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Research article

Some fixed point results for fuzzy generalizations of Nadler's contraction in b-metric spaces

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Abstract: The main purpose of this study is to examine the existence of fuzzy fixed points of fuzzy mappings meeting the criteria of some generalized contractions of Nadler's type in the framework of complete b-metric spaces. From the pertinent literature, there are additional previous observations that are provided as corollaries. Our study expands and incorporates several implications that are apparent in this mode and are addressed in considerable literature.

Keywords: Nadler's contraction; fuzzy sets; fuzzy fixed point; fuzzy mapping; Hausdorff metric space
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1. Introduction

With the introduction of fuzzy set (FS) theory, it actually became much easier to solve problems in the real world since it clarified and improved the explanation of fuzziness and faults. The ability to understand confusion emerging in various materialistic conditions is now a widely accepted concept. Fuzzy logic is a statement that may be right or wrong, or even have a value between them. It is intended to deal with the perception of partial truth. The membership function characterized the degree of truth in this logic. A membership function for a set W is any map from W to the unit interval. The value 0 denotes false, the value 1 denotes truth, and the value between 0 and 1 denotes partial truth. FS theory is applied to fuzzy control systems as it is a modest and natural addition to the ordinary set theory.

In 1922, Banach [1] generated an important result in the metric fixed point (FP) theory. For existence and uniqueness theorems in various disciplines of analysis, the Banach contraction principle is a key source and is widely regarded as the influential research of modern exploration. It guarantees the uniqueness of FP while also ensuring its existence. Banach's contraction theorem was expanded in 1969 by Nadler [2], who also demonstrated an FP theorem for multivalued contractions.

The search of FP for multi-valued mapping (MVM) was mainly started by von Neumann [3]. The improvement in FP theory for MVM was basically originated with the work of Nadler [2]. He connected the idea of MVM and Lipschitz mapping and used the perception of Hausdorff metric (HM) to settle the multivalued contraction principal, generally known as Nadler's contraction mapping principle. Many investigations were conducted into the generalizations of the concept of Nadler's contraction mapping principle (see [4–10] and references therein).

The intensification and exploration of many advanced areas such as robotics, artificial intelligence, general system theory and language theory compelled us to participate in the enumeration of ambiguous concepts. In 1965, Zadeh [11] was the first to mention the concept of fuzzy logic. The affiliation of an element to the set in the theory of fuzzy logic is given as a number from the interval $[0,1]$, unlike in the theory of classical logic, where an element either belongs to the set or not. Zadeh has been studying the theory of FS to address the issue of indeterminacy because uncertainty is a crucial component of a genuine problem. Heilpern [12] introduced the theory of fuzzy mappings (FM) and established a theorem on fixed points for FM of contractions in metric linear space, which serves as a fuzzy generalization of Banach's contraction principle. This sparked the interest of numerous authors to investigate various contraction conditions using FM. FP theory produces valuable results which are very useful in solving optimization problems and physics. FP theorems in fuzzy mathematics are gaining popularity as a source of hope and vital confidence. Weiss [13] and Butnariu [14] were the first who interpret the FP and fuzzy mappings.

The idea of b-metric space (b-MS) was first presented by Backhtin [15]. Czerwik [16] extracted the b-MS results in 1993. Many scholars generalized the Banach contractive principle in b-MS by embracing this theory. The existence of FP and common FP of FM satisfying the contractive type criterion is deduced and estimated by several authors (see [17–20] and references therein). Many authors established wonderful results regarding fixed points and common fixed points for fuzzy and non-fuzzy mappings in b-metric spaces and in its various generalizations for example see ([21–26] and references therein).

The structure of paper is as follows:

In Section 2, some necessary concepts are recalled to facilitate the readers. All these prerequisites are collected from previous research articles exist in the literature. Section 3 deals with some theoretical results. In this study, we have established fuzzy FP results of set-valued FM satisfying generalized contractions of Nadler's type in the setting of complete b-metric spaces. The obtained results are furnished with applications. Previous results are given in the form of corollaries of obtained results.

2. Preliminaries

The motivation behind this section is to facilitate the readers to have comprehensive knowledge about the fundamental definitions, examples and lemmas that are necessary to understand our established results. All these essentials are collected from previous research articles exist in the literature.

2.1. b-Metric space [15]

Let Ω be any non-empty set and $s \geq 1$ be any real number. A function $d: \Omega \times \Omega \to \mathbb{R}^+ \cup \{0\}$ is called a b-MS, if it holds these conditions for all $\omega, \xi, \eta, \in \Omega$:

(1)$d(\omega, \xi) = 0 \; if \; and \; only \; if \; \omega = \xi$;

(2)$d(\omega, \xi) = d(\xi, \omega)$;

(3)$d(\omega, \eta) \leq s[d(\omega, \xi) + d(\xi, \eta)]$.

Then, (Ω, d, s) is called as a b-MS.

Example 1. The $l_p(\Omega)$ with $0 < p < 1$, where $l_p(\Omega) = \{\{\mu_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |\mu_n|^p < \infty\}$, together with the function $d: l_p(\Omega) \times l_p(\Omega) \to [0, \infty)$ defined as:

$$d(\mu, \omega) = \left(\sum_{n=1}^{\infty} |\mu_n - \omega_n|^p\right)^{1/p},$$

where $\mu = \{\mu_n\}, \omega = \{\omega_n\} \in l_p(\Omega)$ is a b-MS with $s = 2^{\frac{1}{p}} > 1$. Notice that the above-mentioned results hold with $0 < p < 1$, where Ω is a b-MS.

Example 2. Let $\Omega = \{g, h, \kappa\}$ and $d(g, h) = d(h, g) = d(h, \kappa) = d(\kappa, h) = 1 \; and \; d(g, \kappa) = d(\kappa, g) = m \geq 2$. Then,

$$d(\mu, \omega) \leq \frac{m}{2}[d(\mu, \upsilon) + d(\upsilon, \omega)],$$

$for \; all \; \mu, \upsilon, \omega \in \Omega$. Now $if \; m > 2$, then (Ω, d, s) is b-metric space with $s=m/2$ while ordinary triangle inequality is not satisfied for $m > 2$, for example if $m=3$, then $d(g, k) \nleq d(g, h) + d(h, k), i.e., 3 \nleq 1 + 1$.

2.2. Completeness in b-metric space

Let (Ω, d, s) be a b-MS and $\{\omega_n\}$ be a sequence in Ω. Then

(1) $\{\omega_n\}$ is called a convergent sequence iff $\exists \, z \in \Omega$, such that for all $\varepsilon > 0 \; \exists \, n(\varepsilon) \in \mathbb{N}$ s.t. for all $n \geq n(\varepsilon)$, we have $d(\omega_n, z) < \varepsilon$. Then, we can write $\lim_{n\to\infty} \omega_n = z$.

(2) $\{\omega_n\}$ is said to be a CS iff for all $\varepsilon > 0 \; \exists \, n(\varepsilon) \in \mathbb{N}$ s.t $\forall \, m, n \geq n(\varepsilon)$, we have $d(\omega_n, \omega) < \varepsilon$.

Ω is called complete if every CS in Ω is convergent in it.

Note: $CB(\Omega)$ denotes the family of closed and bounded subsets of metric space Ω.

2.3. Hausdorff metric space [2]

Let (Ω, d) be a MS. We define the Hausdorff metric on $CB(\Omega)$ induced by d as:

$$H(A, B) = max\{\sup_{\mu \epsilon A} d(\mu, B), \sup_{\nu \epsilon B} d(A, \nu)\},$$

for all $A, B \in CB(\Omega)$. Here $d(\mu, B) = inf\{d(\mu, \eta): \eta \in B\}$.

Lemma 1. Let $G, \; K \in CB(\Omega)$. If $\mu \in G$ then, $d(\mu, K) \leq H(G, K)$, for all $\mu \in G$.

Lemma 2. Let $P, \; Q \in CB(\Omega)$ and $0 < \sigma \in \mathbb{R}$. Then, for all $i \in P$, there exists $\zeta \in Q$ such that

$d\,(i,\zeta) \le H\,(P,Q) + \sigma$.

Lemma 3. If $P, Q \in CB(\Omega)$ with $H\,(P,Q) < \varepsilon$, then for all $\mu \in P$ there exists $\upsilon \in Q$ such that $d\,(\mu,\upsilon) < \varepsilon$.

Lemma 4. [13] If $P \in CB\{\Omega\}$ then $d(\mu,P) \le d\,(\mu,\nu)$, for all $\nu \in P$.

Lemma 5. [5] Let (Ω,d,s) be a b-MS. For any $P,Q,R \in CB(\Omega)$ and any $\mu,\upsilon \in \Omega$, we have following properties:

(1) $d(\mu,Q) \le H(P,Q)$ for all $\mu \in P$;
(2) $\delta(P,Q) \le H(P,Q)$; where $\delta(P,Q) = \inf\{d(\mu,\nu): \mu \,\epsilon\, P \text{ and } \nu \,\epsilon\, Q\}$;
(3) $H(P,P) = 0$;
(4) $H(P,Q) = H(Q,P)$;
(5) $H(P,R) \le s[H(P,Q) + H(Q,R)]$;
(6) $d(\mu,P) \le s[d(\mu,\upsilon) + d(\upsilon,P)$.

Lemma 6. [5] Let (Ω,d,s) be a b-MS. For $Z \in CB(\Omega)$ and $\mu \in \Omega$, we have $d(\mu,Z) = 0$ iff $\mu \in Cl\,(Z) = Z$, where $Cl(Z)$ is the closure of the set Z in Ω.

Lemma 7. [27] Consider $d_2 \le \lambda\, max\{d_1,d_2\} + \xi s(d_1 + d_2)$, where $\lambda,\xi,s,d_1$and d_2 are non-negative reals such that $\lambda + 2\xi s < 1$. Then we have,

$$d_2 \le \max\{\frac{\xi s}{1-(\lambda+\xi s)}, \frac{\lambda+s}{1-\xi s}\}d_1.$$

2.4. Closed mapping [28]

Let (Ω,d,s) be a b-MS. A mapping $G: \Omega \to CB(\Omega)$ is called closed if for all sequences $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ of elements from Ω and $\eta,\zeta \in \Omega$ such that $\lim_{n\to\infty} \eta_n = \eta$, $\lim_{n\to\infty} \zeta_n = \zeta$ and $\zeta_n \in G(\zeta_n)$ for all $n \in \mathbb{N}$, we have $\zeta \in G(\zeta)$.

2.5. ∗-Continuity of b-metric [28]

Let (Ω,d,s) be a b-MS. The b-metric d is called ∗-continuous if for all $A \in CB(\Omega)$, $\eta \in \Omega$ and each sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of elements from Ω such that $\lim_{n\to\infty} \eta_n = \eta$, we have $\lim_{n\to\infty} d(\eta_n, A) = d(\eta, A)$.

Lemma 8. [29] Every sequence $\{v_n\}$ of a b-MS (Ω,d,s), having the property that there exists $\mu \in [0,1)$ such that $d(v_{n+1},v_n) \le \mu d(v_n,v_{n-1})$, for all $n \in \mathbb{N} \cup \{0\}$, is Cauchy.

2.6. Fuzzy set [11]

Let Ω be a universal set. A mapping $G: \Omega \to [0,1]$ is called a fuzzy set in Ω. The value $G(u)$ of G at $u \in \Omega$ stands for the degree of membership of u in G. The set of all fuzzy sets in Ω will be denoted by $F(\Omega)$.

$G(u) = 1$ means full membership, $G(u) = 0$ means no membership and intermediate values between 0 and 1 mean partial membership.

Example 3. Let *A* denotes the old and *B* denotes the young and Ψ *= [0, 100].* Then *A* and *B* both are fuzzy sets that are defined by

$$A(x) = \begin{cases} \left[1 + \left(\frac{x - 50}{5}\right)^{-2}\right]^{-1}, & if\ 50 < x \leq 100 \\ 0, & otherwise \end{cases}$$

$$B(x) = \begin{cases} \left[1 + \left(\frac{x - 25}{5}\right)^{2}\right]^{-1}, & if\ 25 < x \leq 100 \\ 0, & otherwise. \end{cases}$$

The α-level set of G is denoted by $[G]_\alpha$ and defined as

$$[G]_{\alpha} = \{u \in \Omega: G(u) \geq \alpha; \alpha \in (0,1]\}.$$

2.7. Fuzzy mapping [30]

Let Ψ_1 be any set and Ψ_2 be a metric space. A function $g: \Psi_1 \to F(\Psi_2)$ is called an FM. An FM g is an FS on $\Psi_1 \times \Psi_2$ with membership function $g(x)(y)$. The image $g(x)(y)$ is the grade of membership of y in $g(x)$.

2.8. Fuzzy fixed point [17]

Suppose (Ψ, d) is a MS and $T: \Psi \to F(\Psi)$. A point $z \in \Psi$ is a fuzzy FP of T if $z \in [Tz]_\alpha$ for some $\alpha \in (0,1]$.

2.9. Nadler type contraction for fuzzy mappings

Let (Ω, d, s) be a complete b-MS and $G: \Omega \to F(\Omega)$ be an FM. For all $u, v \in \Omega$, then,

$$H([G(u)]_\alpha, [G(v)]_\alpha) \leq N_{a,k,b,c}(u, v), \text{ for all } u, v \in \Omega,$$

where,

$$N_{a,k,b,c}(u,v) = a \max\{d(u,v), kd(u,[G(u)]_\alpha), kd(v,[G(v)]_\alpha\} + bd(u,[G(v)]_\alpha) + cd(v,[G(u)]_\alpha),$$

where $a, b, c \geq 0, k \in [0,1], a + 2\,s\min\{b, c\} < 1, \alpha \in (0,1]$, and $[G(u)]_\alpha, [G(v)]_\alpha \in CB\ (\Omega)$.

3. Fixed points of fuzzy mappings for Nadler's type contractions in b-metric spaces

In the framework on b-metric spaces some existence theorems regarding fuzzy fixed points of fuzzy mappings satisfying various types of contraction conditions are established and some results for multi-valued mappings are incorporated. Moreover, other direct consequences are obtained as well. These existence results will provide an appropriate environment to approximate operator equations in applied sciences.

The following theorem guaranties the existence of fixed points of fuzzy contractive mapping defined on a complete b-MS which satisfies any one of the conditions:

G is closed; d is $*$ continuous; $s(ak + min\{b, c\}) < 1$.

Theorem 1. Let (Ω, d, s) be a complete b-MS and $G: \Omega \to F(\Omega)$ be a mapping satisfying

$$H([G(u)]_\alpha, [G(v)]_\alpha) \leq N_{a,k,b,c}(u,v), \; u, v \in \Omega, \tag{1}$$

where,

$$N_{a,k,b,c}(u,v) = a \max\{d(u,v), kd(u, [G(u)]_\alpha), kd(v, [G(v)]_\alpha\} + bd(u, [G(v)]_\alpha) + cd(v, [G(u)]_\alpha),$$

for all $a, b, c, k \geq 0$, such that $k \in [0,1], a + 2s \min\{b, c\} < 1, \alpha \in (0,1]$, and $[G(u)]_\alpha, [G(v)]_\alpha$ *are in CB (Ω)*. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in Ω converges to some point $u^* \in \Omega$ such that $u_{n+1} \in [G(u_n)]_\alpha$, *for every* $n \in \mathbb{N}$. Also, u^* is an FP of G if one of the following conditions is satisfied:

(1) G is closed;
(2) d is $*$ continuous;
(3) $s(ak + \min\{b, c\}) < 1$.

Proof. Let $u_0 \in \Omega$. Choose $u_1 \in [G(u_0)]_\alpha$. Let

$$\varepsilon = \frac{1-q}{1+q} H([G(u_1)]_\alpha, [G(u_0)]_\alpha),$$

where $q = a + 2s \min\{b, c\}$. If,
$H([G(u_1)]_\alpha, [G(u_0)]_\alpha) = 0$ then we obtain $[G(u_1)]_\alpha = [G(u_0)]_\alpha$ and $u_1 \in [G(u_1)]_\alpha$.
In this case the proof is completed.
So, we may assume $\varepsilon > 0$. By Lemma 2, there is a point $u_2 \in [G(u_1)]_\alpha$ such that

$$d(u_1, u_2) \leq H([G(u_0)]_\alpha, [G(u_1)]_\alpha) + \varepsilon = \frac{2}{1+q} H([G(u_0)]_\alpha, [G(u_1)]_\alpha).$$

Similarly, there is a point $u_3 \in [G(u_2)]_\alpha$ such that

$$d(u_2, u_3) \leq H([G(u_1)]_\alpha, [G(u_2)]_\alpha) + \varepsilon,$$

where,

$$\varepsilon = \frac{1-q}{1+q} H([G(u_2)]_\alpha, [G(u_1)]_\alpha).$$

If $H([G(u_2)]_\alpha, [G(u_1)]_\alpha) = 0$. Then we infer that

$$[G(u_2)]_\alpha = [G(u_1)]_\alpha.$$

In this case the proof is completed.
So, we may assume that $\varepsilon > 0$. Hence

$$d(u_2, u_3) \leq \frac{2}{1+q} H([G(u_1)]_\alpha, [G(u_2)]_\alpha).$$

Continuing in this process, we construct a sequence $\{u_n\}_{n\in\mathbb{N}}$ of points of Ω such that

$$u_{n+1} \in [G(u_n)]_\alpha, \forall n \in \mathbb{N} \cup \{0\}, \tag{2}$$

$$d(u_n, u_{n+1}) \leq \tfrac{2}{1+q} H([G(u_{n-1})]_\alpha, [G(u_n)]_\alpha), \forall n \in \mathbb{N}. \tag{3}$$

Case 1.
Consider that $c = min\{b, c\}$.
Using given contraction in (1) we get

$$H([G(u_n)]_\alpha, [G(u_{n-1})]_\alpha) \leq a\, max \begin{Bmatrix} d(u_n, u_{n-1}), kd(u_n, [G(u_n)]_\alpha), \\ kd(u_{n-1}, [G(u_{n-1})]_\alpha) \end{Bmatrix}$$

$$+bd(u_n, [G(u_{n-1})]_\alpha) + cd(u_{n-1}, [G(u_n)]_\alpha).$$

$$H([G(u_n)]_\alpha, [G(u_{n-1})]_\alpha) \leq a\, max\{d(u_n, u_{n-1}), kd(u_n, u_{n+1}), kd(u_{n-1}, u_n)\}$$

$$+bd(u_n, u_n) + cd(u_{n-1}, u_{n+1}).$$

$$H([G(u_n)]_\alpha, [G(u_{n-1})]_\alpha) \leq a\, max\{d(u_n, u_{n-1}), kd(u_n, u_{n+1}),\} + cd(u_{n-1}, u_{n+1}),$$

$$H([G(u_n)]_\alpha, [G(u_{n-1})]_\alpha) \leq a\, max\{d(u_n, u_{n-1}), kd(u_n, u_{n+1}),\}$$

$$+cs[d(u_{n-1}, u_n) + d(u_n, u_{n+1})].$$

Using (3) we get

$$d(u_n, u_{n+1}) \leq \frac{2}{1+q}[a\, max\{d(u_n, u_{n-1}), kd(u_n, u_{n+1})\} + cs\big(d(u_{n-1}, u_n) + d(u_n, u_{n+1})\big)].$$

Now, since

$$\frac{2}{1+q}(a + 2cs) = \frac{2q}{1+q} < 1,$$

using Lemma 7, we get

$$d(u_{n+1}, u_n) \leq q_1 d(u_n, u_{n-1}), \tag{4}$$

where $q_1 = \max\{\frac{a+q}{a+1}, \frac{q-a}{1-a}\} < 1$.

Case 2.
Consider that $b = min\{b, c\}$.
Using case 1, we get

$$H([G(u_{n-1})]_\alpha, [G(u_n)]_\alpha) \leq a\, max\{d(u_{n-1}, u_n), kd(u_n, u_{n+1})\} + bs[d(u_{n-1}, u_n) + d(u_n, u_{n+1})].$$

Using (3), we get

$$d(u_n, u_{n+1}) \leq \frac{2}{1+q}[amax\{d(u_{n-1}, u_n), kd(u_n, u_{n+1})\} + bs\big(d(u_{n-1}, u_n) + d(u_n, u_{n+1})\big)].$$

We know that

$$\frac{2}{q+1}(2bs + a) = \frac{2q}{q+1} < 1,$$

using Lemma 7, we get

$$d(u_{n+1}, u_n) \leq q_2 d(u_n, u_{n-1}), \tag{5}$$

where $q_2 = \max\{\frac{a+q}{a+1}, \frac{q-a}{1-a}\} < 1.$

As $q_1 < 1$ *and* $q_2 < 1,$ using Lemma 8 together with (4) and (5) we get the sequence $\{u_n\}_{n\in\mathbb{N}}$ is a CS. Since (Ω, d, s) is a complete b-MS, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to some point $u^* \in \Omega$.

(1) Suppose that G is closed.

Using the definition of a closed mapping and (2) we get

$$u^* \in [G(u^*)]_\alpha.$$

(2) Consider that d is $*$ continuous.

Then, we can write

$$\lim_{n\to\infty} d(u_n, [G(u^*)]_\alpha) = d(u^*, [G(u^*)]_\alpha), where\ [G(u^*)]_\alpha \in CB(\Omega). \tag{6}$$

Using Lemma 5 and (2) we get

$$\begin{aligned} d(u_{n+1}, [G(u^*)]_\alpha) &\leq H([G(u_n)]_\alpha, [G(u^*)]_\alpha) \\ &\leq a\, max\{d(u_n, u^*), kd(u_n, [G(u_n)]_\alpha), kd(u^*, [G(u^*)]_\alpha)\} \\ &\quad + bd(u_n, [G(u^*)]_\alpha) + cd(u^*, u_{n+1}) \\ &\leq a\, max\{d(u_n, u^*), kd(u_n, u_{n+1}), kd(u^*, [G(u^*)]_\alpha)\} \\ &\quad + bd(u_n, [G(u^*)]_\alpha) + cd(u^*, u_{n+1}), \end{aligned}$$

and

$$\begin{aligned} d([G(u^*)]_\alpha, u_{n+1}) &\leq H([G(u^*)]_\alpha, [G(u_n)]_\alpha) \\ &\leq a\, max\{d(u^*, u_n), kd(u_n, [G(u_n)]_\alpha), kd(u^*, [G(u^*)]_\alpha)\} \\ &\quad + cd(u_n, [G(u^*)]_\alpha) + bd(u^*, [G(u_n)]_\alpha) \\ &\leq a\, max\{d(u^*, u_n), kd(u_n, u_{n+1}), kd(u^*, [G(u^*)]_\alpha)\} \\ &\quad + cd(u_n, [G(u^*)]_\alpha) + bd(u^*, u_{n+1}). \end{aligned}$$

Using (6) we get

$$d(u^*, [G(u^*)]_\alpha) \leq (ak + b) d(u^*, [G(u^*)]_\alpha), \tag{7}$$

$$d([G(u^*)], u^*) \leq (ak + c) d([G(u^*)], u^*). \tag{8}$$

Since,

$$ak + b < a + 2s \min\{b, c\}, or, ak + c < a + 2s \min\{b, c\},$$

we conclude that $d(u^*, [G(u^*)]_\alpha) = 0$ and from Lemma 6, we get $u^* \in [G(u^*)]_\alpha$.

(3) Suppose $s(ak + \min\{b, c\}) < 1$.

Case (a)

Let *min{b, c}= b* then $s(ak + b) < 1$. Since,

$$d(u_{n+1}, [G(u^*)]_\alpha) \leq a\, max\{d(u_n, u^*), kd(u_n, u_{n+1}), kd(u^*, [G(u^*)]_\alpha)\}$$

$$+bd(u_n, [G(u^*)]_\alpha) + cd(u^*, u_{n+1}), \tag{9}$$

and,

$$d(u_n, [G(u^*)]_\alpha) \leq s[d(u_n, u_{n+1}) + d(u_{n+1}, [G(u^*)]_\alpha), \tag{10}$$

so, we have

$$(1 - bs)d(u_{n+1}, [G(u^*)]_\alpha) \leq a\, max\{d(u_n, u^*), kd(u_n, u_{n+1}), kd(u^*, [G(u^*)]_\alpha)\}$$
$$+ bsd(u_n, u_{n+1}) + cd(u^*, u_{n+1}). \tag{11}$$

As, we know that $1 - bs > 0$, inequalities (9), (10) and using triangular inequality, we get

$$d(u^*, [G(u^*)]_\alpha)$$
$$\leq sd(u^*, u_{n+1})$$
$$+ \frac{s}{1 - sb}[a\, max\{d(u_n, u^*), kd(u_n, u_{n+1}), kd(u^*, [G(u^*)]_\alpha)\} + bsd(u_n, u_{n+1})$$
$$+ cd(u^*, u_{n+1})].$$

Taking $\lim_{n\to\infty} we\ get$

$$d(u^*, [G(u^*)]_\alpha) \leq \left(\frac{ksa}{1-bs}\right) d(u^*, [G(u^*)]_\alpha). \tag{12}$$

As, we know that $\left(\frac{ksa}{1-bs}\right) < 1$ and from (12), we get

$$d(u^*, [G(u^*)]_\alpha) = 0, i.e.,$$
$$u^* \in [G(u^*)]_\alpha.$$

Case (b)

Consider *min {b, c} =c,* then $s(ak + c) < 1$. Then proof will be completed similar as in Case (a). Hence from all above cases we conclude that there exists a fixed point of *G* if any one condition from (1)–(3) is satisfied.

Application

Theorem 2. (Fixed points of multi-valued mappings)

Let (Ω, d, s) be a complete b-MS and $A: \Omega \to CB(\Omega)$ be a multi-valued-valued mapping satisfying

$$H(A(u), A(v)) \leq N_{a,k,b,c}(u, v), for\ all\ u, v \in \Omega,$$

where,

$$N_{a,k,b,c}(u, v) = a \max\{d(u, v), kd(u, A(u)), kd(v, A(v)\} + bd(u, A(v)) + cd(v, A(u)),$$

for all $a, b, c, k \geq 0$, such that $k \in [0,1]$ and $a + 2s \min\{b, c\} < 1$.
Then there exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ in Ω converges to some point $u^* \in \Omega$ such that $u_{n+1} \in A(u_n), for\ every\ n \in \mathbb{N}$. Also, u^* is an FP of $'A'$ if one of the following conditions is

satisfied:

(1) A is closed;

(2) d is $*$ continuous;

(3) $s(ak+\min\{b,c\})<1$.

Proof. Consider an arbitrary mapping $S:\Omega\longrightarrow(0,1]$ and a fuzzy mapping $G:\Omega\rightarrow F(\Omega)$ defined by

$$G(x)(t)=\begin{cases}Sx & t\in Ax\\ 0 & t\notin Ax.\end{cases}$$

Then for $x\in\Omega$,

$$[Gx]_\alpha=\{t:G(x)(t)\geq\alpha\}=Ax.$$

Therefore, Theorem 1 can be applied to obtain $u^*\in\Omega$ such that $u^*\in[Gu^*]_\alpha$=Au^*.

Corollary 1. Let (Ω,d) be a complete MS and $G:\Omega\rightarrow F(\Omega)$ be a fuzzy mapping satisfying

$$H([G(u)]_\alpha,[G(v)]_\alpha)\leq N_{a,k,b,c}(u,v),\ u,v\in\Omega,$$

where,

$$N_{a,k,b,c}(u,v)=a\max\{d(u,v),kd(u,[G(u)]_\alpha),kd(v,[G(v)]_\alpha\}+bd(u,[G(v)]_\alpha)+cd(v,[G(u)]_\alpha),$$

for all $a,b,c,k\geq0$, such that $k\in[0,1],a+2\min\{b,c\}<1,\alpha\in(0,1]$, and $[G(u)]_\alpha,[G(v)]_\alpha\ in$ *CB(Ω)*. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in Ω converges to some point $u^*\in\Omega$ such that $u_{n+1}\in[G(u_n)]_\alpha$, for every $n\in\mathbb{N}$. Also, u^* is an FP of G if one of the following conditions is satisfied:

(1) G is closed;

(2) d is $*$ continuous;

(3) $s(ak+\min\{b,c\})<1$.

Theorem 3. Consider (Ω,d,s) be a complete b-MS. Assume that a fuzzy mapping $G:\Omega\rightarrow F(\Omega)$ having the property that there exist $c,d\in[0,1]$ and $\rho\in[0,1)$ such that:

(i) ρds < 1;

(ii) $H([G(u)]_\alpha,[G(v)]_\alpha)\leq\rho N_{c,d}(u,v) for\ all\ u,v\in\Omega,$ where $\alpha\in(0,1],[G(u)]_\alpha\ and\ [G(v)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(u,v)=\max\{d(u,v),cd(u,[G(u)]_\alpha),cd(v,[G(v)]_\alpha),\tfrac{d}{2}(d(u,[G(v)]_\alpha)+d(v,[G(u)]_\alpha))\}.$$

Then, for every $u_0\in\Omega$, there exists $\gamma\in[0,1)$ and a sequence $\{u_n\}$ of elements from Ω such that

(a) $u_{n+1}\in[G(u_n)]_\alpha\ for\ every\ n\in\mathbb{N}$;

(b) $d(u_{n+1},u_n)\leq\gamma d(u_n,u_{n-1})\ for\ every\ n\in\ \mathbb{N}$;

(c) $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy.

Proof. Consider $\beta\in(\rho,min(1,\frac{1}{sd})),\gamma=\ max\{\beta,\frac{sd\beta}{2-sd\beta}\}<1,u_0\in\Omega\ and\ u_1\in\ [G(u_0)]_\alpha$.

If $u_1=u_0$ then the sequence $\{u_n\}_{n\in\mathbb{N}}$ given by $u_n=u_0$ for every $n\in\mathbb{N}$ satisfying (a)–(c). Since, using (ii), we can write

$$d(u_1,[G(u_1)]_\alpha)\leq H([G(u_0)]_\alpha,[G(u_1)]_\alpha\leq\rho N_{c,d}(u_0,u_1)<\beta N_{c,d}(u_0,u_1),$$

there exists $u_2 \in [G(u_1)]_\alpha$ such that

$$d(u_1, u_2) < \beta N_{c,d}(u_0, u_1).$$

If $u_1 = u_2$ then the sequence $\{u_n\}_{n\in\mathbb{N}}$ given by $u_n = u_1$ for every $n \in \mathbb{N}$ satisfying (a), (b) and (c).
By continuing this process we get a sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements from Ω such that

$$u_{n+1} \in [G(u_n)]_\alpha \text{and } 0 < d_n < \beta N_{c,d}(u_{n-1}, u_n), for\ every\ n \in \mathbb{N}, n \geq 1.$$

Because

$$d(u_{n-1}, [G(u_{n-1})]_\alpha) \leq d_{n-1},$$

$$d(u_n, [G(u_n)]_\alpha) \leq d_n,$$

$$d(u_{n-1}, [G(u_n)]_\alpha) \leq d(u_{n-1}, u_{n+1}),$$

$$d(u_n, [G(u_{n-1})]_\alpha) = 0,$$

so, we have

$$0 < d_n < \beta N_{c,d}(u_{n-1}, u_n),$$

$$\beta N_{c,d}(u_{n-1}, u_n) \leq \beta \max\left\{d_{n-1}, cd_n, cd_{n-1}, \frac{d}{2} d(u_{n-1}, u_{n+1})\right\}$$

$$\leq \beta \max\left\{d_{n-1}, cd_n, cd_{n-1}, \frac{ds}{2} d(d_{n-1} + d_n)\right\}$$

$$\leq \beta \max\left\{d_{n-1}, \frac{ds}{2} d(d_{n-1} + d_n)\right\},$$

for every $n \in \mathbb{N}$, where the explanation of the last inequality is stated as:
if by reduction and absurdum, $\max\left\{d_{n-1}, cd_n, cd_{n-1}, \frac{ds}{2} d(d_{n-1} + d_n)\right\} = cd_n$, then
we required that

$$0 < d_n < \beta c d_n \leq \beta d_n.$$

So, we get the contradiction $1 < \beta$.
Consequently,

$$d_n < \beta d_{n-1} \text{ or } d_n < \beta \frac{ds}{2} d(d_{n-1} + d_n),$$

$$\text{i.e., } d_n < \beta d_{n-1} \text{ or } d_n < \frac{s\beta d}{2 - s\beta d} d_{n-1}, \forall\, n \in \mathbb{N}.$$

Hence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfied (a) and (b).
By using Lemma 8, we conclude that it also satisfied (c).

Application

Theorem 4. Consider (Ω, d, s) be a complete b-MS. Let a multi-valued mapping $A: \Omega \rightarrow CB(\Omega)$ having the property that there exist $c, d \in [0,1]$ and $\rho \in [0,1)$ such that:

(1) ρds < 1;
(2) $H(A(u), A(v)) \leq \rho N_{c,d}(u, v) for\ all\ u, v \in \Omega,$

$$N_{c,d}(u, v) = \max\{d(u, v), cd(u, A(u)), cd(v, A(v)), \tfrac{d}{2}(d(u, A(v)) + d(v, A(u)))\}.$$

Then, for every $u_0 \in \Omega$, there exist $\gamma \in [0,1)$ and a sequence $\{u_n\}$ of elements from Ω such that

(a) $u_{n+1} \in A(u_n)\ for\ every\ n \in \mathbb{N}$;
(b) $d(u_{n+1}, u_n) \leq \gamma d(u_n, u_{n-1})\ for\ every\ n \in \mathbb{N}$;
(c) $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy.

Proof. Consider an arbitrary mapping $S: \Omega \longrightarrow (0, 1]$ and a fuzzy mapping $G: \Omega \rightarrow F(\Omega)$ defined by

$$G(x)(t) = \begin{cases} Sx & t \in Ax \\ 0 & t \notin Ax. \end{cases}$$

Then for $x \in \Omega$,

$$[Gx]_\alpha = \{t: G(x)(t) \geq \alpha\} = Ax.$$

Therefore, Theorem 3 can be applied to obtain the required sequence in Ω and $\in [0,1)$.

Corollary 2. Consider (Ω, d, s) be a complete MS. Let a fuzzy mapping $G: \Omega \rightarrow F(\Omega)$ having the property that there exist $c, d \in [0,1]$ and $\rho \in [0,1)$ such that:

(1) ρds < 1;
(2) $H([G(u)]_\alpha, [G(v)]_\alpha) \leq \rho N_{c,d}(u, v) for\ all\ u, v \in \Omega,$ where $\alpha \in (0,1], [G(u)]_\alpha\ and\ [G(v)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(u, v) = \max\{d(u, v), cd(u, [G(u)]_\alpha), cd(v, [G(v)]_\alpha), \tfrac{d}{2}(d(u, [G(v)]_\alpha) + d(v, [G(u)]_\alpha))\}.$$

Then, for every $u_0 \in \Omega$, there exist $\gamma \in [0,1)$ and a sequence $\{u_n\}$ of elements from Ω such that

(a) $u_{n+1} \in [G(u_n)]_\alpha$ for every $n \in \mathbb{N}$;
(b) $d(u_{n+1}, u_n) \leq \gamma d(u_n, u_{n-1})$ for every $n \in \mathbb{N}$;
(c) $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy.

Theorem 5. Consider a complete b-MS (Ω, d, s). A fuzzy mapping $G: \Omega \rightarrow F(\Omega)$ has an FP provided that it satisfied the following conditions.

(1) $d\ is\ a * continous$;
(2) there exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that $H([G(r_1)]_\alpha), [G(r_2)]_\alpha) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega,$ where $G(r_1)]_\alpha\ and\ [G(r_2)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(r_1, r_2) = max\left\{d(r_1, r_2), cd(r_1, [G(r_1)]_\alpha), cd(r_2, [G(r_2)]_\alpha), \frac{d}{2}\big(d(r_1, [G(r_2)]_\alpha) + d(r_2, [G(r_1)]_\alpha)\big)\right\};$$

(3) $\mu sd < 1$.

Proof. By Theorem 3 part (i) and part (ii) there exists a CS $\{r_n\}$ of elements from Ω such that

$$r_{n+1} \in [G(r_n)]_\alpha, \text{for all n} \in \mathbb{N} \tag{13}$$

As the b-MS (Ω, d, s) is complete, so there exists $w \in \Omega$ such that $\lim_{n\to\infty} r_n = w$.
Then, from [2] and (13), with notation $d(r_n, w) = \delta_n$, we can write

$$d(r_{n+1}, [G(w)]_\alpha) \leq H([G(r_n)]_\alpha, [G(w)]_\alpha) \leq \mu N_{c,d}(r_n, w)$$

$$= \mu \max\left\{\delta_n, cd(r_n, [G(r_n)]_\alpha), cd(w, [G(w)]_\alpha), \frac{d}{2}\left(d(r_n, [G(w)]_\alpha) + d(w, [G(r_n)]_\alpha)\right)\right\}$$

$$\leq \mu \max\left\{\delta_n, cd_n, cd(w, [G(w)]_\alpha), \frac{d}{2}\left(s(\delta_n + d(w, [G(w)]_\alpha)\right) + \delta_{n+1}))\right\}, \tag{14}$$

for all $n \in \mathbb{N}$. As $d\ is * $ continous and $\lim_{n\to\infty} r_{n+1} = w$, so we have

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d_n = 0 \text{ and } \lim_{n\to\infty} d(r_{n+1}, [G(w)]_\alpha) = d(w, [G(w)]_\alpha).$$

Now by applying $\lim_{n\to\infty} in$ (14), we get

$$d(w, [G(w)]_\alpha) \leq \max\left\{\mu c, \frac{\mu sd}{2}\right\} d(w, [G(w)]_\alpha). \tag{15}$$

As $\max\left\{\mu c, \frac{\mu sd}{2}\right\} < 1$, by [3] and using (15) we conclude that
$d(w, [G(w)]_\alpha) = 0, i.e.,$ $w \in [G(w)]_\alpha$. Hence G has a FP.

Application

Theorem 6. Consider a complete b-MS (Ω, d, s). A multi-valued mapping $A: \Omega \to CB(\Omega)$ has an FP provided that it satisfied the following conditions.

(1) $d\ is\ a * continous$;

(2) there exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that
$H(A(r_1), A(r_2)) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega$, where

$$N_{c,d}(r_1, r_2) = max\left\{d(r_1, r_2), cd(r_1, A(r_1)), cd(r_2, A(r_2)), \frac{d}{2}\left(d(r_1, A(r_2)) + d(r_2, A(r_1))\right)\right\};$$

(3) $\mu sd < 1$.

Proof. Consider an arbitrary mapping $S: \Omega \longrightarrow (0, 1]$ and a fuzzy mapping $G: \Omega \to F(\Omega)$ defined by

$$G(x)(t) = \begin{cases} Sx & t \in Ax \\ 0 & t \notin Ax. \end{cases}$$

Then for $x \in \Omega$,

$$[Gx]_\alpha = \{t: G(x)(t) \geq \alpha\} = Ax.$$

Therefore, Theorem 5 can be applied to obtain $w \in \Omega$ such that $w \in [Gw]_\alpha = Aw$

Corollary 3. Consider a complete b-MS (Ω, d, s). A fuzzy mapping $G: \Omega \rightarrow F(\Omega)$ has an FP provided that it satisfied the following conditions.

(1) $d\ is\ a * continous;$

(2) there exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that
$H([G(r_1)]_\alpha), [G(r_2)]_\alpha) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega,$ where $\alpha \in (0,1]$, $[G(r_1)]_\alpha\ and\ [G(r_2)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(r_1, r_2) = max\left\{d(r_1, r_2), cd(r_1, [G(r_1)]_\alpha), cd(r_2, [G(r_2)]_\alpha), \frac{d}{2}\big(d(r_1, [G(r_2)]_\alpha) + d(r_2, [G(r_1)]_\alpha)\big)\right\};$$

(3) $\mu sd < 1$.

Theorem 7. Consider that (Ω, d, s) be a complete b-MS. A fuzzy mapping $G: \Omega \rightarrow F(\Omega)$ has an FP provided that it satisfied the following two conditions.

a) There exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that
$H([G(r_1)]_\alpha), [G(r_2)]_\alpha) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega,$ where $\alpha \in (0,1]$, $[G(r_1)]_\alpha\ and\ [G(r_2)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(r_1, r_2) = max\left\{d(r_1, r_2), cd(r_1, [G(r_1)]_\alpha), cd(r_2, [G(r_2)]_\alpha), \frac{d}{2}\big(d(r_1, [G(r_2)]_\alpha) + d(r_2, [G(r_1)]_\alpha)\big)\right\};$$

b) $\max\{\mu cs, \mu ds\} < 1$.

Proof. By Theorem 3 part (i) and part (ii) there exists a CS $\{r_n\}$ of elements from Ω such that $r_{n+1} \in [G(r_n)]_\alpha, for\ all\ n \in \mathbb{N}$.

As the b-MS (Ω, d, s) is complete. Therefore, there exists $w \in \Omega$ such that

$$\lim_{n\to\infty} r_n = w. \tag{16}$$

Then, from (a) and equation (16), with notation $d(r_n, w) = \delta_n$, we can write

$$d(r_{n+1}, [G(w)]_\alpha) \leq H([G(r_n)]_\alpha, [G(w)]_\alpha) \leq \mu N_{c,d}(r_n, w)$$

$$= \mu \max\left\{\delta_n, cd(r_n, [G(r_n)]_\alpha), cd(w, [G(w)]_\alpha), \frac{d}{2}\big(d(r_n, [G(w)]_\alpha) + d(w, [G(r_n)]_\alpha)\big)\right\},$$

$$\leq \mu \max\left\{\delta_n, cd_n, cd(w, [G(w)]_\alpha), \tfrac{d}{2}\big(s(\delta_n + d(w, [G(w)]_\alpha)) + \delta_{n+1}\big)\right\}, \tag{17}$$

for all $n \in \mathbb{N}$.

We divided the discussion into two cases:

Case 1: $d(w, [G(w)]_\alpha) \leq \overline{\lim_{n\to\infty}} d(r_n, [G(w)]_\alpha)$.

Case 2: $d(w,[G(w)]_\alpha) > \overline{\lim_{n\to\infty}} d\,(r_n,[G(w)]_\alpha)$.

In this Case 1, there exists a subsequence $\{r_{n_k}\}_{k\in\mathbb{N}}$ of $\{r_n\}_{n\in\mathbb{N}}$ having the property that

$\lim_{k\to\infty} d(\,r_{n_{k+1}},[G(w)]_\alpha) \geq d(w,[G(w)]_\alpha)$, so for every $\varepsilon > 0$ *there exists* $k_\varepsilon \in \mathbb{N}$ such that

$$d(w,[G(w)]_\alpha) - \varepsilon \leq d\left(r_{n_{k+1}},[G(w)]_\alpha\right), for\ all\ k \in \mathbb{N} \text{ and } k \geq k_\varepsilon.$$

Thus, using (17) we get

$$d(w,[G(w)]_\alpha) - \varepsilon \leq \mu \max\left\{\delta_{n_k}, cd_{n_k}, cd(w,[G(w)]_\alpha), \tfrac{d}{2}\left(s(\delta_{n_k} + d(w,[G(w)]_\alpha)\right) + \delta_{n_{k+1}}))\right\},$$
$$for\ all\ k \in \mathbb{N} \text{ and } k \geq k_\varepsilon. \tag{18}$$

By, applying $\lim_{k\to\infty} in$ (18) we get

$$d(w,[G(w)]_\alpha) - \varepsilon \leq \ \mu \max\left\{cd(w,[G(w)]_\alpha), \frac{ds}{2} d(w,[G(w)]_\alpha)\right\},$$

$$= d(w,[G(w)]_\alpha) \max\left\{\ \mu c, \tfrac{\mu ds}{2}\right\}, \forall \varepsilon > 0, \text{ so}$$

$$d(w,[G(w)]_\alpha) \leq d(w,[G(w)]_\alpha)\ max\left\{\ \mu c, \tfrac{\mu ds}{2}\right\}. \tag{19}$$

Since,

$max\left\{\ \mu c, \tfrac{\mu ds}{2}\right\} < 1$, and using (19) we conclude that

$d(w,[G(w)]_\alpha) = 0, i.e.,\ w \in [G(w)]_\alpha$. Hence G has a FP.

In this case 2, there exists $n_0 \in \mathbb{N}$ such that

$$d(r_n,[G(w)]_\alpha) \leq \ d(w,[G(w)]_\alpha), for\ all\ n \in \mathbb{N}, n \geq n_0. \tag{20}$$

Since,

$$d(w,[G(w)]_\alpha) \leq (s(\delta_{n+1} + d\left(d(r_n,[G(w)]_\alpha)\right), \text{ i.e.,}$$

$$\tfrac{d(w,[G(w)]_\alpha)}{s} - \delta_{n+1} \leq d(r_{n+1},[G(w)]_\alpha),$$

using (17) and (18), we get

$$\frac{d(w,[G(w)]_\alpha)}{s} - \delta_{n+1} \leq \ d(r_{n+1},[G(w)]_\alpha),$$

$$d(r_{n+1},[G(w)]_\alpha) \leq \mu \max\left\{\delta_n, cd_n, cd(w,[G(w)]_\alpha), \frac{d}{2}(\delta_n + d(r_n,[G(w)]_\alpha) + \delta_{n+1})\right\},$$

$$\leq \mu \max\left\{\delta_n, cd_n, cd(w,[G(w)]_\alpha), \frac{d}{2}(\delta_n + d(w,[G(w)]_\alpha) + \delta_{n+1})\right\}, \tag{21}$$

$for\ all\ n \in \mathbb{N}, n \geq n_0$.

By applying $\lim_{n\to\infty}$ in (21) we get

$$d(w,[G(w)]_\alpha) \leq \mu \max\left\{cd(w,[G(w)]_\alpha), \frac{d}{2}d(w,[G(w)]_\alpha)\right\},$$

$$= \mu s\ max\left\{c, \frac{d}{2}\right\} d(w,[G(w)]_\alpha). \tag{22}$$

As we know that

$$\mu s\ max\left\{c, \frac{d}{2}\right\} < 1,$$

using (22) we find that $d(w,[G(w)]_\alpha) = 0$, i.e., $w \in [G(w)]_\alpha$. Hence G has a FP.

Application

Theorem 8. Consider that (Ω, d, s) be a complete b-MS. A multi-valued mapping $A: \Omega \to CB(\Omega)$ has an FP provided that it satisfied the following two conditions.

a) There exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that
$H(A(r_1), A(r_2)) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega$,
where

$$N_{c,d}(r_1,r_2) = max\left\{d(r_1,r_2), cd(r_1,A(r_1)), cd(r_2,A(r_2)), \frac{d}{2}\left(d(r_1,A(r_2)) + d(r_2,A(r_1))\right)\right\},$$

b) $\max\{\mu cs, \mu ds\} < 1$.

Proof. Consider an arbitrary mapping $S: \Omega \longrightarrow (0,1]$ and a fuzzy mapping $G: \Omega \to F(\Omega)$ defined by

$$G(x)(t) = \begin{cases} Sx & t \in Ax \\ 0 & t \notin Ax. \end{cases}$$

Then for $x \in \Omega$,

$$[Gx]_\alpha = \{t: G(x)(t) \geq \alpha\} = Ax.$$

Therefore, Theorem 7 can be applied to obtain the required fixed point.

Corollary 4. Consider that (Ω, d, s) be a complete MS. A fuzzy mapping $G: \Omega \to F(\Omega)$ has an FP provided that it satisfies the following two conditions.

a) There exist $c, d \in [0,1]$ and $\mu \in [0,1)$ such that
$H([G(r_1)]_\alpha), [G(r_2)]_\alpha) \leq \mu N_{c,d}(r_1, r_2)\ for\ all\ r_1, r_2 \in \Omega$, where $\alpha \in (0,1]$,
$[G(r_1)]_\alpha\ and\ [G(r_2)]_\alpha$ are closed and bounded subsets of Ω and

$$N_{c,d}(r_1,r_2) = max\Big\{d(r_1,r_2), cd(r_1,[G(r_1)]_\alpha), cd(r_2,[G(r_2)]_\alpha), \frac{d}{2}\big(d(r_1,[G(r_2)]_\alpha) + d(r_2,[G(r_1)]_\alpha)\big)\Big\};$$

b) $\max\{\mu cs, \mu ds\} < 1$.

4. Conclusions

In mathematics and the branches of science, such as engineering, game theory, optimization, economic theories, and numerous other disciplines, FP-theory plays a major role. Imprecision has a major influence on a human's life. The implementation of FS theory to deal with ambiguity in logistical considerations has been remarkably effective and well-liked. Outstanding advancements in science and technology have been accomplished by fuzzy techniques, and this has made a huge difference in how problems in daily life are solved. In this study, fuzzy fixed points are investigated using contemporary fuzzy approaches in the context of a complete b-MS. Various generalized Nadler's type contractions are used for this purpose. We have generalized a lot of helpful and applicable findings from the body of literature in this way.

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Conflicts of interest

The authors declare the there are no conflicts of interest regarding the publications.

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