,1]} \int_0^1 |r(\omega) - h(\omega)|d\omega + \sup_{\mu\in[0,1]} \int_0^1 |r(\omega)|d\omega) \\
&\leq c|\lambda|( \sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| \int_0^1 d\omega + \sup_{\mu\in[0,1]} |r(\mu)| \int_0^1 d\omega) \\
&\leq c|\lambda|( \sup_{\mu\in[0,1]} |r(\mu) - h(\mu)| + \sup_{\mu\in[0,1]} |r(\mu)|) \\
&= c|\lambda|\zeta(r(\mu), h(\mu)).
\end{aligned} \tag{5.3}$$

If $c|\lambda| < 1$, we see that $F$ is a contraction. Therefore, by Theorem 4.1, there is a unique function $r \in C[0,1]$ such that $Fr = r$. This implies that the integral equation (5.2) has a unique solution. □

**Example 5.4.** Consider the set $\mathcal{X} = \mathbb{R}$ of all ordered $n$-tuples of real numbers. Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$, and $\zeta(x,y) : \mathcal{X} \times \mathcal{X} \longrightarrow [0,\infty)$ be defined as

$$\zeta(x,y) = \max_{1\leq i\leq n} |x_i - y_i| + \max_{1\leq i\leq n} |y_i|.$$

It is easy to verify that $(\mathcal{X}, \zeta)$ is a complete quasi $M$-metric space.

**Theorem 5.2.** *Let $\mathcal{X} = \mathbb{R}$ be the complete quasi $M$-metric space given in Example* 5.4. *Let $x = (x_1, x_2, ..., x_n)$, and consider the following linear system of equations in equivalent matrix form:*

$$Cx = r, \tag{5.4}$$

*where $C = (c_{ij})$ is a fixed $n \times n$ real matrix, and $r = (r_1, r_2, ..., r_n) \in \mathcal{X}$ a fixed vector. If*

$$\lambda = \max_{1\leq i\leq n} \left( \sum_{j=1, j\neq i}^{n} \left|c_{ij}\right| + |1 + c_{ii}| \right) < 1, \tag{5.5}$$

*then the matrix system given by* (5.4) *has a unique solution.*

*Proof.* The matrix system (5.4) can be equivalently written in the form $(C + I_n)x - r = x$, where $I_n$ is the $n \times n$ identity matrix.

Define a map $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $Tx = (C + I_n)x - r$. We shall prove that $T$ has a unique fixed point, thereby establishing the unique solution of the matrix system (5.4).

We define

$$\tilde{C} = C + I_n = \left(\tilde{c}_{ij}\right), \quad i, j = 1, \dots, n,$$

with $\tilde{c}_{ij} = \begin{cases} c_{ij}, j \neq i, \\ 1 + c_{ii}, j = i \end{cases}$. Hence,

$$\max_{1\leq i\leq n} \sum_{j=1}^{n} \left|\tilde{c}_{ij}\right| = \max_{1\leq i\leq n} \left( \sum_{j=1, j\neq i}^{n} \left|c_{ij}\right| + |1 + c_{ii}| \right) = \lambda < 1.$$

Let $y = (y_1, y_2, ..., y_n) \in \mathcal{X}$. We can write $Tx$ and $Ty$ in the component form as

$$Tx = \sum_{j=1}^{n} \tilde{c}_{ij} x_j - r_j, \ Ty = \sum_{j=1}^{n} \tilde{c}_{ij} y_j - r_j.$$

Using the definition quasi $M$-metric on $\mathcal{X}$ defined in the Example 5.4, we obtain

$$\begin{aligned}
\zeta(Tx, Ty) &= \max_{1\leq i\leq n} |\sum_{j=1}^{n} \tilde{c}_{ij} x_j - r_j - \left( \sum_{j=1}^{n} \tilde{c}_{ij} y_j - r_j \right)| + \max_{1\leq i\leq n} |\sum_{j=1}^{n} \tilde{c}_{ij} y_j - r_j| \\
&= \max_{1\leq i\leq n} |\sum_{j=1}^{n} \tilde{c}_{ij}(x_j - y_j)| + \max_{1\leq i\leq n} |\sum_{j=1}^{n} \tilde{c}_{ij} y_j - r_j| \\
&\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \max_{1\leq k\leq n} |(x_k - y_k)| + \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \max_{1\leq k\leq n} |y_k - r_k| \\
&\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \max_{1\leq k\leq n} |(x_k - y_k)| + \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}(| \max_{1\leq k\leq n} |y_k| + \max_{1\leq k\leq n} |r_k|) \\
&\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \max_{1\leq k\leq n} |(x_k - y_k)| + \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \max_{1\leq k\leq n} |y_k| \\
&\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| \left( \max_{1\leq k\leq n} |(x_k - y_k)| + \max_{1\leq k\leq n} |y_k| \right) \\
&= \lambda \zeta(x, y),
\end{aligned} \tag{5.6}$$

where $\lambda = \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{c}_{ij}| < 1$. Therefore, all the conditions of Theorem 4.1 are satisfied. This implies that $T$ has a unique fixed point. □

## 6. Conclusions and open problems

We developed the idea of quasi $M$-metric space and established fixed point results of the Shukla and Banach types. Many well-known theorems in the literature related to partial metric spaces and $M$-metric spaces are generalized by our results.

It is an problem to establish the Banach type fixed point results in quasi $M$-metric space $(X, \zeta)$ for other types of contraction mappings like Ciric contraction mapping, Riech contraction mapping, Hardy-Roger contraction mapping and Caristi contraction mapping. The applications of the established fixed point results are always of significant interest. As a future work, our results can be applied to shape-memory alloys [12] and questions about the existence and uniqueness of Hamilton-Jacobi equations [13]. Future studies in this direction is highly suggested.

**Acknowledgements**

The authors I. Ayoob and N. Mlaiki would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

**Conflict of interest**

The authors declare that they have no competing interests.

**References**

1. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations intgrales, *Fund. Math.*, **3** (1922), 133–181.
2. S. G. Matthews, Partial metric topology, *Ann. New York Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x
3. M. Asadi, E. Karapınar, P. Salimi, New extension of $p$-metric spaces with some fixed point results on $M$-metric spaces, *J. Inequal. Appl.*, **2014** (2014), 1–9. https://doi.org/10.1186/1029-242X-2014-18
4. N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer, T. Abdeljawed, Fixed point theorems in $M_b$-metric spaces, *J. Math. Anal.*, **7** (2016), 1–9.
5. N. Mlaiki, N. Y. Özgür, A. Mukheimer, N. Taş, A new extension of the $M_b$-metric spaces, *J. Math. Anal.*, **9** (2018), 118–133.
6. M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, *Fixed Point Theory Appl.*, **2012** (2012), 1–7. https://doi.org/10.1186/1687-1812-2012-210
7. W. A. Wilson, On quasi metric spaces, *Amer. J. Math.*, **53** (1931), 675–684.
8. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087
9. M. Asim, A. R. Khan, M. Imdad, Fixed point results in partial symmetric spaces with an application, *Axioms*, **8** (2019), 1–15. https://doi.org/10.3390/axioms8010013
10. M. Asim, M. Imdad, Partial JS-metric spaces and fixed fixed point results, *Indian J. Math.*, **61** (2019), 175–186.
11. N. Y. Özgür, N. Mlaiki, N. Tas, N. Souayah, A new generalization of metric spaces: rectangular $M$-metric spaces, *Math. Sci.*, **12** (2018), 223–233. https://doi.org/10.1007/s40096-018-0262-4

12. A. Mielke, T. Roubcek, A rate-independent model for inelastic behavior of shape-memory alloys, *Multiscale Model. Simul.*, **1** (2003), 571–597. https://doi.org/10.1137/S1540345903422860

13. A. C. G. Mennucci, On asymmetric distances, *Anal. Geom. Metr. Space.*, **1** (2013), 200–231.

14. W. L. Woon, S. Madnick, Asymmetric information distances for automated taxonomy construction, *Knowl. Inform. Syst.*, **21** (2009), 91–111. https://doi.org/10.1007/s10115-009-0203-5

15. M. O. Rieger, J. Zimmer, Young measure flow as a model for damage, *Z. Angew. Math. Phys.*, **60** (2009), 1–32. https://doi.org/10.1007/s00033-008-7016-3

16. S. J. O'Neill, Partial metrics, valuations and domain theory, *Ann. New York Acad. Sci.*, **806** (1996), 304–315. https://doi.org/10.1111/j.1749-6632.1996.tb49177.x

17. M. A. Bukatin, J. S. Scott, Towards computing distances between programs via Scott domains, In: *Logical foundations of computer sicence*, Berlin, Heidelberg: Springer, 1997, 33–43. https://doi.org/10.1007/3-540-63045-7_4

18. M. A. Bukatin, S. Y. Shorina, Partial metrics and co-continuous valuations, In: *Foundations of software science and computation structures*, 1998, 125–139. https://doi.org/10.1007/BFb0053546

19. M. H. Escardo, PCF extended with real numbers, *Theor. Comput. Sci.*, **162** (1996), 79–115. https://doi.org/10.1016/0304-3975(95)00250-2

20. M. P. Schellekens, A characterization of partial metrizability: domains are quantifiable, *Theor. Comput. Sci.*, **305** (2003), 409–432. https://doi.org/10.1016/S0304-3975(02)00705-3

21. M. P. Schellekens, The correpondence between partial metrics and semivaluations, *Theor. Comput. Sci.*, **315** (2004), 135–149. https://doi.org/10.1016/j.tcs.2003.11.016

22. P. Waszkierwicz, Quantitative continuous domains, *Appl. Categor. Struct.*, **11** (2003), 41–67. https://doi.org/10.1023/A:1023012924892

23. P. Waszkierwicz, The local triangle axiom in topology and domain theory, *Appl. Gen. Topol.*, **4** (2003), 47–70. https://doi.org/10.4995/agt.2003.2009

24. M. Asadi, Fixed point theorems for Meir-Keeler type mappings in $M$-metric space with applications, *Fixed Point Theory Appl.*, **2015** (2015), 1–10. https://doi.org/10.1186/s13663-015-0460-9

25. M. Asadi, E. Karapınar, P. Salimi, New extension of $p$-metric spaces with some fixed-point results on $M$-metric spaces, *J. Inequal. Appl.*, **2014** (2014), 1–9. https://doi.org/10.1186/1029-242X-2014-18

26. H. Monfared, M. Asadi, M. Azhini, Coupled fixed point theorems for generalized contractions in ordered $M$-metric spaces, *Results Fixed Point Theory Appl.*, **341** (2018), 1241–1252.

27. H. Monfared, M. Azhini, M. Asadi, Fixed point results on $M$-metric spaces, *J. Math. Anal.*, **7** (2016), 85–101.

28. S. Shukla, Partial b-metric spaces and fixed point theorems, *Mediterr. J. Math.*, **11** (2014), 703–711. https://doi.org/10.1007/s00009-013-0327-4

29. W. Shatanawi, On $w$-compatible mappings and common coupled coincidence point in cone metric spaces, *Appl. Math. Lett.*, **25** (2012), 925–931, https://doi.org/10.1016/j.aml.2011.10.037

30. W. Shatanawi, V. C. Rajic, S. Radenovic, A. Al-Rawashhdeh, Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 1–7. https://doi.org/10.1186/1687-1812-2012-106
31. A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sehmim, W. Shatanawi, On common fixed points for $\alpha - F$-contractions and applications, *J. Nonlinear Sci. Appl.*, **9** (2016), 3445–3458. http://dx.doi.org/10.22436/jnsa.009.05.128
32. W. Shatanawi, Z. Mustafa, N. Tahat, Some coincidence point theorems for nonlinear contraction in ordered metric spaces, *Fixed Point Theory Appl.*, **2011** (2011), 1–15. https://doi.org/10.1186/1687-1812-2011-68
33. W. Shatanawi, Some fixed point results for a generalized $\psi$-weak contraction mappings in orbitally metric spaces, *Chaos Solitons Fract.*, **45** (2012), 520–526. https://doi.org/10.1016/j.chaos.2012.01.015